Derivation Rules

$\frac{d}{dx}(x^a) = a * x^{a-1}$ given: $x, a \in \mathbb{R}$ & $x > 0$

subexamples:

$\frac{d}{dx}x = 1 \rightarrow \frac{d}{dx}(x^1) = 1 * x^{1-1}$

$\frac{d}{dx}x^2 = 2x \rightarrow \frac{d}{dx}(x^2) = 2 * x^{2-1}$

$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2} \rightarrow \frac{d}{dx}(x^{-1} = -1 * x^{-1-1})$

$\frac{d}{dx}\sqrt{x} = \frac{1}{2*\sqrt{x}} \rightarrow \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2*x^{\frac{1}{2}}}$

$\frac{d}{dx}(c) = 0$ given: $c \in \mathbb{R}$ & c is constant & c != factor

$\frac{d}{dx}(e^x) = e^x \rightarrow \frac{d}{dx}(e^x) = ln(e) * e^x * x' = 1 * 1 * e^x$

$\frac{d}{dx}(a^x) = ln(a) * a^x \rightarrow \frac{d}{dx}(a^x) = x' * ln(x) * a^x$ because: $e^{x*ln(a)} = a^x$

Note for this rule:

$\frac{d}{dx}(2^{2x+1}) = ln(2x+1) * 2^{2x+1} * (2x+1)' = ln(2x+1) * 2^{2x+1} * 2$

$\frac{d}{dx}(ln(x)) = \frac{1}{x}$

$\frac{d}{dx}log_b(x) = \frac{1}{ln(b)*x} \rightarrow$ special case for $\frac{d}{dx}(\frac{ln(x)}{ln(b)})$

this is the case because of base change in lograrithmic functions!

$log_a(x) = \frac{ln(x)}{ln(a)} = \frac{log_c(x)}{log_c(a)} \rightarrow \frac{d}{dx}log_a(x) = \frac{ln(x)}{ln(a)} -> \frac{d}{dx}ln(x) = \frac{\frac{1}{x}}{ln(a)} = \frac{1}{ln(a) * x}$

c can be any number!

$\frac{d}{dx}sin(x) = cos(x)$

$\frac{d}{dx}cos(x) = -sin(x)$

$\frac{d}{dx}tan(x) = \frac{1}{cos^2(x)}$

$\frac{d}{dx}tan(x) = 1 + tan^2(x)$

$\frac{d}{dx}(ax) = a \rightarrow \frac{d}{dx}(ax) = a * 1 \rightarrow$ we derive x NOT a!!

$\frac{d}{dx}(3x) = a \rightarrow \frac{d}{dx}(3x) = 3 * 1 \rightarrow$ 3 is a factor!

All of these derive from:

$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Sum Rule

$(f + g)' = f' + g'$

Difference Rule

$(f - g)' = f' - g'$

Product Rule

$(f * g)' = f * g' + f' * g$

Quotient Rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f' * g - f * g'}{g^2}$$

Chain Rule

$[f(g(x))]' = f'(g(x)) * g'(x)$

Example:

$\frac{d}{dx}\sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}} = \frac{1}{2} * (x^2+1)^{-\frac{1}{2}} * 2x$

$\frac{1}{2} * \frac{1}{\sqrt{x^2+1}} * 2x = \frac{2x}{2*\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$

Note: $(x^2 + 1)$ is g(x), while f is the exponent function

More examples:

$\frac{d}{dx}sin(x^2) = sin(x^2)' * (x^2)' = cos(x^2) * 2x$

$\frac{d}{dx}sin^2(x) = \frac{d}{dx}(sin(x))^2 = 2 * sin(x) * cos(x)$

$$\frac{d}{dx}\left(\frac{x-1}{x+1}\right)^2 = 2 * \left(\frac{x-1}{x+1}\right) * \left(\frac{x-1}{x+1}\right)'$$

$\frac{d}{dx}(x+2)^3(x)^4 = (x+2)^3 * ((x)^4)' + ((x+2)^3)' * (x)^4$

$((x+2)^3)' = 3 * (x+2)^2 * (x+2)' = 3 * (x+2)^2 * 1$

$$\frac{d}{dx}sin(cos[tan(x)]) = cos(cos[tan(x)]) * -sin(tan(x)) * \frac{1}{cos^2(x)}$$

Implicit Differentiation

$\frac{d}{dx}(x^2 + y^2 = 9) \rightarrow 2x + \frac{d}{dx}((y)^2) * \frac{dy}{dx}(y) = 0 \rightarrow 2x + (2y * y') = 0 \rightarrow y' = \frac{-2x}{2y}$

!! Remember that this is only necessary if y needs to be derived !!

Higher Derivatives

The best idea for higher derivatives is distance s, velocity v and acceleration a.

$\frac{d}{dt}(s(t)) = v(t) = s'(t) \mid\mid \frac{d}{dt}(v(t)) = a(t) = s''(t) = v'(t)$

This is why the acceleration on earth -> gravity is constant!! HOLY FUCK

Taking Derivations higher than 3

$1 : f' \rightarrow 2 : f'' \rightarrow 3 : f''' \rightarrow 4 : f^{(4)} \rightarrow n : f^{(n)}$

Related Rates

In a Sphere, the rate of change of V is $100cm^3/s$

calculate the rate of change in r = 25cm given rate of change in V

$\frac{dV}{dt} = 100cm^3/s$, $r = 25cm$

$$\frac{dV}{dt} = (4 * \pi * r^2) * \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{100cm^2/s}{4 * \pi * (25cm)^2} = \frac{dr}{dt} = \frac{1}{25 * \pi}cm/s$$

Local Maximum and Minimum

The first derivative of local maximum and minimum MUST be 0!

This includes turning points, aka slope is just 0!

Local Minimum lMin:

$f''(x) > 0 \rightarrow$ given f(x) = 0

OR

$f'(lMin - 1) < 0$ && $f'(lMin) = 0$ && $f'(lMin + 1) \geq 0$

Essentially the minimum is where the slope goes from negative to positive with the turning point being the minimum with slope 0

Local Maximum lMax:

$f''(x) < 0 \rightarrow$ given f(x) = 0

OR

$f'(lMax - 1) \geq 0$ && $f'(lMax) = 0$ && $f'(lMax + 1) < 0$

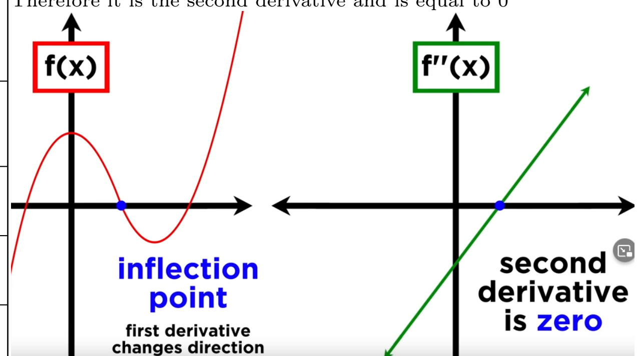
Essentially the maximum is where the slope goes from positive to negative with the turning point being the maximum with slope 0

Absolute minimum and maximum will never be exceeded -> sine absolute-max = 1

Inflection Point

This is the point where the function stops its increase or decrease in slope.

Therefore it is the second derivative and is equal to 0



Use of Maxima

Building a fence adjacent to a river. length l = 2x + y!

Given length of 2400m how big do x and y need to be for the maximum area A?

$l = y + 2x \rightarrow y = 2400m - 2x \rightarrow A = (2400m - 2x) * x$

Remember when you had to use the UI function on the calculator? Yeah, no more!

$$Max \rightarrow f'(A) = f'((244m - 2x) * x = 0 \rightarrow x = \frac{2400m}{4} = 600m \rightarrow y = 1200m$$

Limits:

The limit expresses that a variable is approaching a value

$\lim_{x \to \infty}$ x approaching infinity

This is often used when trying to determine functions that might give an invalid result at x

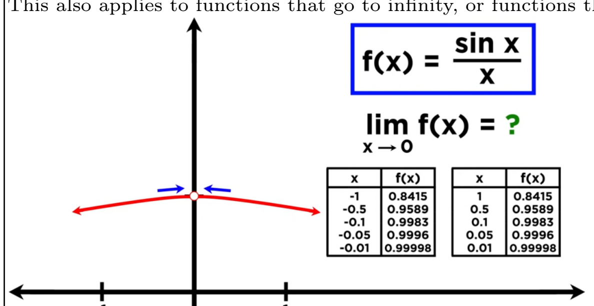
$f(x) = \frac{x-1}{x^2-1} \rightarrow f(1) = ??$

With limit we can say what we would expect the value to be, if the function would continu

aka what is the value of f(1) if the function would not show this abnormality?

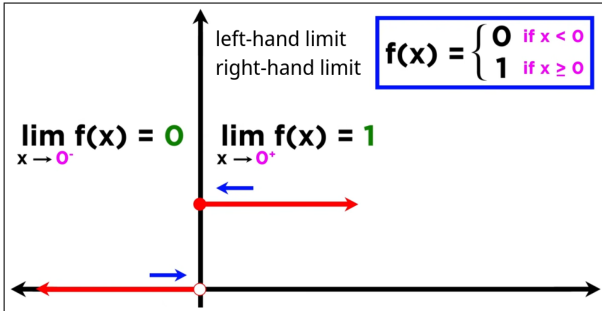
$\lim_{x \to 1} f(1) = 0.5$

This also applies to functions that go to infinity, or functions that are constant for a range.



x	f(x)
-1	0.8415
-0.5	0.9589
-0.1	0.9983
-0.05	0.9996
-0.01	0.99998

x	f(x)
1	0.8415
0.5	0.9589
0.1	0.9983
0.05	0.9996
0.01	0.99998



Limit Rules
Addition:

$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

Subtraction:

$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

Multiplication:

$\lim_{x\to a}[f(x)*g(x)] = \lim_{x\to a} f(x) * \lim_{x\to a} g(x)$

Division:

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \to \text{ given lim != 0}$$

Multiplication by constant:

$\lim_{x\to a}[c*f(x)] = c*\lim_{x\to a} f(x) \to$ given c is constant

Exponent:

$\lim_{x\to a}[f(x)]^2 = [\lim_{x\to a} f(x)]^2$

Root:

$\lim_{x\to a} \sqrt[n]{[f(x)]} = \sqrt[n]{\lim_{x\to a} f(x)}$

x to a:

$\lim_{x\to a}(x) = a$

x to a with exponent:

$\lim_{x\to a}(x^n) = a^n$

x to a with root:

$\lim_{x\to a}(\sqrt[n]{x}) = \sqrt[n]{a}$

limit of a constant:

$\lim_{x\to a}(c) = c \to$ given c is constant

Examples:

$$\lim_{x\to -2}\left(\frac{x^3+2x^2-1}{5-3x}\right) = \frac{\lim_{x\to -2}(x^3+2x^2-1)}{\lim_{x\to -2}(5-3x)}$$

$$\frac{\lim_{x\to -2}(x^3) + \lim_{x\to -2}(2x^2) - \lim_{x\to -2}(1)}{\lim_{x\to -2}(5) - \lim_{x\to -2}(3x)} = \frac{-8+8-1}{5+6} = -\frac{1}{11}$$

Sometimes we need to eliminate terms in order to move on

limit and differntiation -> L'Hospital's Rule:

If either the left side -> $\frac{f(x)}{g(x)}$ is indeterminate
then we can use this rule! Otherwise it doesn't work, and doesn't make sense!

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right)$$

Examples:

$$\lim_{x\to 0}\left(\frac{sinx}{x}\right) = \lim_{x\to 0}\left(\frac{cosx}{1}\right) = \lim_{x\to 0}(cos(x)) = 1$$

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$
indeterminate !!!!

$$\lim_{x\to\infty}\frac{e^x}{x^3} = \frac{\infty}{\infty}$$
indeterminate

$$\lim_{x\to\infty}\frac{e^x}{x^3} = \lim_{x\to\infty}\frac{e^x}{3x^2}$$

we can take the derivatives multiple times

$$\lim_{x\to\infty}\frac{e^x}{x^3} = \frac{\infty}{\infty}$$
indeterminate

$$\lim_{x\to\infty}\frac{e^x}{x^3} = \lim_{x\to\infty}\frac{e^x}{3x^2} = \lim_{x\to\infty}\frac{e^x}{6x} = \lim_{x\to\infty}\frac{e^x}{6} = \infty$$

this rule only applies to indeterminate forms (0/0 or ∞/∞)

$$\lim_{x\to 0}\frac{x+\sin x}{x+\cos x} = 0$$

Using the rule would have given us the wrong answer!

Reforming terms for L'Hospital
reforming a product

$$\lim_{x\to 0^+} x\cdot\ln x = \lim_{x\to 0^+}\frac{(\ln x)'}{(x^{-1})'} = \lim_{x\to 0^+}\frac{1/x}{-x^{-2}}$$

$$\lim_{x\to 0^+}\frac{1}{x}\cdot\frac{-x^2}{1} = \lim_{x\to 0^+} -x = 0$$

reforming an exponent with logarithm

$$\lim_{x\to 0^+} x^x = 0^0$$

$$y = x^x \to \ln y = \ln x^x \to \ln y = x\cdot\ln x$$

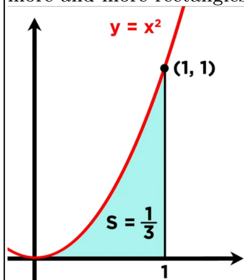
$$\lim_{x\to 0^+} e^{(x\cdot\ln x)} = e^0 = 1$$

Infinity calculation rules

$\infty + c = \infty$	$\infty + \infty = \infty$	$\infty - \infty = NaN$
$\infty * c = \infty \to c \neq 0$	$\infty * \infty = \infty$	$\infty * 0 = NaN$
$\frac{c}{0} = \pm\infty \to c \neq 0$	$\frac{c}{\infty} = 0$	$\frac{\infty}{c} = \infty \to c \neq 0$
$\frac{\infty}{0} = \infty$	$\frac{0}{0} = NaN$	$\frac{\infty}{\infty} = NaN$
$0^c \to c > 0 \backslash (c = 1) = 0$	$0^0 = 1$ or NaN	$\infty^0 = NaN$
$0^c \to c < 0 = \infty$	$k^\infty \to k > 1 = \infty$	$k^\infty\ to 0 < k < 1 = 0$
$0^\infty = 0$	$\infty^\infty = \infty$	$1^\infty = NaN$

Integration

Similarity to limit
Just like limit, you can do it by intuition, by simply adding more and more rectangles into a function to get the area of said function.



let's find the area under the curve from zero to one

number of rectangles	area under the curve
4	0.46875
10	0.385
100	0.33835
1000	0.33383
∞	0.33333...

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

But just like with limit, there is a more elegant and generalized way. -> $\int_a^b x\,dx$

Definite Integrals and Integral Terms

$\int_a^b f(x)\,dx = F(b) - F(a)$

- The weird symbol is called integral sign
- The a and b are the upper and lower limits respectively
- f(x) is the integrand, the function to be integrated
- F(a) or F(b) is the antiderivative -> opposite calculation to derivation
- dx is the infinitesimal, no real use, but is required for notation

An integral with specific limits -> range is called a Definite Integral
Here the range is a to b

Indefinite Integrals

Since we can't put in values with infinite integrals, we instead just evaluate the antiderivative F(x), which in itself is yet another function

$\int f(x)\,dx = F(x) + C \rightarrow$ look at that, the holy constant C

Note that the C always has to be written, as the integral function covers a range of values with F(x) plus some constant! Hence + C!

hence we can also go back again -> reversibility of integrals and derivations
In other words, we differentiate the antiderivative!

$F'(x) = f(x) \rightarrow [F(x) + C]' = f(x) \rightarrow$ C vanishes -> constant!

One might ask now, why do we not consider it with definite integrals?
Check how the C would affect a - b:

$\int_a^b x^2\,dx = \left(\frac{b^3}{3} + C\right) - \left(\frac{a^3}{3} + C\right) = \frac{b^3}{3} - \frac{a^3}{3} \rightarrow C - C = 0$

as one can see, the C simply gets canceled.

Integral Rules

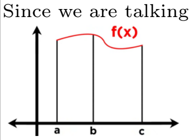
The Integral of a to b is the same as the negative integral of b to a

$\int_b^a x\,dx = -\int_a^b x\,dx$

The Integral of a to a is 0 -> as the area would be 0. a - a = 0

$\int_a^a x\,dx = 0$

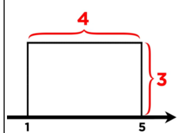
Since we are talking about areas, 2 areas in the same function add up:



$\int_a^b x\,dx + \int_b^c x\,dx = \int_a^c x\,dx$

If we integrate a constant, then the constant will multiple with x = 1:

$\int_a^b c\,dx = c * (b - a) \rightarrow \int_1^5 3\,dx = 3 * (5 - 1) = 12 \rightarrow$ given c is constant



$\int c * f(x)\,dx = c * \int f(x)\,dx \rightarrow$ given c is constant

multiplying the integrand with a function can be done outside of the integral!

$\int c * f(x)\,dx = c * \int f(x)\,dx$

Sum of Integrals

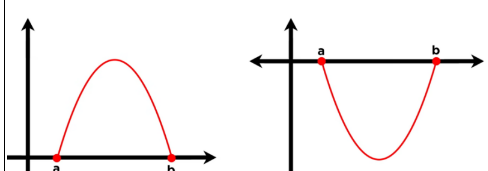
$\int [f(x) + g(x)] = \int f(x) + \int g(x)$

Difference of Integrals

$\int [f(x) - g(x)] = \int f(x) - \int g(x)$

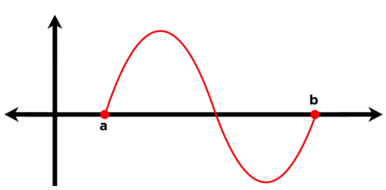
Integrals with y >= 0 and y < 0

$\int_a^b f(x)dx > 0$ $\qquad$ $\int_a^b g(x)dx < 0$



Area of integrals with y below and above 0 at some point

$\int_a^b f(x)dx$ = area above axis - area below axis



often used:

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

And lastly the most important function!

$\frac{d}{dx} \int f(x)\,dx = f(x)$

Example for Integral calculation

$\int_0^3 (x + 5)\,dx = F(3) - F(0) \rightarrow F(x) = (\frac{x^2}{2} + 5x)$

$\rightarrow F(3) - F(0) = \left(\frac{3^2}{2} + 5 * 3\right) - \left(\frac{0^2}{2} + 5 * 0\right) = \frac{39}{2}$

Integrals do not have the product rule

This means that we need to find a different way to remove factors
In fact, Integrals can only be taken over sums and differences

$\int \sqrt{x}\,(x - 2)dx$ $\qquad$ $\int (x^{3/2} - 2x^{1/2})dx$

we must manipulate this a little bit first $\qquad$ **now it's easy to find the antiderivative**

Substitution Rule

This turns complicated nested integrands into smaller pieces
This is the correspondent technique to the chain rule!

$$\int f[g(x)] * g'(x)\,dx = \int f(u)du$$

$$u = g(x) \mid\mid du = g'(x)dx$$

$$u = x^2 + 1$$

- $\frac{d}{dx}(x^2 + 1) = 2x$
- $d(x^2 + 1) = 2x\,dx$
- $du = 2x\,dx$

Example:

$\int 2x cos(x^2 + 1)\,dx \rightarrow 2x = g'(x) \rightarrow cos(x^2 + 1) = f[g(x)] \rightarrow (x^2 + 1) = g(x)$

$(x^2 + 1) = u \rightarrow (x^2 + 1)' = 2x$ this means we can use subtitution!

$\int cos(u)2x\,dx = \int cos(u)\,du = sin(u) + C = sin(x^2 + 1) + C$

Second example with factors

$\int x^2\sqrt{x^3 + 1}\,dx \rightarrow x^2 = g'(x) \rightarrow f(g(x)) = \sqrt{x^3 + 1} \rightarrow g(x) = (x^3 + 1)$

$(x^3 + 1)' = \frac{1}{3}x^2$ note 1/3 is a factor, it can be removed from the term.

$\frac{1}{3} * \int \sqrt{u} * x^2\,dx = \frac{1}{3} * \int u^{\frac{1}{2}}\,du = \frac{1}{3} * -\frac{x^{\frac{3}{2}}}{\frac{3}{2}}$

$\frac{1}{3} * \frac{2}{3} * u^{\frac{3}{2}} + C = \frac{2}{9} * (x^3 + 1)^{\frac{3}{2}} + C$

Integration by Parts

This turns complicated nested integrands into smaller pieces
This is the correspondent technique to the product rule!

$$\int [f(x) * g'(x) + f'(x) * g(x)]\,dx = f(x) * g(x)$$

$$\int [f(x) * g'(x)]\,dx + \int [f'(x) * g(x)]\,dx = f(x) * g(x)$$

$$\int [f(x) * g'(x)]\,dx = f(x) * g(x) - \int [f'(x) * g(x)]\,dx$$

Simply the opposite to the Product rule!!
Unlike the substitution method, this works for EVERY product!

$$\int u\,dv = uv - \int v\,du$$

The same as before, just with a simplified view.
Please note: this technique doesn't always simplify the term, sometimes a different method is better!

$\int x \underline{\sin x}\,dx$ $\qquad$ choosing sin(x) as f(x) would not yield a good result, just another trig function....

one is f(x)
one is g'(x)

$\boxed{\begin{array}{ll} f(x) = x \longrightarrow & \text{the one that becomes much} \\ f'(x) = 1 & \text{simpler upon differentiation} \end{array}}$

$\int x \sin x\,dx = x(-\cos x) - \int(-\cos x)(1)dx$

$u = x$ $\qquad$ $v = -\cos x$
$du = dx$ $\qquad$ $dv = \sin x\,dx$

$\boxed{\int u\,dv = uv - \int v\,du}$

$$\int (\ln x)^2\, dx$$

(ln x)^2 * 1 * dx !!!!!!

better:
dv = 1 * dx
this equal,
but the one
helps with
understanding
this method.

$u = (\ln x)^2$

$du = (2 \ln x / x) dx$

$v = x$

$dv = dx$

$$\int (\ln x)^2\, dx = x(\ln x)^2 - 2\underbrace{\left[\int \ln x\, dx\right]}$$

$$\int \ln x\, dx = x \ln x - \int dx$$

$u = \ln x,\ du = dx/x$
$v = x,\ dv = dx$

$$\int (\ln x)^2\, dx = x(\ln x)^2 - 2x\ln x + 2x + C$$

Example Integration by Parts:

$$\int \frac{ln(x)}{x^2}\, dx \rightarrow ln(x) = f(x) \rightarrow x^2 = g'(x)$$

$$\int \frac{ln(x)}{x^2}\, dx = -\left(\frac{ln(x)}{x}\right) - \int \frac{1}{x} * -\frac{1}{x}$$

$$\int \frac{ln(x)}{x^2}\, dx = -\left(\frac{ln(x)}{x}\right) + \int \frac{1}{x^2}$$

$$\int \frac{ln(x)}{x^2}\, dx = -\left(\frac{ln(x)}{x}\right) - \frac{1}{x} + C$$

Integration by Trigonometric Substitution

This can be done with the following 3 situations

$$\sqrt{a^2 - x^2}\ ||\ \sqrt{a^2 + x^2}\ ||\ \sqrt{x^2 - a^2}$$

Where a can be any positive number

$\sqrt{a^2 - x^2}$	$\sqrt{a^2 + x^2}$	$\sqrt{x^2 - a^2}$
$x = a \sin\theta$	$x = a\tan\theta$	$x = a \sec\theta$
$\sqrt{a^2(1 - \sin^2\theta)}$	$\sqrt{a^2(1 + \tan^2\theta)}$	$\sqrt{a^2(\sec^2\theta - 1)}$
$\sqrt{a^2\cos^2\theta}$	$\sqrt{a^2\sec^2\theta}$	$\sqrt{a^2\tan^2\theta}$
$\boxed{a\cos\theta}$	$\boxed{a\sec\theta}$	$\boxed{a\tan\theta}$

$1 - \sin^2\theta = \cos^2\theta$

$1 + \tan^2\theta = \sec^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$