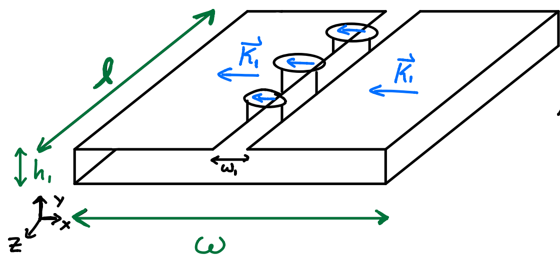
Flux Linkage

Given the following diagram:

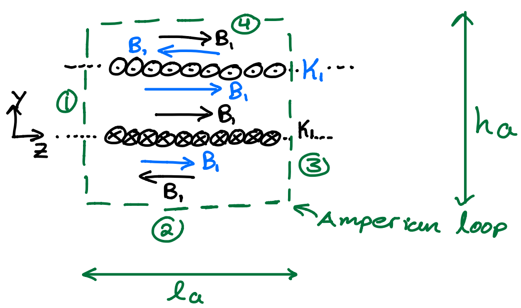


Assume the conductor is thin enough that current is similar to flowing on a sheet

9.1.1.

Assuming $\omega \gg h_1$ & $l \gg h_1$:

Lookin at y-z cross-section (currents coming into and out of page) at



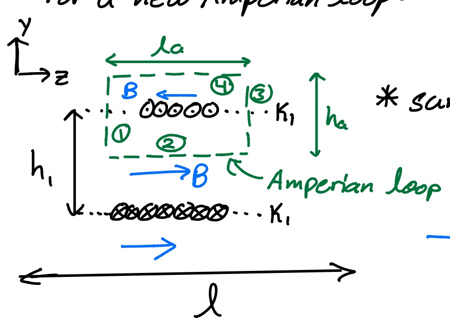
* because $l \gg h_1$ we can approximate that magnetic field is only in z-axis direction.

$$\oint B \cdot dl = \mu_0 I_{encl}$$

When outside of duct, $I_{encl} = Kl - Kl = 0$

$\therefore B = 0$

For a new Amperian loop:



* same argument can be used to say $\vec{B}$ is only in z-axis

$\rightarrow$ = B-field lines due to upper wire

Using Ampere's Law:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$$

$$\int_{①} \cancel{\vec{B}\cdot d\vec{l}_1} + \int_{②} \vec{B}\cdot d\vec{l}_2 + \int_{③} \cancel{\vec{B}\cdot d\vec{l}_3} + \int_{④} \vec{B}\cdot d\vec{l}_4 = \mu_0 I_1$$

* B is only in z-direction & is then parallel to $d\vec{l}_2$ & $d\vec{l}_4$

$$\int B\,dl_2 + \int B\,dl_4 = \mu_0 l_a K_1$$

$$2B\cancel{l_a} = \mu_0 \cancel{l_a} K_1$$

$$B = \frac{\mu_0 K_1}{2}$$ * due to top sheet

* if consider loop going CCW, loop moves in direction of B for both loops, w/ both having constant B field due to $l >> h_1$

$$\therefore \quad \vec{B}_{top} = \begin{cases} -\frac{\mu_0 K_1}{2}\hat{z} & \text{above sheet} \\ \frac{\mu_0 K_1}{2}\hat{z} & \text{below sheet} \end{cases}$$

* if $h_1 << w$, can ignore fields produced to left and right in comparison

Similar arguments can be made for a 2^{nd} loop enclosing Bottom sheet such that

$$\vec{B}_{top} = \begin{cases} \frac{\mu_0 K_1}{2}\hat{z} & \text{above sheet} \\ -\frac{\mu_0 K_1}{2}\hat{z} & \text{below sheet} \end{cases}$$

$$\vec{B}_{Tot} = \vec{B}_{top} + \vec{B}_{bot} = \begin{cases} \mu_0 K_1 \hat{z} & \text{* inside duct} \\ 0 & \text{* outside duct since B's cancel} \end{cases}$$

9.1.2

Because duct has relatively small enough thickness, internal magnetic field of conductor is 0. Leaving only an external magnetic field that exists inside the cross sectional area of duct $A_1 = h_1 w$

$$\therefore L_{int} \approx 0$$

EMF is given by the following

$$\mathcal{E}_1 = -\frac{\partial \Phi_m}{\partial t}, \text{ rewrite this as: } \mathcal{E} = -L_1 \frac{\partial I}{\partial t}$$

Recall $\Phi_m = L_1 I_1$

$$\mathcal{E}_1 = -\frac{\partial}{\partial t}(L_1 I_1)$$

$$\mathcal{E}_1 = -L_1 \frac{\partial I_1}{\partial t}$$

$$\Phi_m = \int \vec{B}\cdot d\vec{A}_1$$

$$= \int (\mu_0 K_1 \hat{z})(dA_1 \hat{z})$$

$$\Phi_m = \mu_0 K_1 \int dA_1 = \mu_0 K_1 h_1 \omega$$

Substituting:

$\mathcal{E} = -\frac{d}{dt}(\mu_0 K_1 h_1 \omega)$ * since μ_0, h_1 & ω are constants

① $\mathcal{E} = -\mu_0 h_1 \omega \frac{dK_1}{dt}$

$\mathcal{E} = -L_1 \frac{\partial I_1}{\partial t}$ * $I_1 = K_1 \ell$

② $\mathcal{E} = -L_1 \cdot \ell \frac{dK_1}{\partial t}$

Relating both eqn ① & ②:

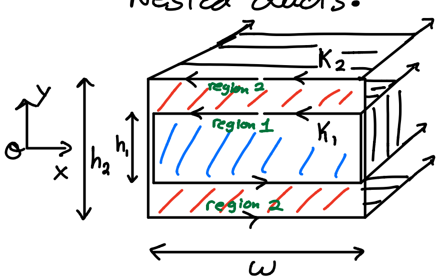
$$\cancel{-}\mu_0 h_1 \omega \cancel{\frac{dK_1}{dt}} = \cancel{-}L_1 \cdot \ell \cancel{\frac{dK_1}{\partial t}}$$

$$\therefore L_1 = \frac{\mu_0 h_1 \omega}{\ell}$$

$$\boxed{L_1 \equiv \frac{\mu_0 A_1}{\ell}}$$

9.1.3. Given the following nested ducts:

Nested ducts:



Assume $\omega_1 = \omega_2 = \omega$

$A_1 = h_1 \omega$ and $A_2 = h_2 \omega$

For region 1, $\vec{B}$ is composed of magnetic field produced by both ducts.

$$\vec{B} = \vec{B}_1 + \vec{B}_2$$

Because physical symmetry arguments are unchanged, Ampere's Law yield:

$$B_2 = \begin{cases} \mu_0 K_2 & \text{, inside duct 2 (in cross-section } A_2) \\ 0 & \text{, outside duct 2} \end{cases}$$

$\therefore$ inside region 1

$\vec{B} = \mu_0 \vec{K}_1 + \mu_0 \vec{K}_2 = \mu_0 (K_1 + K_2)\hat{z}$ *currents in same direction

Then flux for region 2:

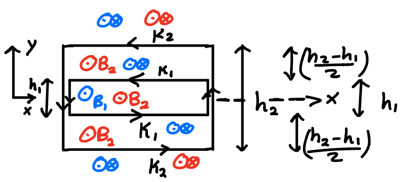
$$\Phi_m = \int \vec{B} \cdot d\vec{A}_1 \quad * \, d\vec{A}_1 \parallel \vec{B} \ \text{w/} \ \vec{B} \text{ being constant due to } l \gg h_1 \ \& \ \omega \gg h_1$$

$$= B \int dA_1 \quad * \, B = B_2 + B_1$$

$$\Phi_m = B A_1 = h_1 \omega \mu_0 (K_1 + K_2)$$

$$\therefore \ \mathcal{E}_1 = - h_1 \omega \mu_0 \frac{d}{dt}(K_1 + K_2)$$

For $\mathcal{E}_2$, must account for B-field in regions 1 & 2 (all of A_2):



B_1 & B_2 are in $+\hat{z}$ direction inside ducts

⊙⊗ means equal and opposite fields produced then cancel

$$\vec{B} = \begin{cases} \vec{B}_1 + \vec{B}_2 & \text{for } |y| < h_1/2 \\ \vec{B}_2 & |y| < h_2/2 \\ 0 & |y| > h_2/2 \ \text{ outside of ducts} \end{cases}$$

Flux can then be:

$$\Phi_m = \int \vec{B} \cdot dA_2$$

$$= \int_{w=0}^{w} \int_{-h_1/2}^{h_1/2} (B_1 + B_2) \cdot d\vec{A}_2 + \int_{0}^{w} \int_{-h_2/2}^{-h_1/2} \vec{B}_2 \cdot d\vec{A}_2 + \int_{0}^{w} \int_{h_1/2}^{h_2/2} \vec{B}_2 \cdot d\vec{A}_2$$

recall: B_1 & B_2 is constant inside due to $h_2 \ll \omega$ & $h_2 \ll l$

$$= (B_1 + B_2) h_1 \omega + B_2 (h_2 - h_1) \frac{\omega}{2} + B_2 (h_2 - h_1) \omega/2$$

$$= (B_1 + B_2) h_1 \omega + B_2 (h_2 - h_1) \omega$$

$$= B_1 A_1 + \cancel{B_2 A_1} + B_2 A_2 - \cancel{B_2 A_1}$$

$$\Phi_m = B_1 A_1 + B_2 A_2 = \mu_0 K_1 h_1 \omega + \mu_0 K_2 h_2 \omega$$

$$\Phi_m = \mu_0 \omega (K_1 h_1 + K_2 h_2)$$

$$\therefore \ \mathcal{E}_2 = -\mu_0 \omega \left[h_1 \frac{\partial}{\partial t} K_1 + h_2 \frac{\partial}{\partial t} K_2 \right]$$

$* \text{ if } K_1 + K_2 = K$

Thus total EMF is:

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$$

$$= -h_1 \omega \mu_0 \frac{d}{dt}(K_1 + K_2) \;\; -\mu_0 \omega \left[h_1 \frac{\partial}{\partial t} K_1 + h_2 \frac{\partial}{\partial t} K_2\right]$$

* $I_1 = K_1 \cdot l$, $I_2 = K_2 \cdot l$

$$\mathcal{E} = - A_1 \frac{\mu_0}{l} \frac{d}{dt} I - \frac{\mu_0 A_1}{l} \frac{d}{dt} I_1 - \frac{\mu_0 A_2}{l} \frac{d}{dt} I_2$$

$$\mathcal{E} = -\left\{\frac{\mu_0}{l} A_1 \frac{d}{dt}(I + I_1) + \frac{\mu_0 A_2}{l} \frac{d}{dt} I_2\right\} \qquad * I = I_1 + I_2$$

$$= - \left\{\frac{\mu_0}{l} A_1 \, 2 \frac{d}{dt}(I_2) + \frac{\mu_0}{l} A_1 \frac{d}{dt} I_2 + \frac{\mu_0 A_2}{l} \frac{d I_2}{dt}\right\}$$

$$\therefore \mathcal{E} = -\left\{2 \frac{\mu_0}{l} A_1 \frac{d I_1}{dt} + \frac{\mu_0}{l}(A_1 + A_2) \frac{d I_2}{dt}\right\}$$

* $\mathcal{E}_1 = -L_1 \frac{\partial I_1}{\partial t}$ & $\mathcal{E}_2 = -L_2 \frac{\partial I_2}{\partial t}$

then $\mathcal{E} = -\left(L_1 \frac{\partial I_1}{\partial t} + L_2 \frac{\partial I_2}{\partial t}\right)$ * at low frequencies $I_1 = I_2 = I$

Comparing equations:

$$\mathcal{E} = -(L_1 + L_2) \frac{\partial I}{\partial t} \qquad \boxed{* \; L = L_1 + L_2 = \frac{2\mu_0}{l} A_1 + \frac{\mu_0}{l}(A_1 + A_2) = \frac{3\mu_0}{l} A_1 + \frac{\mu_0}{l} A_2}$$

9.1.4) Since calculations were done previously in this manner we can use the results from Φ_m computed for region 1 (used to compute $\mathcal{E}_1$)

In region 1 (A_1 area)

$$B = B_1 + B_2$$

$$\Phi_{m_1} = \int \vec{B} \cdot d\vec{A}_1$$

$$= \int \vec{B}_1 A_1 + \int \vec{B}_2 \vec{A}_1$$

* B_1 & B_2 still remain the same formulas as: $B_i = \mu_0 K_i$ $(i = 1, 2)$ and independent of x or y

$$= \mu_0 K_1 A_1 + \mu_0 K_2 A_1$$

$$\Phi_{B_1} = \mu_0 A_1 (K_1 + K_2) = L_{ext} I$$

$$L_{ext} = \frac{\mu_0 A_1}{l} \frac{\cancel{(K_1 + K_2)}}{\cancel{K_{eff}}}$$ * K_2 & K_1 contribute

In region 2:

$$\Phi_{m_2} = \int_{\text{region 2}} B \cdot dA$$ * $\vec{B} = B_2$ since $B_1 = 0$ outside of A_1

Leading to similar result of:

$$\Phi_{m_2} = B_2 \int_{\substack{region \\ 2}} dA = \mu_0 K_2 (A_2 - A_1)$$

$$L_{int} = \frac{\Phi_{m_2}}{I} = \frac{\mu_0 \cancel{K_2} (A_2 - A_1)}{\ell \cancel{K_{eff}}}$$ * K_2 is the effective magnetic field producing flux

$$L_{int} = \frac{\mu_0 (A_2 - A_1)}{\ell}$$

and if $h_2 = h_1 + \delta h \Rightarrow A_2 \approx A_1$ as $dh \to 0$

$$L_{Tot} = L_{ext} + L_{int}$$

$$= \frac{\mu_0 A_1}{\ell} + \frac{\mu_0}{\ell}(A_2 - A_1)$$

$$L_{TOT} = \frac{\mu_0}{\ell} A_2$$

9.1.5.) Flux linkage accounts for the mutual inductance that occurs between 2 current elements that link one another. Although regions use different magnetic fields, each one contributes an induced magnetic field linking bothe regions. This is seen from the exampl that even though the "internal inductance region had a magnetic field that was only contributed by the outer, we do not consider how the internal loop is also linked to the outer and must be accounted for.