

# Math review

---

COMP 4630 | Winter 2025

Charlotte Curtis

# But first, some stuff about assessments

---

- [Assignment 1](#)
- [Journal club guidelines](#)
- [Example](#) of a math-heavy paper
- Additional references for papers:
  - [Google Scholar](#)
  - [ArXiv](#)
  - [Retraction Watch](#)

# Math review

---

- MATH 1200: Differential calculus
- MATH 1203: Linear algebra
- MATH 2234: Statistics

Further reading:

- Calculus: [notebook](#)
- Linear algebra: [notebook](#), [deep learning book](#)

# Calculus: Notation

---

The **derivative** of a function  $y = f(x)$  is represented as:

$$f'(x) = \frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

The **second derivative** is denoted:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right)$$

and so on.



# Differentiability

For a function to be **differentiable** at a point  $x_A$ , it must be:

- **Defined** at  $x_A$
- **Continuous** at  $x_A$
- **Smooth** at  $x_A$
- **Non-vertical** at  $x_A$

# Select rules of differentiation

|                    | Function $f$                 | Lagrange                | Leibniz   |
|--------------------|------------------------------|-------------------------|---|
| <b>Constant</b>    | $f(x) = c$                   | $f'(x) = 0$             | $\frac{df}{dx} = 0$                             |
| <b>Power</b>       | $f(x) = x^r$ with $r \neq 0$ | $f'(x) = rx^{r-1}$      | $\frac{df}{dx} = rx^{r-1}$                      |
| <b>Sum</b>         | $f(x) = g(x) + h(x)$         | $f'(x) = g'(x) + h'(x)$ | $\frac{df}{dx} = \frac{dg}{dx} + \frac{dh}{dx}$ |
| <b>Exponential</b> | $f(x) = e^x$                 | $f'(x) = e^x$           | $\frac{df}{dx} = e^x$                           |
| <b>Chain Rule</b>  | $f(x) = g(h(x))$             | $f'(x) = g'(h(x))h'(x)$ | $\frac{df}{dx} = \frac{dg}{dh} \frac{dh}{dx}$   |

# Example

---

1. Find  $f'(x)$  for  $f(x) = \sigma(x) = \frac{1}{1 + e^{-x}}$
2. Now, let,  $y = \sigma(x_1)$ , where  $x_1 = wx$ . What is  $\frac{dy}{dx}$ ?

# Partial derivatives

---

For a scalar valued function  $y = f(x_1, x_2)$ , there are two partial derivatives:

$$\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}$$

These are computed by holding the "other" variable(s) **constant**. For example, if  $y = 2x_1 + x_2 + x_1x_2$ , then:

$$\frac{\partial y}{\partial x_1} = 2 + x_2, \frac{\partial y}{\partial x_2} = 1 + x_1$$



# Linear algebra

---

**Vectors** are multidimensional quantities  
(unlike **scalars**):

$$\vec{v} = \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

A common **vector space** is  $\mathbb{R}^2$ , or the 2D Euclidean plane. Example:

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

# Vector operations

---

- **Addition:**  $\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} v_{11} + v_{21} \\ v_{12} + v_{22} \end{bmatrix}$
- **Scalar multiplication:**  $c\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$
- **Dot product:**  $\mathbf{v}_1 \cdot \mathbf{v}_2 = v_{11}v_{21} + v_{12}v_{22}$  (yields a scalar)
  - Can be thought of as the **projection** of one vector onto another, or how much two vectors are aligned in the same direction

# Vector norms

---

- The **norm** of a vector is a measure of its length
- Most common is the **Euclidean norm** (or  $L^2$  norm):

$$\|\mathbf{v}\|_2 = \|\mathbf{v}\| = \sqrt{\left(\sum_{i=1}^n v_i^2\right)}$$

- You might also see the  $L^1$  norm, particularly as a regularization term:

$$\|\mathbf{v}\|_1 = \sum_{i=1}^n |v_i|$$

# Useful vectors

---

- **Unit vector:** A vector with a norm of 1, e.g.  $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- **Normalized vector:** A vector divided by its norm, e.g.  $\mathbf{v} = \hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$
- Dot product can also be written as  $\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \cos(\theta)$

*Yes, a normalized vector is also a unit vector, main difference is in context and notation*

# Matrices

---

A **matrix** is a 2D array of numbers:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Notation: Element  $a_{ij}$  is in row  $i$ , column  $j$ , also written as  $A_{ij}$ .

*Rows then columns!  $M \times N$  matrix has  $M$  rows and  $N$  columns*

# Matrix operations

---

- **Addition:** element-wise *if* dimensions match.  $A + B = B + A$
- **Scalar multiplication:** just like vectors
- **Matrix multiplication:**  $C = AB$  where the elements of  $C$  are:

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

- Multiply and sum rows of  $A$  with columns of  $B$
- Usually,  $AB \neq BA$

# Matrix multiplication examples

---

Matrix times a matrix:

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \\ -4 & 5 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 3 & 7 \end{bmatrix}$$

Matrix times a vector:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

# Matrix transpose

---

- **Transpose:**  $A^T$  swaps rows and columns

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

- **Inverse:** just as  $\frac{1}{x} \cdot x = 1$ ,  $A^{-1}A = I$ , where  $I$  is the identity matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, A^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}, A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

*Not every matrix is invertible!*



# A brief introduction to vector calculus

---

Putting together partial derivatives with vectors and matrices we get:

Scalar-valued  $f(\mathbf{x})$ :

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Vector-valued  $\mathbf{f}(\mathbf{x})$ :

$$\mathbf{J}_{\mathbf{f}} = \begin{bmatrix} \nabla^T f_1 \\ \nabla^T f_2 \\ \vdots \\ \nabla^T f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

*Most of the time we'll just be working with the gradient*

# Statistics: Notation

---

- A **random variable**  $\mathbf{x} \sim P$  is a variable that can take on random variables according to some probability distribution  $P$
- $\mathbf{x}$  may take on **discrete** (e.g. dice rolls) or **continuous** (e.g. age) values
- $X$  or  $\mathbf{x}$  for the random variable and  $x$  or  $x_i$  for a specific value
- $P(\mathbf{x})$  for a discrete distribution and  $p(\mathbf{x})$  for continuous
- $\mathbf{x}_P \equiv \mathbf{x} \sim P$  and  $\mathbf{x}_p \equiv \mathbf{x} \sim p$

*Some textbooks/papers/websites use different notation!*

# Discrete random variables

---

- A discrete **probability mass function** describes the probability of  $x$  taking on a specific value
- Example: for a balanced 6-sided die,  $P(x = 1) = \frac{1}{6}$
- You can add together probabilities, e.g.  $P(x \leq 3) = \sum_{i=1}^3 P(x = i)$
- $\sum_x P(x) = 1$  and  $P(x_i) \geq 0$  for any valid distribution

# Continuous random variables

---

- A continuous **probability density function** gives the probability of being in some tiny interval  $\delta x$  given by  $p(x)\delta x$
- Example: the **uniform distribution**,  $p(x) = \frac{1}{b-a}$  for  $a \leq x \leq b$
- $p(x = x_i) = 0$  for any specific value  $x_i$
- Need to integrate to get a concrete value, e.g.  $p(x \leq a) = \int_a^b p(x)dx$
- $\int_{-\infty}^{\infty} p(x)dx = 1$  and  $\int_a^b p(x)dx \geq 0$  for any valid distribution

# Expectation and variance

---

- The **expectation** or **expected value** is its average value  $\mathbb{E}[\mathbf{x}]$

- $\mathbb{E}[\mathbf{x}_P] = \sum_x xP(\mathbf{x})$  and  $\mathbb{E}[\mathbf{x}_p] = \int_{-\infty}^{\infty} xp(x)dx$

- More generally, for any function  $f(\mathbf{x})$ :

$$\mathbb{E}[f(\mathbf{x})] = \sum_x f(x)P(\mathbf{x}) \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)p(x)dx$$

- The **variance** describes how much the values vary from their mean:

$$\text{Var}[\mathbf{x}] = \mathbb{E}[(\mathbf{x} - \mathbb{E}[\mathbf{x}])^2]$$

# Multiple random variables

---

- **Joint probability**  $P(\mathbf{x}, y)$  is the probability of  $\mathbf{x}$  and  $y$  occurring together
- **Conditional probability**  $P(\mathbf{x} = x \mid y = y)$  is the probability that  $\mathbf{x}$  takes on value  $x$  given that  $y = y$  has already happened
- In general, 
$$P(\mathbf{x} = x \mid y = y) = \frac{P(\mathbf{x} = x, y = y)}{P(y = y)}$$
- For **independent** variables,  $P(\mathbf{x} = x \mid y = y) = P(\mathbf{x} = x)$

# Covariance

---

- The **covariance** between  $f(\mathbf{x})$  and  $g(\mathbf{y})$  gives a sense of how linearly related they are and how much they vary together:

$$\text{Cov}(f(\mathbf{x}), g(\mathbf{y})) = \mathbb{E}[(f(\mathbf{x}) - \mathbb{E}[f(\mathbf{x})])(g(\mathbf{y}) - \mathbb{E}[g(\mathbf{y})])]$$

- Related to correlation as  $\text{Corr}(f(\mathbf{x}), g(\mathbf{y})) = \frac{\text{Cov}(f(\mathbf{x}), g(\mathbf{y}))}{\sqrt{\text{Var}(f(\mathbf{x}))\text{Var}(g(\mathbf{y}))}}$
- The **covariance matrix** of a random vector  $\mathbf{x}$  is a square matrix where the  $(i, j)$  element is the covariance between  $x_i$  and  $x_j$
- The diagonal of the covariance matrix gives  $\text{Var}(x_i)$

# The Normal distribution

---

$$N(x; \mu, \sigma^2) = \sqrt{\frac{1}{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

Good "default choice" for two reasons:

- The **central limit theorem** shows that the sum of many ( $> 30$  ish) independent random variables is normally distributed
- Has the most **uncertainty** of any distribution with the same variance

*We can't easily integrate  $N(x)$ , so numerical approximations are used*



