Darwin Quiroz HW 6.3

1) Given $\vec{E} = E_{ox}\cos(K_z z - \omega t + \delta_x)\hat{x} = E_o(z,t)\hat{x}$

Prove E satisfies:

$$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 \vec{E} = (\nabla^2 E_x \hat{x} + \cancel{\nabla^2 E_y}^{\,0}\hat{y} + \cancel{\nabla^2 E_z}^{\,0}\hat{z})$$

$$\therefore \nabla^2 \vec{E_x} = \cancel{\frac{\partial^2 E_x}{\partial x^2}}^{\,0}\hat{x} + \cancel{\frac{\partial^2 E_x}{\partial y^2}}^{\,0}\hat{x} + \frac{\partial^2 E_x}{\partial z^2}\hat{x} = \frac{\partial^2 E_x}{\partial z^2}\hat{x}$$

* because E is only dependent on z & t: $\frac{\partial^2 \vec{E}}{\partial x^2} = \frac{\partial^2 \vec{E}}{\partial y^2} = 0$

$$\frac{\partial^2}{\partial z^2}\vec{E} = \frac{\partial^2}{\partial z^2}\left(E_{ox}\cos(K_z z - \omega t + \delta_x)\hat{x}\right)$$

$$= -E_{ox}K_z^2\cos(K_z z - \omega t + \delta_x)\hat{x}$$

$$= -K_z^2\vec{E}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 E_{ox}\cos(K_z z - \omega t + \delta_x)\hat{x} = -\omega^2\vec{E}$$

Substituting into wave equation:

$$-K_z^2\vec{E} = \frac{1}{c^2}(-\omega^2)\vec{E}$$

$$\vec{E} = \cancel{\frac{1}{c^2}}\cancel{\frac{\omega^2}{K^2}}\vec{E}$$

$\therefore \vec{E} = \vec{E}$ * $\vec{E}$ satisfies wave equation

For $\vec{B} = B_{ox}\cos(K_z z - \omega t + \delta_x')\hat{x} + B_{oy}\cos(K_z z - \omega t + \delta_y')\hat{y}$

$$\nabla^2\vec{B} = \nabla^2 B_x\hat{x} + \nabla^2 B_y\hat{y} + \nabla^2 B_z\hat{z}$$

* Laplace operator acts onto each B component:

$$= \left(\cancel{\frac{\partial^2 B_x}{\partial x^2}}^{\,0} + \cancel{\frac{\partial^2 B_x}{\partial y^2}}^{\,0} + \frac{\partial^2 B_x}{\partial z^2}\right)\hat{x} + \left(\cancel{\frac{\partial^2 B_y}{\partial x^2}}^{\,0} + \cancel{\frac{\partial^2 B_y}{\partial y^2}}^{\,0} + \frac{\partial^2 B_y}{\partial z^2}\right)\hat{y} + \cancel{\left(\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + \frac{\partial^2 B_z}{\partial z^2}\right)\hat{z}}^{\,0}$$

* $B_z = 0$, B_x & B_y are only dependent on x & t

$$= \frac{\partial^2 B_x}{\partial z^2}\hat{x} + \frac{\partial^2 B_y}{\partial z^2}\hat{y} \quad * \ \frac{\partial^2}{\partial z^2}(\cos(Kz)) = -K^2\cos(Kz)$$

$$= -K_z^2 B_{ox}\cos(K_z z - \omega t - \delta'_x)\hat{x} - K_z^2 B_{oy}\cos(K_z z - \omega t - \delta'_y)\hat{y}$$

$$\nabla^2\vec{B} = -K_z^2(B_x\hat{x} + B_y\hat{y}) = -K_z^2\vec{B}$$

$$\frac{\partial^2\vec{B}}{\partial t^2} = -\omega^2 B_{ox}\cos(K_z z - \omega t - \delta'_x)\hat{x} - \omega^2 B_{oy}\cos(K_z z - \omega t - \delta'_y)\hat{y}$$

$$= -\omega^2(B_x\hat{x} + B_y\hat{y}) = -\omega^2\vec{B}$$

Substituting into wave equation

$$\cancel{-}k_z^2 \vec{B} = \frac{1}{c^2}(\cancel{-}\omega^2 \vec{B})$$

$$\vec{B} = \cancel{\frac{1}{c^2}\frac{\omega^2}{k^2}}\vec{B} \quad * \ \frac{\omega}{k} = v \quad * \text{where } v = c \text{ in free space}$$

$$\vec{B} = \vec{B}$$

$\therefore$ Both functions $\vec{B}$ & $\vec{E}$ satisfy respective wave equations

From Faradays Law, the following relationship is known:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

where:

$$\vec{E} = Re\left\{ \tilde{E}_{ox}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)} \right\} \hat{x}$$

* The physical wave is the Real Part of a plane wave where $\tilde{E}_{ox} = E_{ox}e^{j\delta_x}$

$$\vec{\nabla} \times \vec{E} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{pmatrix} = \frac{\partial E_x}{\partial z}\hat{y} - \cancel{\frac{\partial E_x(z,t)}{\partial y}}^{\,0}$$

$$= \frac{\partial}{\partial z} Re\left\{ \tilde{E}_{ox}\, e^{j(k\hat{r} - \omega t + \delta_x)} \right\} \hat{x}$$

$$\vec{\nabla} \times \tilde{\vec{E}} = jk_z E_{ox}\, e^{j(k_z z - \omega t)}\, \hat{x}$$

Similarly:

$$\vec{B} = Re\{\tilde{\vec{B}}\} = Re\left\{ \tilde{B}_{oy}\, e^{j(k_z z - \omega t)} \right\} \quad * \text{ where } \tilde{\vec{B}} = B_{ox}e^{j\delta_x'}\hat{x} + B_{oy}e^{j\delta_y'}\hat{y}$$

$$-\frac{\partial \vec{B}}{\partial t} = -Re\left\{ \frac{\partial \tilde{\vec{B}}}{\partial t} \right\}$$

$$\frac{\partial \tilde{\vec{B}}}{\partial t} = j\omega \tilde{\vec{B}}_o e^{j(kz - \omega t)}$$

Substituting values:

$$\nabla \times \tilde{\vec{E}} = \frac{\partial \tilde{\vec{B}}}{\partial t}$$

$$\cancel{j}k_z E_{ox}\cancel{e^{j(k_z z - \omega t)}}\,\hat{y} = \cancel{j}\omega \tilde{\vec{B}}_o \cancel{e^{j(kz - \omega t)}}$$

$$\tilde{\vec{B}} = \frac{k_z}{\omega} E_{ox}$$

Comparing components:

$$\tilde{\vec{B}}_{ox} = 0, \quad \tilde{\vec{B}}_{oy} = \frac{1}{c}\tilde{\vec{E}}_{ox}$$

$\therefore\ \delta_x' = 0$ * could be any value but unimportant

$\delta_y' = \delta_x$

& $B_{oy} = \frac{1}{c} E_{ox}$

6.3.2) The relationships developed for 6.3.1 are required for Maxwells Equation to be satisfied. The following physical arguments can be made:

* E&M waves are transverse from

$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$ in free space

then $\frac{\partial E_z}{\partial z} = \frac{\partial B_z}{\partial z} = 0$

this explains why no z-component

* $\vec{E}$ & $\vec{B}$ are $\perp$ to one another. This explains why

$\tilde{B}_{ox} = 0$ & δ_x' is unimportant

* From Faradays Law, $\vec{B}$ & $\vec{E}$ are in phase w/ one another

$\therefore\ \delta_x = \delta_y'$

* From Faraday's Law it shows that B_{oy} & E_{ox} are related as:

$B_{oy} = \frac{1}{c} E_{ox} = \frac{K_z}{\omega} E_{ox}$

3) If $\vec{E} = E_{oy} \cos(K_z z - \omega t + \delta_y)\hat{x}$

then if $\vec{E}$ still propogates in $\hat{x}$ direction, $B_{ox} = 0$ for $\vec{E} \perp \vec{B}$

From Faraday's Law:

$\therefore\ B_{oy} = \frac{1}{c} E_{oy} = \frac{K_z}{\omega} E_{oy}$

Both $\vec{B}$ & $\vec{E}$ must be in phase w/ one another, then:

$\delta_y' = \delta_y$

& δ_x' is still unimportant but most likely 0

6.3.4) Following From Faraday's Law, Proved in 6.3.1 using specific functions but can be generalized into 3-D:

If $\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{j(\vec{k}\cdot\vec{r} - \omega t)}$ * where $\tilde{\vec{E}}_0 = \sum_{l=1}^{3} E_{0l} e^{j\delta_l}$ where $l = 1, 2, 3 \rightarrow \hat{x}, \hat{y}, \hat{z}$

& $e^{j\delta_l}$ represent the phase of each complex vector

* Can use same arguments from 6.4 to prove for the general relationship:

$$\vec{B} = \frac{1}{c} \hat{k} \times \vec{E}$$

where the relations used for 6.3.1, 6.3.2, 6.3.3 are all valid and must be true for $\vec{B} = \frac{1}{c} \hat{k} \times \vec{E}$ to also hold true.