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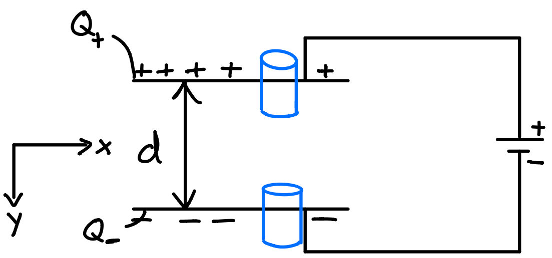
3.1.1. Large Parallel Plates

Gauss Law Derivation:

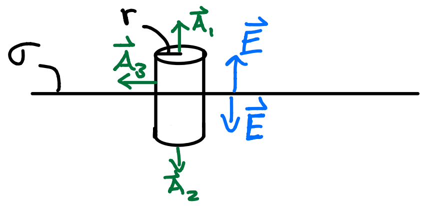
Assume charges distribute uniformly across surfaces



* Each has surface Area = A_{Tot}

2 Gaussian surfaces across each plate

Looking at top surface



$\vec{E}$ Field will be $\perp$ to surface when caps are very close to surface.

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\epsilon_0}$$

$$\int_{top} \vec{E} \cdot d\vec{A}_1 + \int_{bot} \vec{E} \cdot d\vec{A}_2 + \int_{side} E \cdot d\vec{A}_3 = \frac{\int \sigma dA}{\epsilon_0}$$

*charges are uniformly distributed, then σ is constant

* Because $\vec{E}$ is $\perp$ to surface, then $\vec{E} \perp d\vec{A}_3$, $\vec{E} \parallel \vec{A}_1$, $\vec{E} \parallel \vec{A}_2$

$$\int_{top} E dA_1 + \int_{bot} E dA_2 = \frac{\sigma \pi r^2}{\epsilon_0}$$

* Because charge is distributed uniformly, $\vec{E}$ is to be independent of position (const.)

$$E\left(\int dA_1 + \int dA_2\right) = \frac{\sigma \pi r^2}{\epsilon_0}$$

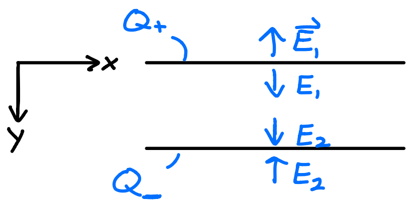
$$E(2\cancel{\pi r^2}) = \frac{\sigma \cancel{\pi r^2}}{\epsilon_0}$$

$$E = \frac{\sigma}{2\epsilon_0}$$ * This $\vec{E}$ field is the magnitude coming out of each side

$$\vec{E} = \frac{Q}{2 A_{Tot} \epsilon_0}$$

* Now looking at bottom plate, both sheets have equal and opposite charges & distributions.

Gauss Law will yield the same magnitude, but inverse directions as such:



* This causes $\vec{E}_{net} = \vec{E}_1 + \vec{E}_2$ to have 2 outcomes:

$$\vec{E}_{net} = \begin{cases} 2E_1\hat{y} & 0<y<d \\ 0 & y \notin (0,d) \end{cases}$$ * because $E_1 = E_2$

$$|\Delta V| = \int_0^d \vec{E}_{net} \cdot d\vec{l}$$

$$= \int_0^d 2E_1 \, dy \, (\hat{y}\cdot\hat{y})$$

$$= \int_0^d \frac{Q\,dy}{\cancel{2}A_{tot}\epsilon_0} \cdot \cancel{2}$$

$$|\Delta V| = \frac{Q\,d}{\epsilon_0 A_{tot}}$$ * Let $A_{tot} = A$

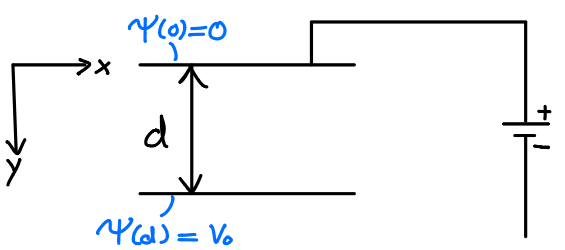
$$\boxed{C = \frac{Q}{|\Delta V|} = \frac{A\epsilon_0}{d}}$$

Boundary Method:

Solving Laplace's Eqn to find Potential Function

$\nabla^2 \psi = 0$

Considering only 2 plates:



$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

Now ψ is only dependent of y:

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial y^2} = C_1 y + C_2$$

Boundary Conditions:

I $\Psi(0) = 0 = C_1(0) + C_2$

$$\therefore C_2 = 0$$

II. $\Psi(d) = V_0 = C_1 d$

$$\therefore C_1 = \frac{V_0}{d}$$

$$\therefore \Psi(y) = \frac{V_0}{d} y$$

The $\vec{E}$ Field is found as:

$$\vec{E} = -\nabla \Psi(y) = -\frac{\partial \Psi(y)}{\partial y} \hat{y}$$

$$\vec{E} = -\frac{V_0}{d} \hat{y}$$

$$|\vec{E}| = \frac{|V_0|}{d}$$

When at the surface of a conductor:

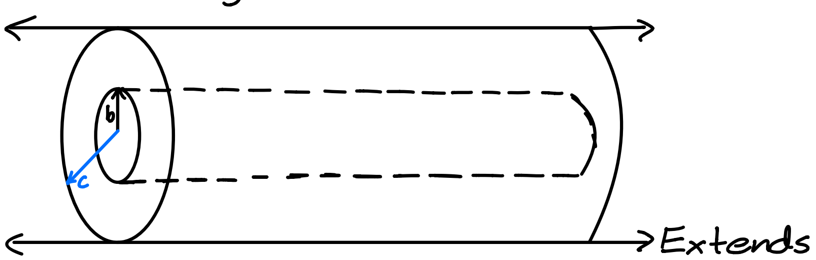
$$E_\perp = \frac{\sigma}{\epsilon_0}$$

$$\sigma = \frac{Q}{A} = \epsilon_0 E_\perp$$

$$\frac{Q}{A} = \epsilon_0 \frac{|V_0|}{d}$$

$$\boxed{\therefore C = \frac{Q}{|V_0|} = \frac{A\epsilon_0}{d}}$$ * which matches Gauss Law result

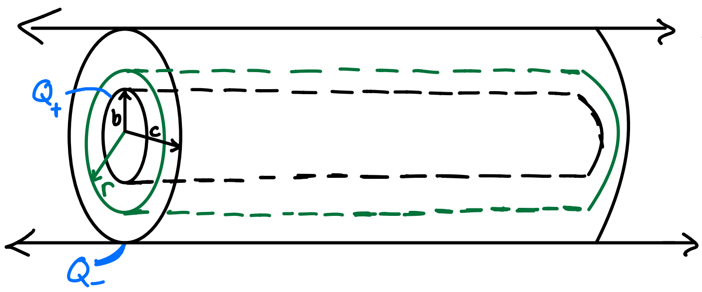
3.1.2 Long Coaxial Cylinders



Gauss Law Derivation:

Let $\begin{cases} Q_+ \text{ at } s=b \\ Q_- \text{ at } s=c \end{cases}$

* Putting a Gaussian cylinder around coaxial cable as such:

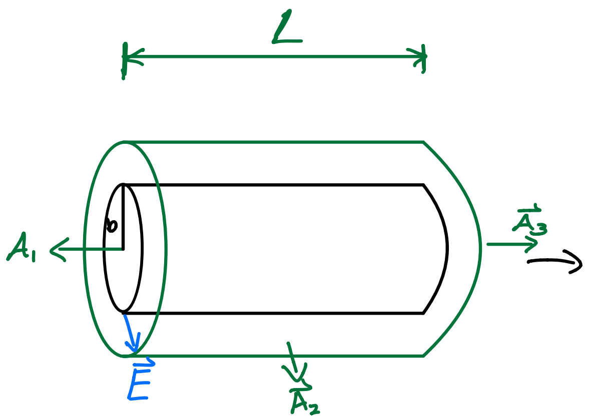


when $r<b$

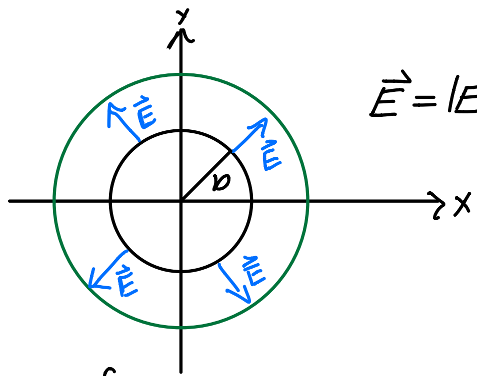
No charged enclosed:

$E=0$

when $b<r<c$



Top-down View



$\vec{E} = |E|\,\hat{s}$

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

$$\int_{top} \vec{E}\cdot dA_1 + \int_{side} \vec{E}\cdot d\vec{A_2} + \int_{bot} \vec{E}\cdot d\vec{A_3} = \frac{\int \sigma\, dA}{\epsilon_0}$$

* because charge is uniformly distributed: λ is constant

* $\vec{E}\perp d\vec{A_1}$, $\vec{E}\perp d\vec{A_3}$, $\vec{E}\parallel d\vec{A_2}$

$$\int_{side} E\, dA_2 = \frac{\sigma\, 2\pi b L}{\epsilon_0}$$

* because charges are uniformly distributed, E is constant

$$E \int_{side} dA_2 = \frac{2\pi b L \sigma}{\epsilon_0}$$

$$E\, 2\cancel{\pi} s \cancel{L} = \frac{\cancel{2\pi} b \cancel{L} \sigma}{\epsilon_0} = \frac{\cancel{b}}{\epsilon_0} \frac{Q}{2\pi \cancel{b} L}$$

$\therefore |E| = \frac{Q}{2\pi s L \epsilon_0}$ due to the inner cylinder

$\vec{E} = \frac{Q}{2\pi s L \epsilon_0} \hat{s}$

When $r > c$

The enclosed charge:

$Q_{encl} = Q_+ + Q_- = 0$

$\therefore E = 0$, for $r > b$

$|\Delta V| = \int_b^c \vec{E} \cdot d\vec{l}$ Let $dl = ds\,\hat{s}$

$|\Delta V| = \int_b^c \frac{Q}{s 2\pi \epsilon_0 L} ds = \frac{Q}{2\pi \epsilon_0 L} \int_b^c \frac{ds}{s}$

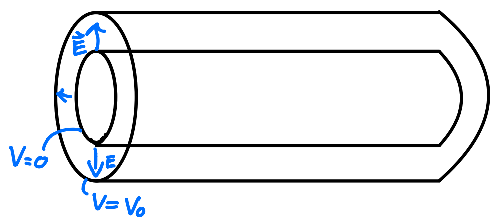
$|\Delta V| = \frac{Q}{2\pi \epsilon_0 L} \ln\left(\frac{c}{b}\right)$

$C = \frac{Q}{|\Delta V|} = \frac{2\pi \epsilon_0 L}{\ln\left(\frac{c}{b}\right)}$

$$\boxed{\therefore \frac{C}{L} = \frac{2\pi \epsilon_0}{\ln(c/b)}}$$

Boundary Value Method:

Assuming outer cylinder at potential V_0 while inner cylinder is considered to be at potential 0.



Must find potential function by solving the Laplace equation in Cylindrical coordinates:

$\nabla^2 \psi = 0$

$\Rightarrow \frac{1}{s} \frac{\partial}{\partial s}\left(s \frac{\partial \psi}{\partial s}\right) + \frac{1}{s^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} = 0$

ψ is the potential function w/ the following conditions:

I. $\Psi(b) = 0$

II. $\Psi(c) = V_0$

With the charges being uniformly distributed, equipotential lines are perpendicular to $\vec{E}$ field lines. Ψ will only depend on s.

$$\Psi(s, \phi, z) \Rightarrow \Psi(s)$$

$$\therefore \nabla^2 \Psi = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial \Psi}{\partial s}\right) = 0$$

then $$s\frac{\partial \Psi}{\partial s} = C_1$$

$$\frac{\partial \Psi}{\partial s} = \frac{1}{s}C_1$$

$$\Psi(s) = \ln(s)C_1 + C_2$$

Applying Boundary Conditions:

I. $\Psi(a) = 0 = C_1 \ln(a) + C_2$

$$C_2 = -C_1 \ln(a)$$

$$\Psi(s) = C_1 \ln(s) - C_1 \ln(a)$$

$$\Psi(s) = C_1 \ln\left(\frac{s}{b}\right)$$

II. $\Psi(c) = V_0 = C_1 \ln(c/b)$

$$C_1 = \frac{V_0}{\ln(c/b)}$$

$$\therefore \Psi(s) = \frac{V_0 \ln(s/b)}{\ln(c/b)} = \frac{V_0}{\ln(c/b)}\left(\ln(s) - \ln(b)\right)$$

$$\vec{E} = -\nabla\Psi(s)$$

$$= -\left(\frac{\partial \Psi}{\partial s}\hat{s} + \frac{1}{s}\cancel{\frac{\partial \Psi}{\partial \phi}}^{0}\hat{\phi} + \cancel{\frac{\partial \Psi}{\partial z}}^{0}\hat{z}\right)$$

$$\vec{E}(s) = -\frac{V_0}{\ln(c/b)}\left(\frac{1}{s}\right)\hat{s}$$

Now when near the surface of the conductor:

$$\vec{E} = E_\perp \hat{s} \quad \& \quad E_\perp = \frac{\sigma}{\epsilon_0}$$

$$\approx \frac{V_0}{\ln(c/b)}\frac{1}{b}$$

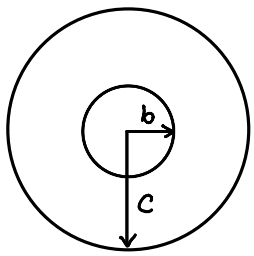
* $s \approx b$ when near the surface of the inner cylinder

$$\sigma = E_\perp \epsilon_0 = \frac{Q}{A}$$

$$\approx \frac{V_0 \epsilon_0}{b \ln(c/b)} = \frac{Q}{2\pi b L}$$

$$\boxed{\therefore \frac{Q}{V_0 L} = \frac{C}{L} = \frac{2\pi\epsilon_0}{\ln(b/c)}}$$

3.1.3. Concentric Spherical Shells



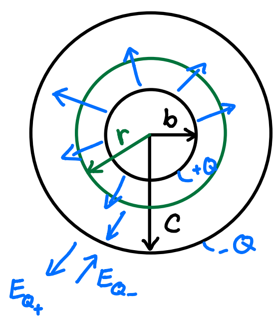
Let 2 concentric spherical shell conductors, Find the capacitance.

Gauss Law Derivation:

Place 2 charges:

Q_- at c
Q_+ at b

Create a Gaussian Sphere w/ radiur r.



when $r<b$:

$Q_{encl} = 0$

$\therefore E = 0$

when $b<r<c$

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

*charges are distributed evenly across surface, causing $\vec{E} = |\vec{E}_{Q_+}|\hat{r}$ & charge density = const

$$\int_{Gauss} E_{Q+}\, dA = \frac{\sigma_{sphere}\int dA}{\epsilon_0}$$

* Because charges are uniformly distributed. E_{Q+} is const everywhere along surface

$$E_{Q+} \int_{Gauss} dA = \frac{\sigma \int_{sphere} dA}{\epsilon_0}$$

$$E_{Q+}\, \cancel{4\pi} r^2 = \frac{\sigma\, \cancel{4\pi} a^2}{\epsilon_0}$$

$$E_{Q+} = \frac{\sigma a^2}{r^2 \epsilon_0} = \frac{Q}{4\pi \cancel{a^2}} \frac{\cancel{a^2}}{r^2 \epsilon_0}$$

$$\therefore E_{Q+} = \frac{Q}{4\pi r^2 \epsilon_0}$$

$$|\Delta V| = \left| \int \vec{E} \cdot d\vec{l} \right|$$

$$= \int_c^b \frac{Q}{4\pi r^2 \epsilon_0}\, dr\ \hat{r} \cdot \hat{r}$$

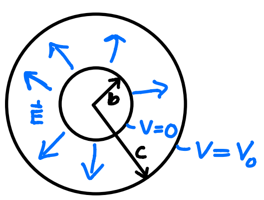
$$= \left| \frac{Q}{4\pi \epsilon_0} \left(\frac{-1}{r}\right)\Big|_b^c \right|$$

$$|\Delta V| = \frac{Q}{4\pi \epsilon_0} \left(\frac{1}{c} - \frac{1}{b}\right)$$

$$\boxed{C = \frac{Q}{|\Delta V|} = \frac{4\pi \epsilon_0}{\left(\frac{1}{c} - \frac{1}{b}\right)}}$$

Boundary Value Method :

Assuming outer sphere has a potential V_0, and inner sphere is at a potential of 0.



Must find potential by solving Laplace Equation in spherical coord.

$$\nabla^2 \Psi(r) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Psi}{\partial \phi^2}$$

Because charges are uniformly distributed

Ψ only varies in r direction, $\Psi(r,\theta,\phi)=\Psi(r)$

$$\therefore \nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)=0$$

then: $$r^2\frac{\partial\Psi}{\partial r}=C_1$$

$$\frac{\partial\Psi}{\partial r}=\frac{C_1}{r^2}$$

$$\Psi=-\frac{C_1}{r}+C_2$$

Applying Boundary Conditions:

I. $$\Psi(b)=-\frac{C_1}{b}+C_2=0$$

$$C_2=\frac{C_1}{b}$$

$$\Psi(r)=-\frac{C_1}{r}+\frac{C_1}{b}=C_1\left(\frac{1}{b}-\frac{1}{r}\right)$$

II. $$\Psi(c)=C_1\left(\frac{1}{b}-\frac{1}{c}\right)=V_0$$

$$\therefore C_1=V_0\left(\frac{1}{b}-\frac{1}{c}\right)^{-1}$$

$$\therefore \Psi(r)=V_0\left(\frac{1}{b}-\frac{1}{c}\right)^{-1}\left(\frac{1}{b}-\frac{1}{r}\right)$$

$$\vec{E}(r)=-\vec{\nabla}\Psi(r)$$

$$=-\frac{\partial}{\partial r}\Psi(r)\,\hat{r}$$

$$=-V_0\left(\frac{1}{b}-\frac{1}{c}\right)^{-1}\left(\frac{1}{r^2}\right)\hat{r}$$

$$|\vec{E}|=\frac{V_0}{\left(\frac{1}{c}-\frac{1}{b}\right)r^2}$$

When looking at the $\vec{E}$ Field near the surface of the conductor:

$|\vec{E}|=E_\perp$ where $r\approx b$

$$\sigma=\epsilon_0 E_\perp$$

$$\sigma=\frac{V_0\epsilon_0}{\left(\frac{1}{c}-\frac{1}{b}\right)\cancel{b^2}}=\frac{Q}{4\pi\cancel{b^2}}$$

$$\boxed{C=\frac{Q}{|V_0|}=\frac{4\pi\epsilon_0}{\left(\frac{1}{c}-\frac{1}{b}\right)}}$$