Spanning Star Forest Problem

Adam Starak

May 12, 2019

Star definition

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A star is a tree of size at least 2 for which at most one vertex has a degree greater than 1.

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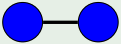

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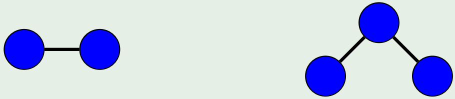

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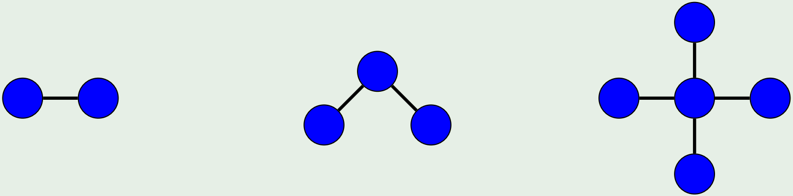
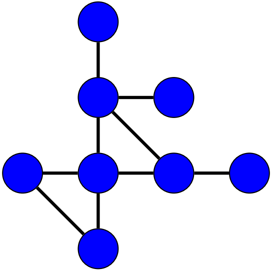


Spanning Star Forest Problem

Definition

Given a graph G decide whether it has a *Spanning Star Forest*.

Example



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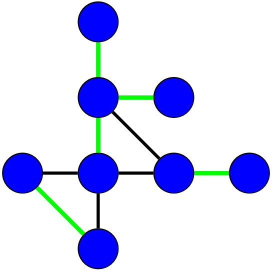

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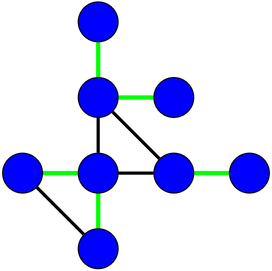

Spanning Star Forest Problem

Lemma

G contains a Spanning Star Forest if and only if it does not contain any isolated vertices.

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Forward implication. Trivial, Every vertex belongs to a tree of size at least 2. Thus, it's degree is at least 1.

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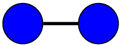

Let $n = 2$.

Graph is a correct Spanning Star Forest.

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Inductive step.

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Assume the condition holds for all graphs of size at most n.

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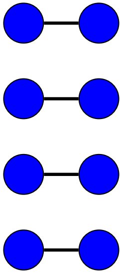

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Let S be a solution for a graph $G \setminus \{v\}$.

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Now, let's try to add vertex v to the solution S.

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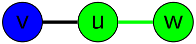

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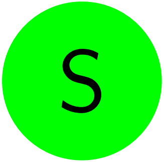

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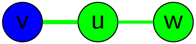

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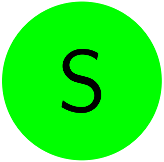

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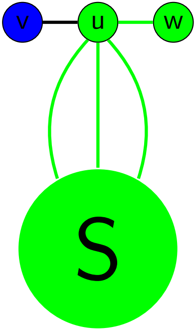

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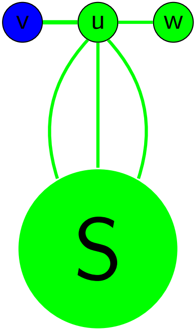

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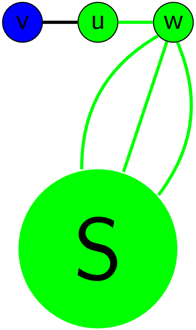

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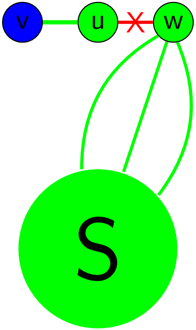

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Given a pair (G, k) find a Spanning Star Forest S such that S has at most k connected components.

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The problem is NP-Complete

NP-completeness - membership in NP

Trivial. Given S check if it is a Spanning Star Forest and if it has at most k connected components.

Construction:

NP-completeness - hardness

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Backward implication.
Set of centers represents a correct dominating set of G.

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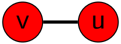

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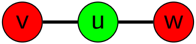

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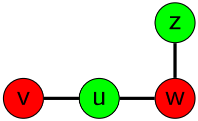

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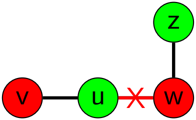


Reduction from SSF to Dom-Set

Construction

Given (G, k) transform the instance as following:

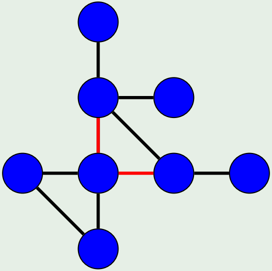
$$(G', k') = \begin{cases} (G, 0), & \text{if } G \text{ contains an isolated vertex.} \\ (G, k), & \text{otherwise.} \end{cases}$$

Spanning Star Forest Extension Problem

Definition

Given a graph G and a set of edges $F \subseteq E(G)$, find a Spanning Star Forest S such that $F \subseteq E(S)$.

Example



Instance normalization

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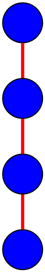

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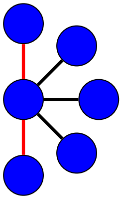
(R3) If G has a path of size 2 made from isolated edges, then remove all the vertices adjacent to the center.



Instance normalization 2

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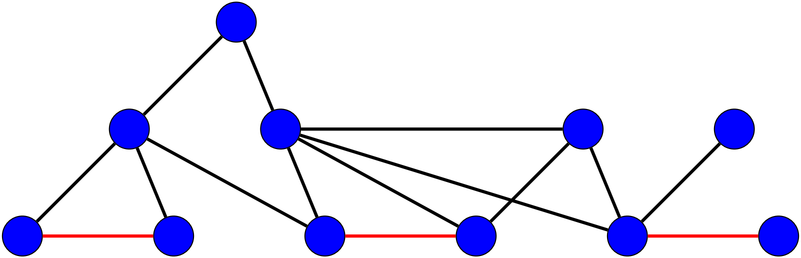
$G_{NP} = G \setminus G_P$.



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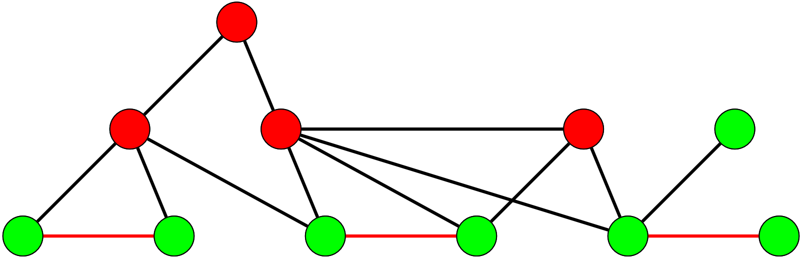

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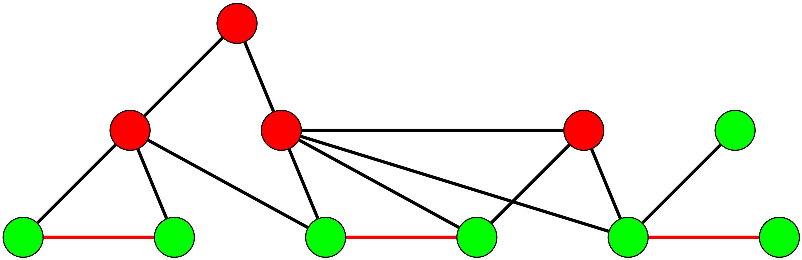
G_P has always a SSF because it does not contain any isolated edges and for all $v \in V(G_P)$ $deg_{G_P}(v) \neq 0$.



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Let S be a solution for G_{NP}.

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Partition G into (G_P, G_{NP}).

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$S \cup S'$ is a solution for G.

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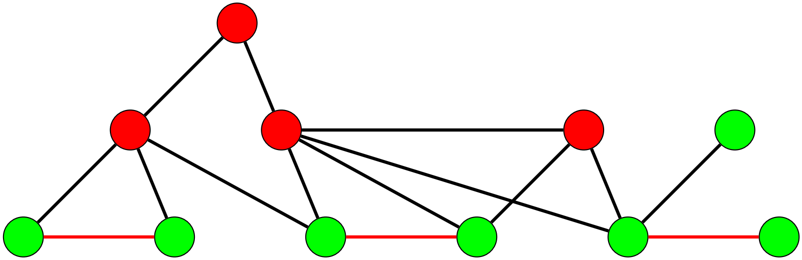

SSFE is NP-complete

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Membership in NP

Trivial. Given a solution S check if all isolated edges are included and if it is a correct SSF.

SSFE - Hardness

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Reduction from 3-SAT: Given formula ϕ do the following.

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$$(x_1 \vee x_2 \vee \neg x_4) \wedge (\neg x_1 \vee x_3 \vee \neg x_5) \wedge (x_4 \vee x_5 \vee \neg x_1)$$

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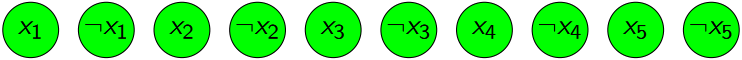

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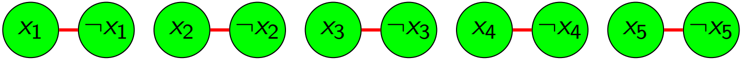

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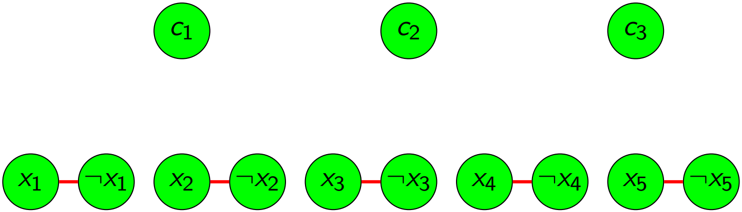

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2. For each v_{x_i}, $v_{\neg x_i}$ introduce an isolated edge.
3. For each clause c_i introduce vertex v_{c_i}.
4. If literal l exists in clause c introduce an edge between them.

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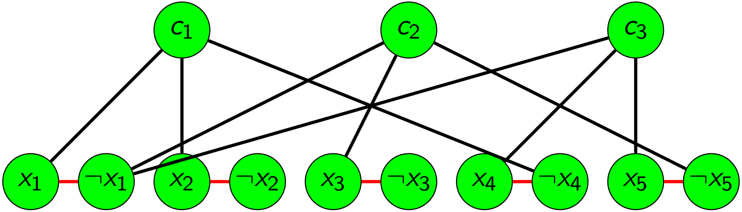

SSFE reverse reduction

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Theorem

There exists a reduction from 3-SAT to SSFE.

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Theorem

There exists a reduction from 3-SAT to SSFE.

Corollary

3-SAT and SSFE are equally hard

Parametrizations

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Number of isolated edges $|F|$.

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Number of vertices $|V(G) \setminus V(F)|$.

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Number of isolated edges $|F|$.

Number of vertices $|V(G) \setminus V(F)|$.

Treewidth.

Number of isolated edges $|F|$

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Simple algorithm

Branch on every isolated edge and check whether chosen vertices form a Spanning Star forest.

Number of isolated edges $|F|$

Simple algorithm

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SETH

let δ_q be the infinimum of the set of constants c for which there exists an algorithm solving q-SAT in time $\mathcal{O}^*(2^{cn})$. Then:

$$\lim_{q \to \infty} \delta_q = 1$$

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For a more general idea, read chapter 15 from Parameterized Algorithms.

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Example

Given instances $(G_1, k), (G_2, k), ..., (G_t, k)$ of k-Path problem return $(\bigcup_{i=1}^{t} G_i, k)$

Proof on the board.

SSFE parameterized by $|V(G)| \setminus |V(F)|$

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Quadratic kernel

If there exists a vertex $v \in V(G) \setminus V(F)$ such that $deg_G(v) > k$, then remove v.

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In graph G $deg_G(v) > k$. So, there exists isolated edge (u, w) such that $deg_S(u) = deg_S(w) = 1$ and either $(v, u) \in E(G)$ or $(v, w) \in E(G)$.

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