Simulation Methods in Medical Engineering, HL2008

Finite Element Method

Your task for this laboratory is to use your knowledge from the lectures in Finite Element Method together with MATLAB to determine the deflection of a fishing rod when a fish is hooked (figure 1). Present your results with a plot of the undeformed and deformed fishing rod plus an addition table of the deflections at the nodes. You can simplify the rod with a beam. The fishing rod is 2 m with a radius of 5 cm. The cross-section of the fishing rod is circular with inertia (I) of $\frac{\pi r^4}{4}$. The Young's modulus (E) is assumed to be 100 MPa.

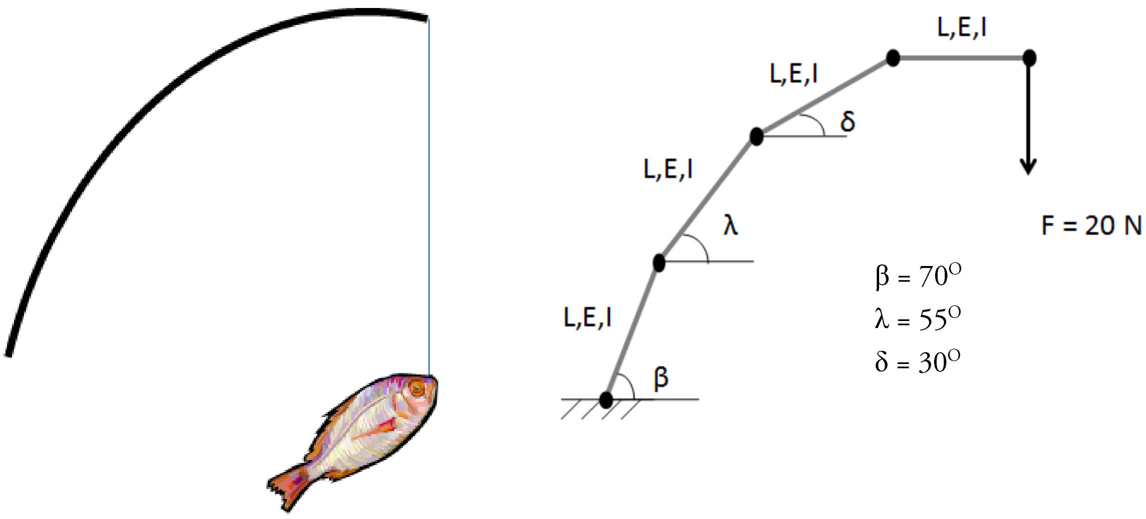

Figure 1. A fishing rod and a model of a fishing rod divided into four elements

Some hints on how to solve the problem is presented in Appendix 1, where the finite element equation for one beam element is presented. The problem can also be solved by using the finite element (FE) programs like COMSOL Multiphysics. The solution to the problem with COMSOL Multiphysics is presented in Appendix 2. When solving the problem, it can also be good to use the command *blkdiag* in MATLAB.

Appendix 1 – Finite Element Equation for one beam element

An example with one beam element will be given as an introduction. The equations are described below. Since the beam is not in the direction of the global coordinate system, a transformation from the local to the global coordinate system is needed. The numbering of the element is presented in figure A1, where the upper-case letters are associated with the global coordinate system and lower-case letters with the local coordinate system.

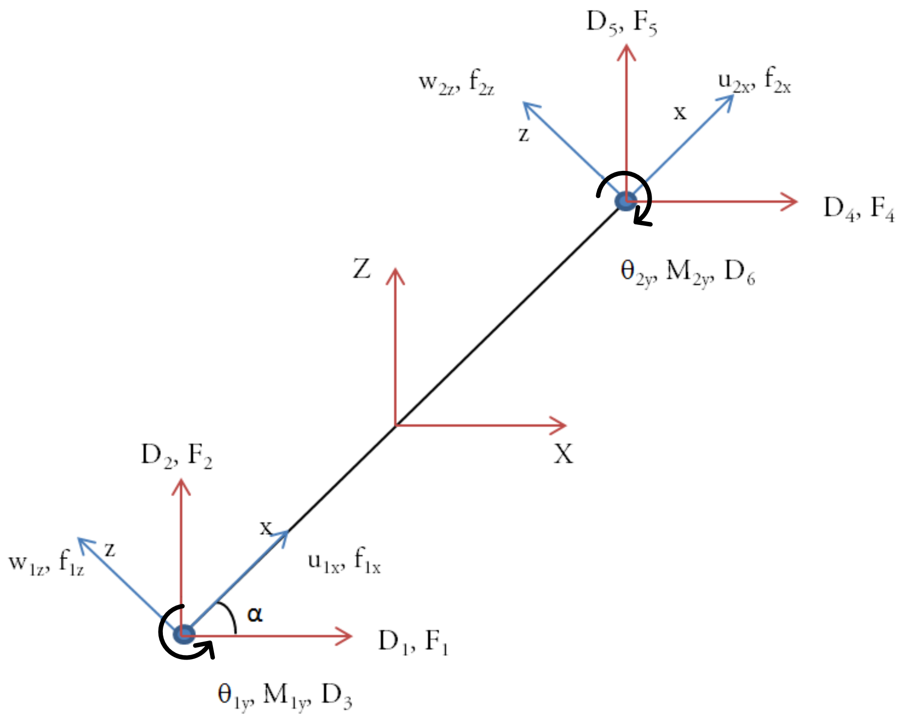

Figure A1. The numbering of the beam and variable for one beam element.

The force equilibrium is

$$f_{1x} = \frac{EA}{L}u_{1x} - \frac{EA}{L}u_{2x} \tag{Eq. 1}$$

$$f_{1z} = \frac{12EI}{L^3}w_{1z} + \frac{6EI}{L^2}\theta_{1y} - \frac{12EI}{L^3}w_{2z} + \frac{6EI}{L^2}\theta_{2y} \tag{Eq. 2}$$

$$M_{1y} = \frac{6EI}{L^3}w_{1z} + \frac{4EI}{L}\theta_{1y} - \frac{6EI}{L^2}w_{2z} + \frac{2EI}{L}\theta_{2y} \tag{Eq. 3}$$

$$f_{2x} = -\frac{EA}{L}u_{1x} + \frac{EA}{L}u_{2x} \tag{Eq. 4}$$

$$f_{2z} = -\frac{12EI}{L^3}w_{1z} - \frac{6EI}{L^2}\theta_{1y} + \frac{12EI}{L^3}w_{2z} - \frac{6EI}{L^2}\theta_{2y} \tag{Eq. 5}$$

$M_{2y} = \frac{6EI}{L^3} w_{1z} + \frac{2EI}{L} \theta_{1y} - \frac{6EI}{L^2} w_{2z} + \frac{4EI}{L} \theta_{2y}$ (Eq. 6)

The force equilibrium (Eq. (1)-Eq. (6)) looks like following on matrix form (the local form)

$$\underbrace{\begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}}_{\mathbf{k_e}} \underbrace{\begin{bmatrix} u_{1x} \\ w_{1z} \\ \theta_{1y} \\ u_{2x} \\ w_{2z} \\ \theta_{2y} \end{bmatrix}}_{\mathbf{d_e}} = \underbrace{\begin{bmatrix} f_{1x} \\ f_{1z} \\ M_{1y} \\ f_{2x} \\ f_{2z} \\ M_{2y} \end{bmatrix}}_{\mathbf{f_e}}$$

(Eq. 7)

$\mathbf{k_e}$ is the stiffness matrix, $\mathbf{d_e}$ deformation vector on local form and $\mathbf{f_e}$ force vector on local form. All three are in the local coordinate system. E is the Young's modulus, A area, I inertia and L length of the element.

Transformation (going from the local coordinate (x,z) system to the global (X,Z))

The relationship between the local and global deformation is defined according to figure A1

$u_{1x} = D_1 \cos\alpha + D_2 \cos\beta = \left\{\cos\beta = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha\right\} = D_1 \cos\alpha + D_2 \sin\alpha$ (Eq. 8)

$w_{1z} = -D_1 \sin\alpha + D_2 \cos\alpha$ (Eq. 9)

$\theta_{1y} = D_3$ (Eq. 10)

Eq. (8) – (10) is also true for node 2 and these equations defines the transformation matrix **T**. $\mathbf{D}_e$ is the deformation in the global coordinate system

$$\underbrace{\begin{bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{bmatrix}}_{\mathbf{d_e}} = \underbrace{\begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{bmatrix}}_{\mathbf{D_e}}$$

(Eq. 11)

The same is for the force

$\mathbf{F}_e = \mathbf{T}^T\,\mathbf{f}_e$ (Eq. 12)

Insertion of Eq. (11) in Eq. (7) gives

$\mathbf{f}_e = \mathbf{k}_e\,\mathbf{T}\mathbf{D}_e$ (Eq. 13)

Eq. (12) and (13) gives the system in global coordinates, where $\mathbf{K}_e$ is the global stiffness matrix

$\mathbf{F}_e = \mathbf{T}^T\,\mathbf{k}_e\,\mathbf{T}\,\mathbf{D}_e = \mathbf{K}_e\,\mathbf{D}_e$ (Eq. 14)

The cosines and sinus of angle α are determined from the geometry presented in Figure A1

$\cos\alpha = \frac{X_2 - X_1}{L}$ (Eq. 15)

$\sin\alpha = \frac{Z_2 - Z_1}{L}$ (Eq. 16)

Appendix 2- Results from COMSOL Multiphysics

How much the fishing rod is deformed when a fish is hooked can also be solved in the FE programs like COMSOL Multiphysics. The same equations are used but you define the problem in another way. In figure A2 and A3 the results are presented.

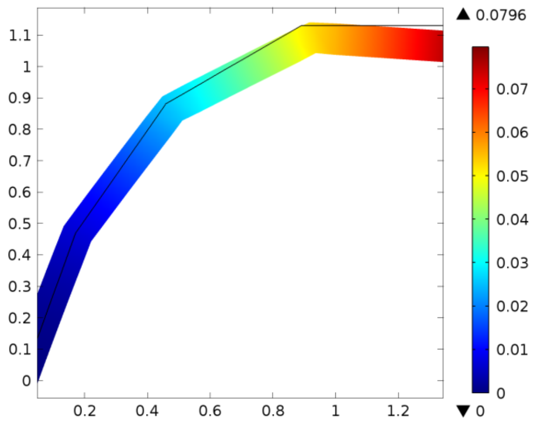

Figure A2. The total linear displacement in meter.

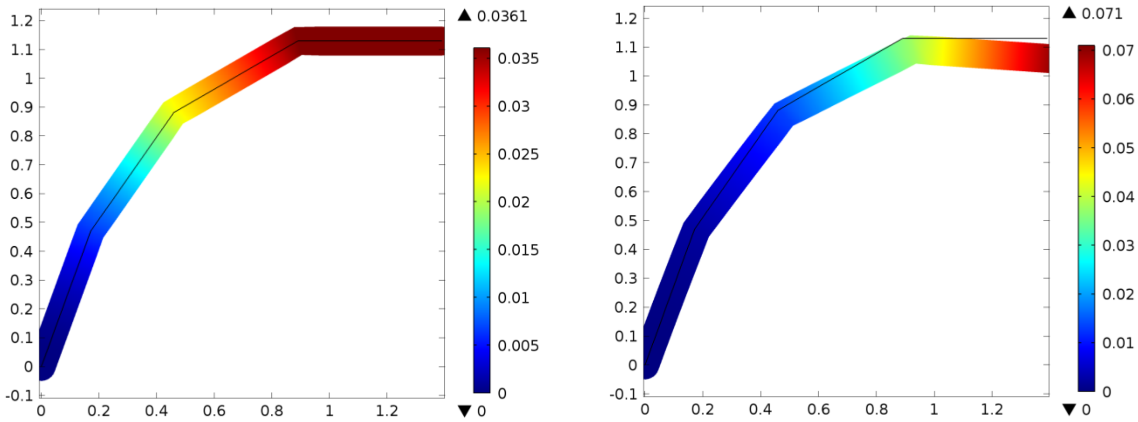

Figure A3. The displacement in x- (left) and z-displacement (right).