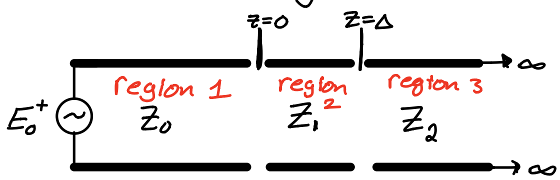
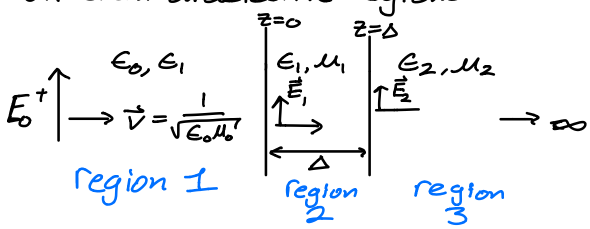
9.2.1

Given the following Transmission line:



where the equivalent region can be described as an electric field entering different diaelectric regions



Let each region be denoted by n $(n=0,1,2)$

Electric potential at each region can be described as a forward & backward travelling voltage:

$$\tilde{V}_n(z) = \tilde{V}_n^+ e^{-j\beta_n z} + \tilde{V}_n^- e^{j\beta_n z}$$

Similarly, the electric field in each can be written in the form of a forward/backward travelling electric field.

$$\tilde{E}_n(z) = \tilde{E}_n^+ e^{-jk_n z} + \tilde{E}_n^- e^{jk_n z}$$

Where the corresponding Real Potential & Electric fields can be given as:

$$E_n(z,t) = Re\left[\tilde{E}_n(z) e^{j\omega t}\right]$$

$$V_n(z,t) = Re\left[\tilde{V}_n(z) e^{j\omega t}\right]$$

If given a $\tilde{V}_0^+$. At $z=0$ and $z=\Delta$, potential & current must be continuous (equivalently E & H must be continuous from Maxwell's B.C.'s). Also assume region 2 extends to infinity such that $V_2^- = 0$

Considering Steady State:

when looking at $z=0$:

Voltage Continuity states:

$$\tilde{V}_0(0) = \tilde{V}_1(0)$$

① $\tilde{V}_0^+ + \tilde{V}_0^- = \tilde{V}_1^+ + \tilde{V}_1^-$

Similarly for Current Continuity equation

$$\tilde{I}_0(0) = \tilde{I}_1(0)$$

at $z=0$

② $\tilde{V}_0^+ - \tilde{V}_0^- = \frac{Z_0}{Z_1}\left(\tilde{V}_1^+ - \tilde{V}_1^-\right)$

Similarly doing this for Boundary 2:

$$\tilde{V}_1(\Delta) = \tilde{V}_2(\Delta)$$

$$\tilde{V}_1^+ e^{-j\beta_1\Delta} + \tilde{V}_1^- e^{j\beta_1\Delta} = \tilde{V}_2^+ e^{j\beta_2\Delta} + \tilde{V}_2^- e^{j\beta_2\Delta}$$

* recall $\tilde{V}_2^- = 0$, no reflection, then:

③ $\tilde{V}_1^+ e^{-j\beta_1\Delta} + \tilde{V}_1^- e^{j\beta_1\Delta} = \tilde{V}_2^+ e^{-j\beta_2\Delta}$

Applying this to current continuity:

$$\tilde{I}_1(\Delta) = \tilde{I}_2(\Delta)$$

④ $\tilde{V}_1^+ e^{-j\beta_1\Delta} - \tilde{V}_1^- e^{j\beta_1\Delta} = \frac{Z_1}{Z_2}\left(\tilde{V}_2^+ e^{-j\beta_2\Delta}\right)$

adding eqns ③ & ④:

$$2\tilde{V}_1 e^{-j\beta_1\Delta} = \tilde{V}_2^+ \left(1+\frac{Z_1}{Z_2}\right) e^{-j\beta_2\Delta}$$

$$\tilde{V}_2^+ = \left(\frac{2}{1+\frac{Z_1}{Z_2}}\right) \tilde{V}_1^+ e^{-j(\beta_1-\beta_2)\Delta}$$

* Let $\tau_2 = \frac{2}{1+\frac{Z_2}{Z_1}}$
* in terms of $\tilde{V}_1^+$

$$\therefore \quad \tilde{V}_2^+ = \tau_2 \tilde{V}_1^+ e^{-j(\beta_1-\beta_2)\Delta}$$

subtracting: ④ $-\left(\frac{Z_1}{Z_2}\right)$ ③

$$\tilde{V}_1^+ e^{-j\beta_1\Delta} - \tilde{V}_1^- e^{j\beta_1\Delta} - \frac{Z_1}{Z_2}\left(\tilde{V}_1^+ e^{-j\beta_1\Delta} + \tilde{V}_1^- e^{j\beta_1\Delta}\right) = 0$$

$$\tilde{V}_1\left(1-\frac{Z_2}{Z_1}\right)e^{-j\beta_1\Delta} = \tilde{V}_1^-\left(1+\frac{Z_2}{Z_1}\right)e^{j\beta_1\Delta}$$

$$\tilde{V}_1^- = \left(\frac{1-\frac{Z_2}{Z_1}}{1+\frac{Z_2}{Z_1}}\right) e^{-j2\beta_1\Delta}\,\tilde{V}_1^+$$

* Let $\rho_2 = \frac{Z_2 - Z_1}{Z_2 + Z_1}$

Then: $\tilde{V}_1^- = \rho_2 e^{-j2\beta_1\Delta} \tilde{V}_1^+$

Can now rewrite eqn ① & ②:

① $\tilde{V}_0^+ + \tilde{V}_0^- = \tilde{V}_1^+(1+\rho_2 e^{-j2\beta\Delta})$

② $\tilde{V}_0^+ - \tilde{V}_0^- = \frac{Z_0}{Z_1}\tilde{V}_1^+(1-\rho_2 e^{-j2\beta\Delta})$

Adding: ① + ②

$$2\tilde{V}_0^+ = \left[(1+\rho_2 e^{-j2\beta\Delta}) + \frac{Z_0}{Z_1}(1-\rho_2 e^{-j2\beta\Delta})\right]\tilde{V}_1^+$$

$$\frac{2\tilde{V}_0^+}{(1+\rho_2 e^{-j2\beta_1\Delta}) + \frac{Z_0}{Z_1}(1-\rho_2 e^{-j2\beta_1\Delta})} = \tilde{V}_1^+$$

$1 + \frac{Z_0}{Z_1} +$

* Let $\alpha = \rho_2 e^{-j2\beta\Delta}$

$$\tilde{V}_1^+ = \frac{2\tilde{V}_0^+}{(1+\frac{Z_0}{Z_1}) + \alpha(1-\frac{Z_0}{Z_1})}$$

then: $\boxed{\tilde{V}_1^+ = \frac{2 V_0^+}{(1+\alpha)+\frac{Z_0}{Z_1}(1-\alpha)}\tilde{V}_0^+}$ * $\tau_1 = \frac{2}{(1+\alpha)+\frac{Z_0}{Z_1}(1-\alpha)}$

$$\tilde{V}_1^+ = \tau_1 \tilde{V}_0^+$$

Similarly: ② $- \frac{Z_0}{Z_1}$ ①

$$\tilde{V}_0^+ - \tilde{V}_0^- - \frac{Z_0}{Z_1}(\tilde{V}_0^+ + \tilde{V}_0^-) = \frac{Z_0}{Z_1}\tilde{V}_1^+(1-\alpha) - \frac{Z_0}{Z_1}\tilde{V}_1^+(1+\alpha)$$

$$\left(1-\frac{Z_0}{Z_1}\right)\tilde{V}_0^+ - \left(1+\frac{Z_0}{Z_1}\right)\tilde{V}_0^- = -2\frac{Z_0}{Z_1}\alpha\tilde{V}_1^+$$

* $\tilde{V}_1^+ = \tau_1 \tilde{V}_0^+$

$$\left[\left(1-\frac{Z_0}{Z_1}\right) + 2\frac{Z_0}{Z_1}\alpha\right]\tilde{V}_0^+ = \left(1+\frac{Z_0}{Z_1}\right)\tilde{V}_0^-$$

$$\tilde{V}_0^- = \underbrace{\left(\frac{1-\frac{Z_0}{Z_1}}{1+\frac{Z_0}{Z_1}}\right)}_{\rho_1}\tilde{V}_0^+ + \underbrace{\left(\frac{2}{1+\frac{Z_0}{Z_1}}\right)}_{\tau_0}\frac{Z_0}{Z_1}\alpha\tilde{V}_0^+$$

$$\tilde{V}_0^- = \rho_1\tilde{V}_0^+ + \tau_0\alpha V_0^+$$

$$\boxed{\tilde{V}_0^- = (\rho_1 + \tau_0\alpha)\tilde{V}_0^+}$$

* Knowing $\tilde{V}_1^+$ in terms of $\tilde{V}_0^+$ we can rewrite as:

$$\tilde{V}_1^+ = \tau_1\tilde{V}_0^+$$

$$\tilde{V}_1^- = \rho_2 e^{-j2\beta_1\Delta}\tilde{V}_1^+ \qquad * \ \alpha = \rho_2 e^{-j2\beta_1\Delta}$$

$$\boxed{\tilde{V}_1^- = \alpha\tau_1\tilde{V}_0^+} \qquad * \ \tilde{V}_1^+ = \tau_1\tilde{V}_0^+$$

And:

$$\tilde{V}_2^+ = \tau_2\tilde{V}_1^+ e^{-j(\beta_1-\beta_2)\Delta}$$

$$\boxed{\tilde{V}_2^+ = \tau_2\tau_1 e^{-j(\beta_1-\beta_2)\Delta}\tilde{V}_0^+}$$

9.2.2

1) Show that $V(z,t) = V^+[\cos(\omega t-\beta z)+\rho\cos(\omega t+\beta z)]$ ①

can be written as: $V(z,t) = A\cos(\omega t)\cos(\beta z) + B\sin(\omega t)\sin(\beta z)$ ②

First from eqn ① can be expanded using the following trig identity:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

Then Eqn ① becomes:

$$V(z,t) = V^+\{[\cos(\omega t)\cos(\beta z) + \sin(\omega t)\sin(\beta z)] + \rho[\cos(\omega t)\cos(\beta z) - \sin(\omega t)\sin(\beta z)]\}$$

$$V(z,t) = V^+\{(1+\rho)\cos(\omega t)\cos(\beta z) + (1-\rho)\sin(\omega t)\sin(\beta z)\}$$

$$\therefore A = V^+(1+\rho)$$

$$B = V^+(1-\rho)$$

2) Code developed to animate V^- w/ amplitude ½

3) From the animation we can see that V^+ & V^- will construct together or be destructive at times.

$$V_1 = V^+ + V^-$$

$$|V_1|_{max} = 1 + 0.5 = 1.5$$

$$|V_1|_{min} = 1 - 0.5 = 0.5$$

Then the standing wave ratio will yield

$$S = \frac{1.5}{0.5} = 3$$

$$\therefore \rho = \frac{S-1}{S+1} = \frac{2}{4} = 1/2$$

Which corresponds to output of system w/ $V^- = \rho V^+ = \frac{1}{2} V^+$

4) Similar things occur when boundary is shifted, position of boundary has no effect of results for Amplitudes

But because waves do not encounter in same spot, a phase shift is then encountered.