University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Adam Starak
Student no. 361021

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Supervisor:
dr Michał Pilipczuk
Instytut Informatyki

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Supervisor's statement

Hereby I confirm that the presented thesis was prepared under my supervision and that it fulfils the requirements for the degree of Master of Computer Science.

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Hereby I declare that the presented thesis was prepared by me and none of its contents was obtained by means that are against the law.

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Abstract

W pracy przedstawiono prototypową implementację blabalizatora różnicowego bazującą na teorii fetorów σ-ρ profesora Fifaka. Wykorzystanie teorii Fifaka daje wreszcie możliwość efektywnego wykonania blabalizy numerycznej. Fakt ten stanowi przełom technologiczny, którego konsekwencje trudno z góry przewidzieć.

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Introduction

Blabalizator różnicowy jest podstawowym narzędziem blabalii fetorycznej. Dlatego naukowcy z całego świata prześcigają się w próbach efektywnej implementacji. Opracowana przez prof. Fifaka teoria fetorów σ-ρ otwiera w tej dziedzinie nowe możliwości. Wykorzystujemy je w niniejszej pracy.

Chapter 1

Basic definitions

1.1. Structures

A simple graph G is a pair (V, E) where V denotes a set of vertices and E denotes a set of undirected edges. Let $deg_G(v)$ denote a degree of vertex v in graph G. Let $G \setminus \{v\}$ be the abbreviation for $G' = (V(G) \setminus \{v\}, E(G) \setminus \{(u,v) : u \in V(G)\})$.A *tree* T is a graph where two vertices are connected by excatly one path. A *spanning tree* T of a graph G is a graph which includes all of the vertices of G, with minimum possible number of edges. A *star* S is a tree of size at least 2 for which at most 1 vertex has a degree greater than 1. A vertex in a *star* that has the greatest degree is called a *center* while the others are called *rays*.

1.2. Parameterized complexity

Definition 1.1. *Parameterized problem*

Definition 1.2. *FPT algorithm*

Definition 1.3. *Kernel*

Definition 1.4. *Kernelization algorithm*

1.3. Graph decomposition

Definition 1.5. *Path decomposition and pathwidth*

Definition 1.6. *Tree decomposition and treewidth*

Definition 1.7. *Nice tree decomposition*

Chapter 2

Spanning Star Forest Problem

For a given graph G, we say that S is a *Spanning Star Forest* if every connected component C is a star. In the *Spanning Star Forest Problem* given a graph G, the objective is to determine whether there exists a *Spanning Star Forest*.

It turns out that the problem formulated in such a way is relatively simple. Although, various parametrizations described in this paper make it more complex. The following lemma easily clarifies all the concerns about it's hardness.

Lemma 2.1. *A graph G has a* Spanning Star Forest *if and only if it does not contain any isolated vertices.*

Proof. If G has a *Spanning Star Forest* S, then trivially for all $v \in V(G)$ $1 \leq deg_S(v) \leq deg_G(v)$. Thus, none of the vertices is isolated.

For the opposite direction, we prove the lemma by induction on $|V(G)|$. Assume $|V(G)| = 2$. The statement trivially holds because a graph representing an edge is a correct *Spanning Star Forest*. Let $|V(G)| > 2$. Suppose that there does not exist a vertex v such that $G \setminus \{v\}$ has no isolated vertices. Then, it holds that for all $v \in V(G)$ $deg_G(v) = 1$ so G itself is a correct *Spanning Star Forest*. Now, suppose that v is a vertex such that $G \setminus \{v\}$ has no isolated vertices. From the inductive assumption, let S be a *Spanning Star Forest* of a graph $G \setminus \{v\}$, u be a vertex such that $(u, v) \in E(G)$ and w be a vertex such that $w \in N_S(u)$. Consider the 3 following cases:

1. $deg_S(u) > 1$. Then, $S' = \big(V(S) \cup \{v\}, E(S) \cup \{(u, v)\}\big)$ is a correct solution for graph G.

2. $deg_S(u) = deg_S(w) = 1$. Then, $S' = \big(V(S) \cup v, E(S) \cup (u, v)\big)$ is a correct solution for graph G.

3. $deg_S(w) > 1$. Then, $S' = \big(V(S) \cup \{v\}, (E(S) \cup \{(u, v)\}) \setminus \{(u, w)\}\big)$ is a correct solution for graph G.

Observe that in graph G there are no isolated vertices. Thus, one can always extend a solution inductively.

□

Application of Lemma 2.1 yields the following result for *Spanning Star Forest Problem*.

Theorem 2.1. *Decision version of* Spanning Star Forest Problem *can be solved in linear time.*

Proof. Given an input $G = (V, E)$ the answer is YES if for all $v \in V(G)$ $deg_G(v) \neq 0$ and NO otherwise. □

2.1. Obtaining a solution

In this section we focus on obtaining an arbitrary solution for a given instance of *Spanning Star Forest Problem*. Firstly, let us introduce 2 claims in order to normalize the instance and make the algorithm look more clear.

Claim 2.1. *Family of disjoint* Spanning Star Forests *is a* Spanning Star Forest.

Claim 2.2. *G has a* Spanning Star Forest *if and only if it's spanning tree T has.*

The first claim can be trivially proven by the definition of *Spanning Star Forest Problem* while the second one follows directly from Lemma 2.1. Equipped with this information, all that is left to do, is to design an algorithm which solves *Spanning Star Forest Problem* for trees.

Data: Graph G
Result: Spanning Star Forest of T
$T \leftarrow SpanningTree(G)$;
$S \leftarrow \emptyset$;
for *v: postorder(T) and* $v \notin$ *V(S)* **do**
  if *v is not a root* **then**
    S $\leftarrow S \cup \{(u, v)\}$ where $u = parent(v)$
  else
    S $\leftarrow S \cup \{(u, v)\}$ where u is any of the root's children
  end
end
return S

Algorithm 1: Obtaining a Spanning Star Forest from a tree.

Lemma 2.2. *Algorithm 1 is correct.*

Proof. Assume contrary, that the algorithm yields an incorrect solution S. Consider the first case: a path $(u, v), (v, w), (w, z)$ exists in S where u is v's child, v is w's child and w is z's child. But, if u is w's grandchild and $(u, v), (v, w) \in S$, then it means that w is a root. Contradiction because w cannot be z's child. Now, suppose the alternative relationship: u is v's child, v and z are w's children. Provided that vertices were visited in postorder, edge (v, w) should not have been added because v was introduced by u and w was introduced by z. □

Theorem 2.2. *A solution for Spanning Star Forest Problem can be found in linear time.*

Proof. Spanning tree of any graph can be found in linear time. The main loop has n iterations (every vertex is visited once), each of which takes constant time. Thus, the total runtime is linear. □

2.2. Spanning Star Forest parameterized by the number of stars

In *Spanning Star Forest Problem* parameterized by the number of stars, given a graph G and a natural number k, the objective is to determine whether there exists a *Spanning Star Forest* S such that the number of connected components is at most k.

It is natural to ask whether one can find a solution that minimizes the number of connected components. The problem formulated in that way looks slightly different than the previous one. From the other hand, the problem resembles *Dominating Set Problem*, which is defined as follows:

Definition 2.1. Dominating Set Problem*: Given a graph G and a positive integer k find a set D such that $|D| \leq k$ and every vertex from the graph is adjacent to one of the vertices from D.*

It turns out, that the second comparison is true and *Spanning Star Forest Problem* parameterized by the number of stars is NP-Complete. But, before we begin, let us introduce one more definition and a lemma that supports a reduction.

Definition 2.2. *Dominating mapping: Given an instance (G, k) of Dominating Set Problem and a solution D, a dominating mapping is a function $m : V(G) \setminus D \to D$ such that satisfies $(x, m(x)) \in E(G)$ for all $x \in Dom(m)$.*

Lemma 2.3. *Let (G, k) be an instance of* Dominating Set Problem *without isolated vertices and let D be a solution of minimal size. Then, there exists a dominating mapping m such that m is surjective.*

Proof. Suppose contrary that such a mapping does not exist i.e. for every mapping m there exists a vertex $v \in D$ such that $v \notin im(m)$. Let us break the proof into 4 cases:

1. Suppose $N_G(v) = \emptyset$. Contradiction, G has no isolated vertices.
2. Suppose $u \in N_G(v) \cap D$. Contradiction, D was said to be a solution of minimal size whereas $D \setminus \{u\}$ is a valid, smaller solution.
3. Suppose $u \in N_G(v) \setminus D$ and $w \in N_G(u) \cap im(m)$. If $|m^{-1}(w)| = 1$, then $((D \setminus \{v, w\}) \cup u)$ is a valid, smaller solution for a graph G. Contradiction.
4. Suppose $u \in N_G(v) \setminus D$ and $w \in N_G(u) \cap im(m)$. If $|m^{-1}(w)| > 1$ then a mapping:

$$m'(x) = \begin{cases} v, & \text{if x = u} \\ m(x), & \text{otherwise} \end{cases}$$

   is a valid mapping that satisfies $im(m) \subset im(m')$. Thus, one can create a new mapping m'' inductively such that m'' is surjective. Contradiction, we assumed that no such mapping exists.

Since all the possible cases led to a contradiction, we may claim that there exists a dominating mapping f such that f is surjective. □

Armed with the lemma, we are ready to prove the main theorem of the chapter.

Theorem 2.3. Spanning Star Forest Problem *parameterized by the number of stars is NP-Complete.*

Proof. Membership in NP: given an oracle (O, k), we check whether the number of components in O is less than k and whether every connected component forms a star. The task can be easily done in polynomial time.

We show hardness by a reduction from *Dominating Set Problem* that completes the proof. Let (G,k) be an instance of it. We create a graph G' as follows: for every isolated vertex $v \in V(G)$ introduce a vertex v' and an edge (v,v'). Now, we claim that (G,k) is a YES-instance for *Dominating Set Problem* if and only if $(G'.k)$ is a YES-instance for *Spanning Star Forest Problem* parameterized by the number of stars.

The backward implication is simple. Suppose S is a solution for (G',k). We claim that a set D representing centers of stars is a correct *Dominating Set*. Obviously $|D| \leq k$ because there are at most k connected components. Every vertex from G' is adjacent to one of the centers. If there exists a vertex $v' \in D$ such that $v' \notin V(G)$ we transform the solution as follows: $D := (D \setminus \{v'\}) \cup \{v\}$.

To prove the forward implication, let D be a solution of minimal size for (G,k). Obviously, D is also a minimal solution for a graph G'. Thus, by lemma 2.3. there exists a mapping m that is surjective. Now, we claim that a graph $S = (V(G'), \{(x, m(x)) : x \in Dom(m)\})$ is a correct solution for *Spanning Star Forest Problem*. Trivially, there are no isolated vertices in S. Moreover, there is no path of length 4 because S consists of edges (v,u) such that $v \in D$, $u \notin D$ and for all $u \in V(S) \setminus D$ $deg_S(u) = 1$.

□

The theorem implies that *Spanning Star Forest Problem* parameterized by the number of stars is as hard as *Dominating Set Problem*. Thus, we can immediately obtain the following corollary.

Corollary 2.1. Spanning Star Forest Problem *parameterized by the number of stars is* W[2]-complete.

The problems look so similar that one could ask whether the reverse reduction is true. Indeed, with a small twist to the previous idea one can prove the reverse reduction instantly.

Theorem 2.4. *There exists a reduction from* Spanning Star Forest Problem *parameterized by the number of stars to* Dominating Set Problem.

Proof. Let (G,k) be an instance of *Spanning Star Forest Problem*. We create an instance (G',k') for *Dominating Set* as follows: let $G' = G$ and if G contains an isolated vertex, then $k' = 0$. Otherwise, the value remains the same. Now, we claim that (G,k) is a YES-instance for *Spanning Star Forest Problem* if and only if (G',k') is a YES-instance for *Dominating Set.*

To prove the following reduction one can use the method which was described in Theorem 2.4 with a little remark: if an instance (G,k) contains an isolated vertex, then obviously it is a NO-instance for *Spanning Star Forest Problem* and so is (G',k') for *Dominating Set* because G' is not an empty graph. □

One can observe now the immediate corollary of the theorem 2.3 and theorem 2.4.

Corollary 2.2. *Every theorem that is true for* Dominating Set Problem *if and only if it is true for a* Spanning Star Forest Problem *parameterized by the number of stars.*

As an example, the following theorem described in can be transfered to *Spanning Star Forest Problem* parameterized by the number of stars.

Theorem 2.5. *Unless* CNF-SAT *can be solved in time $\mathcal{O}^*((2-\epsilon')^n)$ for some $\epsilon' > 0$ there do not exist constant $\epsilon > 0$, $k \geq 3$ and an algorithm solving* Dominating Set Problem *parameterized by the number of stars in time $\mathcal{O}^*(N^{k-\epsilon})$, where N is the number of vertices of the input graph.*

Chapter 3

Spanning Star Forest with isolated edges

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