5.1 Boundary Value Derivation & Maxwell Equations

$\vec{\nabla}\cdot\vec{B}=0$

- originally looking back at the divergence of $\vec{E}$:

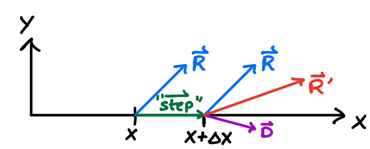
$$\vec{\nabla}\cdot\vec{E}=\rho/\epsilon_0$$

It was seen by only looking one-dimensionally that at a source point (some position in space w/ non-zero charge), the derivative of $\vec{E}$ or change in E along this axis was no longer a constant when approaching the source of charge, and a discontinuity is found at the source

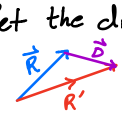
This does not occur for $\vec{B}$, because there are no "magnetic charges", this stems from how no magnetic monopoles have ever been found. Or sources of $\vec{B}$, that would cause a discontinuity. This then shows that $\vec{B}$-Fields are only producud by moving electric charges, and only felt by other charges. This is why B-Fields are a relativistic effect.

Visual:

Vector field tells us the vector in any point inside, at $(x,0)$ we have $\vec{R}$. Moving a "step", $\vec{s}$ from P, yields a new vector $\vec{R}'$ that can be described as a change to $\vec{R}$.

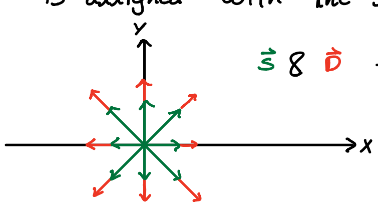


* Let the difference vector be $\vec{D}\equiv\vec{R}'-\vec{R}$



that is applied when taking a "step", $\vec{s}$

* if at any point the divergence is non-zero, then this this difference vector $\vec{D}$ is "alligned" with the "step" vector. Or $(\overrightarrow{\text{Step}})\cdot(\overrightarrow{\text{Difference}})\approx \text{Divergence}_{\text{Ave}}$



$\vec{s}$ & $\vec{D}$ then $\vec{s}\cdot\vec{D}>0$ for "sources" $=+\text{div}$

$\vec{s}\cdot\vec{D}<0$ for "sinks" $=-\text{div}$

Comparing to the $\vec{B}$-Field, if $\vec{\nabla}\cdot\vec{B}\neq 0$, some source must exist for $\vec{B}$-Fields, but no such entity has ever been found.

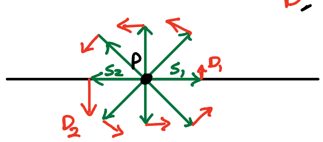
$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Ampere's Law describes where and how magnetic fields are produced. The curl of a vector field is a representation of the "amount of rotation" occurring at every point along the vector field. Unlike $\vec{\nabla} \times \vec{E}$, $\vec{B}$ fields form closed loops. The right hand side of the equation explains what causes this field. Where $\vec{J}$ is the volume current density $(\vec{J} = \frac{I_{encl}}{A_\perp})$ or current enclosed by some cross sectional area. Relating both sides says that B-fields rotate around currents or moving charges.

Visual:

By doing the same procedure as before: at $P = (x, 0)$ a vector $\vec{R}$ is given from a vector field. Moving a "step", $\vec{S}$ from point P generates a new vector $\vec{R}'$, that is different than $\vec{R}$ by some vector $\vec{D}$, making $\vec{D} = \vec{R}' - \vec{R}$. Then the average curl can be represented by $\vec{S} \times \vec{D}$ from point P. $Curl_{Ave}$ = amount of "perpendicularness" between $\vec{S}$ & $\vec{D}$

$\vec{D}$, $\vec{S}$



$\vec{S}_1 \times \vec{D}_1$ = small curl

$\vec{S}_2 \times \vec{D}_2$ = Large curl

This relationship shows that the "amount" of rotation of $\vec{B}$ is dependent on the strength of current density $\vec{J}$ at $\vec{P}$. Unlike $\vec{E}$, $\vec{B}$ has no sources or end goal ($\vec{E}$ field lines go from $\oplus \rightarrow \ominus$) for it's field lines. These field lines are able to form closed loops, and do so for current densities or moving charges.

$\vec{B} = -\vec{\nabla}\psi_m$

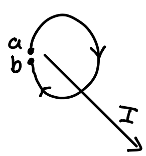
The case for $\vec{B}$ to be defined by some magnetic vector potential. Then $\vec{B}$ is a conservative vector field. Then, this can not hold true for inside the wire, since Ampere's Law defines $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$

But if $\vec{B} = -\vec{\nabla}\psi_m$

$$\vec{\nabla}\times(\vec{\nabla}\psi_m) = 0 \neq \mu_0 J$$

So no closed loops can be done around a current

Using Ampere's integral form of a path around a straight wire from a to b



$$\int_a^b \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$$

$$-\int_a^b \vec{\nabla}\psi_m \cdot d\vec{l} = \mu_0 I_{encl}$$

* Fundamental theorem of gradients say:

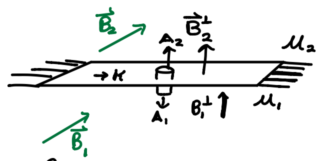
$$\int_a^b \vec{\nabla}U \cdot d\vec{l} = U(b) - U(a)$$

$$\therefore\ \psi_m(b) - \psi_m(a) = -\mu_0 I$$

If b & a are different points, then $\Delta\psi_m$ is non-zero or that the scalar potential cannot be single valued.

Boundary Conditions of $\vec{B}$:

Consider a $\vec{B}$-field going through a Boundary where $\mu_1 = \mu_2$. Using Gauss Law



* where $\vec{K}$ is a surface along a thin Boundary

Referencing the differential form: $\vec{\nabla}\cdot\vec{B} = 0$

then integral form: $\oint \vec{B}\cdot d\vec{A} = 0$

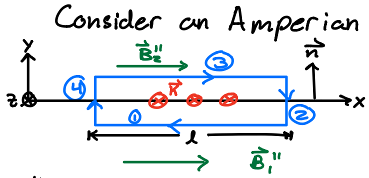
$$-\int_{bot} B_1^\perp dA_1 + \int_{top} B_2^\perp dA_2 = 0$$

$$-B_1^\perp A + B_2^\perp A = 0 \quad \text{* same Gaussian cylinder Areas}$$

$$\therefore \quad B_1^\perp = B_2^\perp$$

This shows how the perpendicular component is unaffected by the Boundary and is continuous. This is due to how no "sources" of B-fields can exist, and become a sudden influence that could increase the B-field spontaneously.

Consider an Amperian loop:



$\vec{K}$ is the surface current along the boundary

* when close enough to surface, the $\vec{B}$-fields will approximate to be perfectly parallel to boundary or $\perp$ to $\hat{y}$.

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$$

$$\int_1 \vec{B}\cdot d\vec{l}_1 + \int_2 \vec{B}\cdot d\vec{l}_2 + \int_3 \vec{B}\cdot d\vec{l}_3 + \int_4 \vec{B}\cdot d\vec{l}_4 = \mu_0 \int \vec{K}\cdot d\vec{l}_\perp$$

for B_1 & B_2 to be parallel to Boundary must require for lengths ② & ④ to be $\ll 1$

& $\int B\cdot dl_2 = \int B\cdot dl_4 = 0$

$$\therefore -\int_1 B_1'' dl_1'' + \int_3 B_2'' dl_3 = \mu_0 K l$$

$$B_2'' - B_1'' = \mu_0 K$$

* This Boundary condition arises from Amperes Law showing the discontinuity that exists for a B-field due to a surface Currents.

Ampere's Law solidifies this idea, by showing that the rotation of the $\vec{B}$-field is dependent on the direction of the current.

This implies that the B-field only curls around moving charges, or $\vec{\nabla} \times \vec{B} = 0$ for directions $\perp$ to $\vec{K}$. Then $B^{\parallel}$ is always $\vec{K} \times \hat{n}$ and is effected across the boundary due to the currents producing a magnetic field.