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- **Hassle-Free Commuting:**
  - Deciding which highways and roads to take to minimize total delay due to traffic ?

Unweighted Shortest Path Problem

Instance: An unweighted graph $G = (V, E)$ with $s \in V$, named as "source"

Goal: To find the shortest path (in terms of length of the path) from s to all vertices in G.

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Solution

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Solution

A BFS can do the job

What if edges have weights

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Does BFS still work for finding minimum cost paths ?

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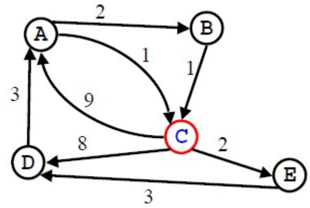

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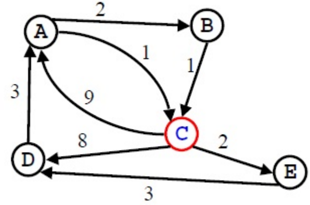

Shortest path from C to A

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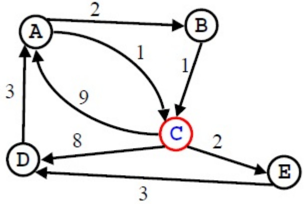

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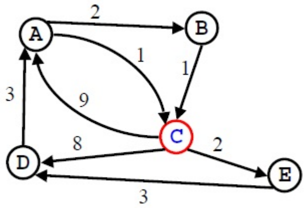

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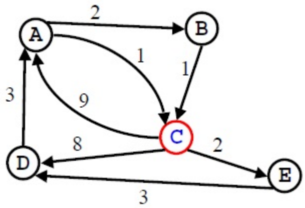

Shortest path from C to A

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- Minimum cost path: C $\rightarrow$ E $\rightarrow$ D $\rightarrow$ A
  Cost is 8

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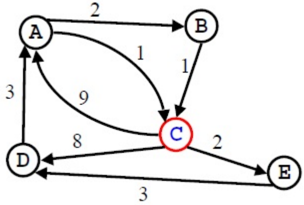

Shortest path from C to A

- BFS: C$\rightarrow$ A and cost is 9.
- Minimum cost path: C $\rightarrow$ E $\rightarrow$ D $\rightarrow$ A
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Therefore, BFS does not work anymore

Dijkstra's Algorithm

From now onwards, if we say shortest path in weighted graphs, then we mean that it is minimum cost path in weighted graph, not the shortest length path

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- Classic algorithm for solving shortest path in weighted graphs (without negative weights)
- The algorithm is based on greedy choice.

Basic Idea

- Each vertex stores a cost for path from source (Initially ∞)
- Greedy Choice: always select the current best vertex
- Update costs of all neighbors of selected vertex

Notations

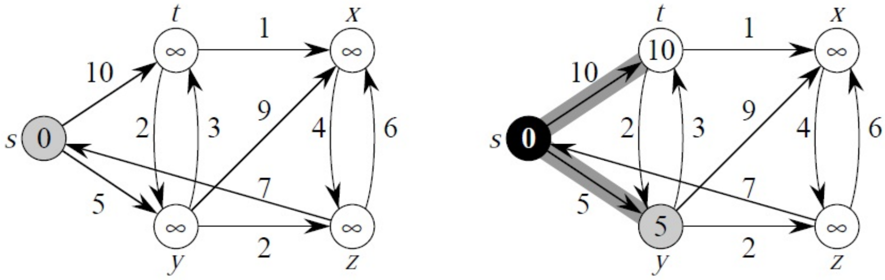
- weighted digraph: A digraph $G = (V, E, w)$, where $w : E \to R$
- weight of a path p: If $p =< v_0, v_1, \ldots, v_k >$ is a path, then $w(p) = \sum_{i=1}^{k} w(v_{i-1} v_i)$
- $\delta(u, v)$: shortest path weight of the paths from u to v and is defined as

  $$\delta(u, v) = \begin{cases} \min\{w(p) : p \text{ is a path from} u \text{ to } v\}, & \text{if there is a path from } u \text{ to } v; \\ \infty, & \text{Otherwise.} \end{cases}$$
- $d[v]$: an upper bound on the weight of a shortest path from source s to v.

Pseudocode of Dijkstra's algorithm

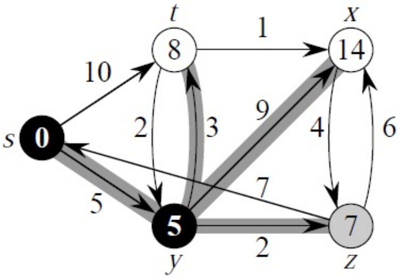
for *each* $v \in V$ **do**
    $d[v] = \infty$;
    $\pi[v] = \mathrm{NIL}$;
end
$d[s] = 0$;
$S = \emptyset$;
$Q = V$;
while $Q \neq \emptyset$ **do**
    $u = \textsc{Extract-Min}(Q)$;
    $S = S \cup \{u\}$;
    for *each* $v \in Adj[u]$ **do**
        if $(d[v] > d[u] + w(u, v))$ **then**
            $d[v] = d[u] + w(u, v)$;
            $\pi[v] = u$;
        end
    end
end

Example

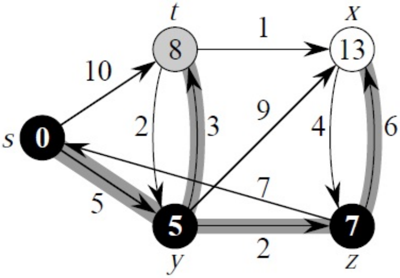


(a) (b)

Example



(c)



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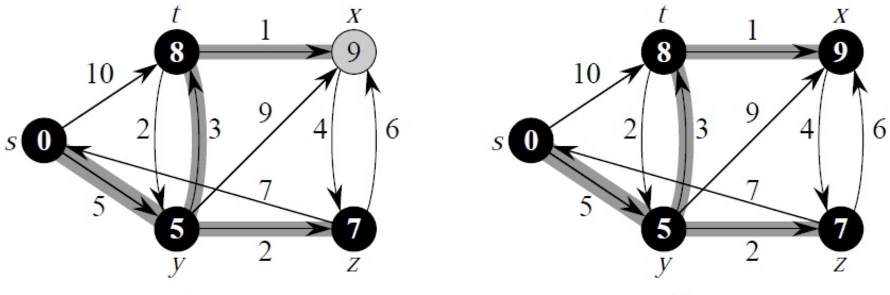

(e) (f)

Analysis

```
for each v ∈ V do
    d[v] = ∞;
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end
d[s] = 0;
S = ∅;
Q = V;
while Q ≠ ∅ do
    u = EXTRACT-MIN(Q);
    S = S ∪ {u};
    for each v ∈ Adj[u] do
        if (d[v] >
          d[u] + w(u, v))
          then
            d[v] =
              d[u] +
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- Lines 1-4 takes $O(|V|)$ time

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$\Theta(|V| \cdot T_{\textsc{Extract-Min}} + |E| \cdot T_{\textsc{extract-key}})$

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Similar to Prim's algorithm.........

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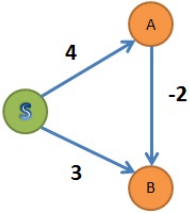

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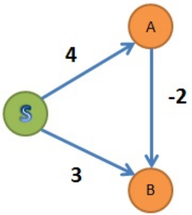

Whether a DP will work ?