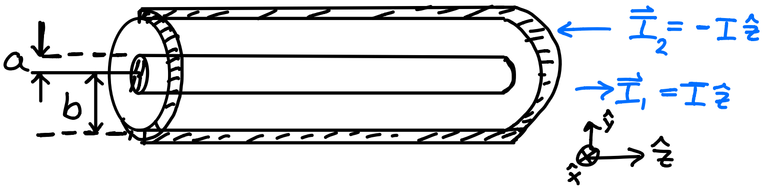
8.1 Inductance Calculations

*Note realized $\vec{B}$ will have a dependence on effective current & not total current

For a Coaxial Cable:



3.16 (6): $\omega L_i = \overline{\sigma\delta}$

$L_i = \frac{1}{\omega\delta\sigma}$

Internal Inductance per unit length formulas were given as:

$\frac{L_{ia}}{2\pi a}$ & $\frac{L_{ib}}{2\pi b}$

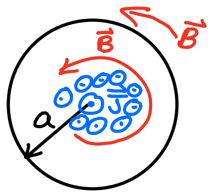
8.1.1.

Assuming current is uniformly distributed throughout the cross section of the conductors. Then derive the inductance per unit length assuming that the current is uniformly distributed only to a depth δ. Compare Results w/ the 2 equations above.

Looking at Inductance per unit lengths at low frequencies, and distributes uniformly only to a depth δ.

Assuming Current flows through the whole conductor of length ℓ

Inner Conductor:

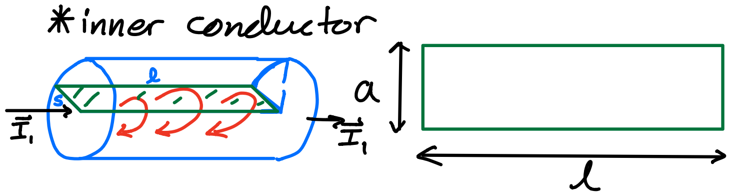


Magnetic flux: $\Phi_B = \int \vec{B} \cdot d\vec{A}$

* $d\vec{A}$ is the rectangular area that is dependent on r
* $dA_\perp$ is the circular cross section of current flowing through the conductor

$$L = \frac{\int \vec{B} \cdot d\vec{A}}{I}$$

*inner conductor



$\vec{A}_1 = a \cdot \ell\,\hat{\phi} \Rightarrow d\vec{A} = dz\,ds\,\hat{\phi}$ *cylindrical coordinates

*current is considered to be constant, creating a constant magnetic field and current

To compute internal inductance, must look at condition when $s < a$

$$L = \frac{\Phi_B}{I_1} = \frac{\int \vec{B}_1 \cdot d\vec{A}_1}{I_1}$$

Must find Magnetic field

* From Ampere's Law, $\vec{B}$ of a long wire is found as

$I = \int J dA_\perp$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$$ * if current is constant, and cylinder is long, B is const.

$$B \int dl = \mu_0 I_{encl}$$

I_{encl} can be represented in terms of the total current as:

$$I_{encl} = I \frac{A_{eff_\perp}}{A_{Tot_\perp}}$$ * A_{Tot} is the total cross sectional area
A_{eff} is the effective cross sectional area

$$I_{encl} = I_{eff} \frac{\pi s^2}{\pi a^2}$$

Substituting: $B \cdot 2\pi s = \mu_0 I \frac{s^2}{a^2} \Rightarrow B = \frac{\mu_0 s I_{eff}}{2\pi a^2}$ *for a long cylinder when computing flux

$$\Phi_B = \int_S \vec{B} \cdot d\vec{A} = \int_{z=0}^{l} \int_{s=0}^{a} \frac{\mu_0 I_{eff} s}{2\pi a^2} dz\, ds$$ * $I_{eff} = I \frac{s^2}{a^2}$

$$\Phi_B = \frac{\mu_0}{2\pi a^4} \frac{s^4}{4} \Big|_0^a$$ * $\int dA$ is found as area of rectangular section

$$\Phi_B = \frac{\mu_0 I \cancel{a^4} l}{8\pi \cancel{a^4}} = \frac{\mu_0 I l}{8\pi}$$

Where:

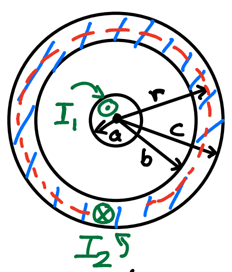
$$\frac{L_i}{l} = \frac{\Phi_B}{l I} \quad H/m$$

$$\therefore \frac{L_i}{l} = \frac{\mu_0}{8\pi}$$

* Not considering when $s > a$ since this will be external inductance

Looking at Co-axial cable referenced in 4.6 a)

B must be computed for the following region:



$b \le s \le c$

$$L = \frac{\Phi_B}{I} = \frac{\int \vec{B} \cdot d\vec{A}}{I}$$

From Ampere's Law:

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} = \int_{Inner} \vec{B}_1 \cdot d\vec{l} + \int_{outer} \vec{B}_2 \cdot d\vec{l} = \mu_0 I_{encl_1} + \mu_0 I_{encl_2}$$

inner conductor is completeley enclosed so:

$$\int_{inner} \vec{B}\cdot d\vec{l} = \mu_0 I_1$$

Outer conductor integral is dependent on value of s

I_{encl_2} can again be expressed in terms of the total current from outer Conductor, multiplied by a ratio of the total area:

$$I_{encl_2} = \left(-I_{eff}\left(\frac{A_{eff_\perp}}{A_{tot_\perp}}\right)\right)$$

* I goes inwards and opposit direction from inner conductor

$$I_{encl_2} = (-I)\left(\frac{\pi s^2 - \pi b^2}{\pi c^2 - \pi b^2}\right)$$

$$\therefore \int \vec{B}\cdot d\vec{l} = \mu_0\left[I - I\left(\frac{s^2-b^2}{c^2-b^2}\right)\right]$$

$$= \mu_0 I\left(1 - \left(\frac{s^2-b^2}{c^2-b^2}\right)\right)$$

$$= \mu_0 I\left(\frac{c^2 - \cancel{b^2} - s^2 + \cancel{b^2}}{c^2-b^2}\right)$$

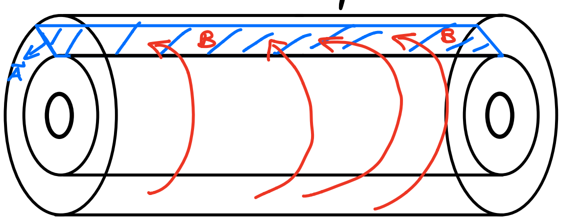
$$\int \vec{B}\cdot d\vec{l} = \mu_0 I\left(\frac{c^2-s^2}{c^2-b^2}\right)$$

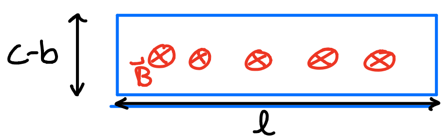
$$B\cdot 2\pi s = \mu_0 I\,\frac{c^2-s^2}{c^2-b^2}$$

$$B = \frac{\mu_0 I_{eff}}{2\pi s}\left(\frac{c^2-s^2}{c^2-b^2}\right)$$

* for $b<r<c$ B is still in the $+\hat{\phi}$ direction
* I_{eff} is still effective current of total

Flux will be computed similarly:





$$\vec{A_2} = cl\,\hat{\phi}$$
$$d\vec{A_2} = ds\,dz\,\hat{\phi}$$

$$\vec{B}(s) = \frac{\mu_0 I}{2\pi s}\left(\frac{c^2-s^2}{c^2-b^2}\right)\hat{\phi} = \frac{\mu_0 I}{2\pi(c^2-b^2)}\left(\frac{c^2}{s} - s\right)\phi$$

Inductance can be found using Energy Method (More Complex to compute using flux)

$$\frac{1}{2}LI^2 = \int_V \frac{\mu_0}{2} H^2\, dV$$

* $H = \frac{B}{\mu}$ ($\mu = \mu_0$ for simple case

$$\frac{1}{2}LI^2 = \int_{s=b}^{c}\int_{\phi=0}^{2\pi}\int_{0}^{l} \frac{\mu_0}{2}\left(\frac{I}{2\pi(c^2-b^2)}\right)^2\left(\frac{c^2}{s} - s\right)^2 s\,ds\,d\phi\,dz$$

*no dependence on ϕ & z

$$\cancel{\frac{1}{2}}L\cancel{I^2} = \frac{\mu_0 \cancel{I^2}(l\,\cancel{2\pi})}{\cancel{2}(2\pi)^{\cancel{2}}(c^2-b^2)^2}\int_b^c\left(\frac{c^4}{s^2} + s^2 - 2\cancel{s}\frac{c^2}{\cancel{s}}\right)s\,ds$$

$$\frac{L}{l} = \frac{\mu_0}{2\pi(c^2-b^2)^2}\int_b^c\left(\frac{c^4}{s} + s^3 - 2c^2 s\right)ds$$

$$\frac{L}{l}=\frac{\mu_0}{2\pi(c^2-b^2)^2}\left(c^4\ln\left(\frac{c}{b}\right)+\frac{c^4-b^4}{4}-c^2(c^2-b^2)\right)$$

$$=\frac{\mu_0}{2\pi}\left[\frac{c^4\ln\left(\frac{c}{b}\right)}{(c^2-b^2)^2}+\frac{(c^2+b^2)\cancel{(c^2-b^2)}}{4(c^2-b^2)^{\cancel{2}}}-\frac{c^2\cancel{(c^2-b^2)}}{(c^2-b^2)^{\cancel{2}}}\right)$$

$$=\frac{\mu_0}{2\pi}\left[\frac{c^4\ln\left(\frac{c}{b}\right)}{(c^2-b^2)^2}+\frac{c^2+b^2-4c^2}{4(c^2-b^2)}\right]$$

$$\therefore\ \frac{L}{l}=\frac{\mu_0}{2\pi}\left[\frac{c^4\ln\left(\frac{c}{b}\right)}{(c^2-b^2)^2}+\frac{b^2-3c^2}{4(c^2-b^2)}\right]$$

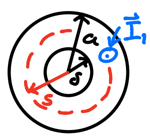
* internal inductance of outer cylinder at low frequencies w/ current distributions uniform over conductor

AT high Frequencies:

* Currents distribute over surface δ

For inner conductor:

$$I_{encl_1}=I\,\frac{A_{eff_1}}{A_{TOT_1}}=I\left(\frac{\pi s^2-\pi(a-\delta)^2}{\pi a^2-\pi(a-\delta)^2}\right)$$



$$A_{eff}=\pi s^2-\pi(a-\delta)^2$$

$$A_{TOT}=\pi a^2-\pi(a-\delta)^2$$

Magnetic field is then computed as:

$$\oint \vec{B}_1\cdot d\vec{l}=\mu_0 I_{encl_1}$$

$$B_1=\frac{\mu_0}{2\pi s}\left[I\left(\frac{s^2-(a-\delta)^2}{a^2-(a-\delta)^2}\right)\right]$$

* rotates in $\hat{\phi}$ direction and only exists for $a-\delta<s<a$

$d\vec{A}_1=ds\,dz\,\hat{\phi}$ *remains the same

$$\Phi_B=\int_{s=a-\delta}^{a}\int_{z=0}^{l}B_1\,ds\,dz\,\cancel{(\hat{\phi}\cdot\hat{\phi})}$$

* B has no dependence on z

$$=\int_{a-\delta}^{a}\frac{\mu_0}{2\pi s}\left[I\left(\frac{s^2-(a-\delta)^2}{a^2-(a-\delta)^2}\right)\right]ds\cdot l$$

$$=\frac{\mu_0 l I}{2\pi[a^2-(a-\delta)^2]}\left[\int_{a-\delta}^{a}s\,ds-\int_{a-\delta}^{a}\frac{(a-\delta)^2}{s}\,ds\right]$$

$$=\frac{\mu_0 l I}{2\pi[a^2-(a-\delta)^2]}\left(\frac{1}{2}(a^2-(a-\delta)^2)-(a-\delta)^2\ln\left(\frac{a}{a-\delta}\right)\right)$$

$$=\frac{\mu_0 l I}{2\pi[\cancel{a^2}-\cancel{a^2}-\delta^2+2a\delta]}\left[\frac{1}{2}(2a\delta-\delta^2)+(a^2+\delta^2-2a\delta)\ln\left(\frac{a-\delta}{a}\right)\right]$$

$$\Phi_B=\frac{\mu_0 l I}{2\pi(2a\delta-\delta^2)}\left[a\delta-\frac{\delta^2}{2}+(a^2+\delta^2-2a\delta)\ln\left(1-\frac{\delta}{a}\right)\right]$$

$$\Phi_B=\frac{\mu_0 l I}{4\pi a\delta\left(1-\frac{\delta}{2a}\right)}\left[a^2\ln\left(1-\frac{\delta}{a}\right)+a\delta\left(1-2\ln\left(1-\frac{\delta}{a}\right)\right)+\delta^2\left(\ln\left(1-\frac{\delta}{a}\right)-\frac{1}{2}\right)\right]$$

* when $\delta<<a\rightarrow\frac{\delta}{a}<<1$, can take first-order approximation
* $\ln(1-x)\approx -x$ when $x<<1$ & $1-\frac{\delta}{2a}\approx 1$

$$\Phi_B = \frac{\mu_0 l I}{4\pi a\delta}\left[a^{\cancel{2}}\left(-\frac{\delta}{\cancel{a}}\right) + a\delta\left(1 \cancel{+} 2\left(\cancel{+}\frac{\delta}{a}\right)\right) + \cancel{\delta^2(\ \ )}^{\,0}\right]$$

$$= \frac{\mu_0 l I}{4\pi a\delta}\left[\cancel{a\delta} - \cancel{a\delta} + 2\delta^2\right]$$

$$= \frac{\mu_0 l I}{\underset{2}{\cancel{4}}\pi a \cancel{\delta}}\,\cancel{2}\delta^{\cancel{2}} = \frac{\mu_0 l I \delta}{2\pi a} \qquad \omega L_i = \frac{1}{\sigma\delta}$$

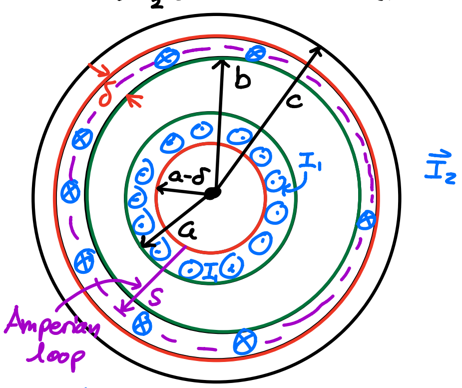
$$\frac{L_{i_a}}{l} = \frac{\Phi_B}{l I} = \frac{\mu_0 \delta}{2\pi a}$$ * as δ decreases, $L_i \approx 0$ (higher frequencies)

*

Considering Outer conductor:

Magnetic field must be found again

$\int \vec{B}_2 \cdot d\vec{l} = \mu_0 I_{encl}$



$|\vec{I}_1| = -|\vec{I}_2| = I$

* Still composed of 2 currents

$\int \vec{B}\cdot d\vec{l} = \int_{inner} \vec{B}\cdot d\vec{l} + \int_{outer} \vec{B}\cdot d\vec{l}$ * although current distribution differs all of current I_1 is enclosed

$\int \vec{B}\cdot d\vec{l} = \mu_0 \overset{I_1 = I}{\cancel{I_{encl_1}}} + \mu_0 I_{encl_2}$

I_{encl_2} can be expressed in terms of total current & effective area

Current distributes to inner radius of outer conductor due to Lenz Law

Then:

$$I_{encl_2} = \left(\frac{s^2 - b^2}{(b+\delta)^2 - b^2}\right) I = \frac{A_{eff}}{A_{TOT}} I$$

substituting:

$$B = \mu_0 I\left(1 - \frac{(s^2 - b^2)}{(b+\delta)^2 - b^2}\right)$$

$$= \mu_0 I\left(\frac{(b+\delta)^2 - b^2 - s^2 + b^2}{(b+\delta)^2 - b^2}\right)$$

$$\vec{B} = \mu_0 I\left(\frac{(b+\delta)^2 - s^2}{(b+\delta)^2 - b^2}\right)$$ * originally: $c \xrightarrow{\text{Now}} (b+\delta)$ as limit

B is still dependent on effective current and thus:

$$\Phi_B = \frac{\mu_0 I l}{(b+\delta)^2 - b^2}\left(\int_b^{b+\delta} (b+\delta)^2\, ds - \int_b^{b+\delta} s^2\, ds\right)$$ * $\int_0^l dz = l$ term computed

$$\Phi_B = \frac{\mu_0 I l}{(b+\delta)^2 - b^2}\left\{(b+\delta)^2\left[(\cancel{b}+\delta) - \cancel{b}\right] - \left[\frac{(b+\delta)^3 - b^3}{3}\right]\right\}$$

* if $b \gg \delta \rightarrow b+\delta \approx b$
$\And\ \delta^2 \approx 0$

$$= \frac{\mu_0 I l}{2b\delta - \delta^2}\left[(b+\delta)^2\delta - \overset{0}{\cancel{\frac{b^3 - b^3}{3}}}\right]$$

$$= \frac{\mu_0 I l}{2b\cancel{\delta}}\left[(b+\delta)^2\cancel{\delta}\right]$$

$$\Phi_B = \frac{\mu_0 I l}{2b}(b+\delta)^2$$

$$\therefore \frac{L_{ib}}{l} = \frac{\mu_0}{2b}(b+\delta)^2$$

8.1.2 External Inductance

The magnetic field in between conductors can be found using Ampere's Law:

$$B = \frac{\mu_0 I_{encl}}{2\pi s}$$ * $I_{encl} = I$, all current enclosed from inner conductor

$$\therefore B = \frac{\mu_0 I}{2\pi s}$$

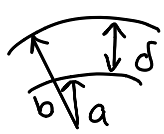
$$\Phi_B = \int \vec{B}\cdot d\vec{A}$$

$$= \int_{s=a}^{b}\int_{z=0}^{l} \frac{\mu_0 I}{2\pi s}\, ds\, dz$$

$$\Phi_B = \frac{\mu_0 l I}{2\pi}\ln\left(\frac{b}{a}\right)$$

$$\therefore \frac{L_e}{l} = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)$$

8.1.3



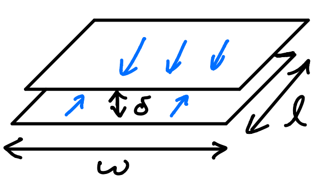
$$2\cos\left(\frac{2\pi x}{\lambda}\right)\cos(\omega t) - 1.5\cos(2\pi\ \underline{(x_1+i)}$$

Originally: $L_e = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)$ * $\delta = b-a$
$b = \delta + a$

$$L_e = \frac{\mu_0}{2\pi}\ln\left(1 + \frac{\delta}{a}\right)$$ * If $x \ll 1$ then $\ln(1+x) \approx x$

$$L_e \approx \frac{\mu_0 \delta}{2\pi a}$$

then consider the external inductance of a parallel plane

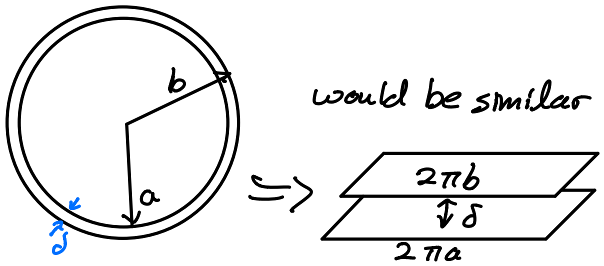


Wiley computes this external inductance as:

$$\frac{L_e}{l} = \frac{\mu_0 \delta}{w}$$

Comparing values:

$$\frac{\mu_0 \delta}{w} = \frac{\mu_0 \delta}{2\pi a}$$ * then if coaxial cables are close together:



would be similar to:

* Where if $\delta \ll a$, $b \approx a$ yielding 2 parallel plates for external inductance.