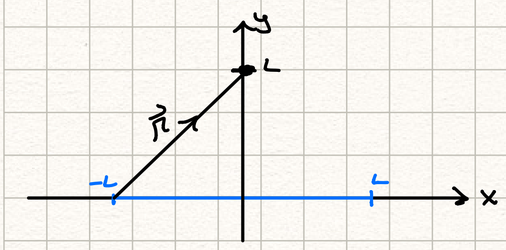
1.31 Exact solution for a finite line of charge



$$\vec{E} = \frac{KQ}{r^2}\hat{r} = \frac{K\int dq\, \vec{\mathscr{r}}}{\mathscr{r}^3}$$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ $\qquad$ $\vec{\mathscr{r}} = (-x')\hat{x} + L\hat{y}$

$\vec{r} = L\hat{y}$ $\qquad$ $|\vec{\mathscr{r}}| = \sqrt{(x')^2 + L^2}$

$\vec{r}\,' = x'\hat{x}$

$dl' = dx'$

$$= \frac{1}{4\pi\epsilon_0}\int_{-L}^{L} \frac{\lambda_0\, dl' \vec{\mathscr{r}}}{\mathscr{r}^3}$$

$$= \frac{\lambda_0}{4\pi\epsilon_0}\int_{-L}^{L} \frac{-x'\hat{x} + L\hat{y}}{((x')^2 + (L^2))^{3/2}}\, dx'$$

$$= \frac{\lambda_0}{4\pi\epsilon_0}\left[\int_{-L}^{L} \frac{-x'\, dx}{((x')^2 + L^2)^{3/2}}\,\hat{x} + \int_{-L}^{L} \frac{L\, dx'}{((x')^2 + L^2)^{3/2}}\,\hat{y}\right]$$

* first integral:

$$\int_{-L}^{L} \frac{-x'\, dx'}{((x')^2 + L^2)^{3/2}}\,\hat{x}$$

* Let $u = x^2 + L^2$, $du = 2x\,dx$
$x\,dx = \frac{du}{2}$

$$= \int_{-L}^{L} \frac{-du}{2u^{3/2}}$$

$$= \frac{(-1)}{\cancel{(2)}} \frac{u^{-1/2}}{\cancel{(-1/2)}}\Bigg|_{-L}^{L}$$

* will convert back u-sub for bounds

$$= \frac{1}{\sqrt{x^2 + L^2}}\Bigg|_{-L}^{L}$$

$= 0$ * no x-component at $\vec{r} = (0, L, 0)$ due to symmetry

2^{nd} integral:

$$= \int_{-L}^{L} \frac{L\, dx'}{((x')^2 + L^2)^{3/2}}$$

$$= \cancel{L}\left(\frac{x'}{L^{\cancel{2}}\sqrt{(x')^2 + L^2}}\right)\Bigg|_{-L}^{L}$$

$$= \left(\frac{2\cancel{L}}{\cancel{L}\sqrt{L^2 + L^2}}\right)$$

$$= \frac{\cancel{2}}{L\cancel{\sqrt{2}}}\frac{\sqrt{2}}{\cancel{\sqrt{2}}} = \frac{\sqrt{2}}{L}$$

$$\therefore\ \vec{E}(y{=}L) = \frac{\lambda}{4\pi\epsilon_0}\frac{\sqrt{2}}{L}\,\hat{y}$$

Generally (From Griffith's 3^{rd} ed. pg. 64)

$$\vec{E}(y) = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{y\sqrt{y^2+L^2}}\hat{y}$$

* for a line of charge of length $2L$, when a distance y from the center

where:

$$E(\alpha) = \frac{2\lambda K}{\alpha\sqrt{1+(\frac{\alpha}{L})^2}}$$

$$E'(y) = 2K\lambda L \frac{\partial}{\partial y}\left(\frac{1}{y\sqrt{y^2+L^2}}\right)$$

$$= 2K\lambda L \frac{\partial}{\partial y}\left[(y^4+y^2L^2)^{-1/2}\right]$$

$$= \cancel{2}K\lambda L(-\tfrac{1}{\cancel{2}})(y^4+y^2L^2)^{-3/2}(4y^3+2yL^2)$$

$$= -K\lambda L\frac{4y(y^2+L^2)}{(y^4+y^2L^2)^{3/2}}$$

$$= -K\lambda L\frac{(2yL^2(1+\frac{2y^2}{L^2})}{(y^2L^2(1+(\frac{y}{L})^2))^{3/2}}$$

$$= -K\lambda\cancel{L}\frac{2y\cancel{L^2}\left[(\frac{\sqrt{2}y}{L})^2+1\right]}{y^3\cancel{L^3}(1+(\frac{y}{L})^2)^{3/2}}$$

$$= \frac{-2K\lambda\left[1+(\frac{y\sqrt{2}}{L})^2\right]}{y^2(1+(\frac{y}{L})^2)(1+(\frac{y}{L})^2)^{1/2}}$$

* Taylor series aproximation of $\sqrt{1+x^2} = 1-\frac{1}{2}x^2+\ldots$ and if $x \ll 1$, ≈ 1

$$\approx \frac{-2K\lambda\left[1+(\frac{y\sqrt{2}}{L})^2\right]}{y^2(1+(\frac{y}{L})^2)}$$

$$E'(y) \approx \frac{-2K\lambda(1+2(\frac{y}{L})^2)}{y^2(1+(\frac{y}{L})^2)}$$

$E'(\alpha)$ where $\alpha \ll L$

$$E'(\alpha) = \frac{-2K\lambda(1+2\cancel{(\frac{\alpha}{L})^2})}{\alpha^2(1+\cancel{(\frac{\alpha}{L})^2})} \approx \frac{-2K\lambda}{\alpha^2}$$

$$E(y) \approx E(\alpha) + E'(\alpha)(y-\alpha)$$

* centered close to origin

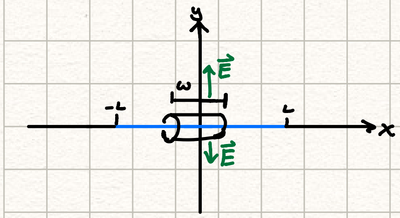
$$= \frac{2K\lambda}{\alpha\sqrt{1+(\frac{\alpha}{L})^2}} + \frac{-2K\lambda}{\alpha^2}(y-\alpha)$$

$$\approx \frac{2K\lambda}{\alpha} - \frac{2K\lambda y}{\alpha^2} + \frac{2K\lambda}{\alpha}$$

Then when at a distance α from the line, approximation shows

$$E(\alpha) = \frac{2K\lambda}{\alpha}$$

Gauss Law Result:



for a Gaussian Cylinder of length ω where $\omega < 2L$ such that the E-Field points radially out (in the y-z plane)

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\epsilon_0}$$

* with the cylinder aligned to the x-axis, $d\vec{A}_{Top\,lid} \perp \vec{E}$ & $d\vec{A}_{bot\,lid} \perp \vec{E}$

$$\int_{Side} E\,da = \frac{\int \lambda\,dl}{\epsilon_0}$$

* when ω is small enough so that field lines are not distorted coming out of the cylinder. Such that E is constant

$$E \int_{side} da = \frac{\lambda \omega}{\epsilon_0}$$

$$E = \frac{\lambda \cancel{\omega}}{2\pi r \cancel{\omega} \epsilon_0}$$

$$E = \frac{\lambda}{2\pi r \epsilon_0}$$

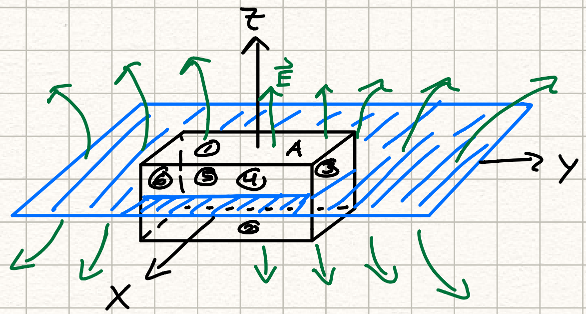
when at a distance $r = \alpha$ from line

$$E = \frac{\lambda}{2\pi \alpha \epsilon_0} = \frac{2K\lambda}{\alpha}$$

* recall exact solution:

$$E(\alpha) = \frac{2K\lambda}{\alpha}$$

1.3.2 Gauss Law Finite Square of length ω & charge σ_0



① & ② represent top & bottom lids

Let the sheet be centered at the origin.

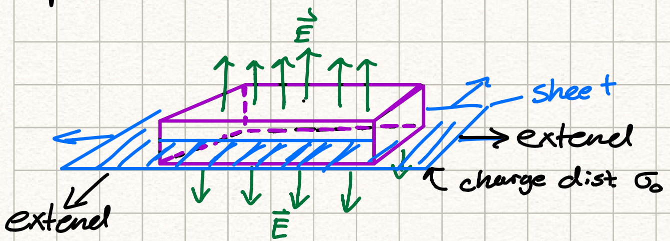
Gauss Law: $\oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\epsilon_0}$

$$\sum_{i=1}^{6} \int_{S_i} \vec{E}\cdot d\vec{A}_i = \frac{\sigma A}{\epsilon_0}$$

* where A is the top surface of the pillbox
* Summation is to account for all sides of Gaussian surface

The $\vec{E}$ field (field lines in Green) will approximately yield (if Gaussian pillbox is small enough in size compared to the sheet's size)

Consider the Gaussian shape small enough that the field lines point in the z-direction.



Then $d\vec{A}_i \perp \vec{E}$ for pill box sides, leaving only top & bot lid (parallel to $\vec{E}$)

$$\int_{S_1} E\,dA + \int_{S_2} E\,dA = \frac{Q_{encl}}{\epsilon_0}$$

* $\vec{E}$ will be constant in this approximation since $\vec{E}$ is constant when looking at a small concentration (relative to sheet) that would show field lines not distorting from z-axis (either above or below)

$$E\int_{S_1} dA_1 + E\int dA_2 = \frac{Q_{encl}}{\epsilon_0} = \frac{\int_S \sigma\,dA}{\epsilon_0}$$

top & bot surface integrals will yield same area

$$E(2\cancel{A}) = \frac{\sigma \cancel{A}}{\epsilon_0}$$

$$\therefore |E| = \frac{\sigma}{2\epsilon_0}$$

exact solution when near the center of the finite sheet

$$E_z(z) = \frac{\sigma_0}{\pi \epsilon_0} \tan^{-1}\left[\frac{\omega^2}{4z} \frac{1}{\sqrt{z^2 + \frac{\omega^2}{2}}}\right]$$

A Taylor series can be taken to show its equivalency to the Gauss law solution.

Where $E(z) = E(a) + \frac{E'(z)}{1!}(z-a) + \frac{E''(a)}{2!}(z-a)^2 + \ldots$

* will center sheet near origin (not at) such that: $a \ll 1$ (very close to sheet)

For Simplicity The First-Order approximation will be used:

$$E_z(z) \approx E_z(a) + E'(a)(z-a)$$

$$E(a) = \frac{\sigma_0}{\pi \epsilon_0} \tan^{-1}\left[\frac{\omega^2}{4a} \frac{1}{\sqrt{a^2 + \frac{\omega^2}{2}}}\right]$$

where $a \ll 1 \rightarrow \frac{1}{a} \approx \infty$

$$E(a) = \frac{\sigma_0}{\pi \epsilon_0} \tan^{-1}\left[\frac{\omega^2}{4} \frac{1}{a} \frac{1}{\sqrt{\frac{\omega^2}{2}}}\right]$$

$$= \frac{\sigma_0}{\pi \epsilon_0} \overset{\pi/2}{\cancel{\tan^{-1}[\infty]}}$$

$$= \frac{\sigma_0}{\pi \epsilon_0}\left(\frac{\pi}{2}\right) = \frac{\sigma_0}{2\epsilon_0}$$

$$E'(z) = \frac{d}{dz}[E(z)]$$

$$= \frac{d}{dz}\left\{\frac{\sigma_0}{\pi \epsilon_0} \tan^{-1}\left[\frac{\omega^2}{4z} \frac{1}{\sqrt{z^2 + \frac{\omega^2}{2}}}\right]\right\}$$

$$= \frac{\sigma_0}{\pi \epsilon_0} \frac{d}{dz} \tan^{-1}(u) \qquad * \text{let } u = \frac{\omega^2}{4z} \frac{1}{\sqrt{z^2 + \frac{\omega^2}{2}}} = \frac{\omega^2}{\sqrt{16z^4 + 8z^2\omega^2}}$$

$$= \frac{\sigma_0}{\pi \epsilon_0}\left(\frac{1}{1+u^2}\right)\frac{du}{dz}$$

$$\frac{du}{dz} = \frac{d}{dz}\left(\frac{\omega^2}{\sqrt{16z^4 + 8z^2\omega^2}}\right)$$

$$= \omega^2 \frac{d}{dz}\left[(16z^4 + 8z^2\omega^2)^{-1/2}\right]$$

$$\frac{du}{dz} = \omega^2 (-1/2)(16z^4 + 8z^2\omega^2)^{-3/2}(64z^3 + 16z\omega^2)$$

$$= \left(-\frac{\omega^2}{2}\right)\frac{16z(4z^2 + \omega^2)}{(16z^4 + 8z^2\omega^2)^{3/2}}$$

$$\frac{1}{1+u^2} = \left[1 + \left(\frac{\omega^2}{\sqrt{16z^4 + 8z^2\omega^2}}\right)^2\right]^{-1}$$

$$= \left(1+\left[\frac{\omega^4}{16z^2+8z^2\omega^2}\right]\right)^{-1}$$

$$= \left(\frac{16z^4+8z^2\omega^2+\omega^4}{16z^4+8z^2\omega^2}\right)^{-1}$$

$$= \frac{16z^4+8z^2\omega^2}{16z^4+8z^2\omega^2+\omega^4}$$

$$\frac{1}{1+u^2} = \frac{16z^4+8z^2\omega^2}{(4z^2+\omega^2)^2}$$

$$\therefore\ E'(z) = \frac{\sigma_0}{\pi\epsilon_0}\left(\frac{\cancel{(16z^4+8z^2\omega^2)}}{(4z^2+\omega^2)^{\cancel{2}}}\right)\left[\left(-\frac{\omega^2}{\cancel{2}_{1}}\right)\frac{\cancel{16}^{8}z\cancel{(4z^2+\omega^2)}}{(16z^4+8z^2\omega^2)^{\cancel{3/2}\ 1/2}}\right]$$

$$= \frac{\sigma_0}{\pi\epsilon_0}\left(\frac{-8z\omega^2}{(4z^2+\omega^2)\sqrt{16z^2\left(z^2+\frac{\omega^2}{2}\right)}}\right)$$

$$= \frac{\sigma_0}{\pi\epsilon_0}\ \frac{-\cancel{8}^{2}\cancel{z}\omega^2}{(4z^2+\omega^2)\cancel{4z}\sqrt{z^2+\left(\frac{\omega^2}{2}\right)}}$$

$$E'(z) = \frac{\sigma_0}{\pi\epsilon_0}\left(\frac{-2\omega^2}{(4z^2+\omega^2)\sqrt{z^2+\frac{\omega^2}{2}}}\right)$$

*recall $a \ll 1$, then $a^2 \approx 0$ (or higher order)

$$E'(a) = \frac{\sigma}{\pi\epsilon_0}\left(\frac{-2\omega^2}{(4\cancel{a^2}_{0}+\omega^2)\sqrt{\cancel{a^2}_{0}+\frac{\omega^2}{2}}}\right)$$

$$= \frac{\sigma}{\pi\epsilon_0}\left(\frac{-2\cancel{\omega^2}}{\cancel{\omega^2}\sqrt{\frac{\omega^2}{2}}}\right)$$

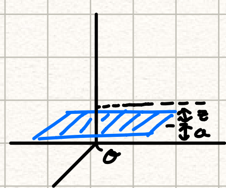
$$E'(a) \approx \frac{\sigma}{\pi\epsilon_0}\left(\frac{-2\sqrt{2}}{\omega}\right)\left(\frac{2}{2}\right) = \frac{\sigma}{2\epsilon_0}\left(\frac{-4\sqrt{2}}{\pi\omega}\right)$$

First-order Taylor series approximation

$$|E_z(z)| \approx E(a) + E'(a)(z-a)$$

$$= \frac{\sigma}{2\epsilon_0} - \frac{\sigma}{2\epsilon_0}\left(\frac{4\sqrt{2}}{\pi\omega}\right)(z-a)$$

*when $z>a$ but $(z-a)\ll 1$ (near the finite sheets center)



$$|E_z(z)| = \frac{\sigma}{2\epsilon_0}\left(1-\frac{4\sqrt{2}}{\pi\omega}(z-a)\right)$$

$$= \frac{\sigma}{2\epsilon_0}\left(1-\frac{4\sqrt{2}}{\pi}\frac{(z-a)}{\omega}\right)$$

* as the distance between z & a decreases (closer) relative to length of sheet (length = ω), $\frac{z-a}{\omega}\approx 0$

$$\boxed{|\vec{E}(z)| \approx \frac{\sigma}{2\epsilon_0}}$$ * if $\frac{z-a}{\omega}\approx 0$