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ECE 513

HW 6.2

Given $\vec{E} = E_{ox}(x,t)\hat{x} + E_{oy}(x,t)\hat{y} + E_{oz}(x,t)\hat{z}$

& $\vec{B} = B_{ox}(x,t)\hat{x} + B_{oy}(x,t)\hat{y} + B_{oz}(x,t)\hat{z}$

6.2.1 Show E_y, E_z, B_y, B_z individually obey the wave equation:

$$\frac{\partial^2 f}{\partial u^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

where $f = E_y(x,t),\ E_z(x,t),\ B_y(x,t),\ B_z(x,t)$

where all possible values of f are only dependent of x & t

$$\therefore \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$$

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f(x,t)}{\partial t^2}$$

*for $f(x,t)$ to obey/satisfy the wave equation, then any function of the form:

$$f(x,t) = g(x \pm vt) = g(u_\pm) \quad * \text{ Let } u = x - vt$$

$$\frac{\partial f}{\partial x} = \frac{dg}{du_\pm}\frac{du_\pm}{dx} \qquad * \frac{du_\pm}{dx} = 1$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{d}{dx}\left(\frac{dg(u_\pm)}{du_\pm}\right) \qquad \text{*chain rule}$$

$$= \frac{d}{du_\pm}\left(\frac{dg}{du_\pm}\right)\cancel{\frac{du}{dx}}^{\,1}$$

$$= \frac{d^2 g}{du_\pm^2}$$

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial g}{\partial u_\pm}\frac{\partial u_\pm}{\partial t} \qquad * \frac{\partial u_\pm}{\partial t} = \pm v$$

$$\frac{\partial f}{\partial t} = \pm v \frac{\partial g}{\partial u_\pm}$$

$$\frac{\partial^2 f}{\partial t^2} = \pm v \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial u_\pm}\right) \qquad \text{*chain rule}$$

$$= \pm v \frac{\partial^2 g}{\partial u_\pm^2}\cancel{\frac{\partial u_\pm}{\partial t}}^{\,\pm v}$$

$$= v^2 \frac{\partial^2 g}{\partial u_\pm^2}, \quad \text{for E\&M waves in free space } v = c$$

$$\therefore \quad \frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

$$\Rightarrow \quad \frac{\partial^2 g}{\partial u^2} = \frac{1}{\cancel{c^2}}\cancel{c^2}\frac{\partial g}{\partial u}$$

$$\frac{\partial^2 g}{\partial u^2} = \frac{\partial^2 g}{\partial u^2}$$

$\therefore$ If $f(x,t) = E_y(x,t), E_z(x,t), B_y(x,t), B_z(x,t)$ are some linear combination of the 2 general solutions:

$$f(x,t) = g(x-vt) + h(x+vt)$$

then $f(x,t)$ is also a solution and satisfies the wave equation.

6.2.2) Because $\vec{E}$ & $\vec{B}$ are waves, and the wave equation given has a velocity of c, then the waves are in free space. This would in turn require no charges or sources, for waves to to propagate at that velocity.

When no source currents or charges:

$$\rho = 0, \quad \vec{J} = 0$$

$$\therefore \quad \vec{\nabla}\cdot\vec{E} = 0$$
$$\vec{\nabla}\cdot\vec{B} = 0$$

From 6.2.1, it is found that the y & z components of E & B satisfy the wave equation.

Because E_y, E_z, B_y, B_z all dependent on x and are waves, then they some linear combination of the general solution:

$f(x,t) = g(x-vt) + h(x+vt)$ *where x is the axis of propagation.

From Maxwell's Eqn

$$\vec{\nabla}\cdot\vec{E} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\frac{\partial E}{\partial x} = \frac{\partial B}{\partial x} = 0$$

Then because $\vec{E}$ & $\vec{B}$ propagate in x-direction:

$E_x = B_x = 0$ * $\vec{E}$ & $\vec{B}$ must be perpendicular to direction of propagation

6.2.3) For waves to propagate at speed of light:

$\rho = 0, \ \vec{J} = 0$ * free space

& $\epsilon = \epsilon_0$ & $\mu = \mu_0$

$$\nabla^2 \vec{E} = (\nabla^2 E_x \hat{x} + \nabla^2 E_y \hat{y} + \nabla^2 E_z \hat{z})$$

* Laplace Operacts acts on each component:

$$\nabla^2 \vec{E} = \left(\frac{\partial^2 E_x}{\partial x^2} + \cancel{\frac{\partial^2 E_x}{\partial y^2}} + \cancel{\frac{\partial^2 E_x}{\partial z^2}}\right)\hat{x} + \left(\frac{\partial^2 E_y}{\partial x^2} + \cancel{\frac{\partial^2 E_y}{\partial y^2}} + \cancel{\frac{\partial^2 E_y}{\partial z^2}}\right)\hat{y} + \left(\frac{\partial^2 E_z}{\partial x^2} + \cancel{\frac{\partial^2 E_z}{\partial y^2}} + \cancel{\frac{\partial^2 E_z}{\partial z^2}}\right)\hat{z}$$

* where each B_x, B_y, B_z component is dependent on only x, t

$$\therefore \nabla^2 \vec{E} = \left(\frac{\partial^2 E_x}{\partial x^2}\hat{x} + \frac{\partial^2 E_y}{\partial x}\hat{y} + \frac{\partial^2 E_z}{\partial x^2}\hat{z}\right)$$

Similarly $\nabla^2 \vec{B}$ simplifies:

$$\nabla^2 \vec{B} = \left(\frac{\partial^2 B_x}{\partial x^2}\hat{x} + \frac{\partial^2 B_y}{\partial y}\hat{y} + \frac{\partial^2 B_z}{\partial z^2}\hat{z}\right)$$

* using arguments obtained from 6.2.2:

since $E_x = B_x = 0$

Then wave equation simplifies too:

$$\nabla^2 \vec{E} = \frac{\partial^2 E_y}{\partial x^2}\hat{y} + \frac{\partial^2 E_z}{\partial x^2}$$

$$\nabla^2 \vec{B} = \frac{\partial^2 B_y}{\partial x^2}\hat{y} + \frac{\partial^2 B_z}{\partial x^2}\hat{z}$$

Then comparing components:

① $$\frac{\partial^2 E_y}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2}$$

② $$\frac{\partial^2 E_z}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

③ $$\frac{\partial^2 B_y}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2}$$

④ $$\frac{\partial^2 B_z}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2}$$

* Can use results 1 to prove solutions again:

If $\vec{E}$ & $\vec{B}$ are some linear combination of general solution to wave equation then ①-④ satisfy the individual wave equations