You don't need to provide a repeat of derivations that exist in most textbooks, but I figure that you did this as practice, which is good. Because the mathematical steps are so similar for the three cases, it makes sense to try to re-use solutions as I mentioned in class.

Lecture 1 ECE 513

Darwin Quiroz

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1 Laplace's equation in 2-D

* assumptions were made that the potential that is to be derived will be independent in the z direction due to being very long.

Contents

1.1 Assume only V_l is nonzero. Find the potential inside of the pipe and call it $\Phi_l(x,y)$.

* This first part will go through most of the derivation thoroughly and will be recited so to not repeat certain solutions or simplifications for further sections (1.2,1.3,and 1.4 specifically)

1.1.1 Solving Laplace's equation for a general solution

Assuming only that V_l is non-zero $V_t = V_r = V_b = 0$ Then a formula for the potential must be derived by solving Laplace's equation for a general formula and then to apply boundary conditions to specify an exact solution. Looking back at the figure the following Boundary Conditions can be stated:

Boundary Conditions:

(i) $\Phi_l(x,0) = 0$

(ii) $\Phi_l(x_0,y) = 0$

(iii) $\Phi_l(x,y_0) = 0$

(iv) $\Phi_l(0, y) = V_l$

*Where $x\epsilon(0, x_0)$ and $y\epsilon(0, y_0)$
beginning at Laplace's equation:

$$\nabla^2\Phi_l(x, y) = 0$$

$$\frac{\partial^2\Phi_l(x, y)}{\partial x^2} + \frac{\partial^2\Phi_l(x, y)}{\partial y^2} = 0 \tag{1}$$

Following along from Griffith's we use Separation of Variables methods:

$$\Phi_l(x, y) = X(x) * Y(y) \tag{2}$$

substituting this back into the equation and dividing equation 1. by Φ_l

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0 \tag{3}$$

Because both equations are independent of one another, we can declare that both equations equate to constants, and can be solved independently, converting the PDE into two ODE's:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2$$

This ODE is has a simple solution of being the sum of two real exponential functions

$$X(x) = Ae^{kx} + Be^{-kx} \tag{4}$$

Solving the second equation:

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2$$

*The reason for these specific constants is that the sum of the two equations must be 0, and therefore must use the same constant k. We also choose the negative constant to be assigned to the Y function because of the boundary conditions that are applied to Y will simplify the equation as a whole when doing so.

Returning back to solving the function Y, this ODE is the one used to solve simple harmonic motion (SHM) and has the solution when using Euler's identity

Use \sin instead of sin in equations. Similar for other trig functions and log.

$$Y(y) = Csin(ky) + Dcos(ky) \tag{5}$$

and substituting equations 4 and 5 back into equation 2

$$\therefore \Phi_l(x, y) = X(x) * Y(y) = (Ae^{kx} + Be^{-kx})(Csin(ky) + Dcos(ky)) \tag{6}$$

Now to simplify and get a more specific solution, boundary conditions must be applied to the equation.

1.1.2 Applying Boundary Conditions

I. $\Phi_l(x, 0) = 0$

$$\Phi_l(x, 0) = (Ae^{kx} + Be^{-kx})(Csin(0) + Dcos(0)) = 0$$

$$\Phi_l(x, 0) = (Ae^{kx} + Be^{-kx})(D) = 0$$

For a non-trivial solution, then $D = 0$
Then by absorbing the constant C into A and B as Where $A' = AC$ and $B' = BC$ the equation simplifies as:

$$\Phi_l(x, y) = (A'e^{kx} + B'e^{-kx})(sin(ky)) \tag{7}$$

II. $\Phi_l(x_0, y) = 0$

$$
\begin{aligned}
(A'e^{kx_0} + B'e^{-kx_0})(sin(ky)) &= 0 \\
(A'e^{kx_0} + B'e^{-kx_0}) &= 0 \\
B'e^{-kx_0} &= -A'e^{kx_0} \\
B' &= -A'e^{2kx_0} \qquad (8)
\end{aligned}
$$

Then substituting equation 8 into equation Φ_l:

$$
\begin{aligned}
\Phi_l(x,y) &= (A'e^{kx} - A'e^{2kx_0}e^{-kx})(sin(ky)) \\
&= sin(ky)A'e^{kx_0}(e^{kx}e^{-kx_0} - e^{-kx}e^{kx_0}) \\
&= sin(ky)A'e^{kx_0}(e^{k(x-x_0)} - e^{-k(x-x_0)})
\end{aligned}
$$

*Then by using the hyperbolic trig functions $sinh(kx) = \frac{1}{2}(e^{k(x-x_0)} - e^{-k(x-x_0)})$ and letting $F = 2A'e^{kx_0}$

$$
\therefore \Phi_l(x,y) = Fsinh(k(x-x_0))sin(ky) \qquad (9)
$$

III.$\Phi_l(x, y_0) = 0$

$$
\begin{aligned}
Fsinh(k(x-x_0))sin(ky_0) &= 0 \\
sin(ky_0) &= 0 \\
ky_0 &= n\pi \\
\therefore k &= \frac{n\pi}{y_0} \qquad (10)
\end{aligned}
$$

* Where n = 1,2,3,...
Then a set of solutions exist depending on the value n, but a general solution can be made by making a linear combination of the n solutions such that:

$$
\Phi_l(x,y) = \sum_{n=1,2,3}^{\infty} F_n sinh(\frac{n\pi}{y_0}(x-x_0))sin(\frac{n\pi}{y_0}y) \qquad (11)
$$

IV. $\Phi_l(0,y) = V_l$

$$
\sum_{n=1,2,3}^{\infty} F_n sinh(\frac{n\pi}{y_0}(-x_0))sin(\frac{n\pi}{y_0}y) = V_l \qquad (12)
$$

To solve for the coefficient, we must use Fourier's trick and integrating from its limits 0 to y_0

$$
\sum_{m=1,2,3,...}^{\infty} \int_0^{y_0} \Phi_l(0,y)sin(\frac{m\pi}{y_0}y)dy = \sum_{m=1,2,3,...}^{\infty} \int_0^{y_0} V_l sin(\frac{m\pi}{y_0}y)dy
$$

$$
\sum_{m=1,2,3,...}^{\infty} \sum_{n=1,2,3}^{\infty} F_n sinh(\frac{n\pi}{y_0}(-x_0)) \int_0^{y_0} sin(\frac{n\pi}{y_0}y)sin(\frac{m\pi}{y_0}y)dy = \sum_{m=1,2,3,...}^{\infty} V_l \int_0^{y_0} sin(\frac{m\pi}{y_0}y)dy \qquad (13)
$$

The first integral can be solved almost immediately by understanding that the sine functions are all orthogonal to each other. This means that the inner product of any two sine function is:

$$
\int_0^{y_0} sin(\frac{ny}{y_0})sin(\frac{my}{y_0})dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{y_0}{2} & \text{if } n = m \end{cases}
$$

Using this definition, all the terms in the series become 0, except for when n=m, reducing the left hand side of equation 13 to:

$$
F_n(\frac{y_0}{2})sinh(\frac{n\pi}{y_0}(-x_0)) = \int_0^{y_0} V_l sin(\frac{n\pi}{y_0}y)dy \qquad (14)
$$

The right side of equation 13 will have the solution:

$$
\begin{aligned}
\int_0^{y_0} V_l sin(\frac{n\pi}{y_0}y)dy &= \frac{y_0 V_l}{n\pi}(-cos(\frac{n\pi}{y_0}y)|_0^{y_0} \\
&= \frac{y_0 V_l}{n\pi}(1 - cos(n\pi)) \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2y_0 V_l}{n\pi} & \text{if } n \text{ is odd} \end{cases}
\end{aligned} \tag{15}
$$

substituting the result from equation 15 to equations 14:

$$
\begin{aligned}
F_n(\frac{y_0}{2})sinh(\frac{n\pi}{y_0}(-x_0)) &= \frac{2y_0 V_l}{n\pi} \\
F_n sinh(\frac{n\pi}{y_0}(-x_0)) &= \frac{4V_l}{n\pi} \\
\therefore F_n &= \frac{-4V_l}{n\pi sinh(\frac{n\pi}{y_0}(x_0))}
\end{aligned} \tag{16}
$$

because of equation 15, the coefficient will only be non-zero when n is an odd integer. Substituting these results back into our original equation for the potential function (equation 11):

$$
\Phi_l(x,y) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{-4V_l}{n\pi sinh(\frac{n\pi}{y_0}(x_0))} sinh(\frac{n\pi}{y_0}(x - x_0)) sin(\frac{n\pi}{y_0}y)
$$

$$
\therefore \Phi_l(x,y) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4V_l sinh(\frac{n\pi}{y_0}(x_0 - x)) sin(\frac{n\pi}{y_0}y)}{n\pi sinh(\frac{n\pi}{y_0}(x_0))}
$$

1.2 Assume only V_b is nonzero. Find the potential inside of the pipe and call it $\Phi_b(x,y)$.

With only V_b being non-zero, then $V_l = V_r = V_t = 0$
The following boundary conditions can then be defined by referring to the figure:
Boundary Conditions:

(i) $\Phi_b(0,y) = 0$

(ii) $\Phi_b(x,y_0) = 0$

(iii) $\Phi_b(x_0,y) = 0$

(iv) $\Phi_b(x,0) = V_b$

*Where $x\epsilon(0,x_0)$ and $y\epsilon(0,y_0)$

1.2.1 Solving Laplace's equation for a general solution

To derive a potential function, we must begin similar to section 1.1 and solve Laplace's equation.

$$
\nabla^2\Phi_b(x,y) = 0
$$

This is to be solved by using separation of variables (to see this steps again, refer to equations 1 - 5. But will yield a different equation for the general potential solutions for Laplace's equation.
It should be noted that Φ can have many different solution forms in its general case (prior to applying boundary conditions). Any 2nd order ODE will have 2 solutions possible (both being equivalent to one another).

X and Y each have two solution forms (this can be seen by applying euler's identity or converting the exponentials to their hyperbolic trig forms). Along with this generalization. The ODE's themselves are dependent on which one is assigned the negative and positive constant, k^2.

This then gives X and Y each 4 alternate forms that can be used. The way to know which form is best suitable, is to analyze the boundary conditions that will be applied to the general function. For this particular case it seemed more beneficial to assign the negative constant ($-k^2$ to the X function ODE and the positive k^2 to the Y function ODE (for the first case the constants were assigned in reverse) and will then change the potential function to the following form :

$$\Phi_b(x, y) = X(x) * Y(y) = (Asin(kx) + Bcos(kx))(Ce^{ky} + De^{-ky})$$

1.2.2 Boundary Conditions

We begin with applying the first B.C.:
I. $\Phi_b(0, y) = 0$

$$(Asin(0) + Bcos(0))(Ce^{ky} + De^{-ky}) = 0$$

As seen from section 1.1 (equations 7), by applying the boundary condition, so to not obtain a trivial solution, B = 0. And by absorbing the A coefficient into the C and D terms (will not add the apostrophe for simplicity and aesthetic reasons), the potential function simplifies too

$$\Phi_b(x, y) = sin(kx)(Ce^{ky} + De^{-ky}) \tag{17}$$

II. $\Phi_b(x, y_0) = 0$

Starting from equation 17

$$sin(kx)(Ce^{ky_0} + De^{-ky_0}) = 0$$

With only the Y function changing, to not have a trivial solution the Y function must be the reason for the 0 to occur:

$$Ce^{ky_0} + De^{-ky_0} = 0$$
$$De^{-ky_0} = -Ce^{ky_0}$$
$$D = -Ce^{2ky_0} \tag{18}$$

Substituting equation 18 into equation 17:

$$\Phi_b(x, y) = sin(kx)(Ce^{ky} - Ce^{2ky_0}e^{-ky})$$

Which can be simplified to:

$$\Phi_b(x, y) = sin(kx)Ce^{ky_0}(e^{k(y-y_0} - e^{-k(y-y_0})$$

$$\Phi_b(x, y) = sin(kx)2Ce^{ky_0}sinh(k(y - y_0))$$

Let $E = 2Ce^{ky_0}$ simplifies the equation to:

$$\Phi_b(x, y) = Esin(kx)sinh(k(y - y_0)) \tag{19}$$

III. $\Phi_b(x_0, y) = 0$

$$Esin(kx_0)sinh(k(y - y_0)) = 0$$

So to not have a trivial solution, this function must be 0 due to the $sin(kx_0)$ term

$$sin(kx_0) = 0$$
$$kx_0 = n\pi, \text{ where n=1,2,3,...}$$
$$\therefore k = \frac{n\pi}{x_0} \tag{20}$$

Then substituting this back to equation 19, and referring to the same argument made in the first section when solving for the potential Φ_l, that a linear combination would create a general solution.

$$\Phi_b(x,y) = \sum_{n=1,2,3...}^{\infty} E_n sin(\frac{n\pi}{x_0}x) sinh(\frac{n\pi}{x_0}(y-y_0)) \tag{21}$$

IV. $\Phi_b(x,0) = V_b$

$$\sum_{n=1,2,3...}^{\infty} E_n sin(\frac{n\pi}{x_0}x) sinh(\frac{n\pi}{x_0}(-y_0)) = V_b$$

With the only term left unknown being the coefficient, we must again apply Fourier's trick to get a solution. This time it will be done by multiplying the sin(x) term:

$$\sum_{m=1,2,3,...}^{\infty} \int_0^{x_0} \Phi_b(x,0) sin(\frac{m\pi}{x_0}x)dx = \sum_{m=1,2,3,...}^{\infty} \int_0^{x_0} V_b sin(\frac{m\pi}{x_0}x)dx$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E_n sinh(\frac{n\pi}{x_0}(-y_0)) \int_0^{x_0} sin(\frac{n\pi}{x_0}x) sin(\frac{m\pi}{x_0}x)dx = \sum_{m=1}^{\infty} \int_0^{x_0} V_b sin(\frac{m\pi}{x_0}x)dx \tag{22}$$

As stated from the previous section, the left hand integral will only give a non-zero value when n=m, dropping the series terms on both sides.
The right hand integral converges to a finite value only for certain n terms (in this case it would be exactly the same as part 1, where n can only be odd for non-zero terms).

And referring back to the solutions obtained for the integrals the following simplification of equation 22 is made:

$$E_n sinh(\frac{n\pi}{x_0}(-y_0))(\frac{x_0}{2}) = V_b \frac{2x_0}{n\pi}$$

$$E_n = \frac{4V_b}{n\pi sinh(\frac{n\pi}{x_0}(-y_0))} \qquad \text{,only when n is odd} \tag{23}$$

substituting this back into the original equation prior to applying the boundary condition (equation 21)

$$\Phi_b(x,y) = \sum_{n=1,3,5...}^{\infty} (\frac{4V_b}{n\pi sinh(\frac{n\pi}{x_0}(-y_0))}) sin(\frac{n\pi}{x_0}x) sinh(\frac{n\pi}{x_0}(y-y_0))$$

$$\therefore \Phi_b(x,y) = \sum_{n=1,3,5...}^{\infty} \frac{4V_b sin(\frac{n\pi}{x_0}x) sinh(\frac{n\pi}{x_0}(y_0-y))}{n\pi sinh(\frac{n\pi}{x_0}(y_0))} \tag{24}$$

1.3 Assume only V_t is nonzero. Find the potential inside of the pipe and call it $\Phi_t(x,y)$

The four B.C.'s are given with each section focusing on one non-zero term. For this case, they are defined as such:

(i) $\Phi_t(0,y) = 0$

(ii) $\Phi_t(x,0) = 0$

(iii) $\Phi_t(x_0,y) = 0$

(iv) $\Phi_t(x,y_0) = V_t$

*Where $x\epsilon(0,x_0)$ and $y\epsilon(0,y_0)$

1.3.1 Solving Laplace's equation for a general solution

We again solve Laplace's equation by using separation of variables.For this time seeing as how we have 2 boundary conditions that would make the function go to 0 based on the X terms. We use the same form that was used for section 1.1.

$$\Phi_t(x, y) = X(x) * Y(y) = (Asin(kx) + Bcos(kx))(Ce^{ky} + De^{-ky})$$

Now to apply Boundary Conditions to get a more specific solution that defines all the variables in the equation

1.3.2 Boundary Conditions

We begin with applying the first B.C.:

I. $\Phi_t(0, y) = 0$
As seen from previous sections (1.1.2 and 1.2.2):

$$(Asin(0) + Bcos(0))(Ce^{ky} + De^{-ky}) = 0$$

This can only hold true when B=0 (for non-trivial solution). Then the equation simplifies to:

$$\Phi_t(x, y) = sin(kx)(Ce^{ky} + De^{-ky}) \tag{25}$$

II. $\Phi_t(x, 0) = 0$

$$sin(kx)(Ce^0 + De^0) = 0$$

Having only the y component change, for a non-trivial solution the Y function must be the cause of the 0.

$$\begin{aligned} C + D &= 0 \\ \therefore D &= -C \end{aligned} \tag{26}$$

Substituting this relation back into Φ_t:

$$\Phi_t(x, y) = sin(kx)C(e^{ky} - e^{-ky})$$

$$\therefore \Phi_t(x, y) = 2Csin(kx)sinh(ky) \tag{27}$$

III. $\Phi_t(x_0, y) = 0$ Now applying the trig function term to be the cause of the 0 only when $x = x_0$, will yield the same result as the previous sections defining k.

$$sin(kx_0) = 0$$

$$k = \frac{n\pi}{x_0} \text{ , where n = 1,2,3,...}$$

This then, means many solutions exist depending on the value n. But a general solution can be defined as a linear combination of infinite terms.

$$\Phi_t(x, 0) = \sum_{n=1,2,3...}^{\infty} 2C_n sin(\frac{n\pi}{x_0}x)sinh(\frac{n\pi}{x_0}y) \tag{28}$$

IV. $\Phi_t(x, y_0) = V_t$

$$\sum_{n=1}^{\infty} 2C_n sin(\frac{n\pi}{x_0}x)sinh(\frac{n\pi}{x_0}y_0) = V_t$$

We again solve for the coefficient by using Fourier's trick:

$$\sum_{m=1,2,3,...}^{\infty} \int_0^{x_0} \Phi_t(x, y_0) sin(\frac{m\pi}{x_0}x)dx = \sum_{m=1,2,3,...}^{\infty} \int_0^{x_0} V_t sin(\frac{m\pi}{x_0}x)dx$$

Substituting the current function of Φ_t

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 2C_n sinh(\frac{n\pi}{x_0}y_0) \int_0^{x_0} sin(\frac{n\pi}{x_0}x) sin(\frac{m\pi}{x_0}x)dx = \sum_{m=1}^{\infty} \int_0^{x_0} V_t sin(\frac{m\pi}{x_0}x)dx$$

Then simplifications can be made by referring back to how the integrals will only yield a non-zero value when n=m and when n is an odd integer. The solutions to the integrals are known and discussed in sections 1.1.2 and 1.2.2. We can jump directly to the simplification then

$$2C_n sinh(\frac{n\pi}{x_0}y_0)(\frac{x_0}{2}) = \frac{2x_0V_t}{n\pi}$$

Then by moving everything to one side we obtain the value of the coefficient as:

$$C_n = \frac{2V_t}{n\pi sinh(\frac{n\pi}{x_0}y_0)} \qquad \text{for when n is odd}$$

Then the equation has all unknowns defined and the Potential Function can be defined as:

$$\Phi_t(x, y) = \sum_{n=1,3,5...}^{\infty} \frac{4V_t sin(\frac{n\pi}{x_0}x) sinh(\frac{n\pi}{x_0}y)}{n\pi sinh(\frac{n\pi}{x_0}y_0)}$$

1.4 Assume only V_r is nonzero. Find the potential inside of the pipe and call it $\Phi_r(x, y)$

The four B.C.'s are given with each section focusing on one non-zero term. For this case, they are defined as such:

(i) $\Phi_r(x, 0) = 0$

(ii) $\Phi_r(0, x) = 0$

(iii) $\Phi_r(x, y_0) = 0$

(iv) $\Phi_r(x_0, y) = V_r$

*Where $x\epsilon(0, x_0)$ and $y\epsilon(0, y_0)$

1.4.1 Solving Laplace's equation for a general solution

Now having dealt with Laplace's equation from previous sections. We will once again find a solution using seperation of variables. The matter of question is which form to use? This will be decided by how the boundary conditions for this case involve setting the function to 0 twice when only the y variable is changed. As seen in previous sections these boundary conditions are easily applied to trig functions. Thus the Y(y) will take the form of trig functions, (which requires setting the ODE's as such):

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2$$

Which will have the following solution.

$$X(x) = Ae^{kx} + Be^{-kx}$$

And setting the Y function to be with the negative constant:

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2$$

Which will yield the following solution:

$$Y(y) = Csin(ky) + Dcos(ky)$$

Then substituting these values into the potential function:

$$\Phi_r(x,y) = (Ae^{kx} + Be^{-kx})(Csin(ky) + Dcos(ky))$$

1.4.2 Boundary Conditions

We begin with applying the first B.C.:
I. $\Phi_r(x,0) = 0$

$$(Ae^{kx} + Be^{-kx})(Csin(0) + Dcos(0)) = 0$$

As seen from previous results this can only hold true if D = 0 Then the function converts by absorbing C into A and B

$$\Phi_r(x,y) = sin(ky)(Ae^{kx} + Be^{-kx})$$

II.$\Phi_r(0,x) = 0$

$$sin(ky)(Ae^0 + Be^0) = 0$$

Because only the X function is changing for a non-trivial solution then (as seen from section 1.3.2 using a similar boundary condition.

$$A = -B$$

This then causes the function to simplify as:

$$\Phi_r(x,y) = 2Asin(ky)sinh(kx)$$

III.$\Phi_r(x,y_0) = 0$

$$2Asin(ky_0)sinh(kx) = 0$$

As seen previously this can only be true because of the Y function

$$sin(ky_0) = 0$$

$$\therefore k = \frac{n\pi}{y_0}$$

substituting this value back into the potential function, and referring back to how a linear combination can create a general solution:

$$\Phi_r(x,y) = \sum_{n=1}^{\infty} 2A_n sin(\frac{n\pi}{y_0}y)sinh(\frac{n\pi}{y_0}x)$$

IV.$\Phi_r(x_0,y) = V_r$

$$\sum_{n=1}^{\infty} 2A_n sin(\frac{n\pi}{y_0}y)sinh(\frac{n\pi}{y_0}x_0) = V_r$$

We find a solution for the coefficient by applying Fourier's trick to the equation as such:

$$\sum_{m=1,2,3,...}^{\infty} \int_0^{x_0} \Phi_r(x_0,y) sin(\frac{m\pi}{x_0}x)dx = \sum_{m=1,2,3,...}^{\infty} \int_0^{x_0} V_r sin(\frac{m\pi}{x_0}x)dx$$

Substituting the current function of Φ_r

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 2A_n sinh(\frac{n\pi}{y_0}x_0) \int_0^{y_0} sin(\frac{n\pi}{y_0}y) sin(\frac{m\pi}{y_0}y)dy = \sum_{m=1}^{\infty} \int_0^{y_0} V_r sin(\frac{m\pi}{y_0}y)dy$$

Where the integrals are again going to only give a non-zero value under certain conditions:

- n = m
- n is an odd integer

Then by completing these requirements and referring back to the simple solutions for these integrals we can see:

$$2A_n sinh(\frac{n\pi}{y_0}x_0)(\frac{y_0}{2}) = V_r(\frac{2y_0}{n\pi})$$

$$A_n = \frac{V_r 2}{n\pi sinh(\frac{n\pi}{y_0}x_0)}$$

Then by substituting this coefficient back into the original series function

$$\Phi_r(x,y) = \sum_{n=1,3,5,...}^{\infty} 2\frac{V_r 2}{n\pi sinh(\frac{n\pi}{y_0}x_0)} sin(\frac{n\pi}{y_0}y) sinh(\frac{n\pi}{y_0}x)$$

$$\therefore \Phi_r(x,y) = \sum_{n=1,3,5,...}^{\infty} \frac{4V_r sin(\frac{n\pi}{y_0}y) sinh(\frac{n\pi}{y_0}x)}{n\pi sinh(\frac{n\pi}{y_0}x_0)}$$

1.5 Justify the claim

Claim: The general solution for this problem is

$$\Phi(x,y) = \Phi_l(x,y) + \Phi_b(x,y) + \Phi_t(x,y) + \Phi_r(x,y)$$

Solution
The potential function derives from the notion that the E-field is a conservative vector field. This means that the line integral from a reference point to a test point produces a unique scalar regardless of the path. This then proves (through vector calculus manipulations) that the potential function of a charge distribution or individual charge is a unique scalar quantity. The energy at a certain position can be comprised as the sum of the component energies (for this case the sides), due to its interaction with each charge distribution. Therefore because the E-field is conservative, the potential has a uniqueness scalar property that allows for the total potential of the system to be simplified as the sum of the individual components that make up this system.

1.6 Can Gauss' law be used in any way at any location to check your analytic answers derived above? If yes, provide details of your check.

Using the relationship the electric field has with the potential. The Electric field can be computed in different ways but most beneficial ways are by using:

- Gauss Law $\vec{E} \cdot \vec{da} = \frac{Q_{encl}}{\epsilon_0}$

Answer is simply superposition the composite diagram is the sum of the individual diagrams and from superposition it follows that one can add the mathematical solutions for the individual solutions to get the composite solution.

- Vector potential $\vec{E} = -\nabla\Phi$

Gauss Law can be used to compute the potential near the surface of any of the borders. This would ultimately yield the following result:

$$|E| = \frac{\sigma}{\epsilon_0}$$

Then by evaluating the potential function at a specific point (essentially making the potential only a function of x or y but not both anymore).

$$\Phi(x, y)|_{x=0}$$

We can then take the gradiant of this potential and obtain the E-field at the surface due to this border. This then can be compared with the E-field formula obtained through Gauss Law (which was obtained through an approximation that we were near the border). When only looking at the perpendicular component to the border.

$$E_x = \vec{E} \cdot \hat{x}$$

$$-\nabla\Phi(x, y)|_{x=0} = \frac{\sigma}{\epsilon_0}$$

We can then obtain the charge configuration which would for this case only depend on the y component. Then the E field can be compared from taking the whole gradient of the field vs that of the magnitude obtained through Gauss law.