Darwin Quiroz
ECE 513

6.4 Complex form

6.4.1) Given:

$\vec{E} \equiv Re\left[\tilde{\vec{E}}_o e^{-j(\omega t - \vec{K}\cdot\vec{r})}\right]$

$\vec{B} \equiv Re\left[\tilde{\vec{B}}_o e^{-j(\omega t - \vec{K}\cdot\vec{r})}\right]$ * where $\tilde{\vec{E}}$ & $\tilde{\vec{B}}$ are vectors w/ complex constants

$\vec{K} \equiv K_x \hat{x} + K_y \hat{y} + K_z \hat{z}$

$\vec{r} = r_x \hat{x} + r_y \hat{y} + r_z \hat{z}$

Faradays law in differential form states:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Left hand side:

Looking at x-component:

$$(\vec{\nabla} \times \vec{E})_x = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x}$$

$$= \left[\frac{\partial}{\partial y}\left(Re\left[\tilde{E}_{o_z} e^{-j(\omega t - \vec{K}\cdot\vec{r})}\right]\right) - \frac{\partial}{\partial z}\left(Re\left[\tilde{E}_{o_y} e^{-j(\omega t - \vec{K}\cdot\vec{r})}\right]\right)\right]\hat{x}$$

* due to derivatives, being a linear, time-invariant operator to signals (all E&M waves are signals), then derivative can be computed first before taking the real part or after (LTI property)

$$= \left\{Re\left[\tilde{E}_{o_z} e^{-j(\omega t)} \frac{\partial}{\partial y}\left(e^{j(K_x x + K_y y + K_z z)}\right)\right] - Re\left[\tilde{E}_{o_y} e^{-j(\omega t)} \frac{\partial}{\partial z}\left(e^{j(K_x x + K_y y + K_z z)}\right)\right]\right\}\hat{x}$$

$$= Re\left[\hat{E}_{o_z} e^{-j(\omega t)} (jK_y) e^{j(\vec{K}\cdot\vec{r})} - \tilde{E}_{o_y} e^{-j(\omega t)} (jK_z) e^{j(\vec{K}\cdot\vec{r})}\right]\hat{x}$$

$(\vec{\nabla} \times \vec{E})_x = Re\left[(j K_y E_{0z} - jK_z \tilde{E}_{oy})\, e^{j(\vec{K}\cdot\vec{r} - \omega t)}\right]\hat{x}$ * where $j = e^{j\pi/2}$ (only adds a phase)

From this we can see a trend that

$$\frac{\partial}{\partial \alpha_i}\left(e^{j K_\alpha \alpha}\right) = jK_\alpha e^{jK_\alpha \alpha} \quad \text{* where } \alpha = \hat{x}, \hat{y}, \hat{z}$$

∴ curl of $\tilde{\vec{E}}$ only acts on exponential term, and the derivatives only multiply the E_i component by jK_i. This's trend will yield:

$\vec{\nabla} \times \vec{E} = Re\left[j(\vec{K} \times \tilde{\vec{E}})\right] = \vec{K} \times \vec{E} = |\vec{K}|(\hat{K} \times \vec{E})$

Right Hand side of Faradays Law:

$-\frac{\partial \vec{B}}{\partial t} = -Re\left[\frac{\partial}{\partial t} \tilde{\vec{B}}_0 e^{-j\omega t} e^{j\vec{K}\cdot\vec{r}}\right]$ * using same LTI argument for derivative

$= -Re\left[\tilde{\vec{B}}(-j\omega) e^{-j(\omega t - \vec{K}\cdot\vec{r})}\right]$

$\frac{\partial \vec{B}}{\partial t} = \omega \vec{B}$

$\therefore \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\Rightarrow \quad \vec{K} \times \vec{E} = \omega \vec{B}$

$\therefore \vec{B} = \frac{|\vec{K}|}{\omega}(\hat{K} \times \vec{E})$ * in free space

$\vec{B} = \frac{1}{c}(\hat{K} \times \vec{E})$

* if not only looking at Real Part, but also complex:

$(\vec{\nabla} \times \tilde{\vec{E}}) = j\vec{K} \times \tilde{\vec{E}}$

$\frac{\partial \tilde{\vec{B}}}{\partial t} = j\omega \tilde{\vec{B}}$

then relationship holds:

$\tilde{\vec{B}} = \frac{1}{c} \hat{K} \times \tilde{\vec{E}}$

Where the physical wave, represented by taking the Real part of these complex vectors. But through eulers identity it is simple to see that the only difference between the Imag & Re part of $\tilde{\vec{E}}$ is the replacement of the cos with a sine. Then both still obey Maxwells equations & thus so will $\tilde{\vec{E}}$ & $\tilde{\vec{B}}$