DSP - Take-home-assignment

ANISH DAS
AD945

1. ~~(a)~~ A: $y_n = |x_{n-1} - x_{n+1}|$

B: $y_n = \frac{1}{2}(x_{2n} + x_{2n+1})$

C: $y_n = \frac{3x_{n-1} + x_{n-2}}{x_{n-3}}$

D: $y_n = \prod_{i=0}^{8} x_{n-i}$

E: $y_n = \sum_{i=-\infty}^{\infty} x_i \delta_{i-n+2} = x_{n-2}$

F: $y_n = x_n \cdot e^{n/14}$

G: $y_n = x_n \cdot u_n$

H: $y_n = x_n + \cos(0.7\pi n)$

c) Causal Systems

→ C; D; E; F; G; H.

Since they only depend on $x_n, x_{n-1}, x_{n-2}, \ldots$

d) Memory less Systems

F; G; H

Since they only depend on x_n.

a) Linear Systems

B; E; F; ~~G~~ G; H.

Suppose the system is T and we have $\{y_i\} = T\{x_i\}$ &
$\{y_i'\} = T\{x_i'\}$

if $T\{a x_i + b x_i'\} = a\{y_i\} + b\{y_i'\}$ then system T is linear.

This doesn't hold for multiplication and division by samples like in C & D.

B may be Linear depending on the ~~is~~ input i.e. iff both $\{x_i'\}$ & $\{x_i\}$ are monotonically increasing or decreasing

~~1)~~ b) Time Invariant System

A; B; C; D; E; ~~F~~; ~~G~~

2.) Using the Polynomial Representation of a ~~series~~ sequence

let $\{a_n\}$ ~~be the~~ ~~F~~ and $\{b_n\}$ be the sequences with finite non-zero values.

For convenience let $a_0 \neq 0$ & $b_0 \neq 0$ i.e. the first non-zero position for both is 0. Following from this, last non-zero positions are $a_{k-1} \neq 0$ & $b_{\ell-1} \neq 0$.

Define Polynomials $A(v)$ & $B(v)$ as

$$A(v) = \sum_{i=0}^{m-1} a_i v^i \quad \&$$

$$B(v) = \sum_{i=0}^{\ell-1} b_i v^i$$

Multiplying the polynomials to convolve the two series

$$C(v) = A(v) \cdot B(v)$$
$$= \left(\sum_{i=0}^{m-1} a_i v^i\right) \cdot \left(\sum_{j=0}^{\ell-1} b_j v^j\right)$$

Clearly, this will be of the form.

$$C(v) = \sum_{i=0}^{m+\ell-2} c_i v^i$$

i.e. the highest power (exponent) will be $(v^{m-1})(v^{\ell-1}) = v^{m+\ell-2}$

and the smallest " " " " $(v^0)(v^0) = v^0$.

Thus the length will be $(m+\ell-2) - 0 + 1 = \underline{\underline{m+\ell-1}}$

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3. a) Consider a sine wave sampled at twice the frequency.

$\{a_n\} = \{ \ldots 0, 1, 0, -1, \ldots \}$ (an infinite sequence)

&

$\{b_n\} = \{0, x, y, x, y\}$

Note: Trivial Proof later on.

Representing the sequence a polynomial.

$$A(v) = \sum_{i=-\infty}^{\infty} a_i v^i$$

since a_i repeats.

$$A(v) = \sum_{i=-\infty}^{\infty} v^{4i} (0 \cdot v^0 + 1 \cdot v^1 + 0 \cdot v^2 - 1 \cdot v^3)$$

$$= \sum_{i=-\infty}^{\infty} v^{4i} (v - v^3)$$

& $B(v) = xv + yv^2 + xv^3 + yv^4$

Let $\{c_n\} = \{a_n\} * \{b_n\}$ and we get polynomial $C(v)$ from $\{c_n\}$.

then

$$C(v) = A(v) \cdot B(v).$$

$$= (xv + yv^2 + xv^3 + yv^4) \cdot \left(\sum_{i=-\infty}^{\infty} v^{4i} (v - v^3)\right)$$

$\forall i \in \mathbb{Z}.$

$$C_{4i} = \text{coeff } v^{4i} \text{ in } \left[(xv + yv^2 + xv^3 + yv^4) \cdot v^{4i-4} \cdot (v - v^3)\right]$$

$$\Rightarrow (xv + yv^2 + xv^3 + yv^4) \cdot (v^{4i-3} - v^{4i-1})$$

$$\Rightarrow (xv^{4i} - xv^{4i} + \ldots)$$

$$\Rightarrow (0 \cdot v^{4i} + \ldots)$$

$$= 0$$

similarly for C_{4i+1} = coeff of v^{4i+1} in $\left[(xv + yv^2 + xv^3 + yv^4)(v^{4i-3} - v^{4i-1})\right]$

$$\Rightarrow -yv^2 \cdot v^{4i-1} + yv^4 \cdot v^{4i-3}$$

$$\Rightarrow 0 \cdot v^{4i+1}$$

$$= 0.$$

for C_{4i+2} = coeff of v^{4i+2} in

$$[(xv+yv^2+xv^3+yv^4)\cdot(v^{4i-3}-v^{4i-1}) + (xv+yv^2+xv^3+yv^4)\cdot(v^{4i+1}-v^{4i+3})]$$

$$\Rightarrow xv^3(-v^{4i-1}) + xv\cdot v^{4i+1}$$

$$\Rightarrow 0\cdot v^{4i+2}$$

$$= 0$$

Similarly for $C_{4i+3} \Rightarrow$ coeff of v^{4i+3} in

$$[(xv+yv^2+xy^3+yv^4)(v^{4i-3}-v^{4i-1}) + (xv+yv^2+xv^3+yv^4)(v^{4i+1}-v^{4i+3})]$$

$$\Rightarrow yv^4(-v^{4i-1}) + yv^2\cdot v^{4i+1}$$

$$\Rightarrow 0\cdot v^{4i+3}.$$

Thus after convolution $\forall i \in \mathbb{Z}$. C_{4i}, C_{4i+1}, C_{4i+2} & C_{4i+3} $= 0$ and therefore $\{C_n\}$ is all zeros.

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Trivial Proof

	$\{a_n\}$	0 1 0 −1 0 1	0 1 0 −1 0 1 0
Reversed	$\{b_n\}$	0 y x y x 0	0 y x y x 0
		$0+y+0+(-y)+0+0 = 0$	$0+0+(-x)+0+x+0 = 0$
	$\{a_n\}$	0 −1 0 1 0 −1	−1 0 1 0 −1 0
Reverse	$\{b_n\}$	0 y x y x 0	0 y x y x 0
		$0+(-y)+0+y+0+0 = 0$	$0+0+x+0+(-x)+0 = 0$

4) a) when we are sampling at frequency f_s, we can only effectively extract frequencies in the interval $\{ n \frac{f_s}{2} < f < (n+1)\frac{f_s}{2}$ for all natural No. n, and this is done to prevent aliasing.

Therefore, for the sampling frequencies provides (16 kHz, 32 KHz & 48 kHz) we want 23 kHz & 25 kHz to lie in the same interval.

This is true for only $\underline{32 \text{ KHz}}$ for the interval for $n = 1$, $f_s = 32000$, $\Rightarrow [16 \text{ KHz} < f < 32 \text{ KHz}]$

b) Intuitively, a DFT is sampling the DTFT at $\frac{0 f_s}{n}$,

$0\frac{f_s}{n}, 1\frac{f_s}{n}, 2\frac{f_s}{n} \ldots \frac{f_s}{2}$ and these correspond to $X_0, X_1, X_2 \ldots X_{n/2}$

and $|X_k| = |X_{n-k}|$

~~Note~~ Note: $f_s = 160$ KHz & $n = 512$

$\therefore$ for the 23 KHz whistle

$\underline{|X_{74}|}$ should have max value as it corresponds to $\frac{74}{512} \cdot 160{,}000 = 23125.0$ Hz.

This is closest to the actual peak at 23 KHz.

& for 25 KHz whistle

$|X_{80}|$ would have the peak & in this case $\frac{80}{512} \, 160000 = 25$ KHz and would be equal to the actual peak.

c) The FFT which is a faster variant of the Discrete ~~Tra~~ Fourier Transform (DFT) simply samples the Discrete Time Fourier Transform (DTFT) at $0 \cdot \frac{f_s}{n}, 1 \cdot \frac{f_s}{n} \dots \frac{f_s}{2}$ frequencies, and these correspond to $X_0, X_1, \dots X_{n/2}$.

The DTFT has a peak where ever the true frequency lies in this case at 23 KHz & 25 KHz respectively. However ~~sta~~ ~~to~~ in the case of 23KHz we aren't quite able to hit it exactly and the ~~bes~~ closest we get is $|X_{74}|$ corresponding to 23125 Hz. Thus the value is lower than that we see at $|X_{80}|$ for 25 KHz where we hit the peak exactly.

There is a big difference in the value mainly because the ~~gr~~ gradient on either side is quite steep causing values to drop significantly if you move in either direction.

Thus, even if both whistles have equal sound-pressure amplitude and the microphone is equally sensitive, still 25kHz tone will show higher peak.

d) In order to reduce the relative amplitude difference in the FFT peaks of the two tones we can apply a hann window $\left\{ w_i = \frac{1-\cos(2\pi \frac{i}{n-1})}{2} \right\}$, where we ~~a~~ multiply every input item x_i with w_i which results in shorter peaks however, the peaks are flatter and thus ~~a~~ even if we don't hit the ~~peak~~ ~~value~~ correct frequency we can still get close to the actual peak value.

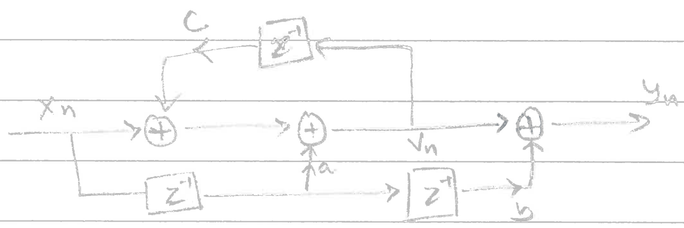
4) e) Q is defined as ~~the~~ a ~~ratio~~ ~~of the~~ quantity that is the ratio of the center (peak) frequency (f_0) to the -3 dB bandwidth bw,

$$Q = \frac{f_0}{bw}.$$

Using $Q = 30$, $f_s = 160$ kHz, $f_0 = 25$ kHz we get a 2nd order peak filter with zeros at $[1, -1]$ & ~~poles at~~ poles at $0.5466 \pm j\,0.8179$, in the z-transform.

&
$$H(z) = \frac{0.0161(1 - z^{-2})}{1 - 1.093z^{-1} + 0.9677z^{-2}}$$

5)



$$v_n = x_n + a x_{n-1} + c v_{n-1}.$$

Applying Z transform.

$$V(z) = X(z) + a z^{-1} X(z) + c z^{-1} V(z)$$

$$\Rightarrow V(z)(1 - cz^{-1}) = X(z)(1 + az^{-1})$$

$$\Rightarrow \frac{V(z)}{X(z)} = \frac{1 + az^{-1}}{1 - cz^{-1}} = \frac{z + a}{z - c} \quad -(*)$$

&
$$y_n = v_n + b x_{n-2}$$

Applying Z-transform.

$$Y(z) = V(z) + b z^{-2} X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{V(z)}{X(z)} + b z^{-2}$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{z+a}{z-c} + bz^{-2}$$ [from (*) & def of $H(z) = \frac{Y(z)}{X(z)}$]

$$\Rightarrow \quad H(z) = bz^{-2} + \frac{1+az^{-1}}{1-cz^{-1}}$$

$$= \frac{1+az^{-1}+bz^{-2}-bcz^{-3}}{1-cz^{-1}}$$

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$h : \{x_n\} \longrightarrow \{y_n\}$ with $y_n + y_{n-1} = x_n - x_{n-4}$

6) a) Using the polynomial representation of sequences.

let $X(v) = \sum_{n=-\infty}^{\infty} x_n v^n$ then $\{x_{n-1}\} \equiv X(v) \cdot v$
& $\{\delta_n\} \equiv 1$. & $\{\delta_{n-4}\} = v^4$

$\therefore$ with $h[\delta_n] \Rightarrow y_n + y_{n-1} = \delta_n - \delta_{n-4}$.

let $Y(v)$ be polynomial for $\{y_n\}$.

then

$$Y(v) + vY(v) = 1 - v^4$$

$$\Rightarrow Y(v) = \frac{1-v^4}{1+v} = -\left(\frac{v^4-1}{v+1}\right)$$

$$= -\frac{(v^2+1)(v-1)\cancel{(v+1)}}{\cancel{(v+1)}}$$

$$= 1 - v + v^2 - v^3$$

$$\therefore y_n = \delta_n - \delta_{n-1} + \delta_{n-2} - \delta_{n-3}$$

$$\cancel{\therefore y_n = \begin{cases} 0, & n<0 \\ 1, & \\ -1, & \\ 1, & \\ -1, & \\ 0, & \end{cases}}$$

$$y_n = \begin{cases} 0, & n<0 \\ 1, & n=0 \\ -1, & n=1 \\ 1, & n=2 \\ -1, & n=3 \\ 0, & n>3 \end{cases}$$

This is the impulse response

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6) b) let $U(v)$ be polynomial repr for $u_n = \begin{cases} 0, & n<0 \\ 1, & n \geq 0 \end{cases}$

$$\therefore \; U(v) = 1 + v + v^2 \ldots$$
$$= \sum_{n=0}^{\infty} v^n$$

applying $\{u_n\}$ to $h \Rightarrow \; y_n + y_{n-1} = u_n - u_{n-4}$.

let $Y(v)$ be the polynomial representing the output ~~eq~~ seq.

then

$$Y(v) + v(Y(v)) = U(v) - v^4 U(v)$$
$$\Rightarrow \; Y(v)(1+v) = U(v)(1-v^4)$$
$$\Rightarrow \; Y(v) = U(v) \cdot \left(\frac{1-v^4}{1+v}\right)$$

$$= U(v) \cdot (1 - v + v^2 - v^3)$$
$$= \sum_{n=0}^{\infty} v^n \cdot (1 - v + v^2 + v^3)$$

$$= \sum_{n=0}^{\infty} v^n - \sum_{n=0}^{\infty} v^{n+1} + \sum_{n=0}^{\infty} v^{n+2} - \sum_{n=0}^{\infty} v^{n+3}$$
$$= 1 + \cancel{\sum_{n=0}^{\infty} v^{n+1}} - \cancel{\sum_{n=0}^{\infty} v^{n+1}} + v^2 + \cancel{\sum_{n=0}^{\infty} v^{n+3}} - \cancel{\sum_{n=0}^{\infty} v^{n+3}}$$

$$\boxed{\sum_{n=0}^{\infty} v^{i+n} = v^i + v^{i+1} + \ldots = v^i + \sum_{n=0}^{\infty} v^{i+1+n}}$$

$$= 1 + v^{n+2}$$
$=$ ~~8~~

$$\therefore \; \{y_n\} = \delta_n + \delta_{n-2}$$

or $y_n = \begin{cases} 0, & n<0 \\ 1, & n=0 \\ 0, & n=1 \\ 1, & n=2 \\ 0, & n>2 \end{cases}$ The step response.

c) $H(z) = \dfrac{Y(z)}{X(z)}$

given $y_n + y_{n-1} = x_n - x_{n-4}$.

applying Z transform.

$$Y(z) + z^{-1} Y(z) = X(z) - z^{-4} X(z)$$

$$\Rightarrow Y(z) \cdot (1 + z^{-1}) = X(z)(1 - z^{-4})$$

$$\Rightarrow \frac{Y(z)}{X(z)} = H(z) = \frac{1 - z^{-4}}{1 + z^{-1}}$$

d) $H(z) = \dfrac{(1+z^{-2})(1-z^{-1})(1+z^{-1})}{(1+z^{-1})}$

we can eleminate the factor $(1+z^{-1})$

$$H(z) = (1+z^{-2})(1-z^{-1})$$

~~This means that the system have overlapping zeros & po~~
~~value is of defin.~~

$H(z)$ is no longer undefined at $z = -1$, since we eleminated the factors.

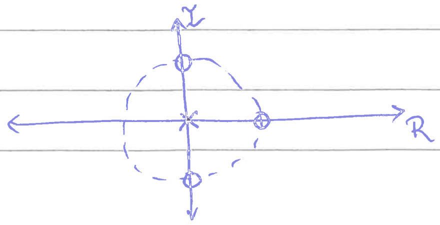
~~e) H~~

e) $H(z) = (1+z^{-2})(1-z^{-1})$

$$= \cancel{(1+z^{-1})(1-z^{-1})}$$

$$= (1 + jz^{-1})(1 - jz^{-1})(1 - z^{-1})$$

Thus $H(z)$ has zero at ~~(two)~~ $1, \pm j$ on the unit circle and 3 poles at $z = 0$.



f) we have zeros at $\theta = \frac{\pi}{2}, 0, -\frac{\pi}{2}$

$\therefore$ any DC value will be rejected & .

$\theta = \frac{2\pi f}{f_s} = \frac{\pi}{2} \Rightarrow \frac{2\pi}{8000} f = \frac{\pi}{2} \Rightarrow f = 2000\,Hz$

& 2000Hz freq will be rejected

Trivially we can see that $H(-1) = 4$

$\therefore$ for $\frac{2\pi f}{f_s} = \pi$ the amplitude will be quadrupled.

& $f = \frac{\pi \cdot 8000}{2\pi} = 4\,kHz$.

3) b) No. Not all ~~any~~ non-zero LTI systems T can have an inverse LTI system T^{-1}. We know that LTI systems ~~being~~ being applied to a sequence is ~~akin~~ ~~to~~ the same as convolution with its impulse response.

i.e. let $\{a_n\} = T\{\delta_n\}$ then for any arbitrary sequence $\{x_n\}$, $T\{x_n\} = \{a_n\} * \{x_n\}$

By this, assume we have an LTI system with impulse response as one of the sequences from part (a).
And let $\{x_n\}$, the input, be the other sequence.
As shown ~~a~~ before this will ~~lead~~ result in an all zero sequence which isn't invertible.

Further, the zeros and poles of the LTI system T & the inverse LTI system T^{-1} has be within the Unit Circle else the inverse ~~isn't~~ can't be feasibly built.