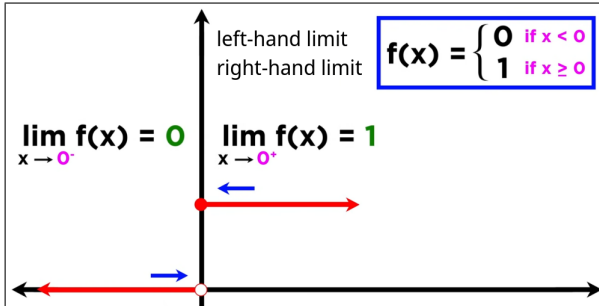


Derivation Rules $\frac{d}{dx}(x^a) = a * x^{a-1}$ given: $x, a \in \mathbb{R} \ \& \ x > 0$ subexamples: $\frac{d}{dx} x = 1 \rightarrow \frac{d}{dx} (x^1) = 1 * x^{1-1}$ $\frac{d}{dx} x^2 = 2x \rightarrow \frac{d}{dx} (x^2) = 2 * x^{2-1}$ $\frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2} \rightarrow \frac{d}{dx} (x^{-1}) = -1 * x^{-1-1}$ $\frac{d}{dx} \sqrt{x} = \frac{1}{2*\sqrt{x}} \rightarrow \frac{d}{dx} (x^{\frac{1}{2}}) = \frac{1}{2*x^{\frac{1}{2}}}$ $\frac{d}{dx} (c) = 0$ given: $c \in \mathbb{R} \ \& \ c$ is constant & $c \neq$ factor $\frac{d}{dx} (e^x) = e^x \rightarrow \frac{d}{dx} (e^x) = \ln(e) * e^x * x' = 1 * 1 * e^x$ $\frac{d}{dx} (a^x) = \ln(a) * a^x \rightarrow \frac{d}{dx} (a^x) = x' * \ln(x) * a^x$ because: $e^{x*\ln(a)} = a^x$ Note for this rule: $\frac{d}{dx} (2^{2x+1}) = \ln(2x+1) * 2^{2x+1} * (2x+1)' = \ln(2x+1) * 2^{2x+1} * 2$ $\frac{d}{dx} (\ln(x)) = \frac{1}{x}$ $\frac{d}{dx} \log_b(x) = \frac{1}{\ln(b)*x} \rightarrow$ special case for $\frac{d}{dx} (\frac{\ln(x)}{\ln(b)})$ this is the case because of base change in logarithmic functions! $\log_a(x) = \frac{\ln(x)}{\ln(a)} = \frac{\log_c(x)}{\log_c(a)} \rightarrow \frac{d}{dx} \log_a(x) = \frac{\ln(x)}{\ln(a)} - > \frac{d}{dx} \ln(x) = \frac{\frac{1}{x}}{\ln(a)} = \frac{1}{\ln(a) * x}$ c can be any number! $\frac{d}{dx} \sin(x) = \cos(x)$ $\frac{d}{dx} \cos(x) = -\sin(x)$ $\frac{d}{dx} \tan(x) = \frac{1}{\cos^2(x)}$ $\frac{d}{dx} \tan(x) = 1 + \tan^2(x)$ $\frac{d}{dx} (ax) = a \rightarrow \frac{d}{dx} (ax) = a * 1 \rightarrow$ we derive x NOT a !! $\frac{d}{dx} (3x) = a \rightarrow \frac{d}{dx} (3x) = 3 * 1 \rightarrow 3$ is a factor!	Implicit Differentiation $\frac{d}{dx} (x^2 + y^2 = 9) \rightarrow 2x + \frac{d}{dx} ((y)^2) * \frac{dy}{dx} (y) = 0 \rightarrow 2x + (2y * y') = 0 \rightarrow y' = \frac{-2x}{2y}$!! Remember that this is only necessary if y needs to be derived !! Higher Derivatives The best idea for higher derivatives is distance s , velocity v and acceleration a . $\frac{d}{dt} (s(t)) = v(t) = s'(t) \parallel \frac{d}{dt} (v(t)) = a(t) = s''(t) = v'(t)$ This is why the acceleration on earth -> gravity is constant!! HOLY FUCK Taking Derivations higher than 3 $1 : f' \rightarrow 2 : f'' \rightarrow 3 : f''' \rightarrow 4 : f^{(4)} \rightarrow n : f^{(n)}$ Related Rates In a Sphere, the rate of change of V is $100\text{cm}^3/\text{s}$ calculate the rate of change in $r = 25\text{cm}$ given rate of change in V $\frac{dV}{dt} = 100\text{cm}^3/\text{s}, r = 25\text{cm}$ $\frac{dV}{dt} = (4 * \pi * r^2) * \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{100\text{cm}^3/\text{s}}{4 * \pi * (25\text{cm})^2} = \frac{dr}{dt} = \frac{1}{25 * \pi} \text{cm/s}$ Local Maximum and Minimum The first derivative of local maximum and minimum MUST be 0! This includes turning points, aka slope is just 0! Local Minimum lMin: $f''(x) > 0 \rightarrow$ given $f(x) = 0$ OR $f'(lMin - 1) < 0 \ \&\& \ f'(lMin) = 0 \ \&\& \ f'(lMin + 1) \geq 0$ Essentially the minimum is where the slope goes from negative to positive with the turning point being the minimum with slope 0 Local Maximum lMax: $f''(x) < 0 \rightarrow$ given $f(x) = 0$ OR $f'(lMax - 1) \geq 0 \ \&\& \ f'(lMax) = 0 \ \&\& \ f'(lMax + 1) < 0$ Essentially the maximum is where the slope goes from positive to negative with the turning point being the maximum with slope 0 Absolute minimum and maximum will never be exceeded -> sine absolute-max = 1 Inflection Point This is the point where the function stops its increase or decrease in slope. Therefore it is the second derivative and is equal to 0 Use of Maxima Building a fence adjacent to a river. length $l = 2x + y$! Given length of 2400m how big do x and y need to be for the maximum area A ? $l = y + 2x \rightarrow y = 2400\text{m} - 2x \rightarrow A = (2400\text{m} - 2x) * x$ Remember when you had to use the UI function on the calculator? Yeah, no more! $Max \rightarrow f'(A) = f'((244\text{m} - 2x) * x = 0 \rightarrow x = \frac{2400\text{m}}{4} = 600\text{m} \rightarrow y = 1200\text{m}$ Limits: The limit expresses that a variable is approaching a value $\lim_{x \rightarrow \infty} x$ approaching infinity This is often used when trying to determine functions that might give an invalid result at x $f(x) = \frac{x-1}{x^2-1} \rightarrow f(1) = ??$ With limit we can say what we would expect the value to be, if the function would continue aka what is the value of $f(1)$ if the function would not show this abnormality? $\lim_{x \rightarrow 1} f(1) = 0.5$ This also applies to functions that go to infinity, or functions that are constant for a range. <table><tr><th>x</th><th>f(x)</th><th>x</th><th>f(x)</th></tr><tr><td>-1</td><td>0.8415</td><td>1</td><td>0.8415</td></tr><tr><td>-0.5</td><td>0.9589</td><td>0.5</td><td>0.9589</td></tr><tr><td>-0.1</td><td>0.9983</td><td>0.1</td><td>0.9983</td></tr><tr><td>-0.05</td><td>0.9996</td><td>0.05</td><td>0.9996</td></tr><tr><td>-0.01</td><td>0.99998</td><td>0.01</td><td>0.99998</td></tr></table>	x	f(x)	x	f(x)	-1	0.8415	1	0.8415	-0.5	0.9589	0.5	0.9589	-0.1	0.9983	0.1	0.9983	-0.05	0.9996	0.05	0.9996	-0.01	0.99998	0.01	0.99998
x	f(x)	x	f(x)																						
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-0.05	0.9996	0.05	0.9996																						
-0.01	0.99998	0.01	0.99998																						



Limit Rules
Addition:

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

Subtraction:

$$\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

Multiplication:

$$\lim_{x \rightarrow a} [f(x) * g(x)] = \lim_{x \rightarrow a} f(x) * \lim_{x \rightarrow a} g(x)$$

Division:

$$\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \rightarrow \text{given } \lim \neq 0$$

Multiplication by constant:

$$\lim_{x \rightarrow a} [c * f(x)] = c * \lim_{x \rightarrow a} f(x) \rightarrow \text{given } c \text{ is constant}$$

Exponent:

$$\lim_{x \rightarrow a} [f(x)]^2 = [\lim_{x \rightarrow a} f(x)]^2$$

Root:

$$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$$

x to a:

$$\lim_{x \rightarrow a} (x) = a$$

x to a with exponent:

$$\lim_{x \rightarrow a} (x^n) = a^n$$

x to a with root:

$$\lim_{x \rightarrow a} (\sqrt[n]{x}) = \sqrt[n]{a}$$

limit of a constant:

$$\lim_{x \rightarrow a} (c) = c \rightarrow \text{given } c \text{ is constant}$$

Examples:

$$\lim_{x \rightarrow -2} \left(\frac{x^3 + 2x^2 - 1}{5 - 3x} \right) = \frac{\lim_{x \rightarrow -2} (x^3 + 2x^2 - 1)}{\lim_{x \rightarrow -2} (5 - 3x)}$$

$$\frac{\lim_{x \rightarrow -2} (x^3) + \lim_{x \rightarrow -2} (2x^2) - \lim_{x \rightarrow -2} (1)}{\lim_{x \rightarrow -2} (5) - \lim_{x \rightarrow -2} (3x)} = \frac{-8 + 8 - 1}{5 + 6} = -\frac{1}{11}$$

Sometimes we need to eliminate terms in order to move on

limit and differentiation -> L'Hospital's Rule:

If either the left side -> $\frac{f(x)}{g(x)}$ is indeterminate

then we can use this rule! Otherwise it doesn't work, and doesn't make sense!

$$\lim_{x \rightarrow a} \left(\frac{f(x)}{g(x)} \right) = \lim_{x \rightarrow a} \left(\frac{f'(x)}{g'(x)} \right)$$

Examples:

$$\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) = \lim_{x \rightarrow 0} \left(\frac{\cos x}{1} \right) = \lim_{x \rightarrow 0} (\cos(x)) = 1$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

indeterminate !!!!

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^3} = \frac{\infty}{\infty}$$

indeterminate

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^3} = \lim_{x \rightarrow \infty} \frac{e^x}{3x^2}$$

we can take the derivatives **multiple times**

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^3} = \frac{\infty}{\infty}$$

indeterminate

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^3} = \lim_{x \rightarrow \infty} \frac{e^x}{3x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{6x} = \lim_{x \rightarrow \infty} \frac{e^x}{6} = \infty$$

this rule only applies to **indeterminate forms** (0/0 or ∞/∞)

$$\lim_{x \rightarrow 0} \frac{x + \sin x}{x + \cos x} = 0$$

Using the rule would have given us the wrong answer!

Reforming terms for L'Hospital
reforming a product

$$\lim_{x \rightarrow 0^+} x \cdot \ln x = \lim_{x \rightarrow 0^+} \frac{(\ln x)'}{(x^{-1})'} = \lim_{x \rightarrow 0^+} \frac{1/x}{-x^{-2}}$$

$$\lim_{x \rightarrow 0^+} \frac{1}{x} \cdot \frac{-x^2}{1} = \lim_{x \rightarrow 0^+} -x = 0$$

reforming an exponent with logarithm

$$\lim_{x \rightarrow 0^+} x^x = 0^0$$

$$y = x^x \rightarrow \ln y = \ln x^x \rightarrow \ln y = x \cdot \ln x$$

$$\lim_{x \rightarrow 0^+} e^{(x \cdot \ln x)} = e^0 = 1$$

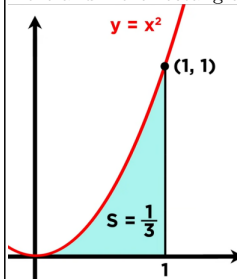
Infinity calculation rules

$\infty + c = \infty$	$\infty + \infty = \infty$	$\infty - \infty = NaN$
$\infty * c = \infty \rightarrow c \neq 0$	$\infty * \infty = \infty$	$\infty * 0 = NaN$
$\frac{c}{0} = \pm \infty \rightarrow c \neq 0$	$\frac{c}{\infty} = 0$	$\frac{\infty}{c} = \infty \rightarrow c \neq 0$
$\frac{\infty}{0} = \infty$	$\frac{0}{0} = NaN$	$\frac{\infty}{\infty} = NaN$
$0^c \rightarrow c > 0 \setminus (c = 1) = 0$	$0^0 = 1 \text{ or } NaN$	$\infty^0 = NaN$
$0^c \rightarrow c < 0 = \infty$	$k^\infty \rightarrow k > 1 = \infty$	$k^\infty \text{ to } 0 < k < 1 = 0$
$0^\infty = 0$	$\infty^\infty = \infty$	$1^\infty = NaN$

Integration

Similarity to limit

Just like limit, you can do it by intuition, by simply adding more and more rectangles into a function to get the area of said function.



let's find the area under the curve from zero to one

number of rectangles	area under the curve
4	0.46875
10	0.385
100	0.33835
1000	0.33383
∞	0.33333...

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \int_a^b f(x) dx$$

But just like with limit, there is a more elegant and generalized way. -> $\int_a^b x dx$

Definite Integrals and Integral Terms

$$\int_a^b f(x) dx = F(b) - F(a)$$

- The weird symbol is called integral sign
- The a and b are the upper and lower limits respectively
- f(x) is the integrand, the function to be integrated
- F(a) or F(b) is the antiderivative -> opposite calculation to derivation
- dx is the infinitesimal, no real use, but is required for notation

An integral with specific limits -> range is called a **Definite Integral**
Here the range is a to b

Indefinite Integrals

Since we can't put in values with infinite integrals, we instead just evaluate the antiderivative $F(x)$, which in itself is yet another function

$\int f(x) dx = F(x) + C \rightarrow$ look at that, the holy constant C

Note that the C always has to be written, as the integral function covers a range of values with $F(x)$ plus some constant! Hence $+ C$!

hence we can also go back again \rightarrow reversibility of integrals and derivations
In other words, we differentiate the antiderivative!

$F'(x) = f(x) \rightarrow [F(x) + C]' = f(x) \rightarrow C$ vanishes \rightarrow constant!

One might ask now, why do we not consider it with definite integrals?
Check how the C would affect a b - a :

$$\int_a^b x^2 dx = \left(\frac{b^3}{3} + C \right) - \left(\frac{a^3}{3} + C \right) = \frac{b^3}{3} - \frac{a^3}{3} \rightarrow C - C = 0$$

as one can see, the C simply gets canceled.

Integral Rules

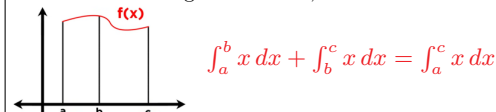
The Integral of a to b is the same as the negative integral of b to a

$$\int_b^a x dx = - \int_a^b x dx$$

The Integral of a to a is $0 \rightarrow$ as the area would be 0 . $a - a = 0$

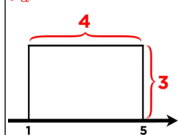
$$\int_a^a x dx = 0$$

Since we are talking about areas, 2 areas in the same function add up:



If we integrate a constant, then the constant will multiply with $x = 1$:

$$\int_a^b c dx = c * (b - a) \rightarrow \int_1^5 3 dx = 3 * (5 - 1) = 12 \rightarrow \text{given } c \text{ is constant}$$



$$\int c * f(x) dx = c * \int f(x) dx \rightarrow \text{given } c \text{ is constant}$$

multiplying the integrand with a function can be done outside of the integral!

$$\int c * f(x) dx = c * \int f(x) dx$$

Sum of Integrals

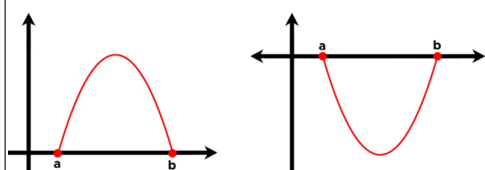
$$\int [f(x) + g(x)] = \int f(x) + \int g(x)$$

Difference of Integrals

$$\int [f(x) - g(x)] = \int f(x) - \int g(x)$$

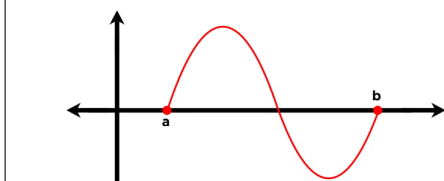
Integrals with $y \geq 0$ and $y < 0$

$$\int_a^b f(x) dx > 0 \quad \int_a^b g(x) dx < 0$$



Area of integrals with y below and above 0 at some point

$$\int_a^b f(x) dx = \text{area above axis} - \text{area below axis}$$



often used:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

And lastly the most important function!

$$\frac{d}{dx} \int f(x) dx = f(x)$$

Example for Integral calculation

$$\int_0^3 (x + 5) dx = F(3) - F(0) \rightarrow F(x) = \left(\frac{x^2}{2} + 5x \right)$$

$$\rightarrow F(3) - F(0) = \left(\frac{3^2}{2} + 5 * 3 \right) - \left(\frac{0^2}{2} + 5 * 0 \right) = \frac{39}{2}$$

Integrals do not have the product rule

This means that we need to find a different way to remove factors
In fact, Integrals can only be taken over sums and differences

$$\int \sqrt{x} (x - 2) dx \quad \int (x^{3/2} - 2x^{1/2}) dx$$

we must manipulate this a little bit first **now it's easy to find the antiderivative**

