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ECE 513:

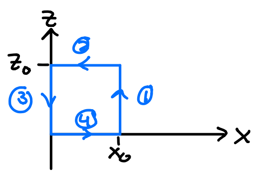
Faradays Law states:

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t}(\Phi_B)$$

Given an electric field:

$$\vec{E} = E_{ox} \cos(K_z z - \omega t)\hat{x}$$

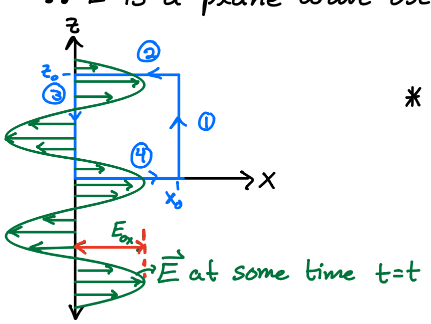
For rectangle 'e' given, let the path of integration in Faraday's Law be defined as:



E-field is a wave, given by the function satisfying the wave equation:

* Any function as $f(\vec{K} \cdot \vec{r} - \omega t)\hat{n}$ satisfies the wave-equation. Where $\vec{K}$ is direction of propagation of the wave, and $\hat{n}$ is the polarization vector (plane of vibration).

$\therefore$ $\vec{E}$ is a plane wave oscillating along x-axis & propogating along z-axis



* $\vec{E} \perp \hat{y}$ & paths ① & ③ are $\perp$ to $\vec{E}$, while paths ② & ④ are $\parallel$ to $\vec{E}$

as E propogates, $|\vec{E}| \in [-E_{ox}, E_{ox}]$

at $z=0$ & $z=z_o$

$$\vec{E}(0,t) = E_{ox}\cos(\omega t)\hat{x} \qquad * \cos(-\omega t) = \cos(\omega t)$$

$$E(z_o,t) = E_{ox}\cos(z_o K_z - \omega t)\hat{x}$$

Fadays Law left hand side becomes:

$$\oint \vec{E}\cdot d\vec{l} = \int_2 E(z_o,t)dx + \int_4 E(o,t)dx$$ * where $E(z,t)$ is independent of x

$$= -X_o E(z_o,t) + X_o E(o,t)$$

$$= X_o \left(E_{ox}\cos(\omega t) - E_{ox}\cos(\omega t - K_z z_o)\right)$$ * using Euler's formula:

$$= X_o E_{ox}\left(\tfrac{1}{2}\right)\left[e^{j\omega t} + e^{-j\omega t} - e^{jK_z z_o}e^{-j\omega t} - e^{-jK_z z_o}e^{j\omega t}\right]$$ $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$

$$= X_o E_{ox}\left(\tfrac{1}{2}\right)\left[e^{j\omega t}(1 - e^{-jK_z z_o}) + e^{-j\omega t}(1 - e^{jK_z z_o})\right]$$

$$= X_o E_{ox}\left(\tfrac{1}{2}\right)\left[e^{j\omega t}e^{-j\frac{K_z z_o}{2}}\left(e^{j\frac{K_z z_o}{2}} - e^{-j\frac{K_z z_o}{2}}\right) - e^{-j\omega t}e^{j\frac{K_z z_o}{2}}\left(e^{j\frac{K_z z_o}{2}} - e^{-j\frac{K_z z_o}{2}}\right)\right]$$ * $\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

$$= jX_o E_{ox}\left[e^{j(\omega t - \frac{K_z z_o}{2})}\sin\left(\tfrac{K_z z_o}{2}\right) - e^{-j(\omega t - \frac{K_z z_o}{2})}\sin\left(\tfrac{K_z z_o}{2}\right)\right]\left(\tfrac{2j}{2j}\right)$$

$$= jX_o E_{ox}\sin\left(\tfrac{K_z z_o}{2}\right)\left[e^{j(\omega t - \frac{K_z z_o}{2})} - e^{-j(\omega t - \frac{K_z z_o}{2})}\right]\frac{2j}{2j}$$

$$= -2X_o E_{ox}\sin\left(\omega t - \tfrac{K_z z_o}{2}\right)\sin\left(\tfrac{K_z z_o}{2}\right)$$

$$\therefore \oint \vec{E}\cdot d\vec{l} = 2X_o E_{ox}\sin\left(\tfrac{K_z z_o}{2} - \omega t\right)\sin\left(\tfrac{K_z z_o}{2}\right)$$

6.1.2) Continuing w/ Faradays Law in Differential form

$$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$$

where $\vec{\nabla}\times\vec{E}$ can be computed in cartesian coordinates:

where $\vec{E} = E_x\hat{x} + 0\hat{y} + 0\hat{z}$ & $\frac{\partial E}{\partial y} = \frac{\partial E}{\partial x} = 0$ * $E(z,t)$ has no dependence on x & y

then:

$$\vec{\nabla}\times\vec{E} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E(z,t) & 0 & 0 \end{pmatrix} = 0\hat{x} + \frac{\partial E}{\partial z}\hat{y} + 0\hat{z}$$

$$\frac{\partial E}{\partial z} = \frac{\partial}{\partial z}\left(E_{ox}\cos(z K_z - \omega t)\right)$$

$$= -K_z E_{ox}\sin(zK_z - \omega t)$$

$$\therefore \vec{B} = \cancel{-}\int\left(\cancel{-}K_z E_{ox}\sin(zK_z - \omega t)\hat{y}\right)dt$$

$$\vec{B} = K_z E_{ox}\frac{\cos(K_z z - \omega t)}{\omega}$$

$$\boxed{\therefore \vec{B} = \frac{K_z}{\omega} E_{ox} \cos(K_z z - \omega t)\,\hat{y}}$$

Faraday's Integral Law can then be proved using previous results:

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\Phi_B$$

where $\Phi_B = \int \vec{B}\cdot d\vec{a}$ of the same rectangular loop 'e':

$$\int_S \vec{B}\cdot d\vec{a}$$

$$= \int_0^{z_0}\int_0^{x_0} B\,dx\,dz\,(\hat{y}\cdot\hat{y})$$

$$= \int_0^{z_0}\int_0^{x_0} \frac{K_z}{\omega} E_{ox}\cos(K_z z - \omega t)\,dx\,dz$$

$$= \frac{K_z}{\omega} E_{ox}\, x_0 \int_0^{z_0} \cos(K_z z - \omega t)\,dz$$

$$= \frac{\cancel{K_z}}{\omega} E_{ox}\, x_0 \left.\frac{\sin(K_z z - \omega t)}{\cancel{K_z}}\right|_0^{z_0}$$

$$\Phi_B = \frac{E_{ox} x_0}{\omega}\left[\sin(K_z z_0 - \omega t) \cancel{-} \sin(\cancel{-}\omega t)\right]$$

$$\frac{\partial \Phi_B}{\partial t} = \frac{\partial}{\partial t}\left[\frac{x_0 E_{ox}}{\omega}\left(\sin(K_z z_0 - \omega t) + \sin(\omega t)\right)\right.$$

$$-\frac{\partial \Phi_B}{\partial t} = \cancel{-}(\cancel{-}\omega)\frac{E_{ox} x_0}{\cancel{\omega}}\left[\cos(K_z z_0 - \omega t) + \cos(\omega t)\right]$$

$$\therefore\ -\frac{\partial \Phi_B}{\partial t} = x_0 E_{ox}\left[\cos(K_z z_0 - \omega t) + \cos(\omega t)\right]$$

which is equivalent to results obtained from path integral.

6.1.3) From Ampere's Law in Integral form when $\vec{J} = 0$

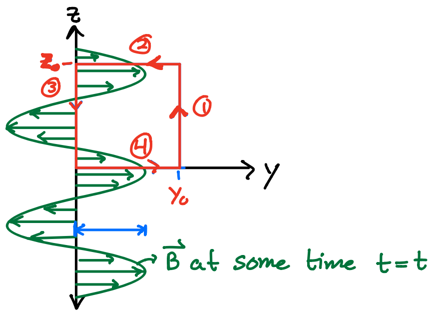
$$\oint \vec{B}\cdot d\vec{l} = \frac{1}{c^2}\frac{\partial \Phi_E}{\partial t}$$

From the $\vec{B}$ found from 6.1.2:

$$\vec{B} = \frac{K_z}{\omega} E_{ox}\cos(K_z z - \omega t)\,\hat{y}$$

we can see that $\frac{K_z}{\omega} = v$, where v is the velocity of the wave, and in free space represented by $v = c$ (speed of light)

Similar to E-field, $\vec{B}$ is a plane wave (from satisfying wave equation by having the argument $(\vec{k}\cdot\vec{r}-\omega t)$. $\vec{B}$ then also propagates in z-direction & $\vec{B}\perp\vec{E}$, so it oscillates in $\hat{y}$-direction. Then using similar arguments from 6.1.1 used for Faradays Law:



paths ① & ③ are $\perp$ to $\vec{B}$ (because $\vec{B}\perp\vec{k}$) & ② & ④ are $\parallel$ to $\vec{B}$

$$\oint \vec{B}\cdot d\vec{l} = \int_{②} B\,dy + \int_{4} B\,dy$$

similar Faradays Law, $\vec{B}(z,t)$, at time t=t, during path ②: $z=0$ & ④: $z=z_0$

$$\int_0^{y_0} B(y_0,t)\,dy + \int_{y_0}^{0} B(0,t)\,dy$$

$$= y_0\left(\frac{1}{c}E_{ox}(\cos(k_z z_0-\omega t)-\frac{1}{c}E_{ox}\cos(\omega t)\right)$$

$$\oint \vec{B}\cdot d\vec{l} = \frac{y_0}{c}E_{ox}(\cos(k_z z_0-\omega t)-\frac{1}{c}E_{ox}\cos(\omega t))$$

then Right-hand side:

$$\Phi_E = \int_S \vec{E}\cdot d\vec{a}$$

$$= \int_0^{z_0}\int_0^{y_0} E_{ox}\cos(z\,k_z-\omega t)\,dy\,dz\,(\hat{x}\cdot\hat{x})$$

$$= y_0 E_{ox}\int_0^{z_0}\cos(zk_z-\omega t)\,dz$$

$$= \frac{y_0 E_{ox}}{k_z}\sin(zk_z-\omega t)\Big|_0^{z_0}$$

$$= \frac{y_0 E_{ox}}{k_z}\left(\sin(z_0 k_z-\omega t)-\sin(-\omega t)\right)$$

$$\Phi_E = \frac{y_0 E_{ox}}{k_z}\left(\sin(z_0 k_z-\omega t)+\sin(\omega t)\right)$$

$$\frac{\partial(\Phi_E)}{\partial t} = Y_0 E_{0x} \overset{c}{\cancel{\frac{\omega}{K_z}}} \left(\cos(K_z - \omega t) + \cos(\omega t)\right)$$

$$\therefore \frac{1}{c^2}\frac{\partial \Phi_E}{\partial t} = \frac{Y_0}{c} E_{0x}\left(\cos(K_z - \omega t) + \cos(\omega t)\right)$$ * results are equivalent

$$\therefore \oint \vec{B}\cdot d\vec{l} = \frac{1}{c^2}\frac{\partial \Phi_E}{\partial t}$$