# **CmdStan Interface**

# User's Guide

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# Part I Introduction

# 1. Overview

This document is a user's guide for the CmdStan interface to the Stan probabilistic modeling language. CmdStan takes Stan programs and generates executables that can be run directly from the command line. CmdStan is one of several interfaces to Stan; there are also R, Python, Matlab, Julia, and Stata interfaces.

# 1.1. Stan Home Page

For links to up-to-date code, examples, manuals, bug reports, feature requests, and everything else Stan related, see the Stan home page:

http://mc-stan.org/

# 1.2. Licensing

CmdStan, Stan, and the Stan Math Library are licensed under the new BSD license (3-clause). See Appendix A for details, including licensing terms for the dependent packages Boost and Eigen.

# 1.3. Modeling Language User's Guide and Reference

Stan's modeling language is shared across all of its interfaces. Stan's language, along with a programming guide and many example models, is detailed in the *Stan Modeling Language User's Guide and Reference Manual*, which is available from the Stan home page (see Section 1.1).

# 1.4. Example Models

There are many example models for Stan, in addition to those in the user's guide and reference. These are all linked from the Stan home page (see Section 1.1).

#### 1.5. Benefits of CmdStan

Although CmdStan has the least amount of functionality among the Stan interfaces, the minimal nature of CmdStan makes it trivial to install and use the latest development version of the Stan library. It also has the fewest dependencies, which makes it easier to run in limited environments such as clusters. The output generated is in CSV format and can be post-processed using other Stan interfaces or general tools.

# 2. Getting Started

This chapter is designed to help users get acquanited with the CmdStan interface. Later chapters are devoted to expanding on the material in this chapter with full reference documentation. See the Stan user's manual for details about the Stan language.

#### 2.1. Installation

Installation of CmdStan is simple, just download and unpack the source. CmdStan requires:

- The CmdStan source and all its libraries. Everything is included in the release tarball or zip file.
- The make utility.
- · A C++ compiler.

For information about supported versions of Windows, Mac, and Linux platforms see Appendix B. For step-by-step installation instructions of the prerequisites, see

· Windows: Appendix B.3

· Mac: Appendix B.4

· Linux: Appendix B.5.

# 2.2. Building CmdStan

Building CmdStan involves building executable programs for the Stan compiler, stanc, and a basic posterior analysis tool, stansummary. The build process utilizes the make command-line utility and works the same way across all supported platforms.

1. Open a command-line terminal window and change directories to the CmdStan directory. From here on, we'll refer to this location as <cmdstan-home>.

> cd <cmdstan-home>

A listing of the files and directory of this folder should show these files:

> 1s

LICENSE makefile

README.md runCmdStanTests.py

doc src examples stan

make test-all.sh

- 2. *Optional:* Set local make variables by editing the file ~/.config/cmdstan/make.local or <cmdstan-home>/make/local. See Appendix B.6 for a list of available options. For most installations, this step can be skipped. Setting local make variables here will persist between builds.
- 3. Use make to build CmdStan. When multiple CPU cores are available on the system, the call to make can be parallelized. It can either be specified directly when calling make with the -jN option, where N is the number of CPU cores. For instance, to run on 2 cores, use

```
> make build -j2
```

*Warning:* The make program may take 10+ minutes and consume 2+ GB of memory to build CmdStan.

When CmdStan is successfully built, the make program will report

```
--- CmdStan v2.8.0 built ---
```

and there will be two executables in the bin folder:

- stanc, the Stan compiler. The Stan compiler translates a Stan program into C++ code. See Chapter 4 for details.
- · stansummary, a posterior analysis tool. The stansummary command summarizes the comma-separated values files that are generated from Stan program runs. For each parameter within the Stan program, stansummary reports the mean, standard deviation, quantiles,  $\hat{R}$ , and other values. See Chapter 6 for details.

# 2.3. Compiling and Executing a Stan Program

The rest of this quick-start guide explains how to code and run a very simple Bayesian model.

#### A Simple Bernoulli Model

The following is a simple, complete Stan program for a Bernoulli model of binary data.<sup>1</sup>

```
data {
   int<lower=0> N;
   int<lower=0,upper=1> y[N];
}
parameters {
   real<lower=0,upper=1> theta;
}
model {
   theta ~ beta(1,1);
   for (n in 1:N)
      y[n] ~ bernoulli(theta);
}
```

The model assumes the binary observed data y[1], ..., y[N] are i.i.d. with Bernoulli chance-of-success theta. The prior on theta is beta(1,1) (i.e., uniform).

#### Data Set

A data set of N = 10 observations is coded as follows. <sup>2</sup>

```
N \leftarrow 10
y <- c(0,1,0,0,0,0,0,0,0,1)
```

This defines the contents of two variables, N and y, using an R-like syntax (see Chapter C for more information).

#### Change directories to <cmdstan-home>

Before building any Stan program, change directories to <cmdstan-home>.

<sup>&</sup>lt;sup>1</sup>The model is distributed with Stan's example models repository, https://github.com/stan-dev/example-models/blob/master/basic\_estimators/bernoulli.stan, and is also available with the CmdStan distribution at the path examples/bernoulli/bernoulli.stan.

<sup>&</sup>lt;sup>2</sup> The data is also available from the example-models repository on GitHub, https://github.com/stan-dev/example-models/blob/master/basic\_estimators/bernoulli.data.R. It is also included with the CmdStan distribution and can be found at path stan/example-models/basic\_estimators/bernoulli.data.R.

#### Compiling a Stan Program

A single call to make will translate a Stan program to C++, then compile the C++ code to an executable for the command line.

A Stan program must be in a file with the file extension .stan. To create an executable from a Stan program, make will be called with the name of the executable as its argument. For Mac and Linux, it is the name of the Stan program with the .stan omitted. For Windows, replace .stan with .exe.

To build the Bernoulli example, use the following command for Mac and Linux:

> make examples/bernoulli/bernoulli

For Windows, the command is the same with the addition of .exe at the end of the target:

> make examples/bernoulli/bernoulli.exe

The generated C++ code (bernoulli.hpp) and the compiled executable will be placed in the same directory as the Stan program.

*Note:* to build a Stan program, you must be in the <cmdstan-home> directory. The path to the Stan program must not contain a space. If using relative paths, the relative path must not contain a space.

# Sampling from the Stan Program

The program can be executed from the directory in which it resides.

> cd examples/bernoulli

To execute sampling of the model under Linux or Mac, use

> ./bernoulli sample data file=bernoulli.data.R

In Windows, the ./ prefix is not needed, resulting in the following command.

> bernoulli.exe sample data file=bernoulli.data.R

The output is the same across all supported platforms. First, the configuration of the program is echoed to the standard output:

```
method = sample (Default)
  sample
   num_samples = 1000 (Default)
   num_warmup = 1000 (Default)
   save_warmup = 0 (Default)
```

```
thin = 1 (Default)
    adapt
     engaged = 1 (Default)
     gamma = 0.0500000000000000000003 (Default)
     kappa = 0.75 (Default)
     t0 = 10 (Default)
     init_buffer = 75 (Default)
     term buffer = 50 (Default)
     window = 25 (Default)
    algorithm = hmc (Default)
     hmc
       engine = nuts (Default)
         nuts
           max_depth = 10 (Default)
       metric = diag_e (Default)
       stepsize = 1 (Default)
       stepsize_jitter = 0 (Default)
id = 0 (Default)
data
  file = bernoulli.data.R
init = 2 (Default)
random
  seed = 4294967295 (Default)
output
  file = output.csv (Default)
 diagnostic_file = (Default)
  refresh = 100 (Default)
```

After the configuration has been displayed, a short timing message is given.

```
Gradient evaluation took 4e-06 seconds
1000 transitions using 10 leapfrog steps per transition would
take 0.04 seconds.
Adjust your expectations accordingly!
```

Next, the sampler reports the iteration number, reporting the percentage complete.

```
Iteration: 1 / 2000 [ 0%] (Warmup)
Iteration: 100 / 2000 [ 5%] (Warmup)
...
Iteration: 2000 / 2000 [100%] (Sampling)
```

#### Sampler Output

Each execution of the model results in draws from a single Markov chain being written to a file in comma-separated value (CSV) format. The default name of the output file is output.csv.

The first part of the output file records the version of the underlying Stan library and the configuration as comments (i.e., lines beginning with the pound sign (#)).

```
# stan_version_major = 2
# stan_version_minor = 7
# stan_version_patch = 0
# model = bernoulli_model
# method = sample (Default)
# sample
# num_samples = 1000 (Default)
# num_warmup = 1000 (Default)
# save_warmup = 0 (Default)
# thin = 1 (Default)
```

This is followed by a CSV header indicating the names of the values sampled.

```
lp__,accept_stat__,stepsize__,treedepth__,n_leapfrog__,n_divergent__,theta
```

The first column reports the unnormalized log probability of the model. The next columns provide sampler-dependent information; here, it is columns two through five. For basic Hamiltonian Monte Carlo (HMC) and its adaptive variant, the No-U-Turn sampler (NUTS), the sampler-depedent parameters are described in the following table.

Sampler	Parameter	Description
NUTS	accept_stat	Metropolis acceptance probability
		averaged over samples in the slice
NUTS	stepsize	Integrator step size
NUTS	treedepth	Tree depth
NUTS	n_leapfrog	Number of leapfrog calculations
NUTS	n_divergent	Number of divergent iterations
HMC	accept_stat	Metropolis acceptance probability
HMC	stepsize	Integrator step size
HMC	int_time	Total integration time

The remaining columns correspond to model parameters. For the Bernoulli model, it is just the sixth column, theta. The header line is streamed to the output file before warmup begins.

The next section describes the results of adaptation taking place during the warmup phase.

```
# Adaptation terminated
# Step size = 1.81311
# Diagonal elements of inverse mass matrix:
# 0.415719
```

The default sampler is NUTS with an adapted step size and a diagonal inverse mass matrix. For this example, the step size is 1.81311, and the inverse mass contains the single entry 0.415719 corresponding to the parameter theta.

Draws from the posterior distribution are printed out next, each line containing a single draw with the columns corresponding to the header. <sup>3</sup>

```
-7.22574,1,1.09068,1,1,0.383089
-6.95293,0.945991,1.09068,2,3,0.335074
-6.92373,0.938744,1.09068,1,1,0.181194
-6.83655,0.934833,1.09068,2,3,0.304882
-7.01732,1,1.09068,1,1,0.348244
```

The output ends with timing details,

```
# Elapsed Time: 0.006811 seconds (Warm-up)
# 0.011645 seconds (Sampling)
# 0.018456 seconds (Total)
```

# Summarizing Sampler Output

The command-line program bin/stansummary will display summary information about the run (for more information, see Chapter 6). To run stansummary on the output file generated for bernoulli on Linux or Mac, type

```
> <cmdstan-home>/bin/stansummary output.csv
```

For Windows, use backslashes to call the stansummary.exe.

```
> <cmdstan-home>\bin\stansummary.exe output.csv
```

The output of the command will display information about the run followed by information for each parameter and generated quantity. For bernoulli, we ran 1 chain and saved 1000 iterations. The information is echoed to the standard output stream. The output is

<sup>&</sup>lt;sup>3</sup>There are repeated entries due to the Metropolis accept step in the No-U-Turn sampling algorithm.

```
Inference for Stan model: bernoulli_model
1 chains: each with iter=(1000); warmup=(0); thin=(1); 1000 iterations saved.
```

Warmup took (0.014) seconds, 0.014 seconds total Sampling took (0.027) seconds, 0.027 seconds total

	Mean	MCSE	StdDev	5%	50%	95%	N_Eff	N_Eff/s	R_hat
1p	-7.3	4.6e-02	7.4e-01	-8.7	-7.0	-6.8	255	9267	1.0e+00
accept_stat	0.94	2.7e-03	8.5e-02	0.76	0.98	1.0	1000	36395	1.0e+00
stepsize	0.79	1.1e-15	7.8e-16	0.79	0.79	0.79	0.50	18	1.0e+00
treedepth	1.9	2.2e-02	6.5e-01	1.0	2.0	3.0	844	30733	1.0e+00
n_leapfrog	3.1	6.4e-02	1.9e+00	1.0	3.0	7.0	897	32644	1.0e+00
n_divergent	0.00	0.0e+00	0.0e+00	0.00	0.00	0.00	1000	36395	nan
theta	0.26	5.9e-03	1.2e-01	0.091	0.25	0.48	420	15292	1.0e+00

Samples were drawn using hmc with nuts.

For each parameter, N\_Eff is a crude measure of effective sample size, and R\_hat is the potential scale reduction factor on split chains (at convergence, R\_hat=1).

In addition to the general information about the runs, stansummary displays summary statistics for each parameter and generated quantity.

In the bernoulli model, there is a single parameter, theta. The mean, standard error of the mean, standard deviation, the 5%, 50%, and 95% quantiles, number of effective samples (total and per second), and  $\hat{R}$  value are displayed. These quantities and their uses are described in detail in the introductory Markov chain Monte Carlo (MCMC) chapter of the language user's guide and reference manual.

The command bin/stansummary can be called with more than one csv file by separating filenames with spaces. It will also take wildcards in specifying filenames. A typical usage of Stan from the command line would first create one or more Markov chains by calling the model executable, typically in parallel, writing the output CSV file for each into its own directory. After all of the processes are finished, the results would be analyzed using stansummary to assess convergence and inspect the means and quantiles of the fitted variables. Additionally, downstream inferences may be performed using the draws (e.g., to make decisions or predictions for unseen data).

# Optimization

CmdStan can be used for finding posterior modes as well as sampling from the posterior distribution. The executable does not need to be recompiled in order to switch from sampling to optimization, and the data input format is the same. The following is a minimal call to Stan's optimizer using defaults for everything but the location of the data file. See Section 8.3 for more details.

> ./bernoulli optimize data file=bernoulli.data.R

Executing this command prints the following.

```
method = optimize
  optimize
     algorithm = lbfgs (Default)
       1bfas
        init_alpha = 0.001 (Default)
        tol_obj = 9.999999999999998e-13 (Default)
        tol_rel_obj = 10000 (Default)
        tol_grad = 1e-08 (Default)
        tol_rel_grad = 10000000 (Default)
         tol_param = 1e-08 (Default)
        history_size = 5 (Default)
     iter = 2000 (Default)
     save iterations = 0 (Default)
id = 0 (Default)
data
  file = bernoulli.data.R
init = 2 (Default)
 random
   seed = 4294967295 (Default)
output
   file = output.csv (Default)
  diagnostic_file = (Default)
   refresh = 100 (Default)
initial log joint probability = -5.18908
          log prob
 Iter
                          lldxll
                                       ||grad||
                                                   alpha
                                                           alpha0 # evals Notes
          -5.00402
                                    7.80306e-05
                       0.00400907
Optimization terminated normally:
 Convergence detected: relative gradient magnitude is below tolerance
```

The first part of the output reports on the configuration used, here indicating the default L-BFGS optimizer, with default initial stepsize and tolerances for monitoring convergence. The second part of the output indicates how well the algorithm fared, here converging and terminating normally. The numbers reported indicate that it took 4 iterations and 7 gradient evaluations, resulting in a final state state where the change in parameters was roughly 0.004 and the length of the gradient roughly 8e-5. The alpha value is for step size used. This is, not surprisingly, far fewer iterations than required for sampling; even fewer iterations would be used with less stringent user-specified convergence tolerances.

### Optimization Output

The output from optimization is written into the file output.csv by default. The output follows the same pattern as the output for sampling, first dumping the entire set of parameters used.

```
# stan version major = 2
# stan version minor = 7
# stan version patch = 0
# model = bernoulli model
# method = optimize
    optimize
      algorithm = lbfgs (Default)
#
#
        1bfqs
          init_alpha = 0.001 (Default)
          tol_obj = 9.99999999999998e-13 (Default)
          tol_rel_obj = 10000 (Default)
          tol_grad = 1e-08 (Default)
          tol_rel_grad = 10000000 (Default)
#
          tol_param = 1e-08 (Default)
          history_size = 5 (Default)
      iter = 2000 (Default)
      save_iterations = 0 (Default)
# id = 0 (Default)
# data
    file = bernoulli.data.R
# init = 2 (Default)
# random
    seed = 458923754
# output
    file = output.csv (Default)
    diagnostic_file = (Default)
   refresh = 100 (Default)
1p___, theta
-5.00402,0.200008
```

Note that everything is a comment other than a line for the header, and a line for the values. Here, the header indicates the unnormalized log probability with lp\_\_ and the model parameter theta. The maximum log probability is -5.0 and the posterior mode for theta is 0.20. The mode exactly matches what we would expect from the data.<sup>4</sup> Because the prior was uniform, the result 0.20 represents the maximum likelihood estimate (MLE) for the very simple Bernoulli model. Note that no uncertainty is reported.

<sup>&</sup>lt;sup>4</sup>The Jacobian adjustment included for the sampler's log probability function is not applied during optimization, because it can change the shape of the posterior and hence the solution.

#### Variational Inference

CmdStan can also approximate the posterior distribution using variational inference. The executable does not need to be recompiled in order to switch to variational inference, and the data input format is the same. The following is a minimal call to Stan's variational inference algorithm using defaults for everything but the location of the data file. See Section 8.3 for more details.

#### > ./bernoulli variational data file=bernoulli.data.R

Executing this command prints the following.

```
method =
variational
   algorithm = meanfield (Default)
    meanfield
   iter = 10000 (Default)
   grad samples = 1 (Default)
   elbo samples = 100 (Default)
   eta_adagrad = 0.1000000000000001 (Default)
   tol_rel_obj = 0.01 (Default)
   eval_elbo = 100 (Default)
   output_samples = 1000 (Default)
id = 0 (Default)
file = bernoulli.data.R
init = 2 (Default)
random
seed = 3959575278
output
file = output.csv (Default)
diagnostic_file = (Default)
refresh = 100 (Default)
This is Automatic Differentiation Variational Inference.
(EXPERIMENTAL ALGORITHM: expect frequent updates to the procedure.)
Gradient evaluation took 1.1e-05 seconds
1000 iterations under these settings should take 0.011 seconds.
Adjust your expectations accordingly!
iter
           ELBO
                  delta_ELBO_mean
                                    delta_ELBO_med
                                                     notes
            -7
  0
                            1.000
                                             1.000
           -6.3
100
                            0.544
                                             1.000
200
           -6.2
                            0.365
                                             0.087
           -6.4
                                             0.087
300
                            0.280
400
          -6.3
                            0.226
                                             0.026
           -6.2
500
                            0.191
                                             0.026
600
           -6.3
                            0.166
                                             0.016
```

700	-6.1	0.149	0.026	
800	-6.2	0.134	0.016	
900	-6.1	0.123	0.017	
1000	-6.3	0.025	0.017	
1100	-6.2	0.019	0.017	
1200	-6.3	0.020	0.020	
1300	-6.1	0.021	0.020	
1400	-6.2	0.021	0.020	
1500	-6.3	0.021	0.020	
1600	-6.2	0.020	0.020	
1700	-6.3	0.019	0.019	
1800	-6.2	0.019	0.019	
1900	-6.2	0.018	0.019	
2000	-6.2	0.015	0.018	
2100	-6.3	0.013	0.014	
2200	-6.2	0.012	0.014	
2300	-6.3	0.011	0.012	
2400	-6.4	0.011	0.012	
2500	-6.2	0.012	0.014	
2600	-6.2	0.011	0.014	
2700	-6.3	0.010	0.012	MEAN ELBO CONVERGED

The first part of the output reports on the configuration used. Here it indicates the default mean-field setting of the variational inference algorithm. It also indicates the default parameter sizes and tolerances for monitoring the algorithm's convergence. The second part of the output describes the progression of the algorithm. The evidence lower bound (ELBO) is the variational objective function and is evaluated based on a Monte Carlo estimate. The variational inference algorithm in Stan is stochastic, which makes it challenging to assess convergence. That is, while the algorithm appears to have converged in  $\sim 100$  iterations, the algorithm runs for another few thousand iterations until the mean change in ELBO drops below the default tolerance of 1e-2.

# Variational Inference Output

The output from variational is written into the file output.csv by default. The output follows the same pattern as the output for sampling, first dumping the entire set of parameters used.

```
# stan_version_major = 2
# stan_version_minor = 7
# stan_version_patch = 0
# model = bernoulli_model
# method = variational
# variational
# algorithm = meanfield (Default)
```

```
#
        meanfield
#
      iter = 10000 (Default)
      grad samples = 1 (Default)
#
      elbo_samples = 100 (Default)
      eta_adagrad = 0.100000000000001 (Default)
      tol_rel_obj = 0.01 (Default)
      eval_elbo = 100 (Default)
      output_samples = 1000 (Default)
# id = 0 (Default)
# data
    file = bernoulli.data.R
# init = 2 (Default)
# random
    seed = 3959575278
# output
   file = output.csv (Default)
    diagnostic_file = (Default)
   refresh = 100 (Default)
lp__,theta
-6.77173,0.223462
-6.74816,0.252055
-6.92751,0.180521
-7.25717,0.13987
-6.91783,0.182266
```

Note that everything is a comment other than a line for the header, and a line for the values. The header indicates the unnormalized log probability with <code>lp\_\_</code> based on the variational parameters. The ELBO is not stored unless a diagnostic option is given. See Section 8.3 for more details. The first line is special: it is the mean of the variational approximation. The rest of the output contains <code>output\_samples</code> number of samples drawn from the variational approximation.

### **Configuring Command-Line Options**

The command-line options for running a model are detailed in Chapter 8. They can also be printed on the command line using Linux or Mac OS with

```
> ./bernoulli help-all
```

and on Windows with

> bernoulli.exe help-all

# Part II CmdStan Tools

# 3. Overview

CmdStan is the command-line interface for Stan. The next two chapters describes tools that are built as part of CmdStan installation: stanc and stansummary. The process of building a CmdStan executable from a Stan program is as follows:

- 1. A Stan program is written to file with a .stan extension.
- 2. stanc is used to translate the Stan program into a C++ file. This C++ file is not a full program that can be compiled to executable directly, but a translation from the Stan language into a C++ concept. Each interface will generate identical C++ for the same Stan program.
- 3. A CmdStan exectuable is generated from the CmdStan source and the generated C++. Each Stan program will have its own CmdStan executable. The options to the CmdStan executable are described in Chapter 8.

# 3.1. Building the CmdStan Tools

The easy way to build CmdStan is through the use of make. From a command line window, type:

- > cd <cmdstan-home>
- > make build

This will build both stanc and stansummary. If your computer has multiple cores and sufficient ram, the build process can be parallelized by providing the -j option. For example, to build on 2 cores, type:

- > cd <cmdstan-home>
- > make -j2 build

*Warning:* The make program may take 10+ minutes and consume 2+ GB of memory to build CmdStan.

# 4. **stanc**: Translating Stan to C++

# 4.1. Building the stanc Compiler

Before the stanc compiler can be used, it must be built. It can be compiled directly using the makefile as follows. For Mac and Linux:

> make bin/stanc

For Windows:

> make bin/stanc.exe

To change the default compiler or the optimization level, see Appendix B.6.

# 4.2. The stanc Compiler

The stanc compiler converts Stan programs to C++ concepts. The first stage of compilation involves parsing the text of the Stan program. If the parser is successful, the second stage of compilation generates C++ code. If the parser fails, it will provide a diagnostic error message indicating the location in the input where the failure occurred and reason for the failure.

The following example illustrates a fully qualified call to stanc to build the simple Bernoulli model.

For Linux and Mac:

- > cd <cmdstan-home>
- > bin/stanc --name=bernoulli --o=bernoulli.hpp \
   stan/example-models/basic\_estimators/bernoulli.stan

The backslash (\) is a continuation of the same line and can be omitted if the command is on a single line.

For Windows:

- > cd <cmdstan-home>
- > bin\stanc.exe --name=bernoulli --o=bernoulli.hpp ^
   stan/example-models/basic\_estimators/bernoulli.stan

The caret ( $^{\land}$ ) is a continuation on Windows.

This call specifies the name of the model, here bernoulli. This will determine the name of the class implementing the model in the C++ code. Because this name is the name of a C++ class, it must start with an alphabetic character (a-z or A-Z) and

contain only alphanumeric characters (a-z, A-Z, and 0-9) and underscores (\_) and should not conflict with any C++ reserved keyword.

The C++ code implementing the class is written to the file bernoulli.hpp in the current directory. The final argument, bernoulli.stan, is the file from which to read the Stan program.

# 4.3. Command-Line Options for stanc

The model translation program stanc is called as follows.

> stanc [options] model\_file

The argument *model\_file* is a path to a Stan model file ending in suffix .stan. The options are as follows.

#### --help

Displays the manual page for stanc. If this option is selected, nothing else is done.

#### --version

Prints the version of stanc. This is useful for bug reporting and asking for help on the mailing lists.

#### --name=*class\_name*

Specify the name of the class used for the implementation of the Stan model in the generated C++ code.

Default: class\_name = model\_file\_model

# --o=cpp\_file\_name

Specify the name of the file into which the generated C++ is written.

Default: cpp\_file\_name = class\_name.hpp

# 5. print: Output Analysis (deprecated)

print is deprecated, but is still available until CmdStan v3.0. See the next chapter for usage (replace stansummary with print).

# 6. stansummary: Output Analysis

CmdStan is distributed with a posterior analysis utility that is able to read in the output of one or more Markov chains and summarize the posterior fits. This operation mimics the print(fit) command in RStan, which itself was modeled on the print functions from R2WinBUGS and R2jags.

# 6.1. Building the stansummary Command

CmdStan's stansummary command is built along with stanc into the bin directory. It can be compiled directly using the makefile as follows.

- > cd <cmdstan-home>
- > make bin/stansummary

# 6.2. Running the stansummary Command

The stansummary command is executed on one or more output.csv files. These files may be provided as command-line arguments separated by spaces. That means that wildcards may be used, as they will be replaced by space-separated file names by the operating system's command-line interpreter.

Suppose there are three samples files in a directory generated by fitting a negative binomial model to a small data set.

```
> 1s output*.csv
```

output1.csv output2.csv output3.csv

> bin/stansummary output\*.csv

The result of bin/stansummary is displayed in Figure 6.1.<sup>1</sup> The posterior is skewed to the high side, resulting in posterior means ( $\alpha = 17$  and  $\beta = 10$ ) that are a long way away from the posterior medians ( $\alpha = 9.5$  and  $\beta = 6.2$ ); the posterior median is the value listed under 50%, which is the 50th percentile of the posterior values.

For Windows, the forward slash in paths need to be converted to backslashes.

<sup>&</sup>lt;sup>1</sup>RStan's and PyStan's output analysis stansummary may be different than that in the command-line version of Stan.

```
Inference for Stan model: negative_binomial_model
1 chains: each with iter=(1000); warmup=(0); thin=(1); 1000 iterations saved.
```

Warmup took (0.054) seconds, 0.054 seconds total Sampling took (0.059) seconds, 0.059 seconds total

	Mean	MCSE	StdDev	5%	50%	95%	N_Eff	N_Eff/s	R_hat
1p	-14	6.2e-02	1.0e+00	-16	-14	-13	283	4773	1.00
accept_stat	0.88	5.6e-03	1.8e-01	0.51	0.95	1.0	1000	16881	1.00
stepsize	0.30	1.3e-15	8.9e-16	0.30	0.30	0.30	0.50	8.5	1.00
treedepth	1.4	2.6e-02	8.0e-01	0.00	1.0	2.0	946	15978	1.00
n_divergent	1.4	0.0e+00	0.0e+00	0.00	0.0	0.0	1000	16949	1.00
alpha	17	1.8e+00	2.5e+01	1.9	9.5	50	181	3054	1.00
beta	10	1.1e+00	1.4e+01	1.2	6.2	31	181	3057	1.00

Samples were drawn using hmc with nuts. For each parameter, N\_Eff is a crude measure of effective sample size, and R\_hat is the potential scale reduction factor on split chains (at convergence, R\_hat=1).

Figure 6.1: Example output from bin/stansummary. The model parameters are alpha and beta. The values for each quantity are the posterior means, standard deviations, and quantiles, along with Monte-Carlo standard error, effective sample size estimates (per second), and convergence diagnostic statistic. These values are all estimated from samples. In addition to the parameters, the value lp\_ is the total log probability computed by the model (up to an additive constant). The quantity accept\_stat\_ is the NUTS acceptance statistic used by NUTS for slice and Metropolis rejection, stepsize\_ the step size used by NUTS in its Hamiltonian simulation, and treedepth\_ is the depth of tree used by NUTS, which is the log (base 2) of the number of leapfrog steps taken during the Hamiltonian simulation. n\_divergent\_ gives the number of leapfrog iterations with diverging error; because NUTS terminates at the first divergent iteration this should always be either 0 or 1.

#### **Output of stansummary Command**

#### n\_divergent

CmdStan uses a symplectic integrator to approximate the exact solution of the Hamiltonian dynamics and when the step size is too large relative to the curvature of the log posterior this approximation can diverge and threaten the validity of the sampler. n\_divergent counts the number of iterations within a given sample that have diverged and any non-zero value suggests that the samples may be biased in which case the step size needs to be decreased. Note that, because sampling is immediately terminated once a divergence is encountered, n\_divergent should be only 0 or 1.

# 6.3. Command-line Options

In addition to the filenames, stansummary includes three flags to customize the output.

#### help

stansummary usage information No help output by default

### sig\_figs=<int>

Sets the number of significant figures displayed in the output Valid values:  $0 < sig\_figs$  (default = 2)

#### autocorr=<int>

Calculates and then displays the autocorrelation of the specified chain Valid values: Any integer matching a chain index (No autocorrelation output by default)

# Part III CmdStan Executables

# 7. Compiling CmdStan Exectuables

Preparing a Stan program to be run involves two steps,

- 1. translating the Stan program to C++, and
- 2. compiling the resulting C++ to an executable.

This chapter discusses both steps, as well as their encapsulation into a single make target.

# 7.1. Translating and Compiling through make

The simplest way to compile a CmdStan program is through the make build tool, which encapsulates the translation and compilation step into a single command. The commands making up the make target for compiling a model are described in the following sections, and the following chapter describes how to run a compiled model. Before compiling a CmdStan program, change directories to <cmdstan-home>.

#### **Translating and Compiling Test Models**

There are a number of example models distributed with CmdStan which unpack into the path examples. To build the simple example examples/bernoulli/bernoulli.stan, the following call to make suffices. The following call will build an executable form of the Bernoulli estimator. On Windows, replace bernoulli with bernoulli.exe.

> make examples/bernoulli/bernoulli

This will translate the model bernoulli.stan to a C++ file, bernoulli.hpp, and compile a CmdStan program using the generated C++ file, putting the executable in examples/bernoulli/bernoulli(.exe).

# Dependencies in make

When executing a make target, all its dependencies are checked to see if they are up to date, and if they are not, they are rebuilt. If the make target to build the Bernoulli estimator is invoked a second time, it will see that it is up to date, and will not recompile the program.

If the file containing the Stan program is updated, the next call to make will rebuild the CmdStan executable.

#### Getting Help from the makefile

CmdStan's makefile, which contains the top-level instructions to make, provides extensive help in terms of targets and options. Invoke make with the target help:

> make help

#### Options to make

CmdStan allows users to change compilers, library versions for Eigen and Boost, as well as compilation options such as optimization.

For a full list of options, see Appendix B.6

### Clean Targets

A very useful target is clean-all, invoked as

> make clean-all

This removes the CmdStan tools. This step is necessary when changing compilers or other make options.

# 8. Running a CmdStan Program

Once a CmdStan program is compiled, it can be run in many different ways. It can be used to sample or optimize parameters, or to diagnose a model. Before diving into the detailed configurations, the first section provides some simple examples.

# 8.1. Getting Started by Example

Once a CmdStan program has been converted to a C++ program for that model (see Chapter 4) and the resulting C++ program compiled to a platform-specific executable (see Chapter 7), the model is ready to be run.

All of the CmdStan functionality is highly configurable from the command line; the options are defined later in this chapter. Each command option also has defaults, which are used in this section.

#### Sampling

Suppose the executable is in file my\_model and the data is in file my\_data, both in the current working directory. To generate samples from a data set using the default settings, use one of the following, depending on platform.

Mac OS and Linux

> ./my\_model sample data file=my\_data

#### Windows

> my\_model sample data file=my\_data

On both platforms, this command reads the data from file my\_data, runs warmup tuning for 1000 iterations (the values of which are discarded), and then runs the fully-adaptive NUTS sampler for 1000 iterations, writing the parameter (and other) values to the file samples.csv in the current working directory. When no random number seed is specified, a seed is generated from the system time.

# Sampling in Parallel

The previous example executes one chain, which can be repeated to generate multiple chains. However, users may want to execute chains in parallel on a multicore machine.

#### Mac OS and Linux

To sample four chains using a Bash shell on Mac OS or Linux, execute<sup>1</sup>

```
> for i in {1..4}
do
    ./my_model sample random seed=12345 \
    id=$i data file=my_data \
    output file=samples$i.csv &
done
```

The ampersand (&) at the end of the nested command pushes each process into the background, so that the loop can continue without waiting for the current chain to finish. The id value makes sure that a non-overlapping set of random numbers are used for each chain. Also note that the output file is explicitly specified, with the variable \$i being used to ensure the output file name for each chain is unique.

The terminal standard output will be interleaved for all chains running concurrently. To suppress all terminal output, direct the standard output to the "null" device. This is achieved by postfixing > /dev/null to a command, which in the above case, means changing the second-to-last line to

```
output file=samples$i.csv > /dev/null &
```

#### **Windows**

On Windows, the following is functionally equivalent to the Bash snippet above

```
> for /l %x in (1, 1, 4) do start /b model sample ^\
random seed=12345 id=%x data file=my_data ^\
output file=samples%x.csv
```

The caret (^) indicates a line continuation in DOS.

### Combining Parallel Chains

CmdStan has commands to analyze the output of multiple chains, each stored in their own file; see Chapter 6. RStan also has commands to read in multiple CSV files produced by CmdStan's command-line sampler.

To compute posterior quantities, it is sometimes easier to have the chains merged into a single CSV file. If the grep and sed programs are installed, then the following will combine the four comma-separated values files into a single comma-separated values file. The command is the same on Windows, Mac OS, and Linux.

<sup>&</sup>lt;sup>1</sup>Complicated multiline commands such as this one are prime candidates for putting into a script file.

```
> grep lp__ samples1.csv > combined.csv
> sed '/^[#1]/d' samples*.csv >> combined.csv
```

#### Scripting and Batching

The previous examples show how to sample in parallel from the command line. Operations like these can also be scripted, using shell scripts (.sh) on Mac OS and Linux and DOS batch (.bat) files on Windows. A sequence of several such commands can be executed from a single script file. Such scripts might contain stanc commands (see Chapter 4) and stansummary commands (see Chapter 6) can be executed from a single script file. At some point, it is worthwhile to move to something with stronger dependency control such as makefiles.

#### Optimization

CmdStan can find the posterior mode (assuming there is one). If the posterior is not convex, there is no guarantee Stan will be able to find the global mode as opposed to a local optimum of log probability.

For optimization, the mode is calculated without the Jacobian adjustment for constrained variables, which shifts the mode due to the change of variables. Thus modes correspond to modes of the model as written.

#### Windows

> my\_model optimize data file=my\_data

#### Mac OS and Linux

> ./my\_model optimize data file=my\_data

#### Variational Inference

CmdStan can fit a variational approximation to the posterior. The approximation is a Gaussian in the unconstrained variable space. Stan implements two variational algorithms. The algorithm=meanfield option uses a fully factorized Gaussian for the approximation. The algorithm=fullrank option uses a Gaussian with a full-rank covariance matrix for the approximation.

#### Mac OS and Linux

#### Windows

# 8.2. Diagnostics

CmdStan has a basic diagnostic feature that will calculate gradients of the initial state and compare them with those calculated with finite differences. If there are discrepancies, there is a problem with the model or initial states (or a bug in Stan). To run on the different platforms, use one of the following.

#### Mac OS and Linux

> ./my\_model diagnose data file=my\_data

#### Windows

> my\_model diagnose data file=my\_data

# 8.3. Command-Line Options

CmdStan executables are highly configurable, allowing the user to specify and customize not only the calculation method but also the data, output, initialization, and random number generation. The arguments are defined hierarchically so that, for example, optimization settings are not necessary when sampling.

The atomic elements of the hierarchy (i.e., those without corresponding values) are *categorical arguments* (sometimes called "flags") which define self-contained categories of arguments.

CmdStan's commands have more hierarchical structure than is typical of command line executables, which usually have at most two subgroups of commands. Arguments grouped within a category are not ordered with respect to each other. The only ordering is that the global options come before the method argument and

subcommand-specific options after the method argument. For example, the following four commands all define the same configuration:<sup>2</sup>

The categorical arguments output and random can be in any order provided that the subarguments follow their respective parent, here diagnostic\_file and file following output and seed coming after random. These four configurations exhaust all valid combinations.

Categorical arguments may appear is isolation, for example when introducing sample or random, or they may appear as the values for other arguments, such as hmc which not only introduces a category of HMC related arguments but also defines the value of the argument algorithm. A visual diagram of the available categorical arguments is shown in Figure 8.1, with the mutual exclusivity of these arguments as values shown in Figure 8.2. Specifying conflicting arguments causes the execution to immediately terminate.

Note that any valid argument configuration must either specify a method or a help request.

#### Method

All commands other than help must include at least one method, specified explicitly as method=method\_name or implicitly with only method\_name. Currently CmdStan supports the following methods:

<sup>&</sup>lt;sup>2</sup> The backslash (\) is used at the end of a line in a command to indicate that it continues on the next line. The indentation to indicate the structure of the command is for pedagogical purposes only; the same result would be obtained writing each command on one line with single spaces separating the elements.

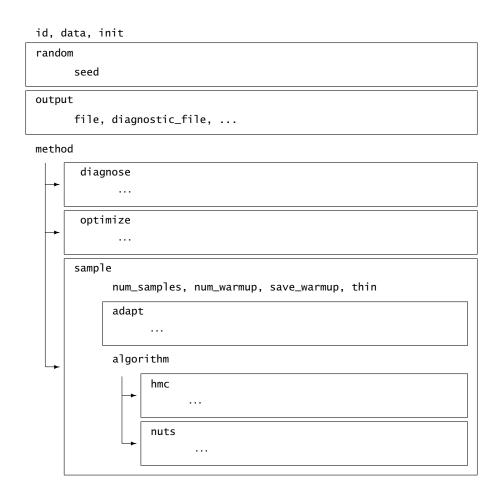


Figure 8.1: In the hierarchical argument structure, certain arguments, such as random and output, introduce new categories of arguments. Categorical arguments may also appear as values of other arguments, such as diagnose, optimize, and sample, which define the mutually exclusive values for the argument method.

### id, data, init

```
random
      seed
output
      file, diagnostic_file, ...
method
      sample
            num_samples, num_warmup, save_warmup, thin
            adapt
            algorithm
                  hmc
```

Figure 8.2: A valid argument configuration defines only one mutually exclusive argument. If conflicting arguments are specified, for example method=optimize method=sample, then execution immediately terminates with a warning message.

Method	Description
sample	sample using MCMC
optimize	find posterior mode using optimization
variational	fit variational approximation (experimental)
diagnose	diagnose models

All remaining configurations are optional, with default values provided for all arguments not explicitly specified.

### Help

Informative output can be retrieved either globally, by requesting help at the toplevel, or locally, by requesting help deeper into the hierarchy. Note that after any help has been displayed the execution immediately terminates, even if a method has been specified.

### Top-Level Help

If help is specified as the only argument then a usage message is displayed. Similarly, specifying help\_all by itself displays the entire argument hierarchy.

# Context-Sensitive Help

Specifying help after any argument displays a description and valid options for that argument. For example,

provides the top-level options for the sample method.

Detailed information on the argument, and all arguments deriving from it, can accessed by specifying help-all instead,

# 8.4. Full Argument Hierarchy

Here we present the full argument hierarchy, along with relevant details. Some typical use-case examples are provided in the next section.

### **Typographical Conventions**

The following typographical conventions are obeyed in the hierarchy.

- arg=<*value*-type>
   Arguments with values; displays the value type, legal values, and default value
- arg
   Isolated categorical arguments: displays all valid subarguments
- value
   Values; describes effect of selecting the value
- · avalue

Categorical arguments that appear as values to other arguments; displays all valid subarguments

# **Top-Level Method Argument**

Every command must have exactly one method specified as the very first argument. The value type of list element means that the valid values are enumerated as a list.

```
method=list element>
    Analysis method (Note that method= is optional)
    Valid values: sample, optimize, variational, diagnose
    (Defaults to sample)
```

# **Sampling-Specific Arguments**

The following arguments are specific to sampling. The method argument sample (or method=sample) must come first in order to enable the subsequent arguments. The other arguments are optional and may appear in any order.

# *⊾ sample*

```
Bayesian inference with Markov Chain Monte Carlo
Valid subarguments: num_samples, num_warmup, save_warmup,
thin, adapt, algorithm
```

L L num\_samples=<int>
Number of sampling iterations
Valid values: 0 ≤ num\_samples
(Defaults to 1000)

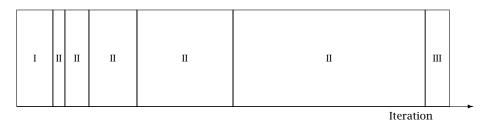


Figure 8.3: Adaptation during warmup occurs in three stages: an initial fast adaptation interval (I), a series of expanding slow adaptation intervals (II), and a final fast adaptation interval (III). For HMC, both the fast and slow intervals are used for adapting the step size, while the slow intervals are used for learning the (co)variance necessitated by the metric. Iteration numbering starts at 1 on the left side of the figure and increases to the right.

- L L num\_warmup=<int>
  Number of warmup iterations
  Valid values: 0 ≤ warmup
  (Defaults to 1000)
- Stream warmup samples to output?
  Valid values: 0, 1
  (Defaults to 0)
- Period between saved samples
  Valid values: 0 < thin
  (Defaults to 1)

# Sampling Adaptation-Specific Parameters

When adaptation is engaged the warmup period is split into three stages (Figure 8.3), with two *fast* intervals surrounding a series of growing *slow* intervals. Here fast and slow refer to parameters that adapt using local and global information, respectively; the Hamiltonian Monte Carlo samplers, for example, define the step size as a fast parameter and the (co)variance as a slow parameter. The size of the the initial and final fast intervals and the initial size of the slow interval are all customizable, although user-specified values may be modified slightly in order to ensure alignment with the warmup period.

The motivation behind this partitioning of the warmup period is to allow for more robust adaptation. In the initial fast interval the chain is allowed to converge to-

wards the typical set,<sup>3</sup> with only parameters that can learn from local information adapted. After this initial stage parameters that require global information, for example (co)variances, are estimated in a series of expanding, memoryless windows; often fast parameters will be adapted here as well. Lastly the fast parameters are allowed to adapt to the final update of the slow parameters.

Currently all Stan sampling algorithms utilize dual averaging to optimize the step size (this optimization during adaptation of the sampler should not be confused with running Stan's optimization method). This optimization procedure is extremely flexible and for completeness we have exposed each option, using the notation of (Hoffman and Gelman, 2011, 2014). In practice the efficacy of the optimization is sensitive to the value of these parameters, and we do not recommend changing the defaults without experience with the dual averaging algorithm. For more information, see the discussion of dual averaging in (Hoffman and Gelman, 2011, 2014).

Variances or covariances are estimated using Welford accumulators to avoid a loss of precision over many floating point operations.

The following subarguments are introduced by the categorical argument adapt. Each subargument must contiguously follow adapt, though they may appear in any order.

### **↓ ↓ adapt**

Warmup Adaptation

Valid subarguments: engaged, gamma, delta, kappa, t0

 ↓
 ↓
 engaged=<boolean>

Adaptation engaged?

Valid values: 0, 1

(Defaults to 1)

# ↓ ↓ gamma=<double>

Adaptation regularization scale

Valid values: 0 < gamma

(Defaults to 0.05)

#### ↓ ↓ ↓ delta=<double>

Adaptation target acceptance statistic

Valid values: 0 < delta < 1

(Defaults to 0.8)

### ↓ ↓ kappa=<doub1e>

Adaptation relaxation exponent

<sup>&</sup>lt;sup>3</sup>The typical set is a concept borrowed from information theory and refers to the neighborhood (or neighborhoods in multimodal models) of significant posterior probability mass through which the Markov chain will travel in equilibrium.

Valid values: 0 < kappa (Defaults to 0.75)

L L t0=<double>
Adaptation iteration offset
Valid values: 0 < t0

(Defaults to 10)

- Width of initial fast adaptation interval Valid values: All (Defaults to 75)
- Use the second of the second of the Width of final fast adaptation interval Valid values: All (Defaults to 50)
- L L window=<unsigned int>
  Initial width of slow adaptation interval
  Valid values: All
  (Defaults to 25)

By setting the acceptance statistic delta to a value closer to 1 (its value must be strictly less than 1 and its default value is 0.8), adaptation will be forced to use smaller step sizes. This can improve sampling efficiency (effective samples per iteration) at the cost of increased iteration times. Raising the value of delta will also allow some models that would otherwise get stuck overcome their blockages; see also the stepsize\_jitter argument.

# Sampling Algorithm- and Engine-Specific Arguments

The following batch of arguments are used to control the sampler used for sampling. The top-level specification is for engine, the only valid value of which is hmc (this will change in the future as we add new samplers).

L L algorithm=<*list element>*Sampling algorithm
Valid values: hmc, fixed\_param
(Defaults to hmc)

Hamiltonian Monte Carlo is a very general approach to sampling that utilizes techniques of differential geometry and mathematical physics to generate efficient MCMC transitions. This generality manifests in a wealth of implementation choices.

#### L L L hmc

Hamiltonian Monte Carlo Valid subarguments: engine, metric, stepsize, stepsize\_jitter

All HMC implementations require at least two parameters: an integration step size and a total integration time. We refer to different specifications of the latter as *engines*.

In the static\_hmc implementation the total integration time must be specified by the user, where as the nuts implementation uses the No-U-Turn Sampler to determine an optimal integration time dynamically.

L L L engine=element>
Engine for Hamiltonian Monte Carlo
Valid values: static, nuts
(Defaults to nuts)

The following options are activated for static HMC.

#### LLLL static

Static integration time Valid subarguments: int\_time

L. L. L. int\_time=<double>Total integration time for Hamiltonian evolution Valid values:  $0 < int_time$ (Defaults to  $2\pi$ )

These options are for NUTS, an adaptive version of HMC.

#### ե ե ե ե **nuts**

The No-U-Turn Sampler Valid subarguments: max\_depth

# Tree Depth

NUTS generates a proposal by evolving the initial system both forwards and backwards in time to form a balanced binary tree. At each iteration of the NUTS algorithm the tree depth is increased by one, doubling the number of leapfrog steps and effectively doubles the computation time. The algorithm terminates in one of two ways: either the NUTS criterion is satisfied for a new subtree or the completed tree, or the depth of the completed tree hits max\_depth.

Both the tree depth and the actual number of leapfrog steps computed are reported along with the parameters in the output as treedepth\_ and n\_leapfrog\_\_,

respectively. Because the final subtree may only be partially constructed, these two will always satisfy

$$2^{\text{treedepth}-1} - 1 < N_{\text{leapfrog}} \le 2^{\text{treedepth}} - 1$$
.

treedepth\_\_ is an important diagnostic tool for NUTS. For example, treedepth\_\_ = 0 occurs when the first leapfrog step is immediately rejected and the initial state returned, indicating extreme curvature and poorly-chosen step size (at least relative to the current position). On the other hand, if treedepth\_\_ = max\_depth then NUTS is taking many leapfrog steps and being terminated prematurely to avoid excessively long execution time. For the most efficient sampling max\_depth should be increased to ensure that the NUTS tree can grow as large as necessary.

For more information on the NUTS algorithm see (Hoffman and Gelman, 2011, 2014).

```
LLLLM max_depth=<int>
Maximum tree depth
Valid values: 0 < max_depth
(Defaults to 10)
```

#### Euclidean Metric

All HMC implementations in Stan utilize quadratic kinetic energy functions which are specified up to the choice of a symmetric, positive-definite matrix known as a *mass matrix* or, more formally, a *metric* (Betancourt and Stein, 2011).

If the metric is constant then the resulting implementation is known as *Euclidean* HMC. Stan allows for three Euclidean HMC implementations: a unit metric, a diagonal metric, and a dense metric. These can be specified with the values unit\_e, diag\_e, and dense\_e, respectively.

Future versions of Stan will also include dynamic metrics associated with *Riemannian* HMC (Girolami and Calderhead, 2011; Betancourt, 2012).

```
Geometry of base manifold
Valid values: unit_e, diag_e, dense_e
(Defaults to diag_e)

L L L L unit_e
Euclidean manifold with unit metric

L L L L diag_e
Euclidean manifold with diag metric
```

L L L dense\_e

Euclidean manifold with dense metric

### Step Size and Jitter

All implementations of HMC also use numerical integrators requiring a step size. We also allow that step size to be "jittered" randomly during sampling to avoid any poor interactions with a fixed step size and regions of high curvature. The maximum amount of jitter is 1, which will cause step sizes to be selected in the range of 0 to twice the adapted step size. Low step sizes can get HMC samplers unstuck that would otherwise get stuck with higher step sizes. The downside is that jittering below the adapted value will increase the number of leapfrog steps required and thus slow down iterations, whereas jittering above the adapted value can cause premature rejection due to simulation error in the Hamiltonian dynamics calculation. See (Neal, 2011) for further discussion of step-size jittering.

```
L L L stepsize=<double>
Step size for discrete evolution
Valid values: 0 < stepsize
(Defaults to 1)

L L L stepsize_jitter=<double>
Uniformly random jitter of the stepsize, in percent
Valid values: 0 ≤ stepsize_jitter ≤ 1
(Defaults to 0)
```

# Fixed Parameter Sampler

The fixed parameter sampler generates a new sample without changing the current state of the Markov chain; only generated quantities may change. This can be useful when, for example, trying to generate pseudo-data using the generated quantities block. If the parameters block is empty (no parameters) then using algorithm=fixed\_param is mandatory.

# L L **fixed\_param**Fixed Parameter Sampler

# **Optimization-Specific Commands**

The following arguments are for the top-level method optimize. They allow control of the optimization algorithm, and some of its configuration. The other arguments may appear in any order.

### **↓** optimize

Point estimation
Valid subarguments: algorithm, iter, save\_iterations

The following options are for the (L-)BFGS optimizer. L-BFGS is the default optimizer and also much faster than the other optimizers.

Convergence monitoring in (L-)BFGS is controlled by a number of tolerance values, any one of which being satisified causes the algorithm to terminate with a solution.

· The log probability is considered to have converged if

$$\left|\log p(\theta_i|y) - \log p(\theta_{i-1}|y)\right| <$$
tol\_obj

or

$$\frac{\left|\log p(\theta_i|y) - \log p(\theta_{i-1}|y)\right|}{\max\left(\left|\log p(\theta_i|y)\right|, \left|\log p(\theta_{i-1}|y)\right|, 1.0\right)} < \texttt{tol\_rel\_obj} * \epsilon.$$

· The parameters are considered to have converged if

$$|| heta_i - heta_{i-1}|| < tollowbreak tollowbreak$$

· The gradient is considered to have converged to 0 if

$$||g_i|| < tol_grad$$

or

$$\frac{g_{i}^{T}\hat{H}_{i}^{-1}g_{i}}{\max\left(\left|\log p(\theta_{i}|y)\right|,1.0\right)} < \texttt{tol\_rel\_grad} * \epsilon.$$

Here, i is the current iteration,  $\theta_i$  is the value of the parameters at iteration i, y is the data,  $p(\theta_i|y)$  is the posterior probability of  $\theta_i$  up to a proportion,  $\nabla_{\theta}$  is the gradient operator with respect to  $\theta$ ,  $g_i = \nabla_{\theta} \log p(\theta_i|y)$  is the gradient at iteration i,  $\hat{H}_i$  is the estimate of the Hessian at iteration i, |u| is absolute value (L1 norm) of u, ||u|| is vector length (L2 norm) of u, and  $\epsilon \approx 2e - 16$  is machine precision. Any of the convergence tests can be disabled by setting its corresponding tolerance parameter to zero.

The other command-line argument for (L-)BFGS is init\_alpha, which is the first step size to try on the initial iteration. If the first iteration takes a long time (and requires a lot of function evaluations), set init\_alpha to be the roughly equal to

the alpha used in that first iteration. init\_alpha has a tiny default value, which is reasonable for many problems but might be too large or too small depending on the objective function and initialization. Being too big or too small just means that the first iteration will take longer (i.e., require more gradient evaluations) before the line search finds a good step length. It's not a critical parameter, but for optimizing the same model multiple times (as you tweak things or with different data) being able to change it can save some real time.

Finally, L-BFGS has a additional command-line argument, history\_size, which controls how much memory is used maintaining the approximation of the Hessian. This should be less than the dimensionality of the parameter space and, in general, relatively small values (5 - 10) are sufficient. If L-BFGS performs badly but BFGS is performing well, then consider increasing this. Note that increasing this will increase the memory usage, although this is unlikely to be an issue for typical Stan models.

# ե ե **(1)bfgs**

(L-)BFGS with linesearch

Valid subarguments: init\_alpha, tol\_obj, tol\_rel\_obj, tol\_grad, tol\_rel\_grad, tol\_param, history\_size (lbfgs only)

 L
 L
 L
 init\_alpha=<double>

Line search step size for first iteration

Valid values:  $0 \le init_alpha$ 

(Defaults to 0.001)

#### ↓ ↓ ↓ tol\_obj=<double>

Convergence tolerance on changes in objective function value

Valid values:  $0 \le tol_obj$ 

(Defaults to 1e-12)

### L L L tol\_rel\_obj=<double>

Convergence tolerance on relative changes in objective function value

Valid values:  $0 \le tol_rel_obj$ 

(Defaults to 1e+4)

# L L L tol\_grad=<double>

Convergence tolerance on the norm of the gradient

Valid values:  $0 \le tol\_grad$ 

(Defaults to 1e-8)

### L L L tol\_rel\_grad=<double>

Convergence tolerance on the relative norm of the gradient

Valid values:  $0 \le tol_rel_grad$ 

(Defaults to 1e+7)

# ↓ ↓ ↓ tol\_param=<double>

Convergence tolerance on changes in parameter value

Valid values:  $0 \le tol_param$ 

(Defaults to 1e-8)

# ↓ ↓ ↓ history\_size=<int>

Number of update vectors to use in Hessian approximations (lbfgs only)

Valid values:  $0 < history_size$ 

(Defaults to 5)

The following argument is for Newton's optimization method; there are currently no configuration parameters for Newton's method, and it is not recommended because of the slow Hessian calculation involving finite differences.

#### L L newton

Newton's method

The remaining arguments apply to all optimizers.

#### ↓ ↓ iter=<int>

Total number of iterations Valid values: 0 < iter

(Defaults to 2000)

### ↳ ↳ save\_iterations=<boolean>

Stream optimization progress to output?

Valid values: 0, 1

(Defaults to 0)

# **Variational Inference-Specific Commands**

The following arguments are for the top-level method variational. They allow control of the variational inference algorithm, and some of its configuration.

#### variationa variationa

Variational inference

Valid subarguments: algorithm, grad\_samples, elbo\_samples, eta\_adagrad, iter, tol\_rel\_obj, eval\_elbo, output\_samples

Variational inference algorithm

Valid values: meanfield, fullrank

(Defaults to meanfield)

### 

Number of samples for Monte Carlo estimate of gradients

Valid values: 0 < grad\_samples

(Defaults to 1)

### ↓ ↓ elbo\_samples=<int>

Number of samples for Monte Carlo estimate of ELBO (objective function)

Valid values:  $0 < elbo_samples$ 

(Defaults to 100)

### 

Stepsize weighting parameter for adaptive stepsize sequence

Valid values:  $0 < eta\_adagrad \le 1.0$ 

(Defaults to 0.1)

#### ե ⊨ iter=<*int*>

Maximum number of iterations

Valid values: 0 < iter (Defaults to 10000)

### ↓ tol\_rel\_obj=<double>

Convergence tolerance on the relative norm of the objective

Valid values: 0 < tol\_rel\_obj

(Defaults to 0.01)

### 

Evaluate ELBO every Nth iteration

Valid values: 0 < eval elbo

(Defaults to 100)

# 

Number of posterior samples to draw and save

Valid values: 0 < output\_samples

(Defaults to 1000)

# Diagnostic-Specific Arguments

The following arguments are specific to diagnostics. As of now, the only diagnostic is gradients of the log probability function.

# **↓ diagnose**

Model diagnostics

Valid subarguments: test

### ↓ \ test=test=test=tement>

Diagnostic test

Valid values: gradient (Defaults to gradient)

### 

Check model gradient against finite differences Valid subarguments: epsilon, error

### L L L epsilon=<real>

Finite difference step size

Valid values: 0 < epsilon

(Defaults to 1e-6)

### ↓ ↓ ↓ error=<rea1>

Error threshold

Valid values: 0 < error

(Defaults to 1e-6)

### **General-Purpose Arguments**

The following arguments may be used with any of the previous configurations. They may come either before or after the other subarguments of the top-level method.

# Process Identifier Argument

# id=<int>

Unique process identifier, used to advance random number generator so that random numbers do not overlap across chains

Valid values: 0 < id

(Defaults to 0)

# Input Data Arguments

#### data

Input data options

Valid subarguments: file

# ↓ file=<string>

Input data file

Valid values: Path to existing file

(Defaults to empty path)

### Initialization Arguments

Initialization is only applied to parameters defined in the parameters block. Any initial values supplied for transformed parameters or generated quantities are ignored.

# init=<string>

Initialization method:

- real number x > 0 initializes randomly bewteen [-x, x];
- 0 initializes to 0;
- non-number interpreted as a data file

Valid values: All (Defaults to 2)

# Random Number Generator Arguments

#### random

Random number configuration Valid subarguments: seed

### seed=<unsigned int>

Random number generator seed

Valid values:

- seed  $\geq 0$  generates seed;
- $\operatorname{\mathsf{seed}} < 0$  uses seed generated from time

(Defaults to -1)

# Output Arguments

# output

File output options

Valid subarguments: file, diagnostic\_file, refresh

# ↓ file=<*string*>

Output file

Valid values: Valid path (Defaults to output.csv)

# ↓ diagnostic\_file=<string>

Auxiliary output file for diagnostic information

Valid values: Valid path (Defaults to empty path)

↓ refresh=<int>

Number of interations between screen updates

Valid values: 0 < refresh

(Defaults to 100)

# 8.5. Command-Line Option Examples

The hierarchical structure of the command-line options can be intimidating, and here we provide an example workflow to help ease the introduction to new users, especially those used to Stan 1.3 or earlier releases. The examples in this section are for Mac OS and Linux; on Windows, just remove the ./ before the executable and change the line-continuation character from Unix's  $\setminus$  to DOS's  $\wedge$ . As in previous sections, the indentation on continued lines is for pedagogical purposes only and does not convey any content to the executable.

Let's say that we've just built our model, model, and are ready to run. We begin by specifying data and init files,

> ./model data file=model.data.R init=model.init.R

but our model doesn't run. Instead, the above command prints

A method must be specified! Failed to parse arguments, terminating Stan

The problem is that we forgot to specify a method.

All Stan arguments have default values, except for the method. This is the only argument that must be specified by the user and a model will not run without it (not to say that the model will run without error, for example a model that requires data will eventually fail unless an input file is specified with file under data). Assuming that we want to draw MCMC samples from our model, we can either specify a method implicitly,

- > ./model sample data file=model.data.R init=model.init.R
  or explicitly,
  - > ./model method=sample data file=model.data.R \
     init=model.init.R

In either case our model now executes without any problem.

Now let's say that we want to customize our execution. In particular we want to set the seed for the random number generator, but we forgot the specific argument syntax. Information for each argument can displayed by calling help,

### > ./model random help

#### which returns

```
random
Random number configuration
Valid subarguments: seed
...
```

before printing usage information. For information on the seed argument we just call help one level deeper,

> ./model random seed help

#### which returns

```
seed=<unsigned int>
  Random number generator seed
  Valid values: seed > 0, if negative seed is generated from time
  Defaults to -1
```

Fully informed, we can now run with a given seed,

```
> ./model method=sample data fle=model.data.R \
    init=model.init.R \
    random seed=5
```

The arguments method, data, init, and random are all top-level arguments. To really see the power of a hierarchical argument structure let's try to drill down and specify the metric we use for HMC: instead of the default diagonal Euclidean metric, we want to use a dense Euclidean metric. Attempting to specify the metric we try

only to have the execution fail with the message

```
metric=unit_e is either mistyped or misplaced.
Perhaps you meant one of the following valid configurations?
  method=sample algorithm=hmc metric=<list_element>
Failed to parse arguments, terminating Stan
```

The argument metric does exist, but not at the top-level. In order to specify it we have to drill down into sample by first specifying the sampling algorithm, as noted in the suggestion,

Unfortunately we still messed up,

```
unit is not a valid value for "metric"

Valid values: unit_e, diag_e, dense_e

Failed to parse arguments, terminating Stan
```

Tweaking the metric name we make one last attempt,

which successfully runs.

Finally, let's consider the circumstance where our model runs fine but the NUTS iterations keep saturating the default tree depth limit of 10. We need to change the limit, but how do we specify NUTS let alone the maximum tree depth? To see how let's take advantage of the help-all option which prints all arguments that derive from the given argument. We know that NUTS is somehow related to sampling, so we try

> ./model method=sample help-all

which returns the verbose output,

```
num warmup=<int>
  Number of warmup iterations
  Valid values: 0 <= warmup
  Defaults to 1000
save_warmup=<boolean>
  Stream warmup samples to output?
  Valid values: [0, 1]
  Defaults to 0
thin=<int>
  Period between saved samples
  Valid values: 0 < thin
  Defaults to 1
adapt
 Warmup Adaptation
  Valid subarguments: engaged, gamma, delta, kappa, t0
  engaged=<boolean>
    Adaptation engaged?
    Valid values: [0, 1]
    Defaults to 1
  gamma=<double>
    Adaptation regularization scale
    Valid values: 0 < gamma
    Defaults to 0.05
  delta=<double>
    Adaptation target acceptance statistic
    Valid values: 0 < delta < 1
    Defaults to 0.65
  kappa=<double>
    Adaptation relaxation exponent
    Valid values: 0 < kappa
    Defaults to 0.75
  t0=<double>
    Adaptation iteration offset
    Valid values: 0 < t0
    Defaults to 10
```

algorithm=<list element> Sampling algorithm Valid values: hmc Defaults to hmc hmc Hamiltonian Monte Carlo Valid subarguments: engine, metric, stepsize, stepsize\_jitter engine=<list element> Engine for Hamiltonian Monte Carlo Valid values: static, nuts Defaults to nuts static Static integration time Valid subarguments: int\_time int\_time=<double> Total integration time for Hamiltonian evolution Valid values: 0 < int\_time Defaults to 2 \* pi nuts The No-U-Turn Sampler Valid subarguments: max\_depth max\_depth=<int> Maximum tree depth Valid values: 0 < max\_depth Defaults to 10 metric=<list element> Geometry of base manifold Valid values: unit\_e, diag\_e, dense\_e Defaults to diag\_e unit e Euclidean manifold with unit metric diag\_e Euclidean manifold with diag metric

```
dense_e
    Euclidean manifold with dense metric

stepsize=<double>
    Step size for discrete evolution
    Valid values: 0 < stepsize
    Defaults to 1

stepsize_jitter=<double>
    Uniformly random jitter of the stepsize, in percent
    Valid values: 0 <= stepsize_jitter <= 1
    Defaults to 0</pre>
```

Following the hierarchy, the maximum tree depth derives from nuts, which itself is a value for the argument engine which derives from hmc. Adding this to our previous call we attempt

which yields

```
-1 is not a valid value for "max_depth"
  Valid values: 0 < max_depth
Failed to parse arguments, terminating Stan</pre>
```

Where did that negative sign come from? Clumsy fingers are nothing to be embarrassed about, especially with such complex argument configurations. Removing the guilty character, we try

which finally runs without issue.

```
> ./my_model sample
                 algorithm=hmc
                     engine=nuts
                       [max_depth=<int>]
                      [metric={unit_e,diag_e,dense_e}]
                     [stepsize=<double>]
                     [stepsize_jitter=<double>]
                 「num_samples=<int>]
                 [num_warmup=<int>]
                 [save_warmup=<boolean>]
                 [thin=<int>]
                 Γadapt
                      [engaged=<boolean>]
                      Γaamma=<doub1e>1
                      [delta=<double>]
                      [kappa=<double>]
                      「t0=<doub1e>1 1
             [data file=<string>]
             [init=<string>]
             [random seed=<int>]
             [output
                  [file=<string>]
                  [diagnostic_file=<string>]
                  [refresh=<int>] ]
```

Figure 8.4: Command skeleton for invoking the no-U-turn sampler (NUTS). This is the same skeleton as that for basic HMC in Figure 8.5. Elements in braces are optional. All arguments and their default values are described in detail in Section 8.4.

# 8.6. Command Templates

This section provides templates for all of the arguments deriving from each of the possible methods: sample, optimize, variational and diagnose. Arguments in square brackets are optional, those not in square brackets are required for the template.

# **Sampling Templates**

The No-U-Turn sampler (NUTS) is the default (and recommended) sampler for Stan. The full set of configuration options is in Figure 8.4.

A standard Hamiltonian Monte Carlo (HMC) sampler with user-specified integration time may also be used. Its set of configuration options are shown in Figure 8.5.

```
> ./mv model sample
                 algorithm=hmc
                     engine=static
                       [int_time=<double>]
                     [metric={unit_e,diag_e,dense_e}]
                     [stepsize=<double>]
                     [stepsize_jitter=<double>]
                 [num_samples=<int>]
                 [num_warmup=<int>]
                 [save_warmup=<boolean>]
                 [thin=<int>]
                 [adapt
                      [engaged=<boolean>]
                      Γaamma=<doub1e>1
                      [delta=<double>]
                      [kappa=<double>]
                      「t0=<doub1e>1 1
             [data file=<string>]
             [init=<string>]
             [random seed=<int>]
             [output
                  [file=<string>]
                  [diagnostic_file=<string>]
                  [refresh=<int>] ]
```

Figure 8.5: Command skeleton for invoking the basic Hamiltonian Monte Carlo sampler (HMC). This is the same as the NUTS command skeleton shown in Figure 8.4 other than for the engine. Elements in braces are optional. All arguments and their default values are described in detail in Section 8.4.

Both NUTS and HMC may be configured with either a unit, diagonal or dense Euclidean metric, with a diagonal metric the default.<sup>4</sup> A unit metric provides no parameter-by-parameter scaling, a diagonal metric scales each parameter independently, and a dense metric also rotates the parameters so that correlated parameters may move together. Although dense metrics offer the hope of superior simulation performance, they require more computation per iteration. Specifically for m samples of a model with n parameters, the dense metric requires  $\mathcal{O}(n^3 \log(m) + n^2 m)$  operations, whereas diagonal metrics require only  $\mathcal{O}(n m)$ . Furthermore, dense metrics are difficult to estimate, given the  $\mathcal{O}(n^2)$  components with complex interdependence.

<sup>&</sup>lt;sup>4</sup>In Euclidean HMC, a diagonal metric emulates different step sizes for each parameter. Explicitly varying step sizes were used in Stan 1.3 and before; Neal (2011) discusses the equivalence.

```
> ./my_model optimize
                 algorithm=1bfgs
                                                          ١
                     [init_alpha=<double>]
                     [tol_obj=<double>]
                     [tol_rel_obj=<double>]
                     [tol_grad=<double>]
                     [tol_rel_grad=<double>]
                     [tol_param=<double>]
                     [history_size=<int>]
                 [iter=<int>]
                 [save_iterations=<boolean>]
             [data file=<string>]
             [init=<string>]
             Frandom seed=<int>1
             [output
                  [file=<string>]
                  [diagnostic_file=<string>]
                  [refresh=<int>] ]
```

Figure 8.6: Command skeleton for invoking the L-BFGS optimizer. All arguments and their default values are described in detail in Section 8.4.

# **Optimization Templates**

CmdStan supports several optimizers. These share many of their configuration options with the samplers. The default optimizer is the the limited memory Broyden-Fletcher-Goldfarb-Shanno (L-BFGS) method; Nocedal and Wright (2006) contains an excellent overview of both the BFGS and L-BFGS algorithms. The command skeleton for L-BFGS is in Figure 8.6 and the one for BFGS is in Figure 8.7. Stan also supports Newton's method; see (Nocedal and Wright, 2006) for more information. This method is the least efficient of the three, but has the advantage of setting its own step size. Other than not having a stepsize argument, the skeleton for Newton's method shown in Figure 8.8 is identical to that for BFGS.

# Variational Inference Templates

CmdStan implements Automatic Differentiation Variational Inference (Kucukelbir et al., 2015). The command skeleton for the meanfield algorithm is in Figure 8.9. The command skeleton for the fullrank algorithm is in Figure 8.10.

```
> ./my_model optimize
                 algorithm=bfgs
                     [init_alpha=<double>]
                     [tol_obj=<double>]
                     [tol_rel_obj=<double>]
                     [tol_grad=<double>]
                     [tol_rel_grad=<double>]
                     [tol_param=<double>]
                 [iter=<int>]
                 [save_iterations=<boolean>]
             [data file=<string>]
             [init=<string>]
             [random seed=<int>]
             [output
                  [file=<string>]
                  [diagnostic_file=<string>]
                  [refresh=<int>] ]
```

Figure 8.7: Command skeleton for invoking the BFGS optimizer. All arguments and their default values are described in detail in Section 8.4.

Figure 8.8: Command skeleton for invoking the Newton optimizer. All arguments and their default values are described in detail in Section 8.4.

```
> ./my_model variational
                 algorithm=meanfield
                     [grad_samples=<int>]
                     [elbo_samples=<int>]
                     [eta_adagrad=<double>]
                     [iter=<int>]
                     [tol_rel_obj=<double>]
                     [eval_elbo=<int>]
                     [output_samples=<int>]
             [data file=<string>]
             [init=<string>]
             [random seed=<int>]
             Foutput
                  [file=<string>]
                  [diagnostic_file=<string>]
                  [refresh=<int>] ]
```

Figure 8.9: Command skeleton for invoking the meanfield variational inference algorithm. All arguments and their default values are described in detail in Section 8.4.

```
> ./my_model variational
                 algorithm=fullrank
                     [grad_samples=<int>]
                     [elbo_samples=<int>]
                     [eta_adagrad=<double>]
                     [iter=<int>]
                     [tol_rel_obi=<double>]
                     「eval_elbo=<int>l
                     Foutput_samples=<int>]
             [data file=<string>]
             [init=<string>]
             [random seed=<int>]
             [output
                  [file=<string>]
                  [diagnostic_file=<string>]
                  [refresh=<int>] ]
```

Figure 8.10: Command skeleton for invoking the fullrank variational inference algorithm. All arguments and their default values are described in detail in Section 8.4.

Figure 8.11: Command skeleton for invoking model diagnostics. All arguments and their default values are described in detail in Section 8.4.

# **Diagnostic Command Skeleton**

Stan reports on gradients for the model at a specified or randomly generated initial value. The command-skeleton in this case is very simple, and shown in Figure 8.11.

# **Appendices**

# A. Licensing

CmdStan, Stan, and Stan's three dependent libraries, Stan Math Library, Boost, and Eigen, are distributed under liberal freedom-respecting<sup>1</sup> licenses approved by the Open Source Initiative.<sup>2</sup>

In particular, the licenses for CmdStan and its dependent libraries have no "copyleft" provisions requiring applications of CmdStan to be open source if they are redistributed.

This chapter describes the licenses for the tools that are distributed with Cmd-Stan. The next chapter explains some of the build tools that are not distributed with CmdStan, but are required to build and run Stan models.

### A.1. CmdStan's License

CmdStan is distributed under the BSD 3-clause license (BSD New).

http://www.opensource.org/licenses/BSD-3-Clause

### A.2. Stan's License

Stan is distributed under the BSD 3-clause license (BSD New).

http://www.opensource.org/licenses/BSD-3-Clause

# A.3. Stan Math Library's License

The Stan Math Library is distributed under the BSD 3-clause license (BSD New).

http://www.opensource.org/licenses/BSD-3-Clause

# A.4. Boost License

Boost is distributed under the Boost Software License version 1.0.

http://www.opensource.org/licenses/BSL-1.0

<sup>&</sup>lt;sup>1</sup>The link http://www.gnu.org/philosophy/open-source-misses-the-point.html leads to a discussion about terms "open source" and "freedom respecting."

<sup>&</sup>lt;sup>2</sup>See http://opensource.org.

# A.5. Eigen License

Eigen is distributed under the Mozilla Public License, version 2.

http://opensource.org/licenses/mpl-2.0

# A.6. Google Test License

CmdStan uses Google Test for unit testing; it is not required to compile or execute models. Google Test is distributed under the BSD 2-clause license.

http://www.opensource.org/licenses/BSD-License

# B. Installation and Compatibility

This appendix describes the hardware and software required to run CmdStan. The software required includes CmdStan and its libraries, as well as a contemporary C++ compiler. CmdStan requires hardware powerful enough to build and execute the models. Ideally, that will be a 64-bit computer with at least 4GB of memory and multiple processor cores.

# **B.1.** Operating System

CmdStan is written in portable C++ without C++11 features, as are the libraries on which it depends. Therefore, CmdStan should run on any machine for which a suitable C++ compiler is available. In practice, CmdStan, like the Boost and Eigen libraries on which it depends, is very hard on the compiler and linker.

CmdStan has been tested on the following operating systems.

- · Linux (Debian, Ubuntu, Red Hat)
- · Mac OS X (10.10 "Yosemite," 10.9 "Mavericks," 10.8 "Mountain Lion," 10.7 "Lion," and 10.6 "Snow Leopard")
- · Windows (XP, 7, 8).

CmdStan should work on other versions of these operating systems if compatible C++ compilers can be found. The plan is to keep up with new versions of these operating systems and gradually phase out testing on older versions.

# **B.2.** Required Software and Tools

The only two absolute requirements for running CmdStan are the CmdStan source code (and dependent libraries) and a C++ compiler.

#### **CmdStan Source**

In order to compile Stan program, the CmdStan source code is required. The CmdStan source code distribution includes CmdStan's source code, Stan's source code, documentation, build scripts, unit tests, documentation and source for the required libraries, Stan Math Library, Boost, and Eigen, and the source for an optional testing library, Google Test.

CmdStan: Stable Releases

The latest release version of CmdStan can be downloaded from the CmdStan home page:

```
http://mc-stan.org/cmdstan.html
```

CmdStan: Development Source Control

The source code repository is hosted by GitHub, and contains the latest versions of CmdStan (and Stan) underdevelopment. See:

```
http://mc-stan.org/source-repos.html
```

Stan Library Source

The source code for Stan's parse and implementation of inference algorithms are in the Stan library.

· Home: http://mc-stan.org/stan.html

· License: BSD

· Tested Version: 2.8.0

The Stan source code is distributed with CmdStan.

Stan Math Library Source

Stan's mathematical functions and inference algorithm rely on the reverse-mode automatic differentiation implemented in the Stan Math Library.

Home: http://mc-stan.org

· License: BSD

· Tested Version: 2.8.0

The Stan Math Library source code is distributed with CmdStan

### Boost C++ Library Source

Stan's parser and some of its mathematical functions and template metaprogramming facilities are implemented with the Boost C++ Library.

Home: http://www.boost.org/users/license.html

· License: Boost Software License

· Tested Version: 1.58.0

The Boost source code is distributed with CmdStan.

Eigen Matrix and Linear Algebra Library Source

Stan's matrix algebra depends on the Eigen C++ matrix and linear algebra library.

Home: http://eigen.tuxfamily.org

· License: Mozilla Public License, version 2.0

· Tested Version: 3.2.4

The Eigen source code is distributed with CmdStan.

# C++ Compiler

Compiling CmdStan programs requires a C++ compiler. CmdStan has been primarily developed with clang++ and g++ and no promises are made for other compilers. The full set of compilers for which CmdStan has been tested is

· q++

Tested Versions: Mac 4.2.1, 4.6, Linux 4.4-4.7 (plus trunk 4.8, 4.9), Windows

4.6.3

Home: http://gcc.gnu.org/

License: GPL3+

· clang++, Mac 2.9-3.1, 6.1, Linux 2.9-3.1

Home: http://clang.llvm.org/

License: BSD

· mingw-64, version 2.0 (Windows 7, cross-compiled from Debian Linux)

# **B.3.** Step-by-Step Windows Install Instructions

CmdStan has been tested on Windows XP, Windows 7, and Windows 8.

CmdStan also runs under Cygwin, which provides a unix-like shell on top of Windows. Instructions for Cygwin installation are provided below in their own subsection.

### **Windows Tips**

Opening a Command Shell

To open a Windows command shell, first open the Start Menu (usually in the lower left of the screen), select option All Programs, then option Accessories, then program Command Prompt.

Alternatively, enter [Windows+r] (both keys together on the keyboard), and enter cmd into the text field that pops up in the Run window, then press [Return] on the keyboard to run.

#### 32-bit Builds

CmdStan defaults to a 64-bit build. On a 32-bit operating system, include BIT=32 in make/local. See Appendix B.6 for more details.

# **Rtools C++ Development Environment**

The simplest way to install a full C++ build environment that will work for CmdStan is to use the Rtools package designed for R developers on Windows (even if you don't plan to use R).

First, download the latest *frozen* (i.e., stable) version of Rtools from the Rtools home page, using

```
http://cran.r-project.org/bin/windows/Rtools/
```

Next, double click on the downloaded file to open the Rtools install wizard, then proceed through its options.

- · Language: select language, click Next,
- · Welcome: click Next,
- · Information: click Next,
- Setup Location: accept default (c:\Rtools), click Next,

- · Select Components: select default, Package Authoring, click Next,
- Select Additional Tasks: check Edit Path and Save Version in Registry, click Next,
- System Path Report: ensure that the paths to c:\Rtools\bin and c:\Rtools\gcc-4.6.3\bin are listed at the beginning of the path and click Next,
- · Ready to Install: click Next, wait for the install to complete, then
- · Finish: click Finish.
- Confirm Path: After the install has completed, open a command prompt and type PATH to ensure that the new path is activated and the Rtools folders are in the system path.

### Checking the Path

Make sure that c:\Rtools\bin has been added to your PATH environment variable, and then open another command window. You should be able to follow the last step, *Comfirm Path*, above.

# **Verifying Tools**

To verify that g++ is installed, use the following command.

This should report version information for g++. Next, verify that make is installed with the following command.

This should print version information for make.

# Downloading and Unpacking CmdStan

The CmdStan source code distribution is named cmdstan-2.8.0.tar.gz; the versions here are major version 2, minor version 7, and patch level 0. Download the latest CmdStan source tarball from the CmdStan downloads page,

https://github.com/stan-dev/cmdstan/releases

to any non-temporary folder. (If in doubt, select My Documents on Windows XP or Documents on Windows 7.)

Change to the download directory (aka folder) using one of the following commands, replacing <username> with a Windows user name.

• *Windows XP*: From the default starting directory, use the following commands (quotes and all):

```
> cd "My Documents"
```

The full path (including quotes) will work from anywhere,

```
> cd "c:\Documents and Settings\<username>\My Documents"
```

· Windows 7: From the default starting directory, use

```
> cd Documents
```

or use the full path, including quotes, from anywhere,

```
> cd "c:\Users\<username>\Documents"
```

To verify that the downloaded CmdStan.tar.gz file is there, list the directory contents using:

```
> dir
```

Finally, unpack the distribution using the tar command (which is installed as part of Rtools).

```
> tar --no-same-owner -xzf cmdstan-2.8.0.tar.gz
```

The -no-same-owner flag is not strictly necessary, but it removes a bunch of irrelevant warnings.

# 64-bit Cygwin Install Instructions

CmdStan can be run under Cygwin, the Unix look-and-feel environment for Windows. Cygwin must have recent versions of make and g++ (part of gcc) installed. Within a Cygwin shell, CmdStan will behave as under other Unixes. Follow the directions in Appendix B.5.

# **B.4.** Step-by-Step Mac Install Instructions

This section provides step-by-step install instructions for the Mac. CmdStan has been tested on Mac OS X versions Mavericks, Snow Leopard, Lion, and Mountain Lion.

#### Install Xcode C++ Development Environment

The easiest (but not the only) way to install a C++ development environment on a Mac is to use Apple's Xcode development environment.

From the Xcode home page,

```
https://developer.apple.com/xcode/
```

click View in Mac App Store.

From the App Store, click Install, enter an Apple ID, and wait for Xcode to finish installing.

Open the Xcode application, click top-level menu Preferences, click top-row button Downloads, click button for Components, click on the Install button to the right of the Command Line Tools entry, then wait for it to finish installing.

Click the top-level menu item Xcode, then click item Quit Xcode to quit.

To test, open the Terminal application and enter

```
> make --version
> g++ --version
```

Verify that make is at version 3.81 or later and g++ is at 4.2.1 or later.

## **Download and Unpack CmdStan Source**

Download the most recent version of cmdstan-2.8.0.tar.gz from the CmdStan downloads list,

```
https://github.com/stan-dev/cmdstan/releases
```

Open the folder containing the download in the Finder (typically, the user's toplevel Downloads folder).

If the Mac OS has not automatically unpacked the .tar.gz file into file cmdstan-2.8.0.tar, double-click the .tar.gz file to unpack.

Double click on the .tar file to unarchive directory cmdstan-2.8.0.

Move the resulting directory to a location where it will not be deleted, henceforth called <cmdstan-home>.

# **B.5.** Step-by-Step Linux Install Instructions

CmdStan has been tested on various Linux installations, including Ubuntu, Debian, and Red Hat.

### **Installing C++ Development Tools**

On Linux, C++ compilers and make are often installed by default.

To see if the g++ compiler and make build system are already installed, use the commands

```
> g++ --version
> make --version
```

If these are at least at g++ version 4.2.1 or later and make version 3.81 or later, no additional installations are necessary. It may still be desirable to update the C++ compiler g++, because later versions are faster.

To install the latest version of these tools (or upgrade an older version), use the commands

```
> sudo apt-get install g++
> sudo apt-get install make
```

A password will likely be required by the superuser command sudo.

### **Downloading and Unpacking CmdStan Source**

Download the most recent stable version of CmdStan, cmdstan-2.8.0.tar.gz, from the CmdStan downloads page,

```
https://github.com/stan-dev/cmdstan/releases
```

to the directory where Stan will reside.

In a command shell, change directories to where the tarball was downloaded, say <download-dir>, with

```
> cd <download-dir>
```

where <download-dir> is replaced with the actual path to the directory.

Then, unpack the distribution into the subdirectory

```
<download-dir>/cmdstan-2.8.0
```

with

```
> tar -xzf cmdstan-2.8.0.tar.gz
```

# **B.6.** make Options

#### Setting a Variable

CmdStan uses make for building the CmdStan tools and compiling Stan programs as executables. Users can customize how the tools are built by editing make/local inside the <cmdstan-home> directory.

Customization is done by setting variables or appending to them. For each variable that needs to be set, write <variable>=<value> on its own line inside make/local. To append to existing variables, write <variable>+=<value>.

#### **Customizing make Options**

#### Compiler Settings

The most common options to set are the compiler options:

- CC. The compiler used to build CmdStan and the Stan executables. The default is CC=g++.
- 0. The optimization level for the compiler. The default is 0=3. Both g++ and clang++ recognize 0,1,2,3, and s.

## Changing Library Locations

The next set of variables allow for easy replacement of dependent libraries. CmdStan depends on Stan, Boost, and Eigen. Although CmdStan is bundled with a particular verison of Stan, other versions can be used. The same is true with Boost and Eigen.

The Stan developers typically replace the tagged version of Stan in <mdstan-home>/stan with an updated version and do not set these variables. That said, it is easy to point CmdStan to other versions of these libraries in other directories.

- STAN. The location of the Stan source directory. The default is STAN=stan/. Note: the trailing forward slash is necessary.
- MATH. The location of the Stan Math Library. The default is MATH=stan/lib/stan\_math\_2.8.0.
- BOOST. The location of the Boost library. If BOOST is not explicitly set, this will default to the lib/boost\_1.58.0 folder within MATH.
- EIGEN. The location of the Eigen library. If EIGEN is not explicitly set, this will default to the lib/eigen\_3.4.0 folder within MATH.

#### Additional Options

These are the additional options and typically do not need to be set.

- · O\_STANC. The optimization level for building stanc. The default is O\_STANC=0.
- CFLAGS. The compiler flags for building CmdStan programs. The default value is dependent on the operating system.
- LDLIBS. The libraries used for linking CmdStan programs. The default value is dependent on the operating system.
- LDLIBS\_STANC. The libraries used for linking the Stan compiler, stanc. The
  default is the value of LDLIBS.
- BIT. The bitness of the executable. This variable only applies to Windows. The
  default is BIT=64. If building on a 32-bit Windows machine, BIT=32 must be
  set.
- TEMPLATE\_DEPTH. The maximum instantiation depth for template classes which is passed to g++ or clang++. This variable only applies to Mac OS X due to the current compilers distributed through Xcode not specifying enough depth by default.
- EXE. The file extension for executables. The default is operating system dependent. On Windows, the default is EXE=.exe. On other operating systems, the default is empty, EXE=.

## **Rebuilding CmdStan**

When compiler flags are changed, CmdStan needs to be rebuilt in order for the changes to take place. The easiest way to do this is to type:

> make clean-all

Then rebuild CmdStan and the Stan programs of interest.

# **B.7.** MKL Compiler Instructions

# Getting the MKL

To purchase a license, see

http://software.intel.com/en-us/intel-mkl

For non-commercial development, see

http://software.intel.com/non-commercial-software-development

### Compiling with MKL

In order to use Intel's math kernel library (MKL) for C++, a few lines need to be added to make/local.

- · CC = icc. Set the compiler to icc.
- MKLROOT = /apps/intel/2013/mkl. Create a new variable with the location of the MKL path.
- CFLAGS += -I \$(MKLROOT)/include -DEIGEN\_USE\_MKL\_ALL. This tells Eigen to use MKL and where to find the necessary header files.
- LDLIBS += -L\$(MKLR00T)/lib/intel64 -lmkl\_intel\_lp64 and LDLIBS += -lmkl\_core -lmkl\_sequential -lpthread -lm. This links in the MKL library for the CmdStan executables. The exact implementation will depend on the particular system. Use the MKL link line advisor for help.

*Note:* Make sure to do the above changes before compiling for the first time - otherwise Stan will be compiled with g++ and you won't see any performance gains.

# **B.8.** Optional Components for Developers

CmdStan is developed using the following set of tools. The various command examples in this manual have assumed they can be found on the command path. The makefile allows precise locations to be plugged in.

#### **GNU Make Build Tool**

CmdStan automates the build, test, documentation, and deployment tasks using scripts in the form of makefiles to run with GNU Make.

Home: http://www.gnu.org/software/make

· License: GPLv3+

• Tested Versions: 3.81 (Mac OS X), 3.79 (Windows 7)

#### **Doxygen Documentation Generator**

CmdStan's API documentation is generated using the Doxygen Tool.

Home: http://www.stack.nl/~dimitri/doxygen/index.html

· License: GPL2

· Tested Version(s): Mac OS X 1.8.2, Windows 1.8.2

#### Git Version Control System

CmdStan uses the Git version control system for its software, libraries, and documentations. Git is required to interact with the most recent versions of code in the version control repository.

Home: http://git-scm.com/

· License: GPL2

• Tested Version(s): Mac versions 1.8.2.3 and 1.7.8.4; Windows version 1.7.9

Google Test C++ Testing Framework

CmdStan's unit testing is based on the Google's googletest C++ testing framework.

Home: http://code.google.com/p/googletest/

· License: BSD

· Tested Version(s): 1.7.0

The Google Test framework is distributed with Stan.

# **B.9.** Tips for Mac OS X

Finding and Opening Mac Applications and Files

To open an application, use [Command-Space] (press both keys at once on the keyboard) to open Spotlight, enter the application's name in the text field, then click on the application in the pop-up menu or [Return] if the right file or application is highlighted.

Spotlight can be used in the same way to find files or folders, such as the default Downloads folder for web downloads.

Open a Terminal for Shell Commands

To run shell commands, open the built-in Terminal application (see the previous subsection for details on how to find and open applications).

#### Install Xcode

Apple's Xcode contains both the clang++ and g++ compilers and make, all of the tools needed to work with CmdStan as a user. The version of Xcode to install depends on the version of Mac OS X.

Alternative, GCC-Only Installer

A stripped down installer for just the GCC package, including the C++ compilers g++ and clang++, available for Mac OS X 10.6 ("Snow Leopard") or later,

```
https://github.com/kennethreitz/osx-gcc-installer/
```

The fill list of tools in this distribution is available at:

```
http://www.opensource.apple.com/release/
developer-tools-41/
```

### **More Recent Compilers**

Alternative compilers to those distributed by Apple as part of Xcode are available at the following locations.

Homebrew

One way to get pre-built binaries for Mac OS X is to use Homebrew, which is available from the following link.

```
http://mxcl.github.com/homebrew/
```

**MacPorts** 

MacPorts hosts recent versions of compilers for the Macintosh.

```
https://distfiles.macports.org/MacPorts/
```

After finding the appropriate .dmg file, clicking on it, then double clicking on the resulting .pkg file, and clicking through some more menus, the following will need to be entered from a terminal window to install it.

```
> sudo port install gccVersion
```

In this command, gccVersion is the name of a compiler version, such as g++-mp-4.6, for version 4.6. Errors may arise during the install such as the following.

```
Error: Target org.macports.activate returned: Image error: /opt/local/include/gmp.h already exists and does not belong to a registered port. Unable to activate port gmp. Use 'port -f activate gmp' to force the activation.
```

This issue can be resolved by running the following command.

```
> sudo port -f activate gmp
```

### **LATEX Typesetting Package**

CmdStan uses the LaTeX typesetting package for generating manuals, talks, and other materials (Doxygen is used for API documentation; see below). The first step is to download the MacTeX .mpkg file from the following URL [warning: the download is approximately 2GB and the installation approximately 3.5GB].

```
http://www.tug.org/mactex/2011/
```

Once it is downloaded, just click on the .mpkg file and then follow the installer instructions. The installer will add the command to the PATH environment variable so that the pdflatex used by Stan is available from the command line.

#### Lucida Console Font

A free TrueType version of Lucida Console for the Mac is available at the following URL.

```
http://www.fontpalace.com/font-details/Lucida+Console/
```

Download the .ttf file, then click on it to install. It will then be available as a preference in the Mac terminal application.

# C. Dump Data Format

For representing structured data in files, CmdStan uses the dump format introduced in S and used in R and JAGS (and in BUGS, but with a different ordering). A dump file is structured as a sequence of variable definitions. Each variable is defined in terms of its dimensionality and its values. There are three kinds of variable declarations, one for scalars, one for sequences, and one for general arrays.

# C.1. Creating Dump Files

Dump files can be created from R using RStan. The function is stan\_rdump in package rstan.

Using R's native dump() function can produce dump files which Stan cannot read in. The underlying cause is that R gets creative in the format it uses for output, only being constrained to something that can be executed in R. So it will write the array containing the values 1, 2, 3, 4 as 1:4 rather than as c(1,2,3,4).

#### C.2. Scalar Variables

A simple scalar value can be thought of as having an empty list of dimensions. Its declaration in the dump format follows the S assignment syntax. For example, the following would constitute a valid dump file defining a single scalar variable y with value 17.2.

A scalar value is just a zero-dimensional array value.

# C.3. Sequence Variables

One-dimensional arrays may be specified directly using the S sequence notation. The following example defines an integer-value and a real-valued sequence.

Arrays are provided without a declaration of dimensionality because the reader just counts the number of entries to determine the size of the array.

Sequence variables may alternatively be represented with R's colon-based notation. For instance, the first example above could equivalently be written as

The sequence denoted by 1:3 is of length 3, running from 1 to 3 inclusive. The colon notation allows sequences going from high to low, as in the first of the following examples, which is equivalent to the second.

# C.4. Array Variables

For more than one dimension, the dump format uses a dimensionality specification. For example,

$$y \leftarrow structure(c(1,2,3,4,5,6), .Dim = c(2,3))$$

This defines a  $2 \times 3$  array. Data is stored in column-major order, meaning the values for y will be as follows.

$$y[1,1] = 1$$
  $y[1,2] = 3$   $y[1,3] = 5$   
 $y[2,1] = 2$   $y[2,2] = 4$   $y[2,3] = 6$ 

The structure keyword just wraps a sequence of values and a dimensionality declaration, which is itself just a sequence of non-negative integer values. The product of the dimensions must equal the length of the array.

If the values happen to form a contiguous sequence of integers, they may be written with colon notation. Thus the example above is equivalent to the following.

$$y \leftarrow structure(1:6, .Dim = c(2,3))$$

The same applies to the specification of dimensions, though it is perhaps less likely to be used. In the above example, c(2,3) could be written as 2:3.

Arrays of more than two dimensions are written in a last-index major form. For example,

$$z \leftarrow structure(1:24, .Dim = c(2,3,4))$$

produces a three-dimensional int (assignable to real) array z with values

$$z[1,1,1] = 1$$
  $z[1,2,1] = 3$   $z[1,3,1] = 5$   
 $z[2,1,1] = 2$   $z[2,2,1] = 4$   $z[2,3,1] = 6$   
 $z[1,1,2] = 7$   $z[1,2,2] = 9$   $z[1,3,2] = 11$   
 $z[2,1,2] = 8$   $z[2,2,2] = 10$   $z[2,3,2] = 12$ 

```
z[1,1,3] = 13 z[1,2,3] = 15 z[1,3,3] = 17

z[2,1,3] = 14 z[2,2,3] = 16 z[2,3,3] = 18

z[1,1,4] = 19 z[1,2,4] = 21 z[1,3,4] = 23

z[2,1,4] = 20 z[2,2,4] = 22 z[2,3,4] = 24
```

#### C.5. Matrix- and Vector-Valued Variables

The dump format for matrices and vectors, including arrays of matrices and vectors, is the same as that for arrays of the same shape.

#### **Vector Dump Format**

The following three declarations have the same dump format for their data.

```
real a[K];
vector[K] b;
row_vector[K] c;
```

#### **Matrix Dump Format**

The following declarations have the same dump format.

```
real a[M,N];
matrix[M,N] b;
```

## Arrays of Vectors and Matrices

The key to undertanding arrays is that the array indexing comes before any of the container indexing. That is, an array of vectors is just that — provide an index and get a vector. See the chapter on array and matrix types in the user's guide section of the languag emanual for more information.

For the dump data format, the following declarations have the same arrangement.

```
real a[M,N];
matrix[M,N] b;
vector[N] c[M];
row_vector[N] d[M];
```

Similarly, the following also have the same dump format.

```
real a[P,M,N];
matrix[M,N] b[P];
vector[N] c[P,M];
row_vector[N] d[P,M];
```

# C.6. Integer- and Real-Valued Variables

There is no declaration in a dump file that distinguishes integer versus continuous values. If a value in a dump file's definition of a variable contains a decimal point (e.g., 132.3) or uses scientific notation (e.g., 1.323e2), Stan assumes that the values are real.

For a single value, if there is no decimal point, it may be assigned to an int or real variable in Stan. An array value may only be assigned to an int array if there is no decimal point or scientific notation in any of the values. This convention is compatible with the way R writes data.

The following dump file declares an integer value for y.

```
y <-
2
```

This definition can be used for a Stan variable y declared as real or as int. Assigning an integer value to a real variable automatically promotes the integer value to a real value.

Integer values may optionally be followed by L or 1, denoting long integer values. The following example, where the type is explicit, is equivalent to the above.

```
y <-
21
```

The following dump file provides a real value for y.

```
y <-
2.0
```

Even though this is a round value, the occurrence of the decimal point in the value, 2.0, causes Stan to infer that y is real valued. This dump file may only be used for variables y declared as real in Stan.

#### Scientific Notation

Numbers written in scientific notation may only be used for real values in Stan. R will write out the integer one million as 1e+06.

#### Infinite and Not-a-Number Values

Stan's reader supports infinite and not-a-number values for scalar quantities (see the section of the reference manual section of the language manual for more information on Stan's numerical data types). Both infinite and not-a-number values are supported by Stan's dump-format readers.

Value	Preferred Form	Alternative Forms
positive infinity	Inf	Infinity, infinity
negative infinity	-Inf	-Infinity, -infinity
not a number	NaN	

These strings are not case sensitive, so inf may also be used for positive infinity, or NAN for not-a-number.

## C.7. Quoted Variable Names

In order to support JAGS data files, variables may be double quoted. For instance, the following definition is legal in a dump file.

### C.8. Line Breaks

The line breaks in a dump file are required to be consistent with the way R reads in data. Both of the following declarations are legal.

Also following R, breaking before the assignment arrow are not allowed, so the following is invalid.

Lines may also be broken in the middle of sequences declared using the c(...) notation., as well as between the comma following a sequence definition and the dimensionality declaration. For example, the following declaration of a  $2 \times 2 \times 3$  array is valid.

```
y <-
structure(c(1,2,3,
4,5,6,7,8,9,10,11,
12), .Dim = c(2,2,
3))
```

Because there are no decimal points in the values, the resulting dump file may be used for three-dimensional array variables declared as int or real.

# C.9. BNF Grammar for Dump Data

A more precise definition of the dump data format is provided by the following (mildly templated) Backus-Naur form grammar.

The template parameters T will be set to either int or real. Because Stan allows promotion of integer values to real values, an integer sequence specification in the dump data format may be assigned to either an integer- or real-based variable in Stan.

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