Darwin Quiroz:
ECE 513:

6.1

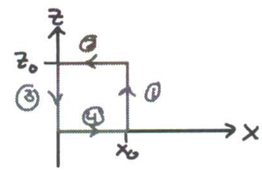
Faradays Law states:

$$\oint \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}(\Phi_B)$$

Given an electric field:

$$\vec{E} = E_{ox}\cos(K_z z - \omega t)\hat{x}$$

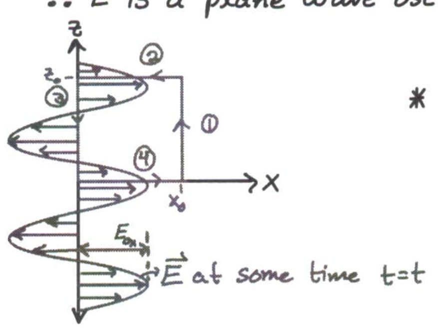
For rectangle 'e' given, let the path of integration in Faraday's Law be defined as:



E-field is a wave, given by the function satisfying the wave equation:

* Any function as $f(\vec{K}\cdot\vec{r} - \omega t)\hat{n}$ satisfies the wave-equation. Where $\vec{K}$ is direction of propagation of the wave, and $\hat{n}$ is the polarization vector (plane of vibration).

$\therefore$ $\vec{E}$ is a plane wave oscillating along x-axis & propogating along z-axis



* $\vec{E} \perp \hat{y}$ & paths ① & ③ are $\perp$ to $\vec{E}$, while paths ② & ④ are $\parallel$ to $\vec{E}$ ✓

as E propogates. $|\vec{E}| \in [-E_{ox}, E_{ox}]$ ✓

at $z=0$ & $z=z_o$ ✓

$\vec{E}(0,t) = E_{ox}\cos(\omega t)\hat{x}$ * $\cos(-\omega t) = \cos(\omega t)$

$E(z_o,t) = E_{ox}\cos(z_o K_z - \omega t)\hat{x}$ ✓

Fadays Law left hand side becomes:

$\oint \vec{E}\cdot d\vec{l} = \int_2 E(z_0,t)\,dx + \int_4 E(0,t)\,dx$ * where $E(z,t)$ is independent of x

$= -X_0 E(z_0,t) + X_0 E(0,t)$

$= X_0\left(E_{ox}\cos(\omega t) - E_{ox}\cos(\omega t - K_z z_0)\right)$ * using Euler's formula:

$= X_0 E_{ox}\left(\frac{1}{2}\right)\left[e^{j\omega t} + e^{-j\omega t} - e^{jK_z z_0}e^{-j\omega t} - e^{-jK_z z_0}e^{j\omega t}\right]$ $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$

$= X_0 E_{ox}\left(\frac{1}{2}\right)\left[e^{j\omega t}(1 - e^{-jK_z z_0}) + e^{-j\omega t}(1 - e^{jK_z z_0})\right]$

$= X_0 E_{ox}\left(\frac{1}{2}\right)\left[e^{j\omega t}e^{-j\frac{K_z z_0}{2}}\left(e^{j\frac{K_z z_0}{2}} - e^{-j\frac{K_z z_0}{2}}\right) - e^{-j\omega t}e^{j\frac{K_z z_0}{2}}\left(e^{j\frac{K_z z_0}{2}} - e^{-j\frac{K_z z_0}{2}}\right)\right]$ * $\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

$= jX_0 E_{ox}\left[e^{j(\omega t - \frac{K_z z_0}{2})}\sin\left(\frac{K_z z_0}{2}\right) - e^{-j(\omega t - \frac{K_z z_0}{2})}\sin\left(\frac{K_z z_0}{2}\right)\right]\left(\frac{2j}{2j}\right)$

$= jX_0 E_{ox}\sin\left(\frac{K_z z_0}{2}\right)\left[e^{j(\omega t - \frac{K_z z_0}{2})} - e^{-j(\omega t - \frac{K_z z_0}{2})}\right]\frac{2j}{2j}$

$= -2X_0 E_{ox}\sin\left(\omega t - \frac{K_z z_0}{2}\right)\sin\left(\frac{K_z z_0}{2}\right)$

$\therefore \oint \vec{E}\cdot d\vec{l} = 2X_0 E_{ox}\sin\left(\frac{K_z z_0}{2} - \omega t\right)\sin\left(\frac{K_z z_0}{2}\right)$

6.1.2) Continuing w/ Faradays Law in Differential form

$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$

where $\vec{\nabla}\times\vec{E}$ can be computed in cartesian coordinates:

where $\vec{E} = E_x\hat{x} + 0\hat{y} + 0\hat{z}$ & $\frac{\partial E}{\partial y} = \frac{\partial E}{\partial x} = 0$ * $E(z,t)$ has no dependence on x & y

then:

$$\vec{\nabla}\times\vec{E} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E(z,t) & 0 & 0 \end{pmatrix} = 0\hat{x} + \frac{\partial E}{\partial z}\hat{y} + 0\hat{z}$$

$\frac{\partial E}{\partial z} = \frac{\partial}{\partial z}\left(E_{ox}\cos(z K_z - \omega t)\right)$

$= -K_z E_{ox}\sin(zK_z - \omega t)$

$\therefore \vec{B} = +\int\left(+K_z E_{ox}\sin(zK_z - \omega t)\hat{y}\right)dt$

$\vec{B} = K_z E_{ox}\frac{\cos(K_z z - \omega t)}{\omega}$

$$\boxed{\therefore \vec{B} = \frac{K_z}{\omega} E_{ox} \cos(K_z z - \omega t)\,\hat{y}}$$

Faraday's Integral Law can then be proved using previous results:

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\Phi_B$$

where $\Phi_B = \int \vec{B}\cdot d\vec{a}$ of the same rectangular loop 'e':

$$\int_S \vec{B}\cdot d\vec{a}$$

$$= \int_0^{z_0}\int_0^{x_0} B\,dx\,dz\,(\hat{y}\cdot\hat{y})$$

$$= \int_0^{z_0}\int_0^{x_0} \frac{K_z}{\omega} E_{ox} \cos(K_z z - \omega t)\,dx\,dz$$

$$= \frac{K_z}{\omega} E_{ox}\, x_0 \int_0^{z_0} \cos(K_z z - \omega t)\,dz$$

$$= \frac{\cancel{K_z}}{\omega} E_{ox}\, x_0 \left. \frac{\sin(K_z z - \omega t)}{\cancel{K_z}} \right|_0^{z_0}$$

$$\Phi_B = \frac{E_{ox} x_0}{\omega}\left[\sin(K_z z_0 - \omega t) + \sin(\cancel{-}\omega t)\right]$$

$$\frac{\partial \Phi_B}{\partial t} = \frac{\partial}{\partial t}\left[\frac{x_0 E_{ox}}{\omega}\left(\sin(K_z z_0 - \omega t) + \sin(\omega t)\right)\right.$$

$$-\frac{\partial \Phi_B}{\partial t} = \cancel{-}(\cancel{-}\omega)\frac{E_{ox} x_0}{\cancel{\omega}}\left[\cos(K_z z_0 - \omega t) + \cos(\omega t)\right]$$

$$\therefore\ -\frac{\partial \Phi_B}{\partial t} = x_0 E_{ox}\left[\cos(K_z z_0 - \omega t) + \cos(\omega t)\right].$$

which is equivalent to results obtained from path integral.

6.1.3) From Ampere's Law in Integral form when $\vec{J} = 0$

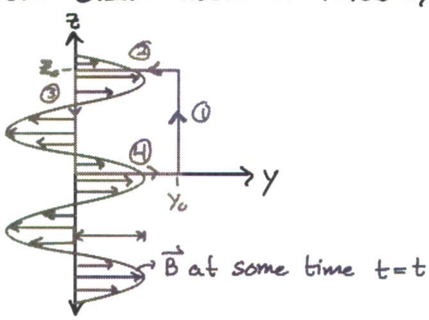
$$\oint \vec{B}\cdot d\vec{l} = \frac{1}{c^2}\frac{\partial \Phi_E}{\partial t}$$

From the $\vec{B}$ found from 6.1.2:

$$\vec{B} = \frac{K_z}{\omega} E_{ox} \cos(K_z z - \omega t)\,\hat{y}$$

we can see that $\frac{K_z}{\omega} = v$, where v is the velocity of the wave, and in free space represented by $v = c$ (speed of light)

Similar to E-field, $\vec{B}$ is a plane wave (from satisfying wave equation by having the argument $(\vec{k}\cdot\vec{r}-\omega t)$. $\vec{B}$ then also propagates in z-direction & $\vec{B}\perp\vec{E}$, so it oscillates in $\hat{y}$-direction. Then using similar arguments from 6.1.1 used for Faradays Law:



paths ① & ③ are $\perp$ to $\vec{B}$ (because $\vec{B}\perp\vec{k}$) & ② & ④ are $\parallel$ to $\vec{B}$

$$\oint \vec{B}\cdot d\vec{l} = \int_{2} B\,dy + \int_{4} B\,dy$$

similar Faradays Law, $\vec{B}(z,t)$, at time $t=t$, during path ②: $z=0$ & ④: $z=z_0$

$$\int_0^{y_0} B(y_0,t)\,dy + \int_{y_0}^{0} B(0,t)\,dy$$

$$= y_0\left(\frac{1}{c}E_{0x}(\cos(k_z z_0-\omega t)-\frac{1}{c}E_{0x}\cos(\omega t)\right)$$

$$\oint \vec{B}\cdot d\vec{l} = \frac{y_0}{c}E_{0x}\left(\cos(k_z z_0-\omega t)-\frac{1}{c}E_{0x}\cos(\omega t)\right)$$

then Right-hand side:

$$\Phi_E = \int_S \vec{E}\cdot d\vec{a}$$

$$= \int_0^{z_0}\int_0^{y_0} E_{0x}\cos(z k_z-\omega t)\,dy\,dz\,(\hat{x}\cdot\hat{x})$$

$$= y_0 E_{0x}\int_0^{z_0}\cos(z k_z-\omega t)\,dz$$

$$= \frac{y_0 E_{0x}}{k_z}\sin(z k_z-\omega t)\Big|_0^{z_0}$$

$$= \frac{y_0 E_{0x}}{k_z}\left(\sin(z_0 k_z-\omega t)-\sin(-\omega t)\right)$$

$$\Phi_E = \frac{y_0 E_{0x}}{k_z}\left(\sin(z_0 k_z-\omega t)+\sin(\omega t)\right)$$

$$\frac{\partial(\Phi_E)}{\partial t} = \gamma_0 E_{0x} \frac{\cancel{\omega}^{\,c}}{\cancel{K_z}} \left(\cos(K_z - \omega t) + \cos(\omega t)\right)$$

$$\therefore \frac{1}{c^2}\frac{\partial \Phi_E}{\partial t} = \frac{\gamma_0}{c} E_{0x}\left(\cos(K_z - \omega t) + \cos(\omega t)\right)$$ * results are equivalent

$$\therefore \oint \vec{B}\cdot d\vec{l} = \frac{1}{c^2}\frac{\partial \Phi_E}{\partial t}$$

✓

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ECE 513

HW 6.2

Given $\vec{E} = E_{ox}(x,t)\hat{x} + E_{oy}(x,t)\hat{y} + E_{oz}(x,t)\hat{z}$

& $\vec{B} = B_{ox}(x,t)\hat{x} + B_{oy}(x,t)\hat{y} + B_{oz}(x,t)\hat{z}$

6.2.1 Show E_y, E_z, B_y, B_z individually obey the wave equation:

$$\frac{\partial^2 f}{\partial u^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

where $f = E_y(x,t),\ E_z(x,t),\ B_y(x,t),\ B_z(x,t)$

where all possible values of f are only dependent of x & t

$$\therefore \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$$

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f(x,t)}{\partial t^2}$$

*for $f(x,t)$ to obey/satisfy the wave equation, then any function of the form:

$$f(x,t) = g(x \pm vt) = g(u_\pm) \quad * \text{ Let } u = x - vt$$

$$\frac{\partial f}{\partial x} = \frac{dg}{du_\pm}\frac{du_\pm}{dx} \quad * \ \frac{du_\pm}{dx} = 1$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{d}{dx}\left(\frac{dg(u_\pm)}{du_\pm}\right) \quad \text{*chain rule}$$

$$= \frac{d}{du_\pm}\left(\frac{dg}{du_\pm}\right)\cancel{\frac{du}{dx}}^{\,1}$$

$$= \frac{d^2 g}{du_\pm^2}$$

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial g}{\partial u_\pm}\frac{\partial u_\pm}{\partial t} \quad * \ \frac{\partial u_\pm}{\partial t} = \pm v$$

$$\frac{\partial f}{\partial t} = \pm v \frac{\partial g}{\partial u_\pm}$$

$$\frac{\partial^2 f}{\partial t^2} = \pm v \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial u_\pm}\right) \quad \text{*chain rule}$$

$$= \pm v \frac{\partial^2 g}{\partial u_\pm^2}\cancel{\frac{\partial u_\pm}{\partial t}}^{\,\pm v}$$

$$= v^2 \frac{\partial^2 g}{\partial u_\pm^2}$$, for E&M waves in free space $v = c$

Need to show how maxwell's eqns lead to wave eqns. Here you are assuming they do at the outset.

$$\therefore \frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 g}{\partial u^2} = \frac{1}{\cancel{c^2}}\cancel{c^2}\frac{\partial^2 g}{\partial u^2}$$

$$\frac{\partial^2 g}{\partial u^2} = \frac{\partial^2 g}{\partial u^2}$$

$\therefore$ If $f(x,t) = E_y(x,t), E_z(x,t), B_y(x,t), B_z(x,t)$ are some linear combination of the 2 general solutions:

$$f(x,t) = g(x-vt) + h(x+vt)$$

then $f(x,t)$ is also a solution and satisfies the wave equation.

6.2.2) Because $\vec{E}$ & $\vec{B}$ are waves, and the wave equation given has a velocity of c, then the waves are in free space. This would in turn require no charges or sources, for waves to to propagate at that velocity.

When no source currents or charges:

$$\rho = 0, \quad \vec{J} = 0$$

$$\therefore \vec{\nabla}\cdot\vec{E} = 0$$
$$\vec{\nabla}\cdot\vec{B} = 0$$

From 6.2.1, it is found that the y & z components of E & B satisfy the wave equation.

Because E_y, E_z, B_y, B_z all dependent on x and are waves, then they some linear combination of the general solution:

$$f(x,t) = g(x-vt) + h(x+vt)$$ *where x is the axis of propagation.

From Maxwell's Eqn

$$\vec{\nabla}\cdot\vec{E} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\frac{\partial E}{\partial x} = \frac{\partial B}{\partial x} = 0$$ which component?

time ~~dep~~ varying part.

Then because $\vec{E}$ & $\vec{B}$ propogate in x-direction: ~~X~~

$E_x = B_x = 0$ * $\vec{E}$ & $\vec{B}$ must be perpenelicular to direction of propagation

6.2.3) For waves to propagate at speed of light:

$\rho = 0, \ \vec{J} = 0$ * free space

& $\epsilon = \epsilon_0$ & $\mu = \mu_0$

$\nabla^2 \vec{E} = (\nabla^2 E_x \hat{x} + \nabla^2 E_y \hat{y} + \nabla^2 E_z \hat{z})$

* Laplace Operacts acts on each component:

$$\nabla^2 \vec{E} = \left(\frac{\partial^2 E_x}{\partial x^2} + \cancel{\frac{\partial^2 E_x}{\partial y^2}} + \cancel{\frac{\partial^2 E_x}{\partial z^2}}\right)\hat{x} + \left(\frac{\partial^2 E_y}{\partial x^2} + \cancel{\frac{\partial^2 E_y}{\partial y^2}} + \cancel{\frac{\partial^2 E_y}{\partial z^2}}\right)\hat{y} + \left(\frac{\partial^2 E_z}{\partial x^2} + \cancel{\frac{\partial^2 E_z}{\partial y^2}} + \cancel{\frac{\partial^2 E_z}{\partial z^2}}\right)\hat{z}$$

* where each B_x, B_y, B_z component is dependent on only x, t

$$\therefore \nabla^2 \vec{E} = \left(\frac{\partial^2 E_x}{\partial x^2}\hat{x} + \frac{\partial^2 E_y}{\partial x}\hat{y} + \frac{\partial^2 E_z}{\partial x^2}\hat{z}\right)$$

Similarly $\nabla^2 \vec{B}$ simplifies:

$$\nabla^2 \vec{B} = \left(\frac{\partial^2 B_x}{\partial x^2}\hat{x} + \frac{\partial^2 B_y}{\partial y}\hat{y} + \frac{\partial^2 B_z}{\partial z^2}\hat{z}\right)$$

* using arguments obtained from 6.2.2:

since $E_x = B_x = 0$ ~~X~~ $\cdot \frac{\partial E_x}{\partial x} = 0$ $\quad \frac{\partial E_x}{\partial t} = 0$ is all that can be shown to be required.

Then wave equation simplifies too:

$$\nabla^2 \vec{E} = \frac{\partial^2 E_y}{\partial x^2}\hat{y} + \frac{\partial^2 E_z}{\partial x^2}$$

$$\nabla^2 \vec{B} = \frac{\partial^2 B_y}{\partial x^2}\hat{y} + \frac{\partial^2 B_z}{\partial x^2}\hat{z}$$

Then comparing components:

① $\frac{\partial^2 E_y}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2}$

② $\frac{\partial^2 E_z}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$

③ $\frac{\partial^2 B_y}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2}$

④ $\frac{\partial^2 B_z}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2}$

* Can use results 1 to prove solutions again:

If $\vec{E}$ & $\vec{B}$ are some linear combination of general solution to wave equation then ①-④ satisfy the individual wave equations ✓

Darwin Quiroz HW 6.3

1) Given $\vec{E} = E_{ox}\cos(K_z z - \omega t + \delta_x)\hat{x} = E_0(z,t)\hat{x}$

Prove E satisfies:

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} = (\nabla^2 E_x \hat{x} + \cancel{\nabla^2 E_y}\,\hat{y} + \cancel{\nabla^2 E_z}\,\hat{z})$$ * because E is only dependent on z & t: $\frac{\partial^2 \vec{E}}{\partial x^2} = \frac{\partial^2 \vec{E}}{\partial y^2} = 0$

$$\therefore \nabla^2 \vec{E}_x = \cancel{\frac{\partial^2 E_x}{\partial x^2}}\hat{x} + \cancel{\frac{\partial^2 E_x}{\partial y^2}}\hat{x} + \frac{\partial^2 E_x}{\partial z^2}\hat{x} = \frac{\partial^2 E_x}{\partial z^2}\hat{x}$$

$$\frac{\partial^2}{\partial z^2}\vec{E} = \frac{\partial^2}{\partial z^2}(E_{ox}\cos(K_z z - \omega t + \delta_x)\hat{x})$$

$$= -E_{ox}K_z^2\cos(K_z z - \omega t + \delta_x)\hat{x}$$

$$= -K_z^2 \vec{E}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 E_{ox}\cos(K_z z - \omega t + \delta_x)\hat{x} = -\omega^2 \vec{E}$$

Substituting into wave equation:

$$-K_z^2 \vec{E} = \frac{1}{c^2}(-\omega^2)\vec{E}$$

$$\vec{E} = \cancel{\frac{1}{c^2}}\,\cancel{\frac{\omega^2}{K^2}}\,\vec{E}$$

$\therefore \vec{E} = \vec{E}$ * $\vec{E}$ satisfies wave equation

For $\vec{B} = B_{ox}\cos(K_z z - \omega t + \delta_x')\hat{x} + B_{oy}\cos(K_z z - \omega t + \delta_y')\hat{y}$

$$\nabla^2 \vec{B} = \nabla^2 B_x \hat{x} + \nabla^2 B_y \hat{y} + \nabla^2 B_z \hat{z}$$

* Laplace operator acts onto each B component:

$$= \left(\cancel{\frac{\partial^2 B_x}{\partial x^2}} + \cancel{\frac{\partial^2 B_x}{\partial y^2}} + \frac{\partial^2 B_x}{\partial z^2}\right)\hat{x} + \left(\cancel{\frac{\partial^2 B_y}{\partial x^2}} + \cancel{\frac{\partial^2 B_y}{\partial y^2}} + \frac{\partial^2 B_y}{\partial z^2}\right)\hat{y} + \cancel{\left(\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + \frac{\partial^2 B_z}{\partial z^2}\right)\hat{z}}$$

* $B_z = 0$, B_x & B_y are only dependent on x & t

$$= \frac{\partial^2 B_x}{\partial z^2}\hat{x} + \frac{\partial^2 B_y}{\partial z^2}\hat{y}$$ * $\frac{\partial^2}{\partial z^2}(\cos(Kz)) = -K^2\cos(Kz)$

$$= -K_z^2 B_{ox}\cos(K_z z - \omega t - \delta_x')\hat{x} - K_z^2 B_{oy}\cos(K_z z - \omega t - \delta_y')\hat{y}$$

$$\nabla^2 \vec{B} = -K_z^2(B_x\hat{x} + B_y\hat{y}) = -K_z^2\vec{B}$$

$$\frac{\partial^2 \vec{B}}{\partial t^2} = -\omega^2 B_{ox}\cos(K_z z - \omega t - \delta_x')\hat{x} - \omega^2 B_{oy}\cos(K_z z - \omega t - \delta_y')\hat{y}$$

$$= -\omega^2(B_x\hat{x} + B_y\hat{y}) = -\omega^2\vec{B}$$

Substituting into wave equation

$$-K_z^2 \vec{B} = \frac{1}{c^2}(-\omega^2 \vec{B})$$

$$\vec{B} = \cancel{\frac{1}{c^2}} \cancel{\frac{\omega^2}{K^2}} \vec{B} \quad * \ \frac{\omega}{K} = v \quad * \text{where } v = c \text{ in free space}$$

$$\vec{B} = \vec{B}$$

∴ Both functions $\vec{B}$ & $\vec{E}$ satisfy respective wave equations

From Faradays Law, the following relationship is known:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Where:

$$\vec{E} = Re\{\tilde{E}_{ox} e^{j(\vec{K}\cdot\vec{r} - \omega t)}\}\hat{x}$$

* The physical wave is the Real Part of a plane wave where $\tilde{E}_{ox} = E_{ox} e^{j\delta_x}$

$$\vec{\nabla} \times \vec{E} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{pmatrix} = \frac{\partial E_x}{\partial z}\hat{y} - \cancel{\frac{\partial E_x(z,t)}{\partial y}}$$

$$= \frac{\partial}{\partial z} Re\{\tilde{E}_{ox} e^{j(K\hat{r} - \omega t + \delta_x)}\}\hat{x}$$

$$\vec{\nabla} \times \tilde{\vec{E}} = jK_z E_{ox} e^{j(K_z z - \omega t)}\hat{x}$$

Similarly:

$$\vec{B} = Re\{\tilde{\vec{B}}\} = Re\{\tilde{B}_{oy} e^{j(K_z z - \omega t)}\} \quad * \text{ where } \tilde{\vec{B}} = B_{ox} e^{j\delta_x'}\hat{x} + B_{oy} e^{j\delta_y'}\hat{y}$$

$$-\frac{\partial \vec{B}}{\partial t} = -Re\left\{\frac{\partial \tilde{\vec{B}}}{\partial t}\right\}$$

$$\frac{\partial \tilde{\vec{B}}}{\partial t} = j\omega \tilde{\vec{B}}_o e^{j(Kz - \omega t)}$$

Substituting values:

$$\nabla \times \tilde{\vec{E}} = \frac{\partial \tilde{\vec{B}}}{\partial t}$$

$$\cancel{j} K_z E_{ox} \cancel{e^{j(K_z z - \omega t)}}\hat{y} = \cancel{j}\omega \tilde{\vec{B}}_o \cancel{e^{j(Kz - \omega t)}}$$

$$\tilde{\vec{B}} = \frac{K_z}{\omega} E_{ox}$$

Comparing components:

$$\tilde{B}_{ox} = 0, \quad \tilde{B}_{oy} = \frac{1}{c}\tilde{E}_{ox}$$

$\therefore\ \delta_x' = 0$ * could be any value but unimportant

$\delta_y' = \delta_x$

& $B_{oy} = \frac{1}{c} E_{ox}$

6.3.2) The relationships developed for 6.3.1 are required for Maxwells Equation to be satisfied. The following physical arguments can be made:

* E&M waves are transverse from

$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$ in free space

then $\frac{\partial E_z}{\partial z} = \frac{\partial B_z}{\partial z} = 0$

this explains why no z-component that depends on Z

* $\vec{E}$ & $\vec{B}$ are $\perp$ to one another. This explains why

$\tilde{\vec{B}}_{ox} = 0$ & δ_x' is unimportant

* From Faradays Law, $\vec{B}$ & $\vec{E}$ are in phase w/ one another

$\therefore\ \delta_x = \delta_y'$

* From Faraday's Law it shows that B_{oy} & E_{ox} are related as:

$$B_{oy} = \frac{1}{c} E_{ox} = \frac{K_z}{\omega} E_{ox}$$

3) If $\vec{E} = E_{oy}\cos(K_z z - \omega t + \delta_y)\hat{x}$

then if $\vec{E}$ still propogates in $\hat{x}$ direction, $B_{ox} = 0$ for $\vec{E} \perp \vec{B}$

From Faraday's Law:

$\therefore\ B_{oy} = \frac{1}{c} E_{oy} = \frac{K_z}{\omega} E_{oy}$

Both $\vec{B}$ & $\vec{E}$ must be in phase w/ one another, then:

$\delta_y' = \delta_y$

& δ_x' is still unimportant but most likely 0

6.3.4) Following From Faraday's Law, Proved in 6.3.1 using specific functions but can be generalized into 3-D:

If $\vec{\tilde{E}} = \vec{\tilde{E}}_0 e^{j(\vec{k}\cdot\vec{r}-\omega t)}$ * where $\tilde{E}_0 = \sum_{l=1}^{3} E_l e^{j\delta_l}$ where $l = 1,2,3 \rightarrow \hat{x}, \hat{y}, \hat{z}$

& $e^{j\delta_l}$ represent the phase of each complex vector

* Can use same arguments from 6.4 to prove for the general relationship:

$\vec{B} = \frac{1}{c}\hat{k}\times\vec{E}$

where the relations used for 6.3.1, 6.3.2, 6.3.3 are all valid and must be true for $\vec{B} = \frac{1}{c}\hat{k}\times\vec{E}$ to also hold true.

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6.4 Complex form

6.4.1) Given:

$$\vec{E} \equiv Re\left[\tilde{\vec{E}}_0 e^{-j(\omega t - \vec{K}\cdot\vec{r})}\right]$$

$$\vec{B} \equiv Re\left[\tilde{\vec{B}}_0 e^{-j(\omega t - \vec{K}\cdot\vec{r})}\right]$$

$$\vec{K} \equiv K_x \hat{x} + K_y \hat{y} + K_z \hat{z}$$

$$\vec{r} = r_x \hat{x} + r_y \hat{y} + r_z \hat{z}$$

* where $\tilde{\vec{E}}_0$ & $\tilde{\vec{B}}_0$ are vectors w/ complex constants

Faradays law in differential form states:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Left hand side:

Looking at x-component:

$$(\vec{\nabla}\times\vec{E})_x = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x}$$

$$= \left[\frac{\partial}{\partial y}\left(Re\left[\tilde{E}_{0z} e^{-j(\omega t - \vec{K}\cdot\vec{r})}\right]\right) - \frac{\partial}{\partial z}\left(Re\left[\tilde{E}_{0y} e^{-j(\omega t - \vec{K}\cdot\vec{r})}\right]\right)\right]\hat{x}$$

$$\frac{\partial}{\partial z}\left[Re\left[\quad\right]\right] = \frac{\partial}{\partial z}\cos(\omega t - K\cdot\vec{r} + \phi)$$
$$= +K_z \sin(\omega t - K\cdot\vec{r} \ldots$$
$$= Re\left[\tilde{E}_0 \frac{\partial}{\partial z} e^{j(\omega t - K\cdot\vec{r} \ldots}\right.$$

* due to derivatives, being a linear, time-invariant operator to signals (all E&M waves are signals), then derivative can be computed first before taking the real part or after (LTI property)

Not sure this argument is needed.

$$= \left\{Re\left[\tilde{E}_{0z} e^{-j(\omega t)}\frac{\partial}{\partial y}\left(e^{j(K_x x + K_y y + K_z z)}\right)\right] - Re\left[\tilde{E}_{0y} e^{-j(\omega t)}\frac{\partial}{\partial z}\left(e^{j(K_x x + K_y y + K_z z)}\right)\right]\right\}\hat{x}$$

$$= Re\left[\tilde{E}_{0z} e^{-j(\omega t)}(jK_y)e^{j(\vec{K}\cdot\vec{r})} - \tilde{E}_{0y} e^{-j(\omega t)}(jK_z)e^{j(\vec{K}\cdot\vec{r})}\right]\hat{x}$$

$$(\vec{\nabla}\times\vec{E})_x = Re\left[(jK_y E_{0z} - jK_z \tilde{E}_{0y})\, e^{j(\vec{K}\cdot\vec{r} - \omega t)}\right]\hat{x}$$ * where $j = e^{j\pi/2}$ (only adds a phase)

From this we can see a trend that

$$\frac{\partial}{\partial \alpha_i}\left(e^{jK_\alpha \alpha}\right) = jK_\alpha e^{jK_\alpha \alpha}$$ * where $\alpha = \hat{x}, \hat{y}, \hat{z}$

$\alpha = x, y, z$ → not a vector

∴ curl of $\tilde{\vec{E}}$ only acts on exponential term, and the derivatives only multiply the E_i component by jK_i. This trend will yield:

$\vec{\nabla} \times \vec{E} = Re\left[j(\vec{K} \times \tilde{\vec{E}})\right] = \vec{K} \times \vec{E} = |\vec{K}|(\hat{K} \times \vec{E})$

Right Hand side of Faradays Law:

$-\frac{\partial \vec{B}}{\partial t} = -Re\left[\frac{\partial}{\partial t}\tilde{\vec{B}}_0 e^{-j\omega t} e^{j\vec{K}\cdot\vec{r}}\right]$ * using same LTI argument for derivative

$= -Re\left[\tilde{\vec{B}}(-j\omega)e^{-j(\omega t - \vec{K}\cdot\vec{r})}\right]$

$\frac{\partial \vec{B}}{\partial t} = \omega \vec{B}$

$\therefore \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\Rightarrow \quad \vec{K} \times \vec{E} = \omega \vec{B}$

$\therefore \vec{B} = \frac{|\vec{K}|}{\omega}(\hat{K} \times \vec{E})$ * in free space

$\vec{B} = \frac{1}{c}(\hat{K} \times \vec{E})$ ✓

* if not only looking at Real Part, but also complex:

$(\vec{\nabla} \times \tilde{\vec{E}}) = j\vec{K} \times \tilde{\vec{E}}$

$\frac{\partial \tilde{\vec{B}}}{\partial t} = j\omega\tilde{\vec{B}}$

then relationship holds:

$\tilde{\vec{B}} = \frac{1}{c}\hat{K} \times \tilde{\vec{E}}$ ✓

Where the physical wave, represented by taking the Real part of these complex vectors. But through eulers identity it is simple to see that the only difference between the Imag & Re part of $\tilde{\vec{E}}$ is the replacement of the cos with a sine. Then both still obey Maxwells equations & thus so will $\tilde{\vec{E}}$ & $\tilde{\vec{B}}$

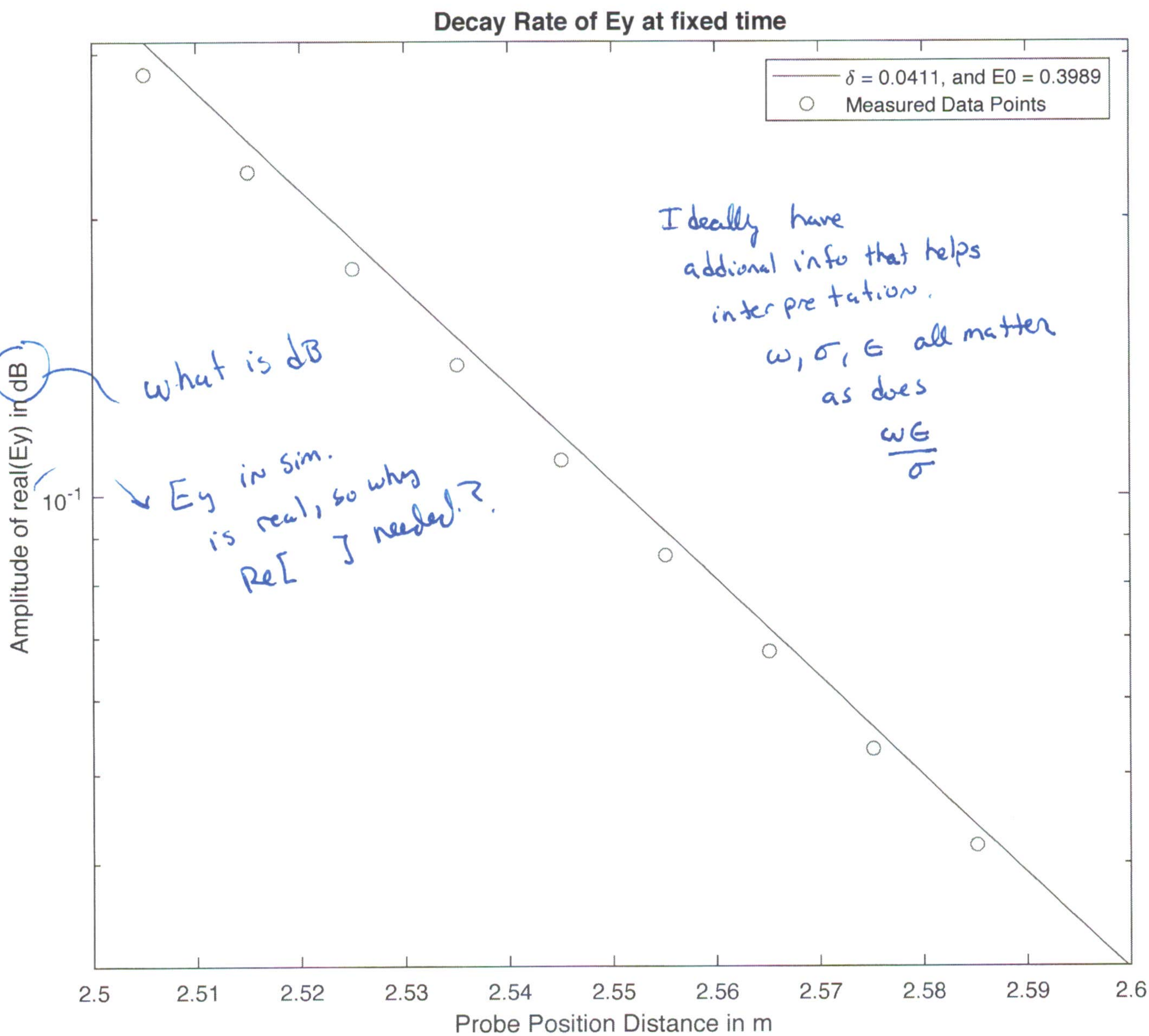
$\tilde{\mathbb{B}}_0 = \mathbb{B}_0^r + j\mathbb{B}_0^i$

$\tilde{\mathbb{E}}_0 = \mathbb{E}_0^r + j\mathbb{E}_0^i$

$\tilde{\mathbb{B}} = \frac{\hat{K}}{c} \times \tilde{\mathbb{E}}$ equate real + imag parts →

$\mathbb{B}_0^r = \frac{\hat{K}}{c} \times \mathbb{E}_0^r \; , \; \mathbb{B}_0^i = \frac{\hat{K}}{c} \times \mathbb{E}_0^i$

but $\mathbb{E}$ + $\mathbb{B}$ are in phase, so both eqns say the same thing.

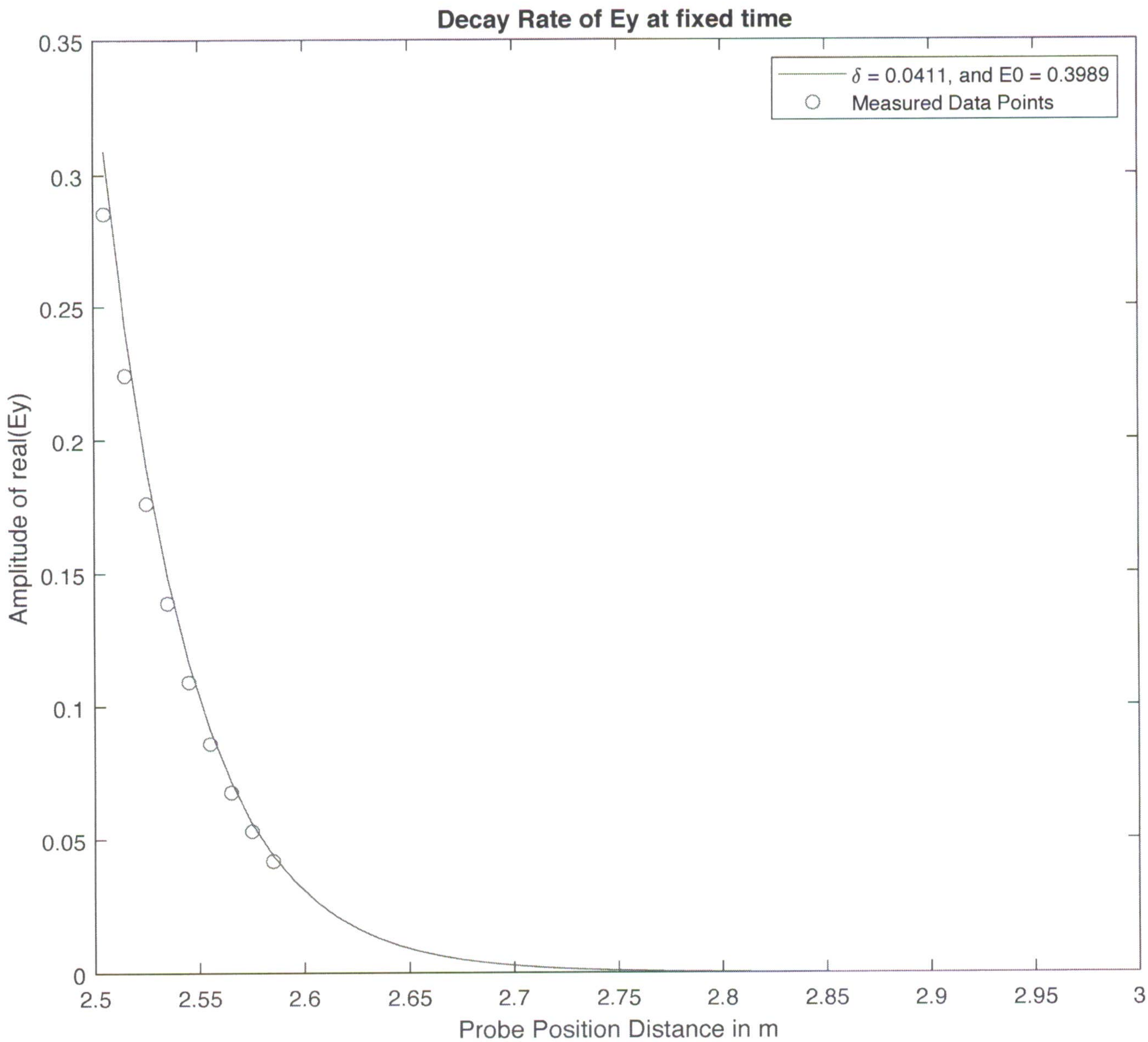


Ideally have addional info that helps interpretation.

ω, σ, ϵ all matter as does $\frac{\omega\epsilon}{\sigma}$

what is dB

Ey in sim. is real, so why is Re[ ] needed.?



Decay Rate of Ey at fixed time
δ = 0.0411, and E0 = 0.3989
Measured Data Points
Amplitude of real(Ey)
Probe Position Distance in m
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
2.5
2.55
2.6
2.65
2.7
2.75
2.8
2.85
2.9
2.95
3

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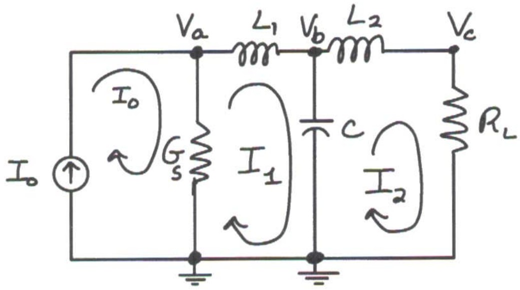
HW 7.2

10/20/20

4.3. b) Replacing Voltage soure and internal resistance to a current source w/ a parallel resistance:



Starting w/ the mesh equations:

Path of I_o:

Not included because it is conneeted to Current source

Path of I_1:

$$\boxed{-L_1\frac{dI_1}{dt}-\frac{1}{C}\int(I_1-I_2)\,dt-\frac{1}{G_S}(I_1-I_o)=0}$$ ✓

Path of I_2:

$$\boxed{-R_L I_2-L_2\frac{dI_2}{dt}-\frac{1}{C}\int(I_2-I_1)\,dt=0}$$ ✓

$\therefore$ Loop/mesh equations derived from KVL still hold true ✓

Node equations:

Current in = '+', Current out = '-'

at Node a:

$$I_o-\frac{1}{L_1}\int(V_b-V_a)\,dt+(V_a-0)G_s=0 \qquad * \; G_s=\frac{1}{R_s}$$

$$\boxed{I_o+\frac{1}{L_1}\int(V_a-V_b)\,dt+\frac{V_a}{R_s}=0}$$ ✓

at node b:

$$\frac{1}{L_1}\int(V_b-V_a)\,dt+C\frac{d}{dt}(V_b-0)-\frac{1}{L_2}\int(V_b-V_c)\,dt=0$$ ✓

$$\boxed{\frac{1}{L_1}\int(V_b-V_a)dt + C\frac{dV_b}{dt} + \frac{1}{L_2}\int(V_c-V_b)dt = 0}$$ ✓

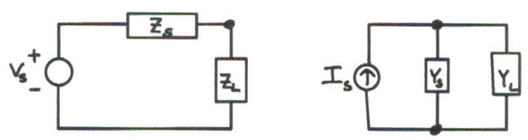
at node c:

$$+\frac{(V_c-0)}{R_L} - \frac{1}{L_2}\int(V_b-V_c)\,dt = 0$$

$$\boxed{\frac{V_c}{R_L} + \frac{1}{L_2}\int(V_c-V_b)\,dt = 0}$$ ✓

Conclusion?

4.3c) Figure 4.3d and e are:



(d) Thevenin

(e) Norton

$* \; Y_s \equiv Z_s^{-1} \rightarrow I_s = V_s Y_s$

From Figure (d)

The current across load Z_L:

$$\boxed{I_L = I_s = \frac{V_s}{Z_s + Z_L}}$$ ✓

Voltage across load is given by voltage divider formula:

$$\boxed{V_L = V_s\left(\frac{Z_L}{Z_L+Z_s}\right)}$$ ✓

From Figure e:

Voltage delivered is given from:

$$V_s = I_s Z_{tot} = I_s(Z_s \parallel Z_L)$$

$$V_L = I_s\left(\frac{Z_L Z_s}{Z_L+Z_s}\right)$$ * due to both in parallel

$$V_L = V_s \cancel{Y_s}\left(\frac{Z_L \cancel{Z_s}}{Z_L+Z_s}\right)$$ * using relationship: $I_s = V_s Y_s$ & $Z_s = \frac{1}{Y_s}$

$$\boxed{V_L = V_s\left(\frac{Z_L}{Z_L+Z_s}\right)}$$ ✓

Current delivered to load is:

$$I_L = \frac{V_L}{Z_L} = \frac{V_s}{\cancel{Z_L}}\left(\frac{\cancel{Z_L}}{Z_L+Z_s}\right) = \boxed{\frac{V_s}{Z_L+Z_s}}$$ ✓

∴ Current & Voltage formulas are equivalent ✓

this is avg power.

4.3.d) For fixed V_s & Z_s, where Z_s is a conjugate match: $Z_L = Z_s^*$

$P = \frac{1}{2} R |I|^2$ for complex circuits. Can you show? (Makes sense, but useful to practice complex notation)

Power delivered to load depends on current delivered to load:

$P_L = R_L |I_L|^2$ inst. real power

Using figure (d) Thevenin equivalent:

$$I_L = I_s = \frac{V_s}{Z_L + Z_s} \qquad * \; Z = R + jX$$

$$I_L = \frac{V_s}{(R_L + R_s) + j(X_s + X_L)}$$

$$I_L^* = \frac{V_s}{(R_L + R_s) - j(X_s + X_L)}$$

$$P_L = \frac{R_L}{2} |I_L|^2 = \frac{R_L}{2}\left(\frac{V_s}{(R_L+R_s)^2 + (X_s + X_L)^2}\right)$$

$*\; Z_L = R_L + jX_L$, minimizing each individually: (b/c $P_L = P_L(R_L, X_L)$ you are finding max/min of surface.)

$$\frac{\partial P_L}{\partial X_L} = \frac{V_s R_L}{2} \frac{\partial}{\partial X_L}\left(\frac{1}{(R_L+R_s)^2 + (X_s+X_L)^2}\right) = 0$$

$$= \frac{V_s R_L}{\cancel{2}}\left(\frac{-\cancel{2}(X_s+X_L)}{[(R_L+R_s)^2 + (X_s+X_L)^2]^2}\right) = 0$$

$$X_L = -X_s \quad ✓$$

Minimizing R_L

$$\frac{\partial P_L}{\partial R_L} = \left(\frac{V_s}{2}\right)\frac{\partial}{\partial R_L}\left(\frac{R_L}{(R_L+R_s)^2 + (X_s+X_L)^2}\right) = 0$$

$$\frac{V_s}{2}\left(\frac{1}{\cancel{(R_L+R_s)^2 + (X_s+X_L)^2}} + \frac{R_L(-1)(2(R_L+R_s))}{[(R_L+R_s)^2 + (X_s+X_L)^2]^2}\right) = 0$$

$$1 - \frac{2R_L(R_L+R_s)}{(R_L+R_s)^2 + (X_s+X_L)^2} = 0$$

$$R_L(R_L+R_s) = [(R_L+R_s)^2 + (\cancelto{0}{X_s+X_L})^2]\tfrac{1}{2} \qquad * \text{ min } X_L \text{ is when } X_L = -X_s$$

$$R_L(\cancel{R_L+R_s}) = (R_L+R_s)^{\cancel{2}}\tfrac{1}{2}$$

$$2R_L = R_L + R_s$$

$$\therefore R_L = R_s \quad ✓$$

$\therefore$ for P_L to be at max value :

$$Z_L = R_S - jX_S = Z_S^*$$

4.6e) Figure 3.17

From Faradays Law

$$\oint \vec{E}\cdot d\vec{l} = -\frac{\partial \Phi_B}{\partial t}$$

$$\oint \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{A}$$

$$\int_A^B \vec{E}\cdot d\vec{l} + \int_B^C \vec{E}\, d\vec{l} + \int_C^D \vec{E}\cdot d\vec{l} + \int_D^A \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{A}$$

in region between coaxial cable :

inductance is a constant.

if considering external inductance independent of time :

$$\vec{B} = \vec{B}_0 e^{j\omega t} \rightarrow -\frac{\partial \vec{B}}{\partial t} = -j\omega \vec{B}$$

$$-\frac{\partial \Phi_B}{\partial t} = -j\omega\, \Phi_B = j\omega L_e \cdot I$$ * external inductance per unit length due to B-field

Left hand integrals can be releated to Potential difference across points:

$$V_{AB} + V_{BC} + V_{CD} + V_{DA} = j\omega L_e I \Delta z$$

Finding V_{CD} :

$$E_z = I\,(Z_{int})_{outer}$$ $*(Z_{int})_{outer}$ = internal impedence of outer conductor

where interal impedence of a conductor is given as

$$\frac{Z_{in}}{\Delta z} = \left(\frac{R_s}{2\pi r_0} + \frac{j\omega L_s}{2\pi r_0}\right)\frac{\Omega}{m}$$

See Piazza for explanation

usually $R = \frac{1}{\sigma}\frac{L}{A}$, so this formula needs explamation.

at path $C \rightarrow D$, $r_0 = b$

$$(Z_{int})_{outer} = \left(\frac{R_{s_b}}{2\pi b} + \frac{j\omega L_{s_b}}{2\pi b}\right)\frac{\Omega}{m}$$

$\therefore$ $V_{CD} = -I(Z_{in})_{outer}\,\Delta z$ * KVL says going w/ path of current, gives neg. Voltage

Similarly one can obtain an expression of impedence for path between $A \rightarrow B$

$$V_{AB} = I(Z_{int})_{inner}\Delta z = -I\left(\frac{R_{s_a}}{2\pi a} + \frac{j\omega L_{s_a}}{2\pi a}\right)\Delta z$$

Substituting into equation:

$$V_{DA} - I\left(\frac{R_{sb} + j\omega L_{ib}}{2\pi b}\right)\Delta z + V_{BC} - I\left(\frac{R_{sa} + j\omega L_{ia}}{2\pi a}\right)\Delta z = (-j\omega L_e I)\Delta z$$

$$I\left(\frac{R_{sb}}{2\pi b} + \frac{R_{sa}}{2\pi a}\right)\Delta z + I\left(\frac{j\omega L_{ib}}{2\pi b} + \frac{j\omega L_{ia}}{2\pi a} + j\omega L_e\right)\Delta z = V_{DA} + V_{BC}$$

$$V_{DA} + V_{BC} = (I R_{eq} + I L_{eq})\Delta z$$

* L_{eq} & R_{eq} hold same formula as circuit diagram
* Inductors & Resistors add in series

where Δz is dependent on length of path, as $\Delta z \to 0$

$V_{DA} + V_{BC} = 0$, *circuit diagram matches voltage gains/drops

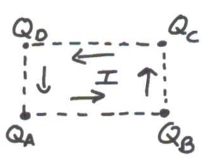
Considering paths $B \to C$ & $D \to A$

- Coaxial cylinder conductors act as capacitors and as such:

$$|V_{BC}| = \frac{Q_{BC}}{C}$$

$$|V_{DA}| = \frac{Q_{DA}}{C}$$

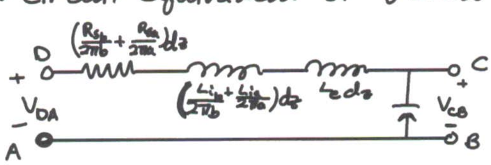
due to conservation of charge:



$Q_{DA} = Q_{BC}$ *current along path is continuous

$$\therefore \; V_{DA} = -V_{BC} \Rightarrow V_{DA} = V_{CB}$$

$\therefore$ Circuit equivalent of $\oint \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int B \cdot dA$



Question is leading you towards

$$V_{DA} - V_{CB} = \Delta V = (\; Z I \;)\Delta z$$

(what I was trying to hint on our zoom session)

$$\frac{dV}{dz} = (\; Z I \;)$$

where $Z = \left(\frac{R_{sa}}{2\pi a} + \frac{R_{sb}}{2\pi b}\right) + j(j\omega L_e + \dots)$