Math review

COMP 4630 | Winter 2025

Charlotte Curtis

But first, some stuff about assessments

- Assignment 1
- Journal club guidelines
- Example of a math-heavy paper
- Additional references for papers:
  - Google Scholar
  - ArXiv
  - Retraction Watch

Math review

- MATH 1200: Differential calculus
- MATH 1203: Linear algebra
- MATH 2234: Statistics

Further reading:

- Calculus: notebook
- Linear algebra: notebook, deep learning book

Calculus: Notation

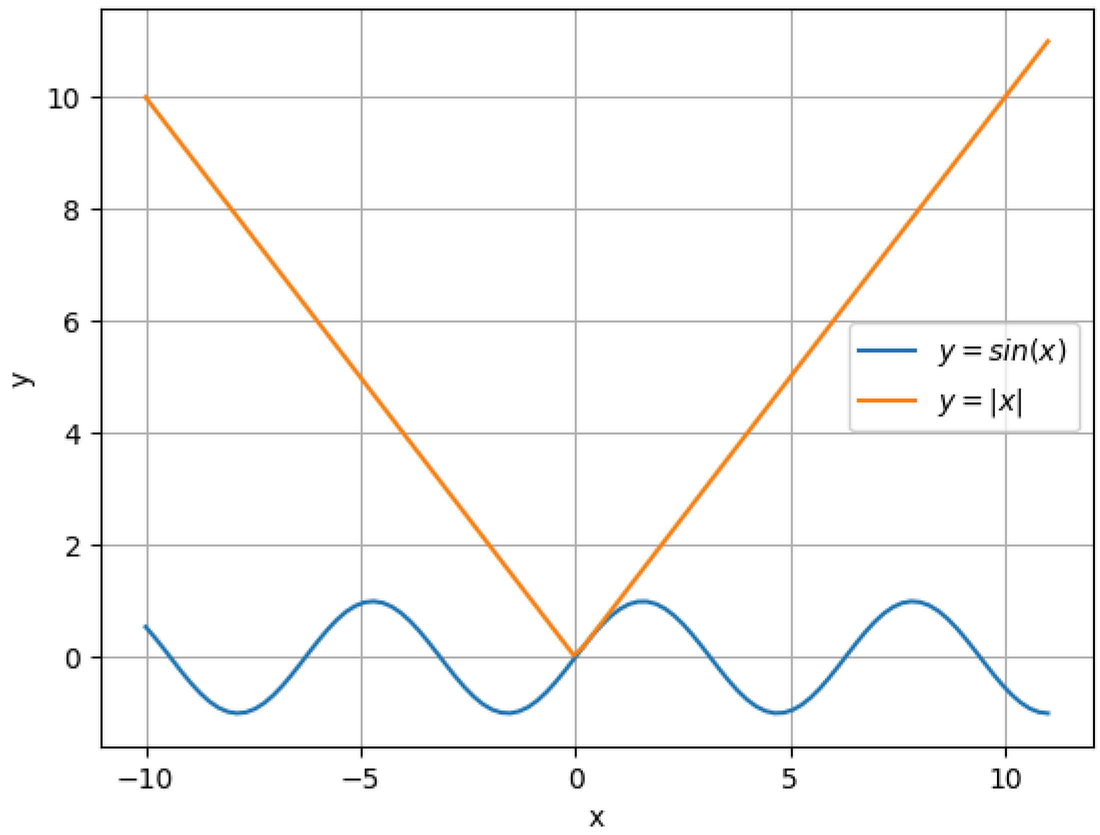
The **derivative** of a function $y = f(x)$ is represented as:

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The **second derivative** is denoted:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

and so on.



Differentiability

For a function to be **differentiable** at a point x_A, it must be:

- **Defined** at x_A
- **Continuous** at x_A
- **Smooth** at x_A
- **Non-vertical** at x_A

Select rules of differentiation

	Function f	Lagrange	Leibniz
Constant	$f(x) = c$	$f'(x) = 0$	$\frac{df}{dx} = 0$
Power	$f(x) = x^r$ with $r \neq 0$	$f'(x) = rx^{r-1}$	$\frac{df}{dx} = rx^{r-1}$
Sum	$f(x) = g(x) + h(x)$	$f'(x) = g'(x) + h'(x)$	$\frac{df}{dx} = \frac{dg}{dx} + \frac{dh}{dx}$
Exponential	$f(x) = e^x$	$f'(x) = e^x$	$\frac{df}{dx} = e^x$
Chain Rule	$f(x) = g(h(x))$	$f'(x) = g'(h(x))h'(x)$	$\frac{df}{dx} = \frac{dg}{dh}\frac{dh}{dx}$

This is the kind of thing I would not expect you to memorize on an exam

Example

1. Find $f'(x)$ for $f(x) = \sigma(x) = \dfrac{1}{1+e^{-x}}$

2. Now, let, $y = \sigma(x_1)$, where $x_1 = wx$. What is $\dfrac{dy}{dx}$?

Partial derivatives

For a scalar valued function $y = f(x_1, x_2)$, there are two partial derivatives:

$$\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}$$

These are computed by holding the "other" variable(s) **constant**. For example, if $y = 2x_1 + x_2 + x_1 x_2$, then:

$$\frac{\partial y}{\partial x_1} = 2 + x_2, \frac{\partial y}{\partial x_2} = 1 + x_1$$

Linear algebra

Vectors are multidimensional quantities (unlike **scalars**):

$$\vec{v} = \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

A common **vector space** is $\mathbb{R}^2$, or the 2D Euclidean plane. Example:

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Vector operations

- **Addition**: $\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} v_{11} + v_{21} \\ v_{12} + v_{22} \end{bmatrix}$
- **Scalar multiplication**: $c\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$
- **Dot product**: $\mathbf{v}_1 \cdot \mathbf{v}_2 = v_{11}v_{21} + v_{12}v_{22}$ (yields a scalar)
  - Can be thought of as the **projection** of one vector onto another, or how much two vectors are aligned in the same direction

Vector norms

- The **norm** of a vector is a measure of its length
- Most common is the **Euclidean norm** (or L^2 norm):

$$\|\mathbf{v}\|_2 = \|\mathbf{v}\| = \sqrt{\left(\sum_{i=1}^{n} v_i^2\right)}$$

- You might also see the L^1 norm, particularly as a regularization term:

$$\|\mathbf{v}\|_1 = \sum_{i=1}^{n} |v_i|$$

Useful vectors

- **Unit vector**: A vector with a norm of 1, e.g. $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- **Normalized vector**: A vector divided by its norm, e.g. $\mathbf{v} = \hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$
- Dot product can also be written as $\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \cos(\theta)$

> *Yes, a normalized vector is also a unit vector, main difference is in context and notation*

Matrices

A **matrix** is a 2D array of numbers:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Notation: Element a_{ij} is in row i, column j, also written as A_{ij}.

> *Rows then columns! $M \times N$ matrix has M rows and N columns*

Matrix operations

- **Addition**: element-wise *if* dimensions match. $A + B = B + A$
- **Scalar multiplication**: just like vectors
- **Matrix multiplication**: $C = AB$ where the elements of C are:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

  - Multiply and sum rows of A with columns of B
  - Usually, $AB \neq BA$

Matrix multiplication examples

Matrix times a matrix:

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \\ -4 & 5 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 3 & 7 \end{bmatrix}$$

Matrix times a vector:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Matrix transpose

- **Transpose**: A^T swaps rows and columns

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

- **Inverse**: just as $\frac{1}{x} \cdot x = 1$, $A^{-1}A = I$, where I is the identity matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, A^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}, A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

> *Not every matrix is invertible!*

A brief introduction to vector calculus

Putting together partial derivatives with vectors and matrices we get:

Scalar-valued $f(\mathbf{x})$:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Vector-valued $\mathbf{f}(\mathbf{x})$:

$$\mathbf{J_f} = \begin{bmatrix} \nabla^T f_1 \\ \nabla^T f_2 \\ \vdots \\ \nabla^T f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

> *Most of the time we'll just be working with the gradient*

Statistics: Notation

- A **random variable** $\mathrm{x} \sim P$ is a variable that can take on random variables according to some probability distribution P
- x may take on **discrete** (e.g. dice rolls) or **continuous** (e.g. age) values
- X or x for the random variable and x or x_i for a specific value
- $P(\mathrm{x})$ for a a discrete distribution and $p(\mathrm{x})$ for continuous
- $\mathrm{x}_P \equiv \mathrm{x} \sim P$ and $\mathrm{x}_p \equiv \mathrm{x} \sim p$

> *Some textbooks/papers/websites use different notation!*

Discrete random variables

- A discrete **probability mass function** describes the probability of x taking on a specific value
- Example: for a balanced 6-sided die, $P(\mathrm{x}=1)=\dfrac{1}{6}$
- You can add together probabilities, e.g. $P(\mathrm{x}\leq 3)=\sum_{i=1}^{3}P(\mathrm{x}=i)$
- $\sum_{x}P(\mathrm{x})=1$ and $P(x_i)\geq 0$ for any valid distribution

Continuous random variables

- A continuous **probability density function** gives the probability of being in some tiny interval δx given by $p(x)\delta x$
- Example: the **uniform distribution**, $p(\mathrm{x}) = \dfrac{1}{b-a}$ for $a \leq x \leq b$
- $p(\mathrm{x} = x_i) = 0$ for any specific value x_i
- Need to integrate to get a concrete value, e.g. $p(\mathrm{x} \leq a) = \displaystyle\int_a^b p(x)dx$
- $\displaystyle\int_{-\infty}^{\infty} p(x)dx = 1$ and $\displaystyle\int_a^b p(x)dx \geq 0$ for any valid distribution

Expectation and variance

- The **expectation** or **expected value** is its average value $\mathbb{E}[\mathrm{x}]$
- $\mathbb{E}[\mathrm{x}_P] = \sum_x xP(\mathrm{x})$ and $\mathbb{E}[\mathrm{x}_p] = \int_{-\infty}^{\infty} xp(x)dx$
- More generally, for any function $f(\mathrm{x})$:

$$\mathbb{E}[f(\mathrm{x})] = \sum_x f(x)P(\mathrm{x}) \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)p(x)dx$$

- The **variance** describes how much the values vary from their mean:

$$\mathrm{Var}[\mathrm{x}] = \mathbb{E}[(\mathrm{x} - \mathbb{E}[\mathrm{x}])^2]$$

Multiple random variables

- **Joint probability** $P(\mathrm{x}, \mathrm{y})$ is the probability of x and y occurring together
- **Conditional probability** $P(\mathrm{x} = x \mid \mathrm{y} = y)$ is the probability that x takes on value x given that $\mathrm{y} = y$ has already happened
- In general, $P(\mathrm{x} = x \mid \mathrm{y} = y) = \dfrac{P(\mathrm{x} = x, \mathrm{y} = y)}{P(\mathrm{y} = y)}$
- For **independent** variables, $P(\mathrm{x} = x \mid \mathrm{y} = y) = P(\mathrm{x} = x)$

Note: I'm using uppercase P here, but it all applies to continuous distributions as well

Covariance

- The **covariance** between $f(\mathrm{x})$ and $g(\mathrm{y})$ gives a sense of how linearly related they are and how much they vary together:

$$\mathrm{Cov}(f(\mathrm{x}), g(\mathrm{y})) = \mathbb{E}[(f(\mathrm{x}) - \mathbb{E}[f(\mathrm{x})])(g(\mathrm{y}) - \mathbb{E}[g(\mathrm{y})])]$$

- Related to correlation as $\mathrm{Corr}(f(\mathrm{x}), g(\mathrm{y})) = \dfrac{\mathrm{Cov}(f(\mathrm{x}), g(\mathrm{y}))}{\sqrt{\mathrm{Var}(f(\mathrm{x}))\mathrm{Var}(g(\mathrm{y}))}}$
- The **covariance matrix** of a random vector $\mathbf{x}$ is a square matrix where the (i, j) element is the covariance between x_i and x_j
- The diagonal of the covariance matrix gives $\mathrm{Var}(x_i)$

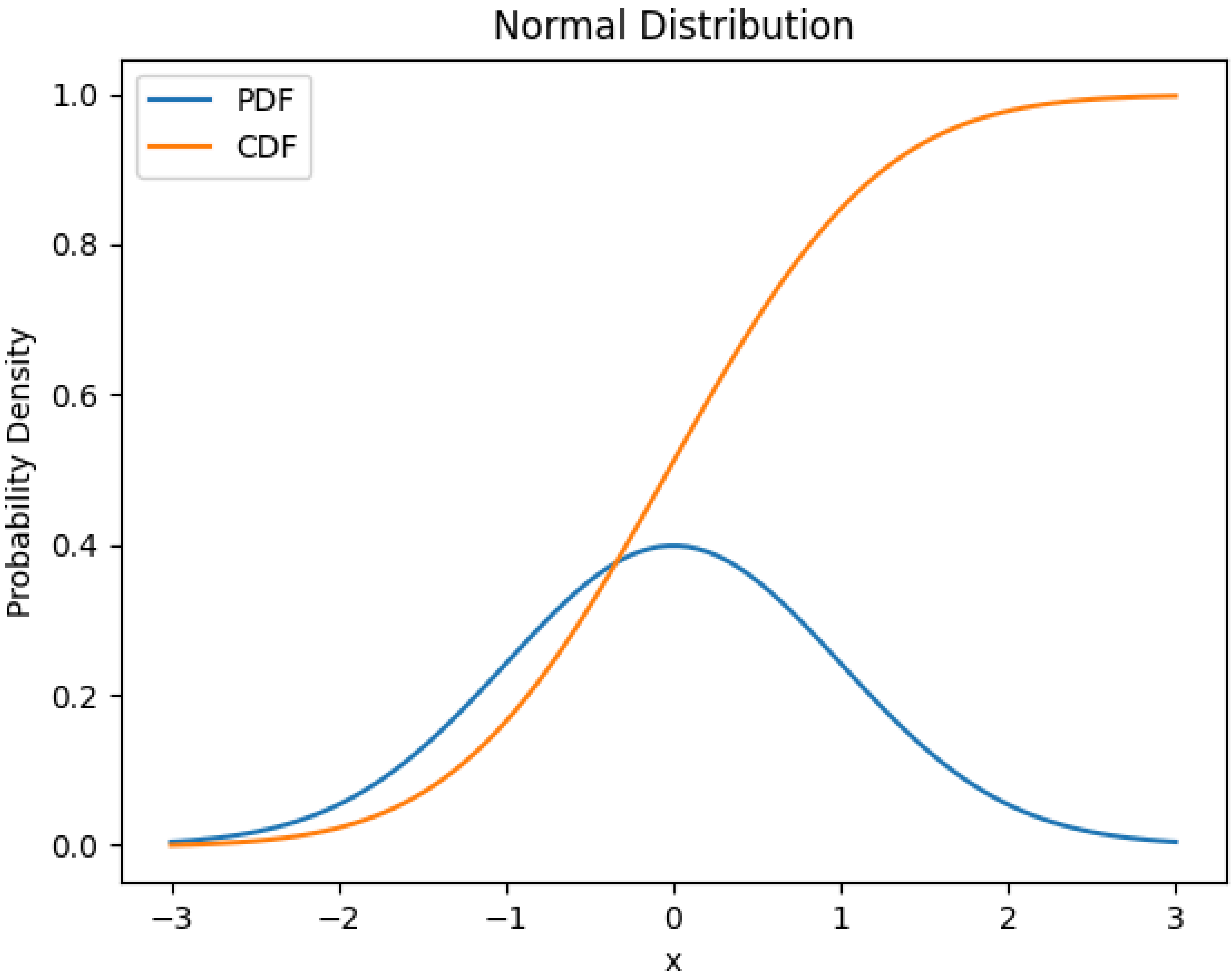
The Normal distribution

$$N(x; u, \sigma^2) = \sqrt{\frac{1}{2\pi\sigma^2}} \exp^{\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)}$$

Good "default choice" for two reasons:

- The **central limit theorem** shows that the sum of many (> 30 ish) independent random variables is normally distributed
- Has the most **uncertainty** of any distribution with the same variance

> *We can't easily integrate $N(x)$, so numerical approximations are used*


Normal Distribution
PDF
CDF
Probability Density
1.0
0.8
0.6
0.4
0.2
0.0
−3
−2
−1
0
1
2
3
x