Derivation Rules

$\frac{d}{dx}(x^a) = a * x^{a-1}$ given: $x, a \in \mathbb{R}$ & $x > 0$

subexamples:

$\frac{d}{dx}x = 1 \rightarrow \frac{d}{dx}(x^1) = 1 * x^{1-1}$

$\frac{d}{dx}x^2 = 2x \rightarrow \frac{d}{dx}(x^2) = 2 * x^{2-1}$

$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2} \rightarrow \frac{d}{dx}(x^{-1} = -1 * x^{-1-1})$

$\frac{d}{dx}\sqrt{x} = \frac{1}{2*\sqrt{x}} \rightarrow \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2*x^{\frac{1}{2}}}$

$\frac{d}{dx}(c) = 0$ given: $c \in \mathbb{R}$ & c is constant & c != factor

$\frac{d}{dx}(e^x) = e^x \rightarrow \frac{d}{dx}(e^x) = ln(e) * e^x * x' = 1 * 1 * e^x$

$\frac{d}{dx}(a^x) = ln(a) * a^x \rightarrow \frac{d}{dx}(a^x) = x' * ln(x) * a^x$ because: $e^{x*ln(a)} = a^x$

Note for this rule:

$\frac{d}{dx}(2^{2x+1}) = ln(2x+1) * 2^{2x+1} * (2x+1)' = ln(2x+1) * 2^{2x+1} * 2$

$\frac{d}{dx}(ln(x)) = \frac{1}{x}$

$\frac{d}{dx}log_b(x) = \frac{1}{ln(b)*x} \rightarrow$ special case for $\frac{d}{dx}(\frac{ln(x)}{ln(b)})$

this is the case because of base change in lograrithmic functions!

$log_a(x) = \frac{ln(x)}{ln(a)} = \frac{log_c(x)}{log_c(a)} \rightarrow \frac{d}{dx}log_a(x) = \frac{ln(x)}{ln(a)} -> \frac{d}{dx}ln(x) = \frac{\frac{1}{x}}{ln(a)} = \frac{1}{ln(a)*x}$

c can be any number!

$\frac{d}{dx}sin(x) = cos(x)$ $\quad \frac{d}{dx}csc(x) = (-1) * csc(x) * cot(x)$

$\frac{d}{dx}cos(x) = -sin(x)$ $\quad \frac{d}{dx}sec(x) = sec(x) * tan(x) =$

$\frac{d}{dx}tan(x) = \frac{1}{cos^2(x)}$ $\quad \frac{d}{dx}cot(x) = (-1) * csc^2(x)$

Note, sin(x) is done via chain rule -> sin(x) * 1 !!

For more than x inside sin, FULL CHAIN RULE!! $sin(x^2) -> sin(x^2) * x$

For integration, you need the substitution rule!

$\frac{d}{dx}tan(x) = 1 + tan^2(x)$

$\frac{d}{dx}(ax) = a \rightarrow \frac{d}{dx}(ax) = a * 1 \rightarrow$ we derive x NOT a!!

$\frac{d}{dx}(3x) = a \rightarrow \frac{d}{dx}(3x) = 3 * 1 \rightarrow$ 3 is a factor!

All of these derive from:

$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

Sum Rule

$(f+g)' = f' + g'$

Difference Rule

$(f-g)' = f' - g'$

Product Rule

$(f*g)' = f*g' + f'*g$

Quotient Rule

$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'*g - f*g'}{g^2}$

Chain Rule

$[f(g(x))]' = f'(g(x)) * g'(x)$

Example:

$\frac{d}{dx}\sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}} = \frac{1}{2} * (x^2+1)^{-\frac{1}{2}} * 2x$

$\frac{1}{2} * \frac{1}{\sqrt{x^2+1}} * 2x = \frac{2x}{2*\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$

Note: (x^2+1) is g(x), while f is the exponent function

More examples:

$\frac{d}{dx}sin^2(x) = \frac{d}{dx}(sin(x))^2 = 2 * sin(x) * cos(x)$

$\frac{d}{dx}\left(\frac{x-1}{x+1}\right)^2 = 2 * \left(\frac{x-1}{x+1}\right) * \left(\frac{x-1}{x+1}\right)'$

$\frac{d}{dx}(x+2)^3(x)^4 = (x+2)^3 * ((x)^4)' + ((x+2)^3)' * (x)^4$

$((x+2)^3)' = 3 * (x+2)^2 * (x+2)' = 3 * (x+2)^2 * 1$

$\frac{d}{dx}sin(cos[tan(x)]) = cos(cos[tan(x)]) * -sin(tan(x)) * \frac{1}{cos^2(x)}$

Implicit Differentiation

$\frac{d}{dx}(x^2+y^2=9) \rightarrow 2x + \frac{d}{dx}((y)^2) * \frac{dy}{dx}(y) = 0 \rightarrow 2x + (2y * y') = 0 \rightarrow y' = \frac{-2x}{2y}$

!! Remember that this is only necessary if y needs to be derived !!

Higher Derivatives

The best idea for higher derivatives is distance s, velocity v and acceleration a.

$\frac{d}{dt}(s(t)) = v(t) = s'(t) \;||\; \frac{d}{dt}(v(t)) = a(t) = s''(t) = v'(t)$

This is why the acceleration on earth -> gravity is constant!! HOLY FUCK

Taking Derivations higher than 3

$1: f' \rightarrow 2: f'' \rightarrow 3: f''' \rightarrow 4: f^{(4)} \rightarrow n: f^{(n)}$

Related Rates

In a Sphere, the rate of change of V is $100cm^3/s$

calculate the rate of change in r = 25cm given rate of change in V

$\frac{dV}{dt} = 100cm^3/s$, $r = 25cm$

$\frac{dV}{dt} = (4*\pi*r^2) * \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{100cm^2/s}{4*\pi*(25cm)^2} = \frac{dr}{dt} = \frac{1}{25*\pi}cm/s$

Local Maximum and Minimum

The first derivative of local maximum and minimum MUST be 0!

This includes turning points, aka slope is just 0!

Local Minimum lMin:

$f''(x) > 0 \rightarrow$ given f(x) = 0

OR

$f'(lMin-1) < 0 \;\&\&\; f'(lMin) = 0 \;\&\&\; f'(lMin+1) \geq 0$

Essentially the minimum is where the slope goes from negative to positive with the turning point being the minimum with slope 0

Local Maximum lMax:

$f''(x) < 0 \rightarrow$ given f(x) = 0

OR

$f'(lMax-1) \geq 0 \;\&\&\; f'(lMax) = 0 \;\&\&\; f'(lMax+1) < 0$

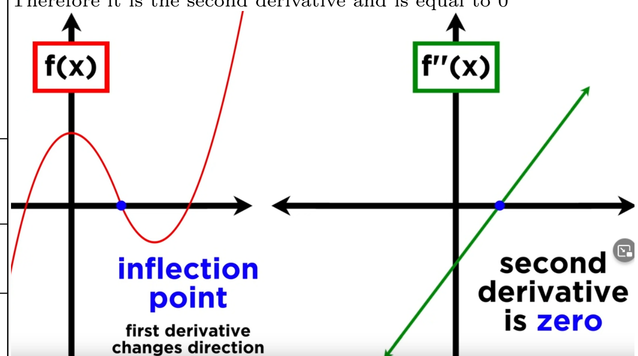
Essentially the maximum is where the slope goes from positive to negative with the turning point being the maximum with slope 0

Absolute minimum and maximum will never be exceeded -> sine absolute-max = 1

Inflection Point

This is the point where the function stops its increase or decrease in slope.

Therefore it is the second derivative and is equal to 0



Use of Maxima

Building a fence adjacent to a river. length l = 2x + y!

Given length of 2400m how big do x and y need to be for the maximum area A?

$l = y + 2x \rightarrow y = 2400m - 2x \rightarrow A = (2400m - 2x) * x$

Remember when you had to use the UI function on the calculator? Yeah, no more!

$Max \rightarrow f'(A) = f'((244m - 2x) * x = 0 \rightarrow x = \frac{2400m}{4} = 600m \rightarrow y = 1200m$

Limits:

The limit expresses that a variable is approaching a value

$\lim_{x \to \infty}$ x approaching infinity

This is often used when trying to determine functions that might give an invalid result at x

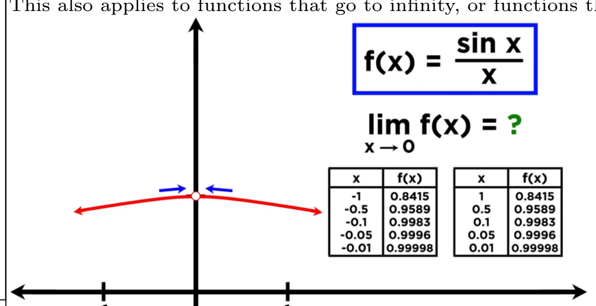
$f(x) = \frac{x-1}{x^2-1} \rightarrow f(1) = ??$

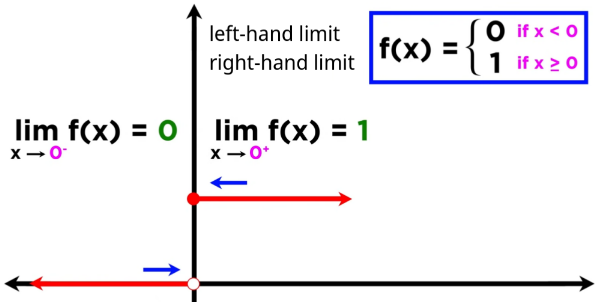
With limit we can say what we would expect the value to be, if the function would continu

aka what is the value of f(1) if the function would not show this abnormality?

$\lim_{x \to 1} f(1) = 0.5$

This also applies to functions that go to infinity, or functions that are constant for a range.





Limit Rules
Addition:

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

Subtraction:

$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

Multiplication:

$$\lim_{x\to a}[f(x)*g(x)] = \lim_{x\to a} f(x) * \lim_{x\to a} g(x)$$

Division:

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \rightarrow \text{ given lim != 0}$$

Multiplication by constant:

$$\lim_{x\to a}[c*f(x)] = c*\lim_{x\to a} f(x) \rightarrow \text{ given c is constant}$$

Exponent:

$$\lim_{x\to a}[f(x)]^2 = [\lim_{x\to a} f(x)]^2$$

Root:

$$\lim_{x\to a}\sqrt[n]{[f(x)]} = \sqrt[n]{\lim_{x\to a} f(x)}$$

x to a:

$$\lim_{x\to a}(x) = a$$

x to a with exponent:

$$\lim_{x\to a}(x^n) = a^n$$

x to a with root:

$$\lim_{x\to a}(\sqrt[n]{x}) = \sqrt[n]{a}$$

limit of a constant:

$$\lim_{x\to a}(c) = c \rightarrow \text{ given c is constant}$$

Examples:

$$\lim_{x\to -2}\left(\frac{x^3+2x^2-1}{5-3x}\right) = \frac{\lim_{x\to -2}(x^3+2x^2-1)}{\lim_{x\to -2}(5-3x)}$$

$$\frac{\lim_{x\to -2}(x^3)+\lim_{x\to -2}(2x^2)-\lim_{x\to -2}(1)}{\lim_{x\to -2}(5)-\lim_{x\to -2}(3x)} = \frac{-8+8-1}{5+6} = -\frac{1}{11}$$

Sometimes we need to eliminate terms in order to move on

limit and differntiation -> L'Hospital's Rule:

If either the left side -> $\frac{f(x)}{g(x)}$ is indeterminate
then we can use this rule! Otherwise it doesn't work, and doesn't make sense!

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right)$$

Examples:

$$\lim_{x\to 0}\left(\frac{sinx}{x}\right) = \lim_{x\to 0}\left(\frac{cosx}{1}\right) = \lim_{x\to 0}(cos(x)) = 1$$

$$\boxed{\underbrace{\lim_{x\to a}\frac{f(x)}{g(x)}}_{\text{indeterminate}} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

!!!!

$$\lim_{x\to\infty}\frac{e^x}{x^3} = \frac{\infty}{\infty}$$

indeterminate

$$\lim_{x\to\infty}\frac{e^x}{x^3} = \lim_{x\to\infty}\frac{e^x}{3x^2}$$

we can take the derivatives multiple times

$$\lim_{x\to\infty}\frac{e^x}{x^3} = \frac{\infty}{\infty}$$

indeterminate

$$\lim_{x\to\infty}\frac{e^x}{x^3} = \lim_{x\to\infty}\frac{e^x}{3x^2} = \lim_{x\to\infty}\frac{e^x}{6x} = \lim_{x\to\infty}\frac{e^x}{6} = \infty$$

this rule only applies to indeterminate forms (0/0 or ∞/∞)

$$\lim_{x\to 0}\frac{x+\sin x}{x+\cos x} = 0$$

Using the rule would have given us the wrong answer!

Reforming terms for L'Hospital
reforming a product

$$\lim_{x\to 0^+} x\cdot\ln x = \lim_{x\to 0^+}\frac{(\ln x)'}{(x^{-1})'} = \lim_{x\to 0^+}\frac{1/x}{-x^{-2}}$$

$$\lim_{x\to 0^+}\frac{1}{x}\cdot\frac{-x^2}{1} = \lim_{x\to 0^+} -x = 0$$

reforming an exponent with logarithm

$$\lim_{x\to 0^+} x^x = 0^0$$

$$y = x^x \rightarrow \ln y = \ln x^x \rightarrow \ln y = x\cdot\ln x$$

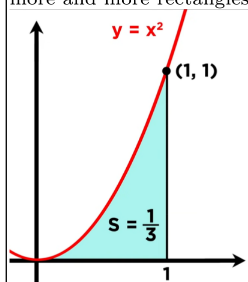
$$\lim_{x\to 0^+} e^{(x\cdot\ln x)} = e^0 = 1$$

Infinity calculation rules

$\infty + c = \infty$	$\infty + \infty = \infty$	$\infty - \infty = NaN$
$\infty * c = \infty \rightarrow c \neq 0$	$\infty * \infty = \infty$	$\infty * 0 = NaN$
$\frac{c}{0} = \pm\infty \rightarrow c \neq 0$	$\frac{c}{\infty} = 0$	$\frac{\infty}{c} = \infty \rightarrow c \neq 0$
$\frac{\infty}{0} = \infty$	$\frac{0}{0} = NaN$	$\frac{\infty}{\infty} = NaN$
$0^c \rightarrow c > 0 \backslash (c=1) = 0$	$0^0 = 1$ or NaN	$\infty^0 = NaN$
$0^c \rightarrow c < 0 = \infty$	$k^\infty \rightarrow k > 1 = \infty$	$k^\infty\ to 0 < k < 1 = 0$
$0^\infty = 0$	$\infty^\infty = \infty$	$1^\infty = NaN$

Integration

Similarity to limit
Just like limit, you can do it by intuition, by simply adding more and more rectangles into a function to get the area of said function.



let's find the area under the curve from zero to one

number of rectangles	area under the curve
4	0.46875
10	0.385
100	0.33835
1000	0.33383
∞	0.33333...

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

But just like with limit, there is a more elegant and generalized way. -> $\int_a^b x\,dx$

Definite Integrals and Integral Terms

$\int_a^b f(x)\,dx = F(b) - F(a)$

- The weird symbol is called integral sign
- The a and b are the upper and lower limits respectively
- f(x) is the integrand, the function to be integrated
- F(a) or F(b) is the antiderivative -> opposite calculation to derivation
- dx is the infinitesimal, no real use, but is required for notation

An integral with specific limits -> range is called a Definite Integral
Here the range is a to b

Indefinite Integrals

Since we can't put in values with infinite integrals, we instead just evaluate the antiderivative F(x), which in itself is yet another function

$\int f(x)\,dx = F(x) + C \to$ look at that, the holy constant C

Note that the C always has to be written, as the integral function covers a range of values with F(x) plus some constant! Hence + C!

hence we can also go back again -> reversibility of integrals and derivations
In other words, we differentiate the antiderivative!

$F'(x) = f(x) \to [F(x) + C]' = f(x) \to$ C vanishes -> constant!

One might ask now, why do we not consider it with definite integrals?
Check how the C would affect a - b:

$$\int_a^b x^2\,dx = \left(\frac{b^3}{3} + C\right) - \left(\frac{a^3}{3} + C\right) = \frac{b^3}{3} - \frac{a^3}{3} \to C - C = 0$$

as one can see, the C simply gets canceled.

Integral Rules

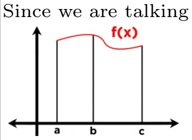
The Integral of a to b is the same as the negative integral of b to a

$\int_b^a x\,dx = -\int_a^b x\,dx$

The Integral of a to a is 0 -> as the area would be 0. a - a = 0

$\int_a^a x\,dx = 0$

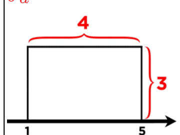
Since we are talking about areas, 2 areas in the same function add up:



$\int_a^b x\,dx + \int_b^c x\,dx = \int_a^c x\,dx$

If we integrate a constant, then the constant will multiple with x = 1:

$\int_a^b c\,dx = c * (b - a) \to \int_1^5 3\,dx = 3 * (5 - 1) = 12 \to$ given c is constant



$\int c * f(x)\,dx = c * \int f(x)\,dx \to$ given c is constant

multiplying the integrand with a function can be done outside of the integral!

$\int c * f(x)\,dx = c * \int f(x)\,dx$

Sum of Integrals

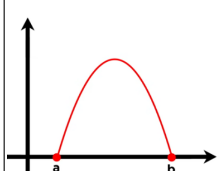
$\int [f(x) + g(x)] = \int f(x) + \int g(x)$

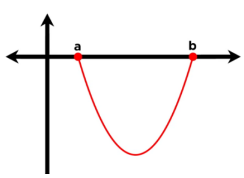
Difference of Integrals

$\int [f(x) - g(x)] = \int f(x) - \int g(x)$

Integrals with y >= 0 and y < 0

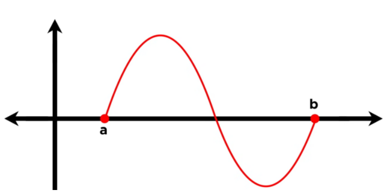
$\int_a^b f(x)dx > 0$ $\qquad$ $\int_a^b g(x)dx < 0$





Area of integrals with y below and above 0 at some point

$\int_a^b f(x)dx$ = **area above axis - area below axis**



often used:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

And lastly the most important function!

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

Example for Integral calculation

$$\int_0^3 (x + 5)\,dx = F(3) - F(0) \to F(x) = \left(\frac{x^2}{2} + 5x\right)$$

$$\to F(3) - F(0) = \left(\frac{3^2}{2} + 5 * 3\right) - \left(\frac{0^2}{2} + 5 * 0\right) = \frac{39}{2}$$

Integrals do not have the product rule

This means that we need to find a different way to remove factors
In fact, Integrals can only be taken over sums and differences

$\int \sqrt{x}\,(x - 2)dx$ $\qquad$ $\int (x^{3/2} - 2x^{1/2})dx$

we must manipulate this a little bit first $\qquad$ **now it's easy to find the antiderivative**

Substitution Rule usually the best!

This turns complicated nested integrands into smaller pieces
This is the correspondent technique to the chain rule! Note f(x) can be f(x) = x

$$\int f[g(x)] * g'(x)\,dx = \int f(u)du$$

$$\text{OR: } \int g(x) * g'(x)\,dx = \int u\,du$$

$$u = g(x) \;||\; du = g'(x)dx$$

$$u = x^2 + 1$$

f(x) = x + 5

$\frac{d}{dx}$ f(x) -> $\frac{d}{dx}$ x + 5 = 1 + 0 = $\frac{df(x)}{dx}$

command, derive this $\qquad$ command, derive this $\qquad$ result!

- $\frac{d}{dx}(x^2 + 1) = 2x$
- $d(x^2 + 1) = 2x\,dx$
- $du = 2x\,dx$

$\boxed{\frac{d}{dx} y^2 = 2y * y' \text{ OR } 2y * \frac{dy}{dx}}$

Example:

$\int 2x * cos(x^2 + 1)\,dx \to 2x = g'(x) \to cos(x^2 + 1) = f[g(x)] \to (x^2 + 1) = g(x)$

$(x^2 + 1) = u \to (x^2 + 1)' = 2x$ this means we can use subtitution!

$\int cos(u) 2x\,dx = \int cos(u)\,du = sin(u) + C = sin(x^2 + 1) + C$

Second example with factors

$\int x^2\sqrt{x^3 + 1}\,dx \to x^2 = g'(x) \to f(g(x)) = \sqrt{x^3 + 1} \to g(x) = (x^3 + 1)$

$(x^3 + 1)' = \frac{1}{3}x^2$ note 1/3 is a factor, it can be removed from the term.

$$\frac{1}{3} * \int \sqrt{u} * x^2\,dx = \frac{1}{3} * \int u^{\frac{1}{2}}\,du = \frac{1}{3} * -\frac{x^{\frac{3}{2}}}{\frac{3}{2}}$$

$$\frac{1}{3} * \frac{2}{3} * u^{\frac{3}{2}} + C = \frac{2}{9} * (x^3 + 1)^{\frac{3}{2}} + C$$

Integration by Parts

This turns complicated nested integrands into smaller pieces
This is the correspondent technique to the product rule!

$$\int [f(x) * g'(x) + f'(x) * g(x)]\,dx = f(x) * g(x)$$

$$\int [f(x) * g'(x)]\,dx + \int [f'(x) * g(x)]\,dx = f(x) * g(x)$$

$$\int [f(x) * g'(x)]\,dx = f(x) * g(x) - \int [f'(x) * g(x)]\,dx$$

Simply the opposite to the Product rule!!
Unlike the substitution method, this works for EVERY product!

$$\int u\,dv = uv - \int v\,du$$

The same as before, just with a simplified view.
Please note: this technique doesn't always simplify the term, sometimes a different method is better!

$\int x \sin x\,dx$

one is f(x)
one is g'(x)

choosing sin(x) as f(x) would not yield a good result, just another trig function....

$\boxed{f(x) = x \to \text{the one that becomes much simpler upon differentiation} \quad f'(x) = 1}$

$\int x \sin x\,dx = x(-\cos x) - \int (-\cos x)(1)dx$

u = x $\qquad$ v = -cos x
du = dx $\qquad$ dv = sin x dx

$\boxed{\int u\,dv = uv - \int v\,du}$

u = f(x)
v = g(x)
du = f'(x)
dv = g'(x)
d just mean derivative!

$$\int (\ln x)^2\, dx$$

(ln x)^2 * 1 * dx !!!!!!

$$u = (\ln x)^2$$
$$du = (2\ln x / x)dx$$
$$v = x$$
$$dv = dx$$

better:
dv = 1 * dx
this equal,
but the one
helps with
understanding
this method.

$$\int (\ln x)^2\, dx = x(\ln x)^2 - 2\Big[\underbrace{\int \ln x\, dx}\Big]$$

$$\int \ln x\, dx = x \ln x - \int dx$$

$u = \ln x,\ du = dx/x$
$v = x,\ dv = dx$

$$\int (\ln x)^2\, dx = x(\ln x)^2 - 2x\ln x + 2x + C$$

Example Integration by Parts:

$$\int \frac{ln(x)}{x^2}\, dx \to ln(x) = f(x) \to x^2 = g'(x)$$

$$\int \frac{ln(x)}{x^2}\, dx = -\left(\frac{ln(x)}{x}\right) - \int \frac{1}{x} * -\frac{1}{x}$$

$$\int \frac{ln(x)}{x^2}\, dx = -\left(\frac{ln(x)}{x}\right) + \int \frac{1}{x^2}$$

$$\int \frac{ln(x)}{x^2}\, dx = -\left(\frac{ln(x)}{x}\right) - \frac{1}{x} + C$$

Integration by Trigonometric Substitution

This can be done with the following 3 situations

$$\sqrt{a^2 - x^2}\ ||\ \sqrt{a^2 + x^2}\ ||\ \sqrt{x^2 - a^2}$$

Where a can be any positive number

$\sqrt{a^2 - x^2}$	$\sqrt{a^2 + x^2}$	$\sqrt{x^2 - a^2}$
$x = a\sin\theta$	$x = a\tan\theta$	$x = a\sec\theta$
$\sqrt{a^2(1 - \sin^2\theta)}$	$\sqrt{a^2(1 + \tan^2\theta)}$	$\sqrt{a^2(\sec^2\theta - 1)}$
$\sqrt{a^2\cos^2\theta}$	$\sqrt{a^2\sec^2\theta}$	$\sqrt{a^2\tan^2\theta}$
$\boxed{a\cos\theta}$	$\boxed{a\sec\theta}$	$\boxed{a\tan\theta}$

$$1 - \sin^2\theta = \cos^2\theta$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\sec^2\theta - 1 = \tan^2\theta$$

ignore the Theta θ, it stands for ANY angle. It is just there for completeness.

$$\boxed{\begin{array}{c} \sin^2\theta + \cos^2\theta = 1 \\ \tan^2\theta + 1 = \sec^2\theta \\ 1 + \cot^2\theta = \csc^2\theta \end{array}}$$

SohCahToa / ChoShaCao

$$sin = \frac{opposite}{hypotenuse} \quad cos = \frac{adjacent}{hypotenuse} \quad tan = \frac{opposite}{adjacent}$$

$$csc = \frac{hypotenuse}{opposite} \quad sec = \frac{hypotenuse}{adjacent} \quad cot = \frac{adjacent}{opposite}$$

$$\int \frac{\sqrt{9 - x^2}}{x^2}\, dx \to \int \frac{3\cos\theta}{9\sin^2\theta}\, dx \to \int \frac{3\cos\theta}{9\sin^2\theta} 3\cos\theta\, d\theta$$

$x = 3\sin\theta$ $\quad \frac{dx}{d\theta} = 3\cos\theta$ $\quad$ we need $d\theta$ now $\quad dx = 3\cos\theta\, d\theta$

$$\int \frac{9\cos^2\theta}{9\sin^2\theta}\, d\theta \to \int \cot^2\theta\, d\theta \to \int (\csc^2\theta - 1)d\theta = -\cot\theta - \theta + C$$

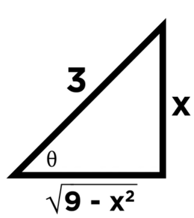
$\cot^2\theta = \csc^2\theta - 1$ $\quad$ $(\cot\theta)' = -\csc^2\theta$

$$\int \frac{\sqrt{9 - x^2}}{x^2}\, dx \to -\cot\theta - \theta + C$$

we need to get back to terms with x in them

$x = 3\sin\theta$

$$\boxed{-\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}(x/3) + C}$$



Steps for Integration by Trigonometric Substitution

1. Check if the term matches one of the 3 possible scenarios
2. Replace all x with one of the signatures
3. Replace the dx with $d\Theta \frac{dx}{d\Theta}$
– This means taking the derivation of x
4. simplify and integrate, don't forget the + C
5. Put in the values of x into the triangle to get the result
6. Simplify if needed.

Integral Lookup table

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} \quad \int \frac{1}{x}\, dx = \ln|x| \quad \int a^x\, dx = \frac{a^x}{\ln a} \quad \int e^x\, dx = e^x$$

$$\int \sin x\, dx = -\cos x \quad \int \cos x\, dx = \sin x \quad \int \sec x \tan x\, dx = \sec x$$

$$\int \sec^2 x\, dx = \tan x \quad \int \csc^2 x\, dx = -\cot x \quad \int \csc x \cot x\, dx = -\csc x$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| \quad \int \csc x\, dx = \ln|\csc x - \cot x|$$

$$\int \tan x\, dx = \ln|\sec x| \quad \int \cot x\, dx = \ln|\sin x|$$

Substitution without 2 functions

$$\int \cos\sqrt{x}\, dx \leftarrow \text{no term to act as } g'(x)dx$$

$$u = \sqrt{x}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \quad dx = 2\sqrt{x}\, du$$

solve for dx instead! This means we don't need the g'(x)!!

$$\int \cos\sqrt{x}\, dx \to \int (\cos u)\, 2\sqrt{x}\, du$$

$$\int \cos\sqrt{x}\, dx \to 2\int (\cos u) u\, du$$

Another example as brainfuck:

$$\int \frac{1}{1 - \cos x}\frac{1 + \cos x}{1 + \cos x}\, dx = \boxed{-\cot x - \csc x + C}$$

$$\int \frac{1 + \cos x}{1 + \cos x - \cos x - \cos^2 x}\, dx$$

$$\int \frac{1 + \cos x}{1 - \cos^2 x}\, dx \to \int \frac{1 + \cos x}{\sin^2 x}\, dx \to \int \left(\csc^2 x + \frac{\cos x}{\sin^2 x}\right) dx$$

$$-\cot x + \int \frac{\cos x}{\sin^2 x}\, dx \to \int u^{-2}\, du = -\frac{1}{u} = -\csc x$$

$u = \sin x,\ du = \cos x\, dx$

Factor plays

$$\int x * sin(x^2)\, dx$$

This seems like there is no way to take the integral.
However, look at x, we only need the factor 2 to make it work.
How about we just slap a 1/2 infront to make up for the missing 2?

$$!!\ \int x * sin(x^2)\, dx == \frac{1}{2} * \int 2x * sin(x^2)\, dx\ !!$$

Now we can just do regular substitution to get this:

$$\frac{-cos(x^2)}{2} + C$$

Another Example:

$$\int x^3 * sin(x^2)\, dx$$

strategy: change term

$$\int x^2 * x * sin(x^2)\, dx$$

strategy: integral by parts

$$\int x^2 \to u * (x sin(x^2)) \to du = uv - \int v * du$$

strategy: integrate dv to v with strategy substitution and factor play

$$\frac{1}{2}\int 2 * x sin(x^2)\, dx = \frac{1}{2} - cos(x^2)$$

$$\int u * du = x^2 * \frac{-1}{2}cos(x^2) - \int 2x * \frac{-1}{2}cos(x^2)$$

strategy: integrate v du with strategy substitution and factor play

$\int 2x * \frac{-1}{2} cos(x^2) = \int x * -cos(x^2) = \frac{1}{2} \int 2x * -cos(x^2) = \frac{-1}{2} sin(x^2)$

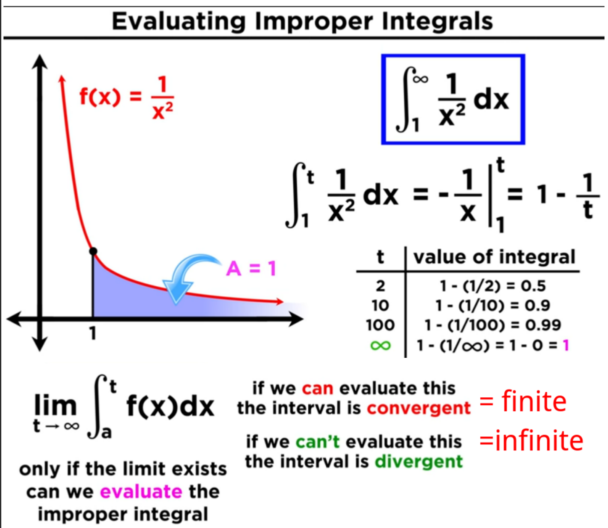
$\int u * du = x^2 * \frac{-1}{2} cos(x^2) + \frac{1}{2} sin(x^2) + \mathbf{C}$

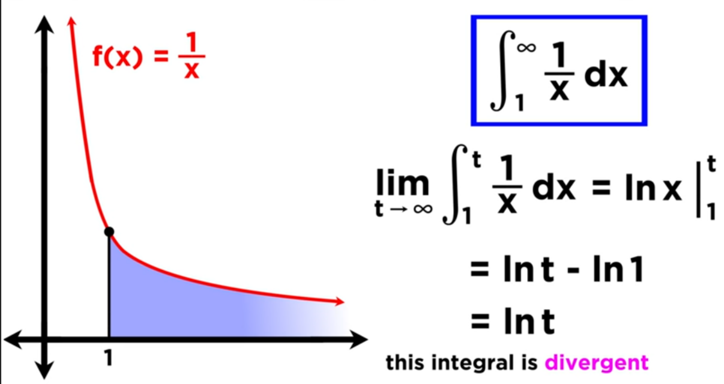
C is only necessary at the END of the integral.
Putting it in before isn't wrong, but there is no point!

Improper Integrals
These are integrals with strange ranges:

$\int_a^\infty f(x)\,dx$ $\quad\quad$ $\int_{-\infty}^b f(x)\,dx$

$\int_{-\infty}^\infty f(x)\,dx$ $\quad\quad$ $\int_a^b f(x)\,dx$
contains a discontinuity





Taylor Series

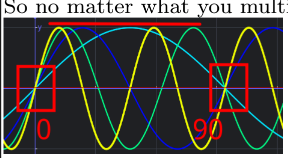
$g(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$

$a_k = \frac{g^{(k)}(x0)}{k!}$

The difference between a taylor series and a fourier series
is that a taylor series can be any polynomial, fourier is made of trig functions

Fourier Series

First we need to understand why the fourier series even works
Every 90 degrees, the sin wave will be 0, -> sin(0), sin(90) ...
This is regardless of how fast the frequency is!.
So no matter what you multiply a sine wave with, 0 will be 0 90 will be 0



Adding to this, the fourier series can already be read out of this.
check the top of these sine waves, the maximum almost looks like
a rectangle, aka a -1 0 1 signal!!, with just 4 different sine waves!

At last, the actual series:

$s_N(x) = a_0 + \sum_{k=1}^{\infty} a_k\, cos(\omega k\, x) + b_k\, sin(\omega k\, x)$

Fourier Terms

p = Period
α = variable to multiply a period -> $f(\alpha x)$ » default 0

T = Calculated Period when $a \geq 1$ -> $\frac{p}{a}$ can also be called the primitive period.

ω = Frequency $\frac{2\pi}{T}$

k/m/n factors for frequency!

smallest period == primitive period
The fastest time that the function repeats itself -> sin(360)
$f(x+p) = f(x) \to sin(x+360) = sin(x)$
Even every multiple of 360 gives the same period, just multiple times.
$f(x+2p) = f(x) \to sin(x+2*360) = sin(x+720) = sin(x))$

Constant periodic function
For a constant function, there is no primitive period.
I mean how, a constant doesn't change so the minimum period is 0....
for a constant function, every p > 0 is a valid period

Multiplying a periodic function
Take a look at sin(x), if we multiply x with a number, what happens to the period?
$sin(2x) \to sin(2*0) = 0 \to sin(2*180) = sin(360) = 0$
As you can see the period gets halved, it is now 180 instead of 360.
This can be calculated by the following
$f(\alpha x) \to f = \frac{p}{a} \to sin(2x) \to f = \frac{360}{2} = 180$

Periodic Function Addition
Adding a function with period p to another function with period p
results in another function with period p!
$h(x) = f(x) + g(x) \to pf = pg$
This is also called a Linear Combination!

Additionterms

$sin(x+y) = sin(x) * cos(y) \pm sin(y) * cos(x)$

$cos(x+y) = cos(x) * cos(y) \pm sin(y) * sin(x)$

for $\omega = \frac{2\pi}{T}$ and m/n $\in \mathbb{N}$

$$\int_0^{2\pi} sin(\omega m\, x)\, sin(\omega n\, x)\, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \quad (1)$$

$$\int_0^{2\pi} cos(\omega m\, x)\, cos(\omega n\, x)\, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \quad (2)$$

$$\int_0^{2\pi} cos(\omega m\, x)\, sin(\omega n\, x)\, dx = 0$$

$\int sin(k*x)\,dx \to f[g(x)] = sin(x*k) \to g'(x) = dx * \mathbf{k}$
the k is artificially added!
$\frac{\mathbf{1}}{\mathbf{k}} \int sin(u)\, du = \frac{1}{k} * -cos(u) = \frac{-cos(x*k)}{k}$

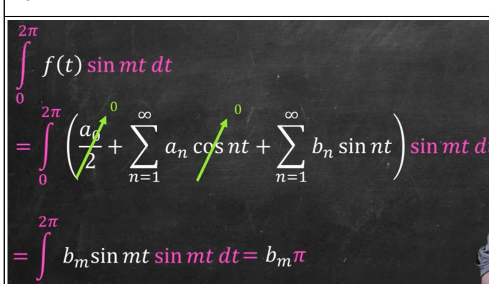
T-Periodic

$a_0 = \frac{1}{T} \int_0^T f(t)\, dt$

$a_m = \frac{2}{T} \int_0^T f(t) cos(\omega l t)\, dt$

$b_m = \frac{2}{T} \int_0^T f(t) sin(\omega l t)\, dt$

$\int_0^\infty (f(t) - S_N(t))^2\, dt \to^{N \to \infty} 0$



$$\int_0^{2\pi} f(t) \cos mt \, dt$$

$$= \int_0^{2\pi} \left(\cancelto{0}{\frac{a_0}{2}} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \cancelto{0}{\sin nt} \right) \cos mt \, dt$$

$$= \int_0^{2\pi} a_m \cos mt \cos mt \, dt = a_m \pi$$

$$\int_0^{2\pi} f(t) 1 dt$$

$$= \int_0^{2\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cancelto{0}{\cos nt} + \sum_{n=1}^{\infty} b_n \cancelto{0}{\sin nt} \right) 1 dt$$

$$= a_0 \pi$$