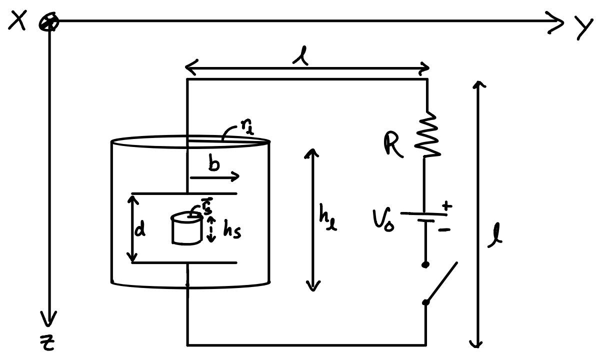
5.2 Poynting Theorem

The RC circuit given:



Poynting theorem from Wiley states:

$$\int_V \left[\frac{\partial}{\partial t}\left(\frac{\vec{B}\cdot\vec{H}}{2}\right) + \frac{\partial}{\partial t}\left(\frac{\vec{D}\cdot\vec{E}}{2}\right) + \vec{E}\cdot\vec{J}\right] dV = \oint_S (\vec{E}\times\vec{H})\cdot d\vec{s}$$

If we consider the distance d between the plates small enough then the $\vec{E}$-field will be $\perp$ to surfaces of the capacitor plates and $b \gg d$:

$$\vec{E} \approx \frac{\sigma}{\epsilon_0}\hat{z} = \frac{Q\hat{z}}{A\epsilon_0}$$

Now when the switch is closed, can assume capacitor is empty then:

$$V_c(t) = V_0(1 - e^{-t/RC})$$

Where $V(t)$ is the voltage drop across the capacitor and will slowly increase over time:

$$\therefore\ C = \frac{Q}{V} \rightarrow Q = C\,V(t)$$

$$Q(t) = CV_0(1 - e^{-\frac{t}{RC}})$$

Next is current across capacitor is:

$\vec{J} = 0$, only current is diffusion current inside

$$\vec{E}(t) = \frac{\sigma}{\epsilon_0}\hat{z} = \frac{Q(t)}{A\epsilon_0}\hat{z} = \frac{CV_0}{A\epsilon_0}(1 - e^{-\frac{t}{RC}})\hat{z}$$

$$\vec{E}(t) = \frac{CV_0}{\pi b^2\epsilon_0}(1 - e^{-\frac{t}{RC}})\hat{z}$$

$$\vec{E}(t) = E_z(t)\hat{z}$$

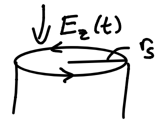
Maxwell's eqn says:

$$\vec{\nabla}\times\vec{B} = \cancelto{0}{\mu_0\vec{J}} + \epsilon_0\mu_0\frac{\partial E}{\partial t}$$

$$\oint\vec{B}\cdot d\vec{l} = \epsilon_0\mu_0\frac{\partial\Phi_E}{\partial t}$$ * flux flowing through cap at radius r_s

$$B\cdot 2\cancel{\pi}\cancel{r_s} = \epsilon_0\mu_0\cancel{\pi}r_s^{\cancel{2}}\frac{\partial E_z(t)}{\partial t}$$

$$B = \frac{r_s}{2c}\frac{\partial E_z(t)}{\partial t}$$



$$\vec{B} = B\hat{\phi}$$

$$-\vec{E}\times\vec{H} = \frac{+1}{\mu_0}|\vec{E}||\vec{B}|\cancelto{+\hat{s}}{(\hat{z}\times\hat{\phi})}$$

$$= \frac{1}{\mu_0}\left(\frac{r_s}{c}\right)\frac{1}{2}E_z(t)\frac{\partial E_z(t)}{\partial t}$$ * where c = speed of light

$$= \frac{1}{\mu_0}\frac{r_s}{2c}\left(\frac{1}{2}\right)\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E})$$

$$\oint(\vec{E}\times\vec{H})\cdot d\vec{s} = \int_{side}\frac{1}{\mu_0}\frac{r_s}{4c}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E})\,dA$$

* Top & bottom cap area integrals are 0.

* $E_z(t)$ is dependent only on time, & $dA_{side} = r\,d\phi\,dz$ for side, then r_s is a const.

$$-\oint(\vec{E}\times\vec{H})\cdot d\vec{s} = \frac{r_s}{\mu_0 \cancelto{2}{4}c}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E})\,\cancel{2}\pi r_s h_s = \frac{\pi r_s^2 h_s}{\mu_0 2c}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E})$$

$$\therefore -\oint(\vec{E}\times\vec{H})\cdot d\vec{s} = \frac{V}{\mu_0 2c}\frac{\partial(E_z^2(t))}{\partial t} = \frac{\epsilon_0}{2}V\frac{\partial}{\partial t}(E_z^2)$$ * V = Volume of Gaussian cylinder

For left hand side 3 expressions must be found

$$\frac{\partial}{\partial t}\left(\frac{\vec{D}\cdot\vec{E}}{2}\right) = \frac{\epsilon_0}{2}\frac{\partial(E^2)}{\partial t}$$

$$\frac{\partial}{\partial t}\left(\frac{\vec{H}\cdot\vec{B}}{2}\right) = \vec{H}\cdot\frac{\partial(\vec{B})}{\partial t} = \frac{B}{\mu_0}\frac{\partial B}{\partial t}$$

$$= \frac{1}{\mu_0}\left(\frac{r_s}{2c}\frac{\partial E_z}{\partial t}\right)\left(\frac{r_s}{2c}\frac{\partial^2 E_z}{\partial t^2}\right)$$

$$= \frac{r_s^2}{4c^2\mu_0}\frac{\partial}{\partial t}\left(E_z\frac{\partial E_z}{\partial t}\right)$$

$$= \frac{r_s^2}{4c^2\mu_0}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left(\frac{E_z^2}{2}\right)\right)$$

$$\therefore \frac{\partial}{\partial t}\left(\frac{\vec{H}\cdot\vec{B}}{2}\right) = \frac{r_s^2}{4c^2\mu_0}\frac{\partial^2}{\partial t^2}\left(\frac{E_z^2}{2}\right)$$

Substituting values

$$\int_V \left[\frac{\partial}{\partial t}\left(\frac{\vec{B}\cdot\vec{H}}{2}\right)+\frac{\partial}{\partial t}\left(\frac{\vec{D}\cdot\vec{E}}{2}\right)+\vec{E}\cdot\vec{J}\right]dV=-\oint_S(\vec{E}\times\vec{H})\cdot d\vec{s}$$

$$\int_V\left[\frac{r_s^2}{8c^3\mu_0}\ \frac{\partial^2}{\partial t^2}(E_z^2)+\frac{\epsilon_0}{2}\frac{\partial}{\partial t}(E_z^2)\right]dV=\frac{\epsilon_0}{2}V\frac{\partial}{\partial t}(E_z^2)$$

* for second term, because E_z is only dependent on time, derivative is a constant

$$\frac{1}{8c^2\mu_0}\ \frac{\partial}{\partial t}\left(\frac{E_z^2}{2}\right)\int_{r=0}^{r_s}\int_{\phi=0}^{2\pi}\int_{z=0}^{h_s} r_s^2\, s\,dr\,d\phi\,dz+\cancel{\frac{\epsilon_0}{2}\left(\frac{\partial E_z^2}{\partial t}\right)V}=\cancel{\frac{\epsilon_0}{2}V\frac{\partial}{\partial t}(E_z^2)}$$

$$\frac{1}{8c^2\mu_0}\ \frac{\partial}{\partial t}(E_z^2)(\pi r^2 h_s)\ =0$$

$$\frac{\pi r_s^2 h_s}{8c^2\mu_0}\left(\frac{C_1 V_0}{\pi b^2}\frac{1}{\epsilon_0}\right)^2\frac{\partial}{\partial t}\left(1-e^{-\frac{t}{RC}}\right)$$

* Let C_1 = Capacitance
c = speed of light

$$\frac{C_1^2V_0^2}{8}\left(\frac{1}{c^2\mu_0\epsilon_0^2}\right)\frac{A_{Gauss}}{A^2_{capacitor\ plate}}$$ where $r_s \ll b \rightarrow \frac{r_s}{b^2}\approx 0$

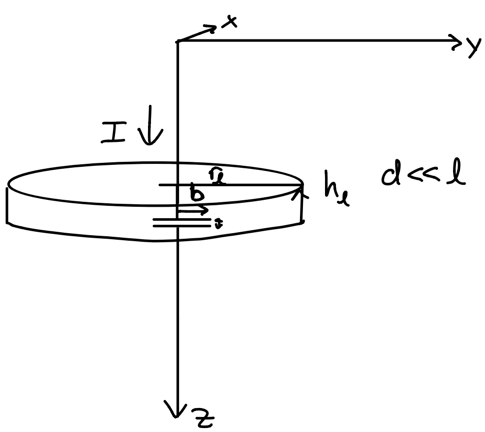
$\therefore$ 1st term approximates to 0 as $t\rightarrow\infty$

Poynting theorem holds inside conductor:

Outside Conductor:

using Gaussian cylinder where:

$r_\ell > b$ & $d<h_\ell<\ell$



as time increases RC circuit current decreases

$I=\frac{V_0}{R}e^{-t/RC}$ at $t=0\rightarrow I=I_0$

$\vec{J}=\frac{\vec{I}}{a_\perp}$ where $a_\perp$ is cross sectional area of wire from Ohms Law:

$\vec{J} = \sigma \vec{E}$

$\vec{E} = \frac{\vec{J}}{\sigma}$ if σ for wire $\gg |\vec{J}|$ then $\vec{E} \approx 0$ (where σ is conductivity)

$\therefore \int \vec{E} \cdot \vec{J} \, dV = 0$ & $\frac{d}{dt}\left(\frac{E^2}{2}\right) = \frac{d}{dt}\left(\frac{J^2}{\sigma^2}\right) \approx 0$

$B_\phi(t) = \frac{\mu_0 I(t)}{2\pi r_2}$

$B_\phi(t) = \frac{\mu_0}{2\pi r_2}\left(\frac{V_0}{R}\right)e^{-t/RC}$ $\quad * \; I_0 = \frac{V_0}{R}$

$$\frac{\partial}{\partial t}\left(\frac{\vec{B}\cdot\vec{B}}{\mu_0}\right) = \frac{1}{\mu_0}\frac{\partial}{\partial t}\left[\left(\frac{\mu_0}{2\pi r_2}\right)^2 I_0^2 \, e^{-\frac{2t}{RC}}\right]$$

$$= \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\left[\frac{I^2(t)}{\pi r_2^2}\right]$$

$* \; I(t) = \vec{J} a_\perp$ if $a_\perp << \pi r_2^2$

then $I^2(t) \approx 0$

$\therefore \frac{1}{\mu_0}\frac{d}{dt}(B^2) \approx 0$

& Because $\vec{E} \approx 0$ then Poynting's Theorem holds as

$$\int \frac{d}{dt}\cancelto{0}{\left(\frac{B^2}{2\mu_0}\right)} + \frac{d}{dt}\cancelto{0}{\left(\frac{\epsilon E^2}{2}\right)} + \cancelto{0}{\vec{E}\cdot\vec{J}} \, dV = \int \cancelto{0}{(\vec{E}\times\vec{H})}\cdot d\vec{A}$$

$\therefore$ all terms approximate to 0