Random Intervals and Confidence Intervals (Ch. 6.1)

Dason Kurkiewicz

Iowa State University

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Outline

Motivation

Confidence Intervals - σ known, Sampling distribution is normal

Sample Size for desired half width

Statistical inference

- **Statistical inference**: using data from the sample to draw formal conclusions about the population
  - Point estimation (confidence intervals): estimating population parameters and specifying the degree of precision of the estimate.
  - Hypothesis testing: testing the validity of statements about the population that are framed in terms of parameters.

Motivation for confidence intervals

- We want information on a population. For example:
  - True mean breaking strength of a kind of wire rope.
  - True mean fill weight of food jars.
  - True mean instrumental drift of a kind of scale.
  - Average number of cycles to failure of a kind of spring.
- We can use point estimates:
  - For example: if we measure breaking strengths (in tons) of 6 wire ropes as 5, 3, 7, 3,10, and 1, we might estimate the true mean breaking strength $\mu \approx \overline{x} = \frac{5+3+7+3+10+1}{6} = 4.83$ tons.
- Or, we can use interval estimates:
  - μ is likely to be inside the interval $(4.83 - 2, 4.83 + 2) = (2.83, 6.83)$.
  - We are confident that the true mean breaking strength, μ, is somewhere in $(2.83, 6.83)$. But how confident can we be?

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Confidence intervals for μ: σ known, Sampling distribution of $\bar{x}$ is normal

- Two-sided $1-\alpha$ confidence interval:

$$\left(\overline{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

Example: fill weight of jars

- Suppose a manufacturer fills jars of food using a stable filling process with a known standard deviation of $\sigma = 1.6g$.
- We take a sample of n =47 jars and measure the sample mean weight $\overline{x} = 138.2$ g.
- A two-sided 90% confidence interval $(\alpha = 0.1)$ for the true mean weight μ is:

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- A two-sided 90% confidence interval ($\alpha = 0.1$) for the true mean weight μ is:

$$
\begin{aligned}
&\left(\overline{x} - z_{1-0.1/2}\frac{\sigma}{\sqrt{n}},\ \overline{x} + z_{1-0.1/2}\frac{\sigma}{\sqrt{n}}\right) \\
&= \left(138.2 - z_{0.95}\frac{1.6}{\sqrt{47}},\ 138.2 + z_{0.95}\frac{1.6}{\sqrt{47}}\right) \\
&= (138.2 - 1.64 \cdot 0.23,\ 138.2 + 1.64 \cdot 0.23) \\
&= (137.82, 138.58)
\end{aligned}
$$

I could have also written the interval as:

$$138.2 \pm 0.38\ g$$

Interpreting the confidence interval: fill weight of jars

- We are 90% confident that the true mean fill weight is between 137.82g and 138.58g.
- If we took 100 more samples of 47 jars each, roughly 90 of those samples would yield confidence intervals containing the true mean fill weight.
- These methods of interpretation generalize to all confidence intervals.

One sided confidence intervals

- What if we just want to be sure that the true mean fill weight is high enough?

Example: fill weight of jars.

- What if we just want to be sure that the true mean fill weight is high enough?
- Then, we would use a one-side lower 90% confidence interval:

$$
\begin{aligned}
&\left(\overline{x} - z_{1-\alpha}\frac{\sigma}{\sqrt{n}},\ \infty\right) \\
&= \left(138.2 - z_{1-\alpha}\frac{\sigma}{\sqrt{n}},\ \infty\right) \\
&= \left(138.2 - z_{0.9}\frac{1.6}{\sqrt{47}},\ \infty\right) \\
&= (138.2 - 1.28 \cdot 0.23,\ \infty) \\
&= (137.91,\ \infty)
\end{aligned}
$$

- We're 90% confident that the true mean fill weight is above 137.91 g.

Your turn: car engines

- Consider a grinding process used to rebuild car engines, which involves grinding rod journals for engine crankshafts.
- Of interest is the deviation of the true mean rod journal diameter from the target diameter.
- Suppose the standard deviation of the individual differences from the target diameter is 0.7×10^{-4} in.
- 32 consecutive rod journals are ground, with a sample mean deviation of -0.16×10^{-4} in from the target diameter.
- Calculate and interpret a two-sided 95% confidence interval for the true mean deviation from the target diameter. Is there enough evidence that we're missing the target on average?

Answer: car engines

- $\alpha = 1 - 0.95 = 0.05$, $n = 32$, $\sigma = 0.7 \times 10^{-4}$, and $\overline{x} = -0.16 \times 10^{-4}$.
- Interval:

$$\left(\overline{x} - z_{1-0.05/2}\frac{\sigma}{\sqrt{n}},\ \overline{x} + z_{1-0.05/2}\frac{\sigma}{\sqrt{n}}\right)$$
$$= \left(-0.16 \times 10^{-4} - z_{0.975}\frac{0.7 \times 10^{-4}}{\sqrt{32}},\ -0.16 \times 10^{-4} + z_{0.975}\frac{0.7 \times 10^{-4}}{\sqrt{32}}\right)$$
$$= (-0.16 \times 10^{-4} - 1.96 \cdot 1.2 \times 10^{-5},\ -0.16 \times 10^{-4} + 1.96 \cdot 1.2 \times 10^{-5})$$
$$= (-4.0 \times 10^{-5}, 7.5 \times 10^{-6})$$

- We are 95% confident that the true mean deviation from the target diameter of the rod journals is between -4.0×10^{-5} in and 7.5×10^{-6} in.
- Since 0 is in the confidence interval, there is not enough evidence to conclude that the rod journal grinding process is off target.

Your turn: hard disk failures

- F. Willett, in the article ?The Case of the Derailed Disk Drives? (Mechanical Engineering, 1988), discusses a study done to isolate the cause of ?blink code A failure? in a model of Winchester hard disk drive.
- For each disk, the investigator measured the breakaway torque (in. oz.) required to loosen the drive's interrupter flag on the stepper motor shaft.
- Breakaway torques for 26 disk drives were recorded, with a sample mean of 11.5 in. oz.
- Suppose you know the true standard deviation of the breakaway torques is 5.1 in. oz.
- Calculate and interpret:
  1. A two-sided 90% confidence interval for the true mean breakaway torque of the relevant type of Winchester drive.
  2. An analogous two-sided 95% confidence interval.
  3. An analogous two-sided 99% confidence interval.
- Is there enough evidence to conclude that the mean breakaway torque is different from the factory's standard of 33.5 in. oz.?

Answers: hard disk failures

- $\sigma = 5.1, \overline{x} = 11.5, n = 26.$
- All three confidence intervals have the form:

$$\begin{aligned}
&\left(\overline{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}},\ \overline{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \\
&= \left(11.5 - z_{1-\alpha/2}\frac{5.1}{\sqrt{26}},\ 11.5 + z_{1-\alpha/2}\frac{5.1}{\sqrt{26}}\right) \\
&= (11.5 - 1.0002 \cdot z_{1-\alpha/2},\ 11.5 + 1.0002 \cdot z_{1-\alpha/2})
\end{aligned}$$

- The confidence intervals are thus:
  1. 90% CI means $\alpha = 0.1$

$$\begin{aligned}
&(11.5 - 1.0002 \cdot z_{1-0.1/2},\ 11.5 + 1.0002 \cdot z_{1-0.1/2}) \\
&= (11.5 - 1.0002 \cdot z_{0.95},\ 11.5 + 1.0002 \cdot z_{0.95}) \\
&= (11.5 - 1.0002 \cdot 1.64,\ 11.5 + 1.0002 \cdot 1.64) \\
&= (9.86, 13.14)
\end{aligned}$$

Answers: hard disk failures

2. 95% CI means $\alpha = 0.05$

$$
\begin{aligned}
&(11.5 - 1.0002 \cdot z_{1-0.05/2},\ 11.5 + 1.0002 \cdot z_{1-0.05/2}) \\
&= (11.5 - 1.0002 \cdot z_{0.975},\ 11.5 + 1.0002 \cdot z_{0.975}) \\
&= (11.5 - 1.0002 \cdot 1.96,\ 11.5 + 1.0002 \cdot 1.96) \\
&= (9.54, 13.46)
\end{aligned}
$$

3. 99% CI means $\alpha = 0.01$

$$
\begin{aligned}
&(11.5 - 1.0002 \cdot z_{1-0.01/2},\ 11.5 + 1.0002 \cdot z_{1-0.01/2}) \\
&= (11.5 - 1.0002 \cdot z_{0.995},\ 11.5 + 1.0002 \cdot z_{0.995}) \\
&= (11.5 - 1.0002 \cdot 2.33,\ 11.5 + 1.0002 \cdot 2.33) \\
&= (9.17, 13.83)
\end{aligned}
$$

Answers: hard disk failures

- Notice: the confidence intervals get wider as the confidence level $1 - \alpha$ increases.
- None of these confidence intervals contains the manufacturer's target of 33.5 in. oz., so there is significant evidence that the process misses this target.
- Hence, there is a design flaw in the manufacturing process of the disk drives that must be corrected.

Confidence intervals for μ: σ known, Sampling distribution of $\bar{x}$ is normal

- Two-sided $1-\alpha$ confidence interval:

$$\left(\overline{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

- One-sided $1-\alpha$ **upper confidence interval**:

$$\left(-\infty,\ \overline{x} + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}\right)$$

- One-sided $1-\alpha$ **lower confidence interval**:

$$\left(\overline{x} - z_{1-\alpha}\frac{\sigma}{\sqrt{n}},\ \infty\right)$$

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Sample Size for desired half width

Controlling the width of a confidence interval

- Suppose you want to estimate the true fill weight of food jars. Your boss wants to know the true weight within .1 grams. Recall the for this process $\sigma = 1.6g$.

Controlling the width of a confidence interval

- Suppose you want to estimate the true fill weight of food jars. Your boss wants to know the true weight within .1 grams with 95% confidence. Recall the for this process $\sigma = 1.6g$.

Our interval estimate is

$$\bar{x} \pm 1.96 \frac{1.6}{\sqrt{n}}$$

So we want $.1 \geq 1.96 \frac{1.6}{\sqrt{n}}$ for some value of n. We can rewrite this as

$$\left(\frac{1.96 * 1.6}{.1} \right)^2 \leq n$$

The first n that meets that criteria is $n = 984$.

Sample size for desired half width

- If we know σ, have a desired level of confidence $(1-\alpha)$, and know we want a half width of δ then the sample size required is

$$n \geq \left(\frac{z_{1-\alpha/2} \cdot \sigma}{\delta}\right)^2$$