Before giving the lemma, we list the definitions of the symbols to be used as follows.

TABLE I: Summary of notation conventions used in the article

δ_X	Dirac measure.
$\Gamma(x)$	The gamma function $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$.
t_v distribution	Student's t distribution with the probability density function $f(t)=\frac{\Gamma((v+1)/2)}{\sqrt{\pi v}\Gamma(v/2)}(1+t^2/v)^{-(v+1)/2}$.

A. An Auxiliary Lemma

Definition 1. *A d-dimensional random vector X is said to follow a spherical distribution if OX and X have the same probability distribution for all $d\times d$ orthogonal matrix O.*

There are many typical examples of spherical distributions, such as the zero-mean normal distribution, the multivariate t-distribution, and so on [1, p.33].

Let $Y\in\mathbb{R}^d$ be a fixed nonzero vector, $X_1,\ldots,X_n$ be random vectors independet chosen with a spherical distribution on $\mathbb{R}^d$ and $P(X_1=X_2)=0$. Let $\psi_{ij}\in[0,\pi]$ denote the angle between X_i-X_j and Y for all $1\le i<j\le n$. Consider the normalized empirical distribution

$$\mu_{n,d}=\frac{1}{\binom{2}{n}}\sum_{1\le i<j\le n}\delta_{\sqrt{d}(\frac{\pi}{2}-\psi_{ij})}.$$

under the case where both n and d grow.

Lemma 1. *Assume $d=d_n$ satisfying $\lim_{n\to\infty}d_n=\infty$, Then, with probability one, $\mu_{n,d}$ converges weakly to $N(0,1)$ as $n\to\infty$.*

Proof. Firstly, denote

$$r_{ij}=\cos\psi_{ij}=\frac{Y\cdot(X_i-X_j)}{\|Y\|\cdot\|X_i-X_j\|},\quad 1\le i<j\le n.$$

Note that X_i-X_j also follows a spherical distribution, by [1, Theorem 1.5.7 (i)],

$$\frac{(d-1)^{1/2}r_{ij}}{(1-r_{ij}^2)^{1/2}} \tag{1}$$

has the t_{d-1} distribution. Then, by [1, p.147 (5)], it follows that the density function of r_{ij} is

$$\frac{\Gamma(d/2)}{\pi^{1/2}\Gamma((d-1)/2)}(1-r^2)^{(d-3)/2},\quad r\in(-1,1). \tag{2}$$

Similar to the proof of [2, Lemma 4.1] and [3, Lemma 12], we obtain that $\{r_{ij}, 1 \le i < j \le n\}$ are pairwise independent. Moreover, by (2) and some direct computations, $\{\psi_{ij} = \cos^{-1} r_{ij}, 1 \le i < j \le n\}$ are pairwise independent with the density function

$$h(\psi) = \frac{\Gamma(d/2)}{\pi^{1/2}\Gamma((d-1)/2)}(\sin\psi)^{d-2}, \quad \psi \in [0, \pi]. \tag{3}$$

Next, borrow the idea from the proof of [3, Lemma 16], since $\{\psi_{ij}, 1 \le i < j \le n\}$ are pairwise independence with common distribution, we can show that for any bounded and continuous function $u(x)$ defined on $\mathbb{R}$,

$$\frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} [u(\varphi_n(\psi_{ij})) - Eu(\varphi_n(\psi_{ij}))] \to 0 \tag{4}$$

with probability one, where $\varphi_n(\psi) = \sqrt{d_n}(\frac{\pi}{2} - \psi)$. Then, if we replace Θ_{ij} with ψ_{ij} in [3, Lemma 16 (ii)], the conclusion still holds. This means that we only need to show that $\varphi_n(\psi_{12})$ converges weakly to $N(0,1)$, and then we can get that $\mu_{n,d}$ converges weakly to $N(0,1)$ by [3, Lemma 16 (ii)].

Finally, by taking account of (3) and [3, equations (41)–(43)], it is straightforward to obtain that $\varphi_n(\psi_{12})$ converges weakly to $N(0,1)$, thus completes the proof. ■

References

[1] R. J. Muirhead, *Aspects of Multivariate Statistical Theory*. Wiley, New York, 1982.

[2] T. T. Cai and T. Jiang, "Phase transition in limiting distributions of coherence of high-dimensional random matrices," *Journal of Multivariate Analysis*, vol. 107, pp. 24–39, 2012.

[3] T. T. Cai, J. Fan, and T. Jiang, "Distributions of angles in random packing on spheres," *Journal of Machine Learning Research*, vol. 14, p. 1837, 2013.