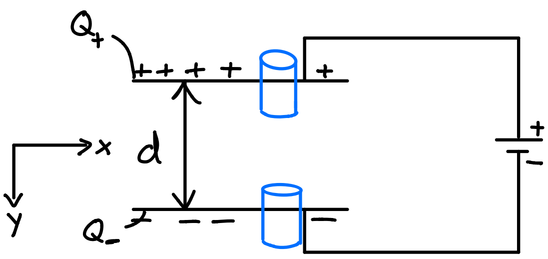
Darwin Quiroz

3.1.1. Large Parallel Plates

Gauss Law Derivation:

Assume charges distribute uniformly across surfaces

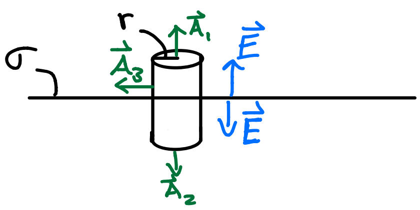


∗ Each has surface Area = A_{Tot}

If the plates are conducting and the top surface of the Gaussian cylinder is as drawn, the electric field will be zero on it. I'll explain in class. This is the reason that I always draw conductors with a tiny thickness. It is to avoid this error.

2 Gaussian surfaces across each plate

Looking at top surface



$\vec{E}$ Field will be ⊥ to surface when caps are very close to surface.

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\epsilon_0}$$

$$\int_{top} \vec{E} \cdot d\vec{A}_1 + \int_{bot} \vec{E} \cdot d\vec{A}_2 + \int_{side} E \cdot d\vec{A}_3 = \frac{\int \sigma \, dA}{\epsilon_0}$$

∗charges are uniformly distributed, then σ is constant

∗ Because $\vec{E}$ is ⊥ to surface, then $\vec{E} \perp d\vec{A}_3$, $\vec{E} \parallel \vec{A}_1$, $\vec{E} \parallel \vec{A}_2$

$$\int_{top} E \, dA_1 + \int_{bot} E \, dA_2 = \frac{\sigma \pi r^2}{\epsilon_0}$$

∗ Because charge is distributed uniformly, $\vec{E}$ is to be independent of position (const.)

$$E \left(\int dA_1 + \int dA_2 \right) = \frac{\sigma \pi r^2}{\epsilon_0}$$

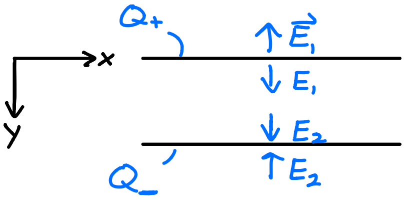
$$E \left(2 \cancel{\pi r^2} \right) = \frac{\sigma \cancel{\pi r^2}}{\epsilon_0}$$

$$E = \frac{\sigma}{2\epsilon_0}$$ ∗ This $\vec{E}$ field is the magnitude coming out of each side

$$\vec{E} = \frac{Q}{2 A_{Tot} \epsilon_0}$$

* Now looking at bottom plate, both sheets have equal and opposite charges & distributions.

Gauss Law will yield the same magnitude, but inverse directions as such:



* This causes $\vec{E}_{net} = \vec{E}_1 + \vec{E}_2$ to have 2 outcomes:

$$\vec{E}_{net} = \begin{cases} 2E_1\,\hat{y} & 0<y<d \\ 0 & y \notin (0,d) \end{cases}$$

* because $E_1 = E_2$

$$|\Delta V| = \int_0^d \vec{E}_{net}\cdot d\vec{l}$$

$$= \int_0^d 2E_1\,dy\,(\hat{y}\cdot\hat{y})$$

$$= \int_0^d \frac{Q\,dy}{\cancel{2}A_{tot}\epsilon_0}\cdot\cancel{2}$$

$$|\Delta V| = \frac{Q\,d}{\epsilon_0 A_{tot}}$$ * Let $A_{tot} = A$

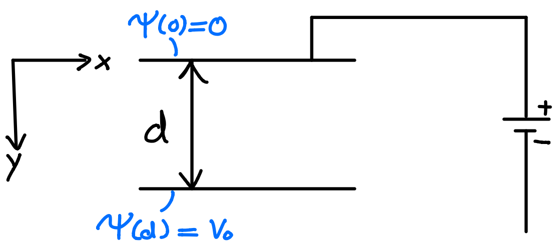
$$\boxed{C = \frac{Q}{|\Delta V|} = \frac{A\epsilon_0}{d}}$$

Boundary Method:

Solving Laplace's Eqn to find Potential Function

$$\nabla^2\psi = 0$$

Considering only 2 plates:



$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

Great explanation of steps.

Now ψ is only dependent of y:

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial y^2} = C_1 y + C_2$$

Boundary Conditions:

I $\Psi(0) = 0 = C_1(0) + C_2$

$\therefore C_2 = 0$

II. $\Psi(d) = V_0 = C_1 d$

$\therefore C_1 = \frac{V_0}{d}$

$\therefore \Psi(y) = \frac{V_0}{d} y$

The $\vec{E}$ Field is found as:

$$\vec{E} = -\nabla \Psi(y) = -\frac{\partial \Psi(y)}{\partial y} \hat{y}$$

$$\vec{E} = -\frac{V_0}{d} \hat{y}$$

$$|\vec{E}| = \frac{|V_0|}{d}$$

When at the surface of a conductor:

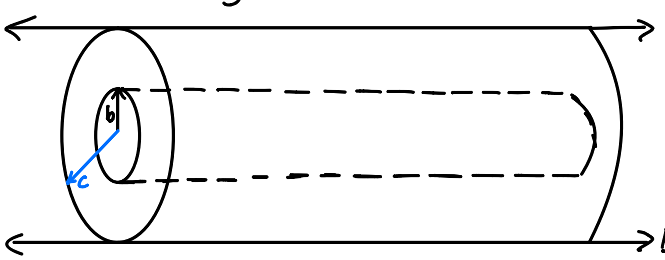
$$E_\perp = \frac{\sigma}{\epsilon_0}$$

$$\sigma = \frac{Q}{A} = \epsilon_0 E_\perp$$

$$\frac{Q}{A} = \epsilon_0 \frac{|V_0|}{d}$$

$\boxed{\therefore C = \frac{Q}{|V_0|} = \frac{A\epsilon_0}{d}}$ * which matches Gauss Law result

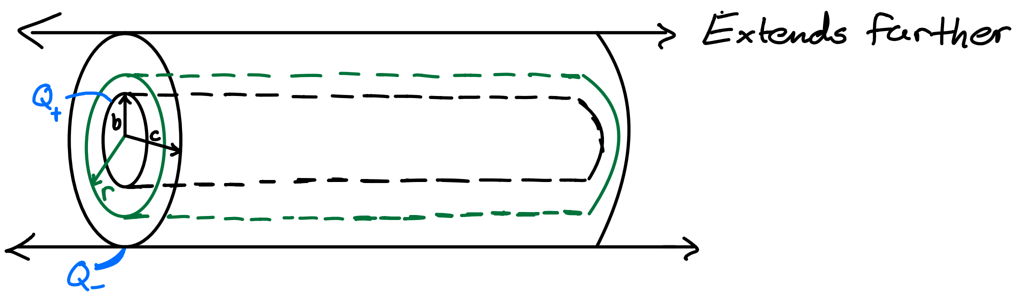
3.1.2 Long Coaxial Cylinders



Gauss Law Derivation:

Let $\begin{cases} Q_+ \text{ at } s=b \\ Q_- \text{ at } s=c \end{cases}$

* Putting a Gaussian cylinder around coaxial cable as such:

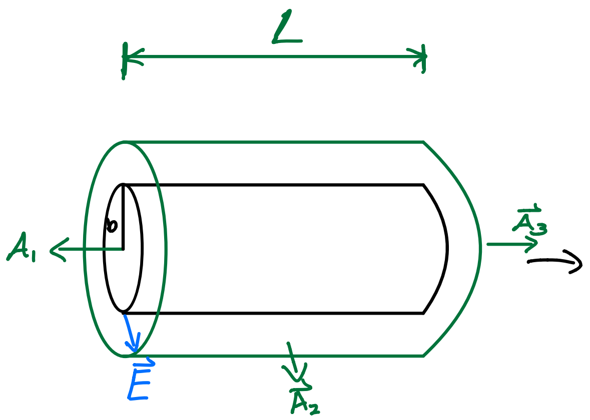


when $r<b$

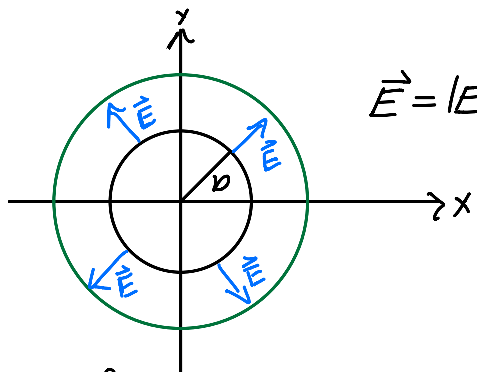
No charged enclosed:

$E=0$

when $b<r<c$



Top-down View



$\vec{E} = |E|\hat{s}$

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

$$\int_{top} \vec{E}\cdot dA_1 + \int_{side} \vec{E}\cdot d\vec{A}_2 + \int_{bot} \vec{E}\cdot d\vec{A}_3 = \frac{\int \sigma dA}{\epsilon_0}$$

* because charge is uniformly distributed: λ is constant

* $\vec{E}\perp d\vec{A}_1$, $\vec{E}\perp d\vec{A}_3$, $\vec{E}\parallel d\vec{A}_2$

This seems "obviously true", but showing it is not easy. Make sure that you know how you justify it to someone who said "it is not obvious to me".

$$\int_{side} E\, dA_2 = \frac{\sigma 2\pi b L}{\epsilon_0}$$

* because charges are uniformly distributed, E is constant

$$E\int_{side} dA_2 = \frac{2\pi b L \sigma}{\epsilon_0}$$

$$E\, \cancel{2\pi} s \cancel{L} = \frac{\cancel{2\pi} b \cancel{L} \sigma}{\epsilon_0} = \frac{\cancel{b}}{\epsilon_0}\ \frac{Q}{2\pi \cancel{b} L}$$

$\therefore |E| = \frac{Q}{2\pi s L \epsilon_0}$ due to the inner cylinder

$\vec{E} = \frac{Q}{2\pi s L \epsilon_0} \hat{s}$

When $r > c$

You don't actually need to know E for r > c to get the capacitance. Probably you knew this and were being thorough.

The enclosed charge:

$Q_{encl} = Q_+ + Q_- = 0$

$\therefore E = 0$, for $r > b$

$|\Delta V| = \int_b^c \vec{E} \cdot d\vec{l}$ Let $dl = ds\,\hat{s}$

$|\Delta V| = \int_b^c \frac{Q}{s 2\pi \epsilon_0 L} ds = \frac{Q}{2\pi \epsilon_0 L} \int_b^c \frac{ds}{s}$

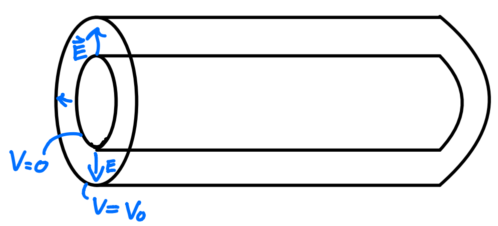
$|\Delta V| = \frac{Q}{2\pi \epsilon_0 L} \ln\left(\frac{c}{b}\right)$

$C = \frac{Q}{|\Delta V|} = \frac{2\pi \epsilon_0 L}{\ln\left(\frac{c}{b}\right)}$

$$\boxed{\therefore \frac{C}{L} = \frac{2\pi \epsilon_0}{\ln(c/b)}}$$

Boundary Value Method:

Assuming outer cylinder at potential V_0 while inner cylinder is considered to be at potential 0.



Must find potential function by solving the Laplace equation in Cylindrical coordinates:

$\nabla^2 \psi = 0$

$\Rightarrow \frac{1}{s} \frac{\partial}{\partial s}\left(s \frac{\partial \psi}{\partial s}\right) + \frac{1}{s^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} = 0$

ψ is the potential function w/ the following conditions:

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I. $\Psi(b) = 0$

II. $\Psi(c) = V_0$

With the charges being uniformly distributed, equipotential lines are perpendicular to $\vec{E}$ field lines. Ψ will only depend on s.

$$\Psi(s,\phi,z) \Rightarrow \Psi(s)$$

$$\therefore \nabla^2\Psi = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\Psi}{\partial s}\right) = 0$$

then $s\dfrac{\partial\Psi}{\partial s} = C_1$

$$\frac{\partial\Psi}{\partial s} = \frac{1}{s}C_1$$

Here you can also argue that because the problem does not change if you rotate coordinate system around z-axis, that psi cannot depend on phi. Because the problem does not change if you move any distance along z, psi cannot depend on z. As a result, psi=psi(s). From this argument and E (vector) =-grad(psi), if follows that E (vector) = E (vector)(s). (Using E (vector) b/c I can't bold an individual letter in my PDF editor.)

$$\Psi(s) = \ln(s)C_1 + C_2$$

Applying Boundary Conditions:

I. $\Psi(a) = 0 = C_1\ln(a) + C_2$

$$C_2 = -C_1\ln(a)$$

$$\Psi(s) = C_1\ln(s) - C_1\ln(a)$$

$$\Psi(s) = C_1\ln\left(\frac{s}{b}\right)$$

II. $\Psi(c) = V_0 = C_1\ln(c/b)$

$$C_1 = \frac{V_0}{\ln(c/b)}$$

$$\therefore \Psi(s) = \frac{V_0\ln(s/b)}{\ln(c/b)} = \frac{V_0}{\ln(c/b)}(\ln(s) - \ln(b))$$

$$\vec{E} = -\nabla\Psi(s)$$

$$= -\left(\frac{\partial\Psi}{\partial s}\hat{s} + \frac{1}{s}\overset{0}{\cancel{\frac{\partial\Psi}{\partial\phi}}}\hat{\phi} + \overset{0}{\cancel{\frac{\partial\Psi}{\partial z}}}\hat{z}\right)$$

$$\vec{E}(s) = -\frac{V_0}{\ln(c/b)}\left(\frac{1}{s}\right)\hat{s}$$

Now when near the surface of the conductor:

$$\vec{E} = E_\perp\hat{s} \quad \& \quad E_\perp = \frac{\sigma}{\epsilon_0}$$

E_perp * s_hat is unusual notation; should use E_perp = E_s * s_hat or E_perp = E dot s_hat

$$\approx \frac{V_0}{\ln(c/b)}\frac{1}{b}$$

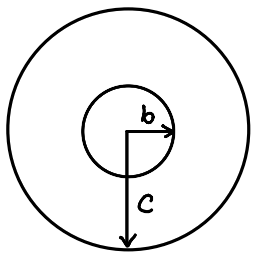
$*$ $s \approx b$ when near the surface of the inner cylinder

$$\sigma = E_\perp \epsilon_0 = \frac{Q}{A}$$

$$\approx \frac{V_0 \epsilon_0}{b \ln(c/b)} = \frac{Q}{2\pi b L}$$

$$\boxed{\therefore \frac{Q}{V_0 L} = \frac{C}{L} = \frac{2\pi \epsilon_0}{\ln(b/c)}}$$

3.1.3. Concentric Spherical Shells



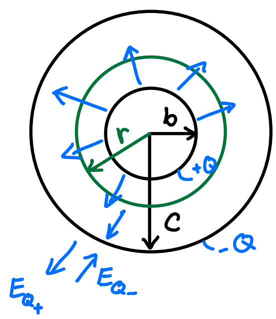
Let 2 concentric spherical shell conductors, Find the capacitance.

Gauss Law Derivation:

Place 2 charges:

Q_- at c
Q_+ at b

Create a Gaussian Sphere w/ radiur r.



when $r<b$:

$Q_{encl} = 0$

$\therefore E = 0$

when $b<r<c$

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

*charges are distributed evenly across surface, causing $\vec{E} = |\vec{E}_{Q_+}| \hat{r}$ & charge density = const

$$\int_{Gauss} E_{Q_+}\, dA = \frac{\sigma_{sphere}\int dA}{\epsilon_0}$$

* Because charges are uniformly distributed. E_{Q_+} is const everywhere along surface

$$E_{Q_+} \int_{Gauss} dA = \frac{\sigma \int_{sphere} dA}{\epsilon_0}$$

$$E_{Q_+}\, \cancel{4\pi} r^2 = \frac{\sigma\, \cancel{4\pi} a^2}{\epsilon_0}$$

$$E_{Q_+} = \frac{\sigma a^2}{r^2 \epsilon_0} = \frac{Q}{4\pi \cancel{a^2}} \frac{\cancel{a^2}}{r^2 \epsilon_0}$$

$$\therefore E_{Q_+} = \frac{Q}{4\pi r^2 \epsilon_0}$$

$$|\Delta V| = \left| \int \vec{E} \cdot d\vec{l} \right|$$

$$= \int_c^b \frac{Q}{4\pi r^2 \epsilon_0}\, dr\ \hat{r} \cdot \hat{r}$$

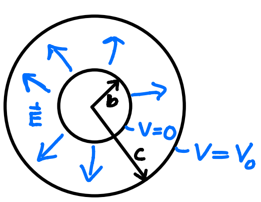
$$= \left| \frac{Q}{4\pi \epsilon_0} \left(\frac{-1}{r}\right)\Big|_b^c \right|$$

$$|\Delta V| = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{c} - \frac{1}{b}\right)$$

$$\boxed{C = \frac{Q}{|\Delta V|} = \frac{4\pi\epsilon_0}{\left(\frac{1}{c} - \frac{1}{b}\right)}}$$

Boundary Value Method:

Assuming outer sphere has a potential V_0, and inner sphere is at a potential of 0.



Must find potential by solving Laplace Equation in spherical coord.

$$\nabla^2 \Psi(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Psi}{\partial \phi^2}$$

Because charges are uniformly distributed

Ψ only varies in r direction, $\Psi(r,\theta,\phi)=\Psi(r)$

$$\therefore \nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Psi}{\partial r}\right)=0$$

then: $r^2\frac{\partial\Psi}{\partial r} = C_1$

$$\frac{\partial \Psi}{\partial r} = \frac{C_1}{r^2}$$

$$\Psi = -\frac{C_1}{r}+C_2$$

Applying Boundary Conditions:

I. $\Psi(b) = -\frac{C_1}{b}+C_2 = 0$

$$C_2 = \frac{C_1}{b}$$

$$\Psi(r) = -\frac{C_1}{r}+\frac{C_1}{b} = C_1\left(\frac{1}{b}-\frac{1}{r}\right)$$

II. $\Psi(c) = C_1\left(\frac{1}{b}-\frac{1}{c}\right) = V_0$

$$\therefore C_1 = V_0\left(\frac{1}{b}-\frac{1}{c}\right)^{-1}$$

$$\therefore \Psi(r) = V_0\left(\frac{1}{b}-\frac{1}{c}\right)^{-1}\left(\frac{1}{b}-\frac{1}{r}\right)$$

$$\vec{E}(r) = -\vec{\nabla}\Psi(r)$$

$$= -\frac{\partial}{\partial r}\Psi(r)\,\hat{r}$$

$$= -V_0\left(\frac{1}{b}-\frac{1}{c}\right)^{-1}\left(\frac{1}{r^2}\right)\hat{r}$$

$$|\vec{E}| = \frac{V_0}{\left(\frac{1}{c}-\frac{1}{b}\right)r^2}$$

when looking at the $\vec{E}$ field near the surface of the conductor:

$|\vec{E}| = E_\perp$ where $r \approx b$

$$\sigma = \epsilon_0 E_\perp$$

$$\sigma = \frac{V_0\epsilon_0}{\left(\frac{1}{c}-\frac{1}{b}\right)\cancel{b^2}} = \frac{Q}{4\pi \cancel{b^2}}$$

$$\boxed{C = \frac{Q}{|V_0|} = \frac{4\pi\epsilon_0}{\left(\frac{1}{c}-\frac{1}{b}\right)}}$$