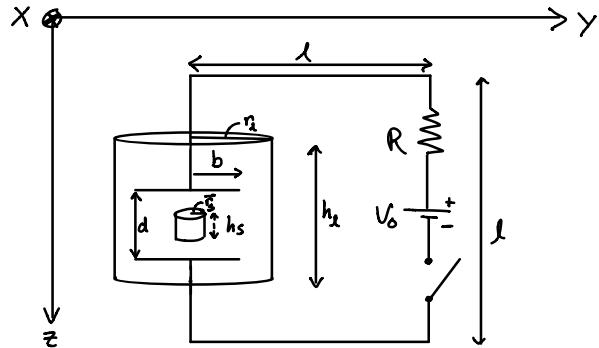


5.2 Poynting Theorem

The RC circuit given:



Poynting theorem from Wiley states:

$$\int_V \left[\frac{\partial}{\partial t} \left(\frac{\vec{B} \cdot \vec{H}}{2} \right) + \frac{\partial}{\partial t} \left(\frac{\vec{D} \cdot \vec{E}}{2} \right) + \vec{E} \cdot \vec{j} \right] dV = \oint_S (\vec{E} \times \vec{H}) \cdot d\vec{s}$$

If we consider the distance d between the plates small enough then the \vec{E} -field will be \perp to surfaces of the capacitor plates and $b \gg d$:

$$\vec{E} \approx \frac{\sigma}{\epsilon_0} \hat{z} = \frac{Q \hat{z}}{A \epsilon_0}$$

Now when the switch is closed, can assume capacitor is empty then:

$$V_c(t) = V_0 (1 - e^{-\frac{t}{RC}})$$

Where $V(t)$ is the voltage drop across the capacitor and will slowly increase over time:

$$\therefore C = \frac{Q}{V} \rightarrow Q = CV(t)$$

$$Q(t) = CV_0 (1 - e^{-\frac{t}{RC}})$$

Next is current across capacitor is:

$\vec{j} = 0$, only current is diffusion current inside

$$\vec{E}(t) = \frac{\sigma}{\epsilon_0} \hat{z} = \frac{Q(t)}{A \epsilon_0} \hat{z} = \frac{CV_0}{A \epsilon_0} (1 - e^{-\frac{t}{RC}}) \hat{z}$$

$$\vec{E}(t) = \frac{CV_0}{\pi b^2 \epsilon_0} (1 - e^{-\frac{t}{RC}}) \hat{z}$$

$$\vec{E}(t) = E_z(t) \hat{z}$$

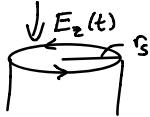
Maxwell's eqn says:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

$$\oint \vec{B} \cdot d\vec{l} = \epsilon_0 \mu_0 \frac{\partial \vec{E}_z}{\partial t} \quad * \text{ flux flowing through cap at radius } r_s$$

$$B \cdot 2\pi r_s = \epsilon_0 \mu_0 \pi r_s^2 \frac{\partial E_z(t)}{\partial t}$$

$$B = \frac{r_s}{2c} \frac{\partial E_z(t)}{\partial t}$$



$$\vec{B} = B \hat{z}$$

$$-\vec{E} \times \vec{H} = \frac{1}{\mu_0} |\vec{E}| |\vec{B}| (\vec{E} \times \hat{z})$$

$$= \frac{1}{\mu_0} \left(\frac{r_s}{c} \right) \frac{1}{2} E_z(t) \frac{\partial E_z(t)}{\partial t} \quad * \text{ where } c = \text{speed of light}$$

$$= \frac{1}{\mu_0} \frac{r_s}{2c} \left(\frac{1}{2} \right) \frac{\partial}{\partial t} (\vec{E} \cdot \vec{E})$$

$$\oint (\vec{E} \times \vec{H}) \cdot d\vec{s} = \int_{\text{side}} \frac{1}{\mu_0} \frac{r_s}{4c} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{E}) dA$$



* Top & bottom cap area integrals are 0.

* $E_z(t)$ is dependent only on time, & $dA_{\text{side}} = r d\phi dz$ for side, then r_s is a const.

$$-\oint (\vec{E} \times \vec{H}) \cdot d\vec{s} = \frac{r_s}{\mu_0 c} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{E}) 2\pi r_s h_s = \frac{\pi r_s^2 h_s}{\mu_0 2c} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{E})$$

$$\therefore \oint (\vec{E} \times \vec{H}) \cdot d\vec{s} = \frac{V}{\mu_0 2c} \frac{\partial (E_z^2(t))}{\partial t} = \frac{r_s}{2} V \frac{\partial}{\partial t} (E_z^2) \quad * V = \text{volume of Gaussian cylinder}$$

for left hand side 3 expressions must be found

$$\frac{\partial}{\partial t} \left(\frac{\vec{D} \cdot \vec{E}}{2} \right) = \frac{\epsilon_0}{2} \frac{\partial (E_z^2)}{\partial t}$$

$$\frac{\partial}{\partial t} \left(\frac{\vec{H} \cdot \vec{B}}{2} \right) = \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{B}{\mu_0} \frac{\partial B}{\partial t}$$

$$= \frac{1}{\mu_0} \left(\frac{r_s}{2c} \frac{\partial E_z}{\partial t} \right) \left(\frac{r_s}{2c} \frac{\partial^2 E_z}{\partial t^2} \right)$$

$$= \frac{r_s^2}{4c^2 \mu_0} \frac{\partial}{\partial t} (E_z \frac{\partial E_z}{\partial t})$$

$$= \frac{r_s^2}{4c^2 \mu_0} \frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \left(\frac{E_z^2}{2} \right) \right)$$

$$\therefore \frac{\partial}{\partial t} \left(\frac{\vec{H} \cdot \vec{B}}{2} \right) = \frac{r_s^2}{4c^2 \mu_0} \frac{\partial^2}{\partial t^2} \left(\frac{E_z^2}{2} \right)$$



Substituting values

$$\int \left[\frac{\partial}{\partial t} \left(\frac{\vec{B} \cdot \vec{H}}{2} \right) + \frac{\partial}{\partial t} \left(\frac{\vec{D} \cdot \vec{E}}{2} \right) + \vec{E} \cdot \vec{J} \right] dV = - \oint_S (\vec{E} \times \vec{H}) \cdot d\vec{s}$$

$$\int \left[\frac{r_s^2}{8\epsilon_0 \mu_0} \frac{\partial^2}{\partial t^2} (E_z^2) + \frac{\epsilon_0}{2} \frac{\partial}{\partial t} (E_z^2) \right] dV = \frac{\epsilon_0}{2} V \frac{\partial}{\partial t} (E_z^2)$$

* for second term, because E_z is only dependent on time, derivative is a constant

$$\frac{1}{8\epsilon_0 \mu_0} \frac{\partial}{\partial t} (E_z^2) \int_{r=0}^{r_s} \int_{\theta=0}^{2\pi} \int_{z=0}^{h_s} r_s^2 \epsilon_0 \mu_0 \frac{\partial E_z^2}{\partial t} V = \frac{\epsilon_0}{2} V \frac{\partial}{\partial t} (E_z^2)$$

$$\frac{1}{8\epsilon_0 \mu_0} \frac{\partial}{\partial t} (E_z^2) (\pi r_s^2 h_s) = 0$$

$$\frac{\pi r_s^2 h_s}{8\epsilon_0 \mu_0} \left(\frac{C}{\pi b^2 \epsilon_0} \right)^2 \frac{1}{\partial t} (1 - e^{-\frac{t}{RC}}) * \text{Let } C = \text{Capacitance}$$

$c = \text{speed of light}$

$$\frac{C^2 V_0^2}{8} \left(\frac{1}{c^2 \mu_0 \epsilon_0} \right) \frac{A_{\text{Gauss}}}{A_{\text{capacitor plate}}} \text{ where } r_s \ll b \rightarrow \frac{r_s}{b} \approx 0$$

$\therefore 1^{\text{st}}$ term approximates to 0
as $t \rightarrow \infty$

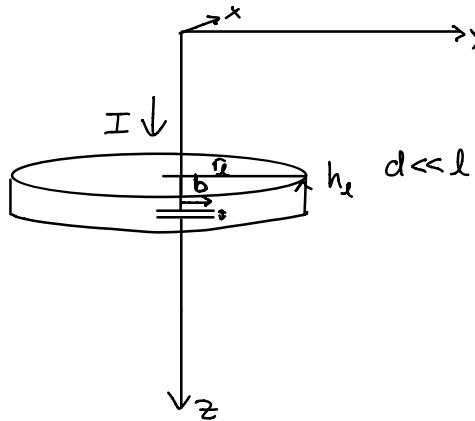
Poynting theorem holds inside conductor :

Outside Conductor :

using Gaussian cylinder where :



$$r_s > b \quad \& \quad d < h_s < l$$



as time increases RC circuit current decreases

$$I = \frac{V_0}{R} e^{-\frac{t}{RC}} \quad \text{at } t=0 \rightarrow I=I_0$$

$$\vec{J} = \frac{\vec{I}}{a_{\perp}} \quad \text{where } a_{\perp} \text{ is cross sectional area of wire}$$

from Ohms Law:

$$\vec{J} = \sigma \vec{E}$$

$\vec{E} = \frac{\vec{J}}{\sigma}$ if σ for wire $\gg |\vec{J}|$ then $\vec{E} \approx 0$ (where σ is conductivity)

$$\therefore \int \vec{E} \cdot \vec{J} dV = 0 \quad \& \quad \frac{d}{dt} \left(\frac{E^2}{2} \right) = \frac{d}{dt} \left(\frac{J^2}{\sigma^2} \right) \approx 0$$

$$B_p(t) = \frac{\mu_0 I(t)}{2\pi r_L}$$

$$B_p(t) = \frac{\mu_0}{2\pi r_L} \left(\frac{V_o}{R} \right) e^{-\frac{t}{RC}} \quad * \quad I_o = \frac{V_o}{R}$$

$$\frac{\partial}{\partial t} \left(\frac{\vec{B} \cdot \vec{B}}{2} \right) = \frac{1}{\mu_0} \frac{\partial}{\partial t} \left[\left(\frac{\mu_0}{2\pi r_L} \right)^2 I_o^2 e^{-\frac{2t}{RC}} \right]$$

$$= \frac{\mu_0}{4\pi} \frac{\partial}{\partial t} \left[\frac{I_o^2 t}{\pi r_L^2} \right]$$

$$* \quad I(t) = \vec{J} a_L \quad \text{if } a_L \ll \pi r_L^2$$

$$\text{then } I^2(t) \approx 0$$

$$\therefore \frac{1}{\mu_0} \frac{d}{dt} (B^2) \approx 0$$

& Because $\vec{E} \approx 0$ then Poynting's Theorem holds as

$$\int \frac{d}{dt} \left(\frac{B^2}{2\mu_0} \right) + \frac{d}{dt} \left(\frac{\epsilon E^2}{2} \right) + \vec{E} \cdot \vec{J} dV = \int (\vec{E} \times \vec{H}) \cdot d\vec{A}$$

\therefore all terms approximate to 0