Spanning Star Forest Problem

Adam Starak

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Star definition

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A star is a tree of size at least 2 for which at most one vertex has a degree greater than 1.

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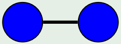

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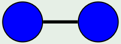

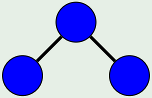

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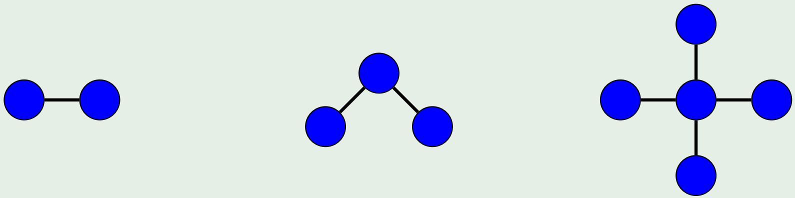
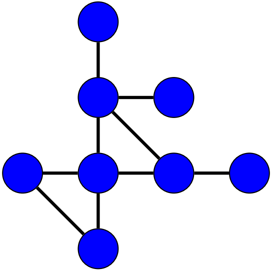


Spanning Star Forest Problem

Definition

Given a graph G decide whether it has a *Spanning Star Forest*.

Example



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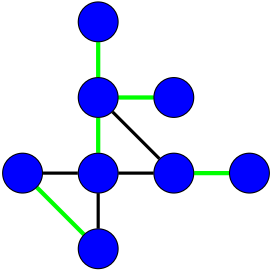

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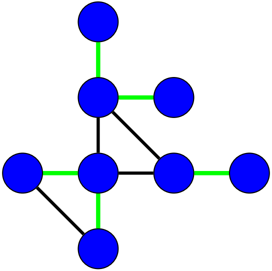

Spanning Star Forest Problem

Lemma

G contains a Spanning Star Forest if and only if it does not contain any isolated vertices.

Spanning Star Forest Problem

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Forward implication. Trivial, Every vertex belongs to a tree of size at least 2. Thus, it's degree is at least 1.

Backward implication

Suppose G has no isolated vertices.

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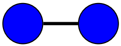

Let $n = 2$.

Graph is a correct Spanning Star Forest.

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Inductive step.

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Assume the condition holds for all graphs of size at most n.

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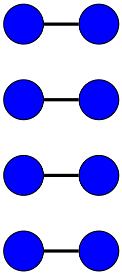

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Let S be a solution for a graph $G \setminus \{v\}$.

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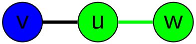

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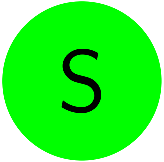

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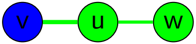

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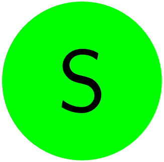

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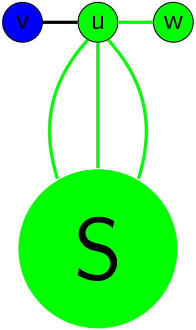

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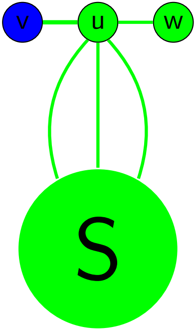

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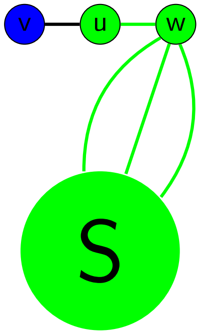

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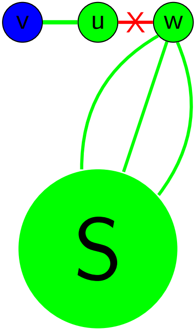

Parametrization by the number of stars

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Given a pair (G, k) find a Spanning Star Forest S such that S has at most k connected components.

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The problem is NP-Complete

NP-completeness - membership in NP

Trivial. Given S check if it is a Spanning Star Forest and if it has at most k connected components.

Construction:

NP-completeness - hardness

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Backward implication.
Set of centers represents a correct dominating set of G.

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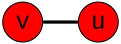

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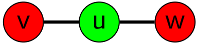

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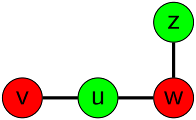

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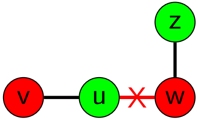


Reduction from SSF to Dom-Set

Construction

Given (G, k) transform the instance as following:

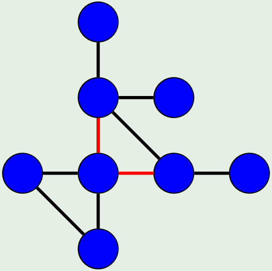
$$(G', k') = \begin{cases} (G, 0), & \text{if } G \text{ contains an isolated vertex.} \\ (G, k), & \text{otherwise.} \end{cases}$$

Spanning Star Forest Extension Problem

Definition

Given a graph G and a set of edges $F \subseteq E(G)$, find a Spanning Star Forest S such that $F \subseteq E(S)$.

Example



Instance normalization

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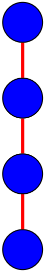

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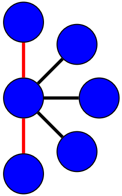
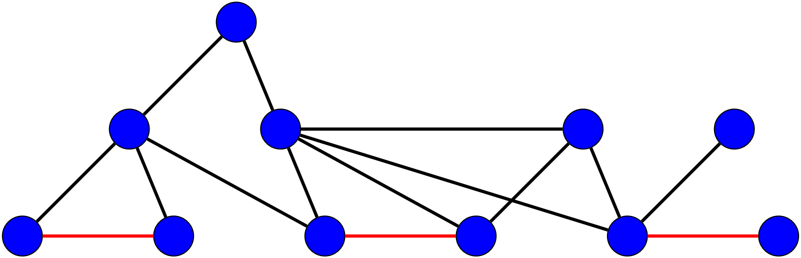
(R3) If G has a path of size 2 made from isolated edges, then remove all the vertices adjacent to the center.



Instance normalization 2

$G_P \;= \{v : u, v \notin V(F) \text{ and } \exists (u,v) \in E(G)\}.$

$G_{NP} \;= G \setminus G_P.$



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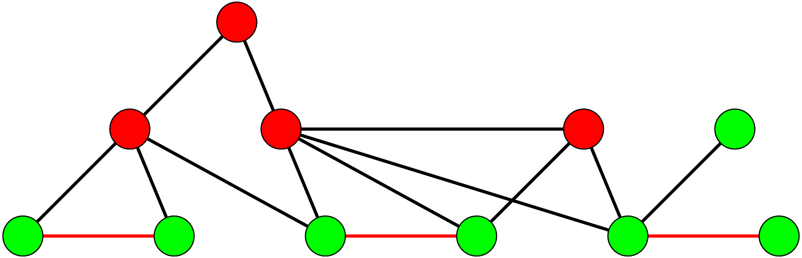

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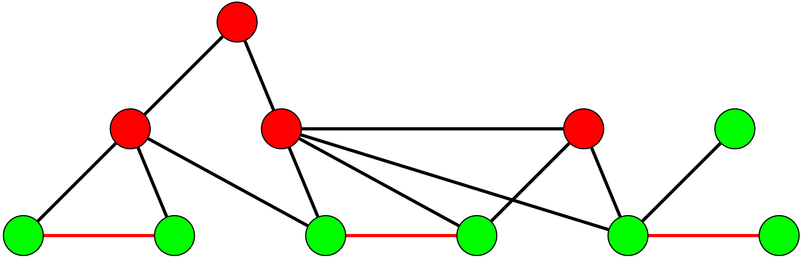
G_P has always a SSF because it does not contain any isolated edges and for all $v \in V(G_P)$ $deg_{G_P}(v) \neq 0$.



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Let S be a solution for G_{NP}.

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Partition G into (G_P, G_{NP}).

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$S \cup S'$ is a solution for G.

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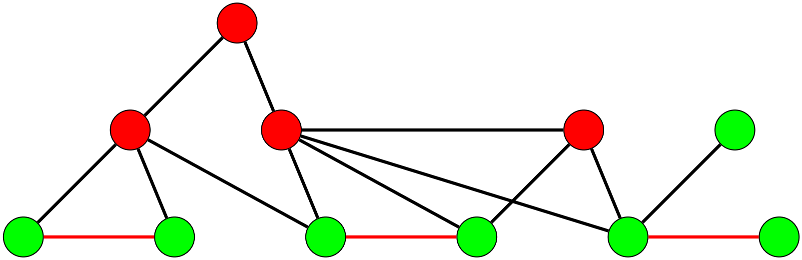

SSFE is NP-complete

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Membership in NP

Trivial. Given a solution S check if all isolated edges are included and if it is a correct SSF.

SSFE - Hardness

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$$(x_1 \vee x_2 \vee \neg x_4) \wedge (\neg x_1 \vee x_3 \vee \neg x_5) \wedge (x_4 \vee x_5 \vee \neg x_1)$$

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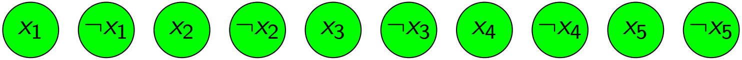

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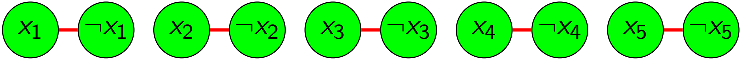

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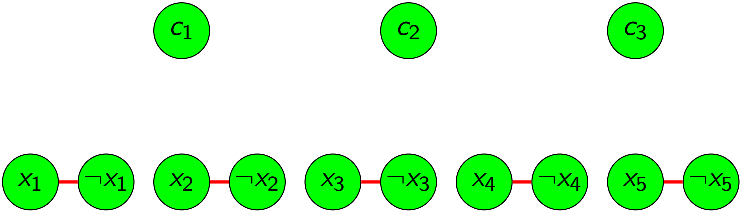

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3. For each clause c_i introduce vertex v_{c_i}.
4. If literal l exists in clause c introduce an edge between them.

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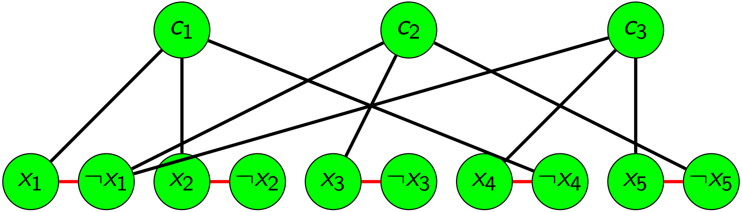

SSFE reverse reduction

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Theorem

There exists a reduction from 3-SAT to SSFE.

SSFE reverse reduction

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There exists a reduction from 3-SAT to SSFE.

Corollary

3-SAT and SSFE are equally hard

Parametrizations

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Number of vertices $|V(G) \setminus V(F)|$.

Parametrizations

Number of isolated edges $|F|$.

Number of vertices $|V(G) \setminus V(F)|$.

Treewidth.

Number of isolated edges $|F|$

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Simple algorithm

Branch on every isolated edge and check whether chosen vertices form a Spanning Star forest.

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Complexity: $\mathcal{O}^*(2^{|F|})$.

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SETH

let δ_q be the infimum of the set of constants c for which there exists an algorithm solving q-SAT in time $\mathcal{O}^*(2^{cn})$. Then:

$$\lim_{q \to \infty} \delta_q = 1$$

SSFE parameterized by $|F|$ doesn't invoke kernel.

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Proving nonexistence of a kernel

Input: $(x_1, k), (x_2, k), ..., (x_t, k)$ where (x_i, k) - instance of a given problem Q.

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For a more general idea, read chapter 15 from Parameterized Algorithms.

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Example

Given instances $(G_1, k), (G_2, k), ..., (G_t, k)$ of k-Path problem return $(\bigcup_{i=1}^{t} G_i, k)$

SSFE parameterized by $|F|$ doesn't invoke kernel - Proof

Proof on the board.

SSFE parameterized by $|V(G)| \setminus |V(F)|$

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Quadratic kernel

If there exists a vertex $v \in V(G) \setminus V(F)$ such that $deg_G(v) > k$, then remove v.

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In graph G $deg_G(v) > k$. So, there exists isolated edge (u, w) such that $deg_S(u) = deg_S(w) = 1$ and either $(v, u) \in E(G)$ or $(v, w) \in E(G)$.

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Number of edges: $\leq k + 2k^2$.

SSFE parameterized by $tw(G)$

Dynamic table. $dp[t, f]$ where t is node from a tree decomposition and $f : X_t \to \{true, false\}$ defined as follows:

$$f(v) = \begin{cases} true, & v \in V(F) \text{ and } v \text{ is a center.} \\ true, & v \notin V(F) \text{ and } v \text{ is a part of a Star Tree.} \\ false, & \text{otherwise.} \end{cases}$$

$$f_{v \to p}(u) = \begin{cases} p, & \text{if u=v.} \\ f(u), & \text{otherwise.} \end{cases}$$

SSFE parameterized by $tw(G)$

Leaf node. An empty graph is a correct Spanning Star Forest.

$$dp[t, f] = true$$

SSFE parameterized by $tw(G)$

Introduce vertex $v \in V(F)$.

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We assign *false* if both vertices of an isolated vertex (u, v) have the same value.

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$$dp[t, f_{v \to p}] = \begin{cases} \mathit{false}, & \text{if } u \in X_t \text{ and } f(u) = p. \\ dp[t', f], & \text{otherwise.} \end{cases}$$

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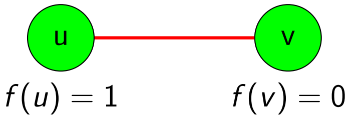

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$f(u) = 0$ $\quad$ $f(v) = 1$

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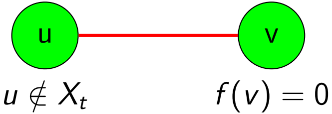

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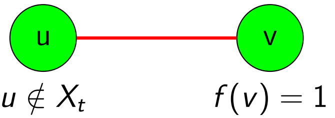

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Vertex is isolated. Thus, the only correct possibility is to assign 0.

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$$dp[t, f_{v \to p}] = \begin{cases} dp[t', f], & \text{if } p = 0. \\ false, & \text{otherwise.} \end{cases}$$

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Introduce edge $(u, v) \in F$

SSFE parameterized by $tw(G)$

Introduce edge $(u, v) \in F$
We can skip this part. We assigned proper values during vertex introduction.

SSFE parameterized by $tw(G)$

Forget $v \in V(F)$

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The vertex is covered anyway. We get the maxial value.

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$$dp[t, f_{|v}] = dp[t', f_{v\to 0}] \vee dp[t', f_{v\to 1}]$$

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The vertex needs to be covered. So, we pass the value if and only if 1 is assigned to vertex v.

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$$dp[t, f_{|v}] = \begin{cases} dp[t', f], & \text{if f(v)=1.} \\ false, & \text{otherwise.} \end{cases}$$

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Introduce edge $(u, v) \notin F$**,** $v \notin V(F)$

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Introduce edge $(u, v) \notin F$, $v \notin V(F)$
We change the value of vertex v from 0 to 1 if and only if u is a center ($f(u) = 1$):

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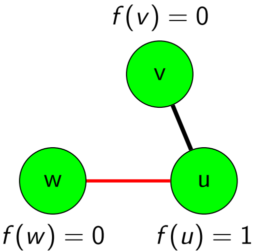
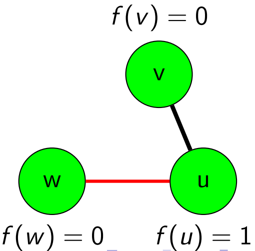
Before



After



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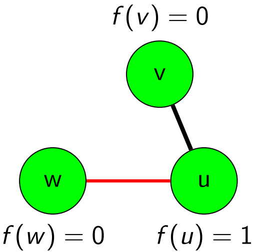
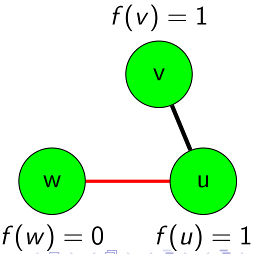


After



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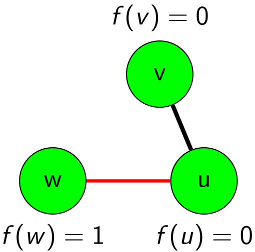
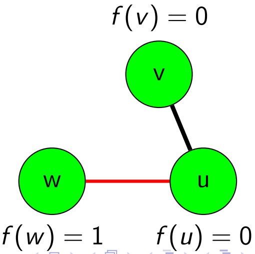


After



SSFE parameterized by $tw(G)$

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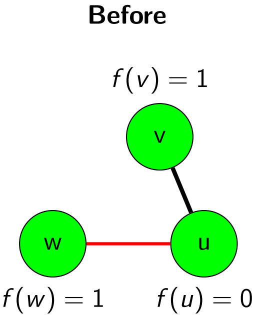
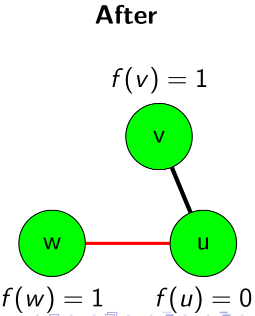


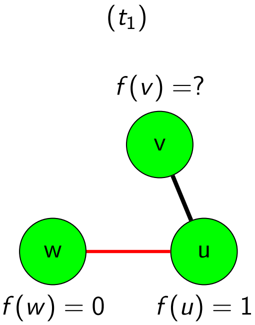


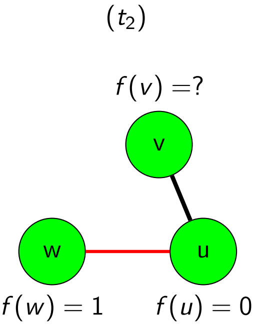
SSFE parameterized by $tw(G)$

Join nodes t_1**,** t_2

SSFE parameterized by $tw(G)$

Join nodes t_1, t_2

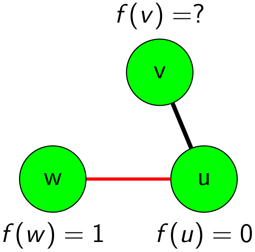




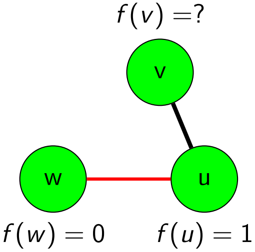
SSFE parameterized by $tw(G)$

Join nodes t_1, t_2

(t_1)



(t_2)



$$\forall v \in V(F) \cap X_t \ f(v) = f_1(v) = f_2(v)$$

SSFE parameterized by $tw(G)$

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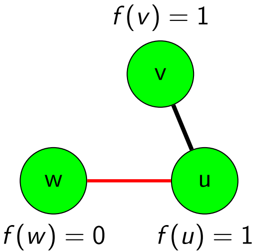
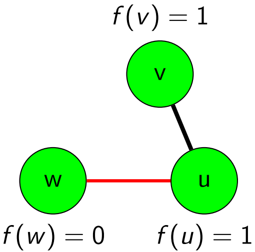
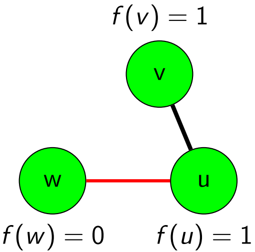
(t_1)



(t_2)



(t)



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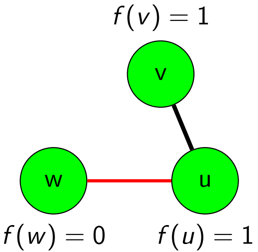
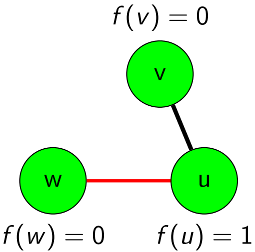
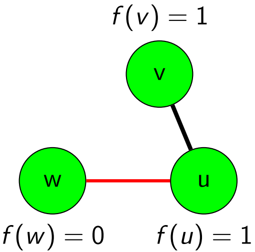
(t_1)



(t_2)



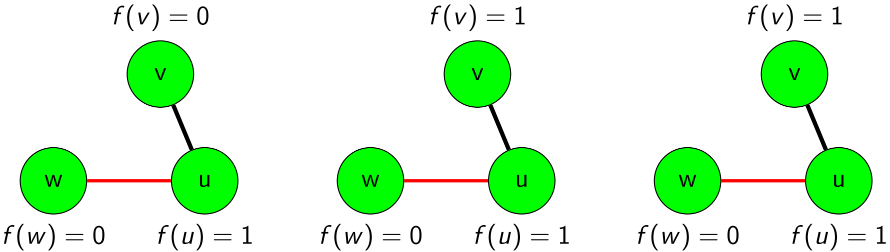
(t)



SSFE parameterized by $tw(G)$

Join nodes t_1**,** t_2

(t_1**)** **(**t_2**)** **(**t**)**



$$C_2 = \forall v ((V(G) \setminus V(F)) \cap X_t) \; f(v) = f_1(v) \vee f_2(v)$$

SSFE parameterized by $tw(G)$

SSFE parameterized by $tw(G)$

$$dp[t, f] = \begin{cases} \bigvee_{f_1, f_2} dp[t_1, f_1] \wedge dp[t_2, f_2], & \text{if } C_1 \text{ and } C_2 \text{ holds.} \\ false, & \text{otherwise.} \end{cases}$$

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There are $2^{|X_t|}$ functions. Each can be calculated naively in time $\mathcal{O}^*(2^{|X_t|})$.

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SSFE parameterized by $tw(G)$

There exists an algorithm solving SSFE parameterized by treewidth in time $\mathcal{O}^*(2^{tw(G)})$.

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SSFE parameterized by $tw(G)$

There exists an algorithm solving SSFE parameterized by treewidth in time $\mathcal{O}^*(2^{tw(G)})$.

Lower bound

Assuming SETH, there does not exist an algorithm solving SSFE parameterized by $tw(G)$ in time $\mathcal{O}^*((2-\epsilon)^{tw(G)})$.