Assumptions and More Inference for Simple Linear Regression (Ch. 9.1)

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Outline

Assumptions for SLR

SLR: Inference for the Mean Response at some x

Simultaneous Confidence Intervals for $\mu_{y|x}$

SLR

- Recall our model:

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

$\epsilon_1, \ldots, \epsilon_n \sim$ iid $N(0, \sigma^2)$

- Under the model, the true mean response at some observed covariate value x_i is:

$$\mu_{y|x_i} = \beta_0 + \beta_1 x_i$$

Assumptions in SLR

- Correct Model (sometimes called 'Linearity')
- Constant Variance
- Normally distributed error terms
- Independence

Correct Model

- We assume that we have specified a correct functional form relating the predictor to the response.
- To check this we can use:
  - Scatterplot
  - Residual by Predicted plot
  - Residual by Predictor (x) plot

Correct Model

- The Residual by Predicted plot is one of the best ways to check this assumptions.
- Things we want to see
  - A horizontal band of points with no discernable pattern left over
- Things we don't want to see
  - Left over pattern!
  - Any curviture left over - implies we should fit a higher order polynomial
  - A sinusoidal pattern
  - ANY pattern

Correct Model

Constant Variance

- It is assumed that $\epsilon_i \sim N(0, \sigma^2)$
- Notice that the error terms all have the same variance
- Once again we can use the residual by predicted plot to check this assumption
- The residuals themselves actually have a slightly different variance so it is typical to 'standardize' them.

$$e_i \sim N\left(0, \sigma^2(1 - \frac{1}{n} - \frac{(x_i - \bar{x})^2}{\sum(x_i - \bar{x})^2})\right)$$

Constant Variance

Normally distributed errors

- It is assumed that $e_i \sim N(0, \sigma^2)$
- We should check if the errors are normally distributed
- We will use the residuals as estimates for the errors and check for approximate normality.
- Make a normal qq-plot for the residuals (or standardized residuals)

Independence

- The most important assumption
- It is also the hardest to check in practice
- Typically have to rely on what you know about the data
- Things to watch for:
  - Data collected through time - consecutive points might be more correlated than points seperated by larger periods of time.
  - Units that have a natural grouping.
    - Maybe you have multiple measurements of multiple experimental units. These measurements shouldn't be considered independent.

Outline

Assumptions for SLR

SLR: Inference for the Mean Response at some x

Simultaneous Confidence Intervals for $\mu_{y|x}$

SLR: mean response at x

- Recall our model:

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

$\varepsilon_1, \ldots, \varepsilon_n \sim$ iid $N(0, \sigma^2)$

- Under the model, the true mean response at some observed covariate value x_i is:

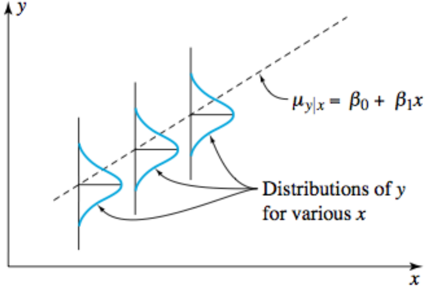
$$\mu_{y|x_i} = \beta_0 + \beta_1 x_i$$

- Now, if some new covariate value x is within the range of the x_i's, we can estimate the true mean response at this new x:

$$\widehat{\mu}_{y|x} = b_0 + b_1 x$$

SLR: mean response at x

- But how good is the estimate?



- That's why we do inference.

SLR: mean response at x

- Under the model, $\widehat{\mu}_{y|x}$ is normally distributed with:

$$E(\widehat{\mu}_{y|x}) = \mu_{y|x} = \beta_0 + \beta_1 x$$

$$Var(\widehat{\mu}_{y|x}) = \sigma^2 \left(\frac{1}{n} + \frac{(x - \overline{x})^2}{\sum_i (x_i - \overline{x})^2} \right)$$

- We can construct a $N(0, 1)$ random variable by standardizing:

$$Z = \frac{\widehat{\mu}_{y|x} - \mu_{y|x}}{\sigma \sqrt{\frac{1}{n} \frac{(x-\overline{x})^2}{\sum_i (x_i - \overline{x})^2}}} \sim N(0, 1)$$

- Replacing σ with $s_{LF} = \sqrt{\frac{1}{n-2} \sum_i (y_i - \widehat{y}_i)^2}$:

$$T = \frac{\widehat{\mu}_{y|x} - \mu_{y|x}}{s_{LF} \sqrt{\frac{1}{n} \frac{(x-\overline{x})^2}{\sum_i (x_i - \overline{x})^2}}} \sim t_{n-2}$$

SLR: mean response at x

- To test $H_0 : \mu_{y|x} = \#$, we can use the test statistic:

$$K = \frac{\widehat{\mu}_{y|x} - \#}{s_{LF}\sqrt{\frac{1}{n} + \frac{(x-\overline{x})^2}{\sum_i (x_i - \overline{x})^2}}}$$

  which has a t_{n-2} distribution if H_0 is true and the model is correct.

- A 2-sided $1 - \alpha$ confidence interval for $\mu_{y|x}$ is:

$$\left(\widehat{\mu}_{y|x} - t_{n-2,\ 1-\alpha/2} \cdot s_{LF}\sqrt{\frac{1}{n} + \frac{(x-\overline{x})^2}{\sum_i (x_i - \overline{x})^2}},\right.$$

$$\left.\widehat{\mu}_{y|x} + t_{n-2,\ 1-\alpha/2} \cdot s_{LF}\sqrt{\frac{1}{n} + \frac{(x-\overline{x})^2}{\sum_i (x_i - \overline{x})^2}}\right)$$

  and the one-sided intervals are analogous.

Pressing pressures and specimen densities for a ceramic compound

A mixture of Al_2O_3, polyvinyl alcohol, and water was prepared, dried overnight, crushed, and sieved to obtain 100 mesh size grains. These were pressed into cylinders at pressures from 2,000 psi to 10,000 psi, and cylinder densities were calculated.

x (pressure in psi)	y (density in g/cc)
2000.00	2.49
2000.00	2.48
2000.00	2.47
4000.00	2.56
4000.00	2.57
4000.00	2.58
6000.00	2.65
6000.00	2.66
6000.00	2.65
8000.00	2.72
8000.00	2.77
8000.00	2.81
10000.00	2.86
10000.00	2.88
10000.00	2.86

Example: ceramics

- First, I'll make a 2-sided 95% confidence interval for the true mean density of the ceramics at 4000 psi.

$$\widehat{\mu}_{y|x} = 2.375 + 0.0000487(4000) = 2.5697 g/cc$$

With $t_{n-2,\ 1-\alpha/2} = t_{13,0.975} = 2.160$, the margin of error in the confidence interval is:

$$t_{n-2,\ 1-\alpha/2} \cdot s_{LF} \sqrt{\frac{1}{n} + \frac{(x - \overline{x})^2}{\sum_i (x_i - \overline{x})^2}}$$

$$= 2.160(0.0199)\sqrt{\frac{1}{15} + \frac{(4000 - 6000)^2}{1.2 \times 10^8}} = 0.0136 g/cc$$

Hence, the 95% CI is:

$$(2.5697 - 0.0136,\ 2.5697 + 0.0136) = (2.5561,\ 2.5833)$$

- We're 95% confident that the true mean density of the ceramics at 4000 psi is between 2.5561 g/cc and 2.5833 g/cc.

Your turn: ceramics

- Calculate and interpret a 2-sided 95% confidence interval for the true mean density at 5000 psi, given:
  - $\widehat{\mu}_{y|x} = 2.375 + 0.0000487x$
  - The margin of error is $t_{n-2,1-\alpha/2} s_{LF} \sqrt{\frac{1}{n} + \frac{(x-\overline{x})^2}{\sum_i (x_i - \overline{x})^2}}$
  - $\sum_i (x_i - \overline{x})^2 = 1.2 \times 10^8$
  - $n = 15$, $\overline{x} = 6000$.
  - $s_{LF} = 0.0199$
  - $t_{13,0.975} = 2.16$
- Test $H_0 : \beta_0 = 0$ vs. $H_a : \beta_0 \neq 0$ at significance level $\alpha = 0.05$ using the method of p-values.

Answers: ceramics

- Make a 2-sided 95% confidence interval for the true mean density of the ceramics at 5000 psi:

$$\widehat{\mu}_{y|x} = 2.375 + 0.0000487(5000) = 2.6183 g/cc$$

With $t_{n-2,\ 1-\alpha/2} = t_{13,0.975} = 2.160$, the margin of error in the confidence interval is:

$$t_{n-2,\ 1-\alpha/2} \cdot s_{LF}\sqrt{\frac{1}{n} + \frac{(x-\overline{x})^2}{\sum_i (x_i - \overline{x})^2}}$$

$$= 2.160(0.0199)\sqrt{\frac{1}{15} + \frac{(5000-6000)^2}{1.2 \times 10^8}} = 0.0118 g/cc$$

Hence, the 95% CI is:

$$(2.6183 - 0.0118,\ 2.6183 + 0.0118) = (2.6065,\ 2.6301)$$

- We're 95% confident that the true mean density of the ceramics at 5000 psi is between 2.6065 g/cc and 2.6301 g/cc.

Answers: ceramics

Now for the hypothesis test:

1. $H_0 : \beta_0 = 0, \;\; H_a : \beta_0 \neq 0$
2. $\alpha = 0.05$
3. β_0 is just $\mu_{y|x=0}$. The test statistic is:

$$K = \frac{b_0 - 0}{s_{LF}\sqrt{\frac{1}{n} + \frac{(0-\overline{x})^2}{\sum_i (x_i - \overline{x})^2}}} = \frac{b_0}{s_{LF}\sqrt{\frac{1}{n} + \frac{\overline{x}^2}{\sum_i (x_i - \overline{x})^2}}}$$

- $K \sim t_{n-2}$ assuming:
  - H_0 is true.
  - The model $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ is correct, with $\varepsilon_1, \ldots \varepsilon_n \sim$ iid $N(0, 1)$.

Answers: ceramics

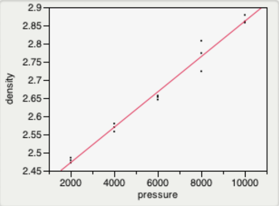
4. The moment of truth:

$$b_0 = 2.375$$
$$K = \frac{2.375}{0.0199\sqrt{\frac{1}{15} + \frac{6000^2}{1.2\times 10^8}}} = 197.09$$
$$\text{p-value} = P(|t_{13}| > 197.09) \ll 0.0001$$

5. With a p-value $\ll 0.0001 < \alpha$, we reject H_0 and conclude H_a.
6. There is overwhelming evidence that the intercept of the "true" line is different from 0.

Ceramics: back to the JMP output

Bivariate Fit of density By pressure



Linear Fit

Linear Fit

density = 2.375 + 4.8667e-5*pressure

Summary of Fit

RSquare	0.982193
RSquare Adj	0.980824
Root Mean Square Error	0.019909
Mean of Response	2.667
Observations (or Sum Wgts)	15

Lack Of Fit

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Ratio
Model	1	0.28421333	0.284213	717.0604
Error	13	0.00515267	0.000396	**Prob > F**
C. Total	14	0.28936600		<.0001*

Parameter Estimates

Term	Estimate	Std Error	t Ratio	Prob>|t|
Intercept	2.375	0.012055	197.01	<.0001*
pressure	4.8667e-5	1.817e-6	26.78	<.0001*

Ceramics: back to the JMP output

Parameter Estimates

Term	Estimate	Std Error	t Ratio	Prob>|t|
Intercept	2.375	0.012055	197.01	<.0001*
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- The test statistic K is under "t Ratio" for the intercept.
- "Prob$>|t|$" for the intercept is the p-value for the significance test you just did.
- "Estimate" for the intercept is b_0.
- "Std Error" for the intercept is:

$$\widehat{SD}(b_0) = s_{LF}\sqrt{\frac{1}{n} + \frac{\overline{x}^2}{\sum_i (x_i - \overline{x})^2}}$$

Be careful with Inference on β_0

- In this case and many others, $\beta_0 = \mu_{y|x=0}$ is beyond the range of our data.
- Estimating beyond the range of our covariate values is called **extrapolation**, which is dangerous for linear regression.
- Only extrapolate when:
  - You know your process or system well, and can describe it with the right differential equations.
  - You estimate the parameters of the resulting model using *nonlinear* regression:
    - Example: special case of the Michaelis-Menten model for enzyme kinetics with reaction speed y and substrate concentration x:

$$Y_i = \frac{\theta_1 x_i}{\theta_2 + x_i} + \varepsilon_i$$

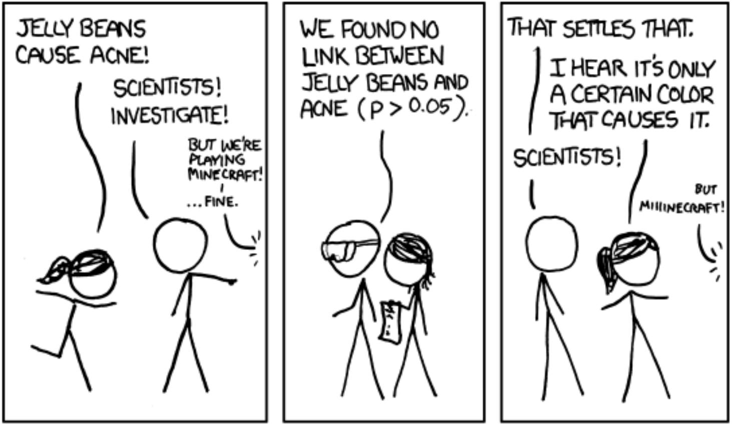
  - See Nonlinear Regression Analysis and Its Applications by Bates and Watts for more information on nonlinear regression.

Outline

Simultaneous confidence intervals

- Situations will arise when you'll want to do inference on $\mu_{y|x=2000}, \mu_{y|x=4000}, \mu_{y|x=6000}, \ldots$, all at once.
- When you compute several confidence intervals at once or do multiple tests at once, you need to account for the simultaneity.
- On average, for every 20 tests you do at $\alpha = 0.05$, we expect 1 of those tests to conclude H_a by chance alone.
  - Remember: $\alpha =$ P(reject H_0 assuming H_0 is true).

Example: http://xkcd.com/882/



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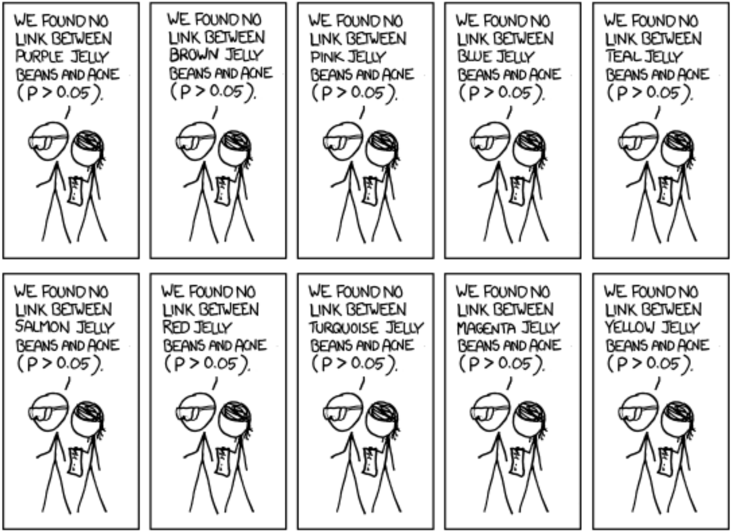

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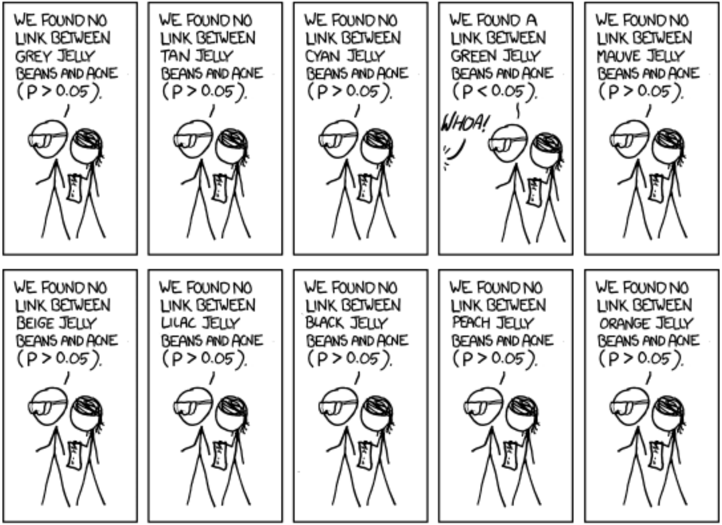

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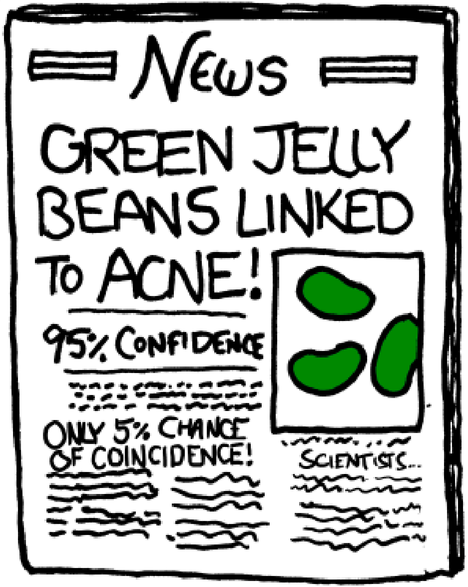


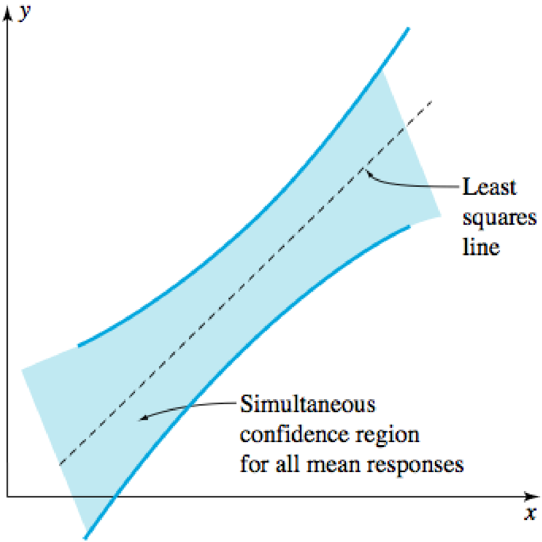
Simultaneous confidence intervals for $\mu_{y|x}$

- For simultaneous confidence intervals for $\mu_{y|x}$ for multiple values of x, use:

$$b_0 + b_1 x \pm \sqrt{2F_{2,n-2,1-\alpha}} \cdot s_{LF} \cdot \sqrt{\frac{1}{n} + \frac{(x-\overline{x})^2}{\sum_i (x_i - \overline{x})^2}}$$

- This formula accounts for the fact that we're computing k confidence intervals at the same time.

Simultaneous confidence intervals for $\mu_{y|x}$



Example: ceramics

- Given:
  - $n = 15$
  - $\overline{x} = 6000$
  - $\sum_i (x_i - \overline{x})^2 = 1.2 \times 10^8$
  - $\widehat{y} = 2.375 + 4.87 \times 10^{-5} x$, $s_{LF} = 0.0199$.
  - The simultaneous confidence interval formula is:

$$b_0 + b_1 x \pm \sqrt{2F_{2,n-2,1-\alpha/2}} \cdot s_{LF} \cdot \sqrt{\frac{1}{n} + \frac{(x - \overline{x})^2}{\sum_i (x_i - \overline{x})^2}}$$

- I will calculate simultaneous 95% confidence intervals for the mean responses $\mu_{y|x}$ at $x = 2000$, 4000, 6000, 8000, and 10000.

Example: ceramics

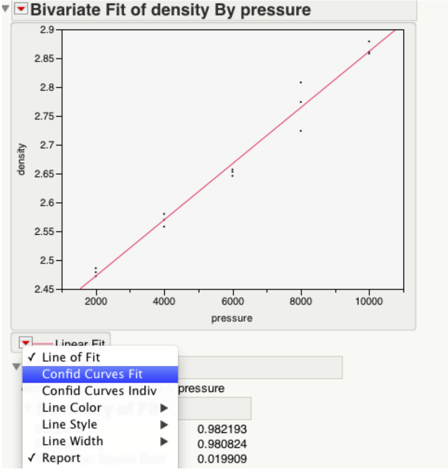
- Using $F_{2,n-2,1-\alpha} = F_{2,13,0.95} = 3.81$, the intervals are of the form:

$$2.375 + 4.87 \times 10^{-5}x \pm \sqrt{2 \cdot 3.81} \cdot 0.0199 \cdot \sqrt{\frac{1}{15} + \frac{(x-6000)^2}{1.2 \times 10^8}}$$

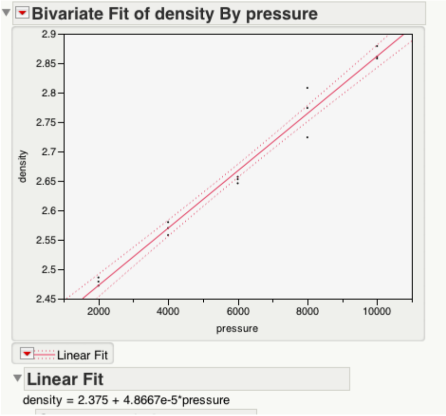
$$= 2.375 + 4.87 \times 10^{-5}x \pm 0.0549\sqrt{0.066 + 8.33 \times 10^{-9}(x-6000)^2}$$

x, pressure	CI, compact form	CI
2000	2.4723 ± 0.0246	(2.4477, 2.4969)
4000	2.5697 ± 0.0174	(2.5523, 2.5871)
6000	2.6670 ± 0.0142	(2.6528, 2.6812)
8000	2.7643 ± 0.0174	(2.7469, 2.7817)
10000	2.8617 ± 0.0246	(2.8371, 2.8863)

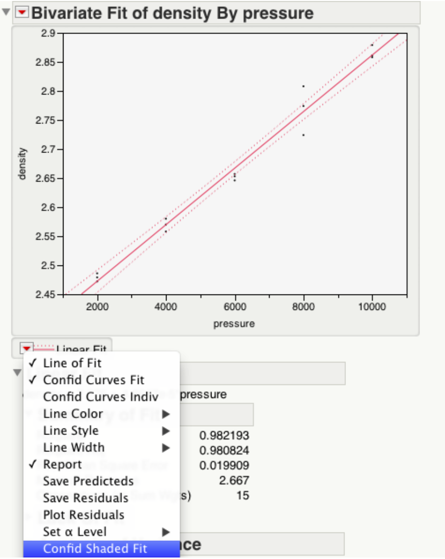
Ceramics; plotting simultaneous confidence regions



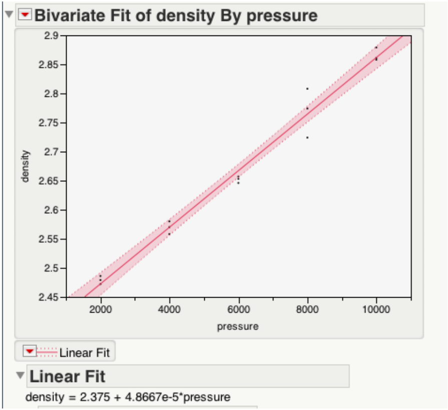
Ceramics; plotting simultaneous confidence regions



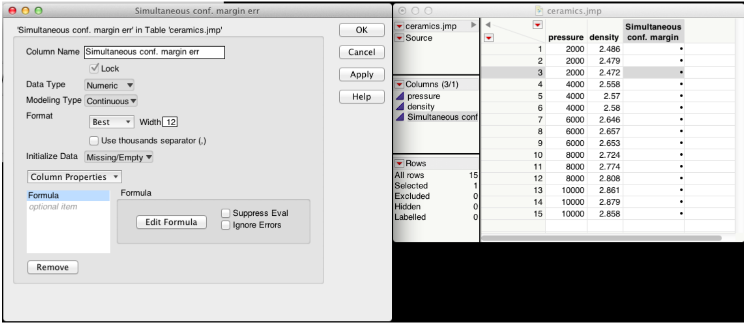
Ceramics; plotting simultaneous confidence regions



Ceramics; plotting simultaneous confidence regions



Ceramics: calculating the margins of error in JMP



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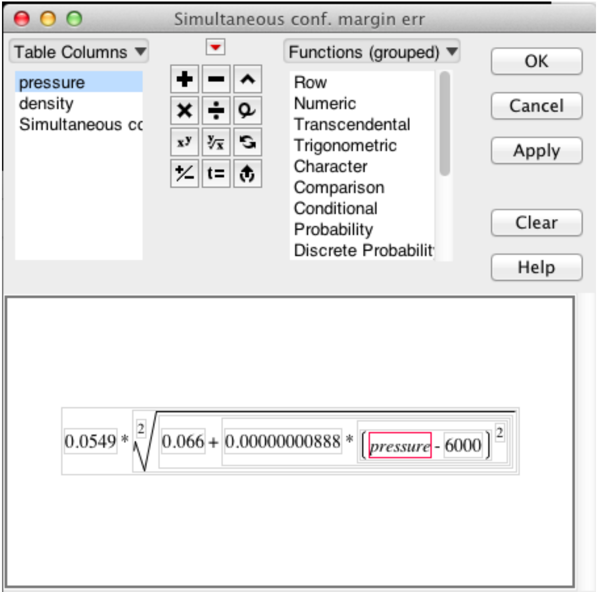

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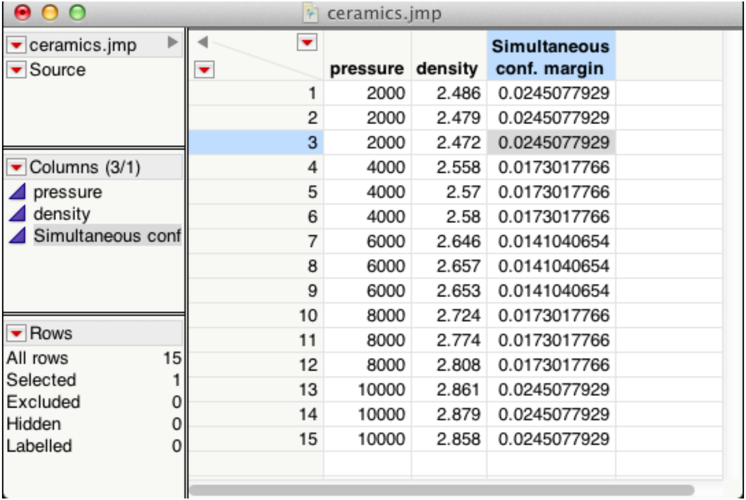


ceramics.jmp

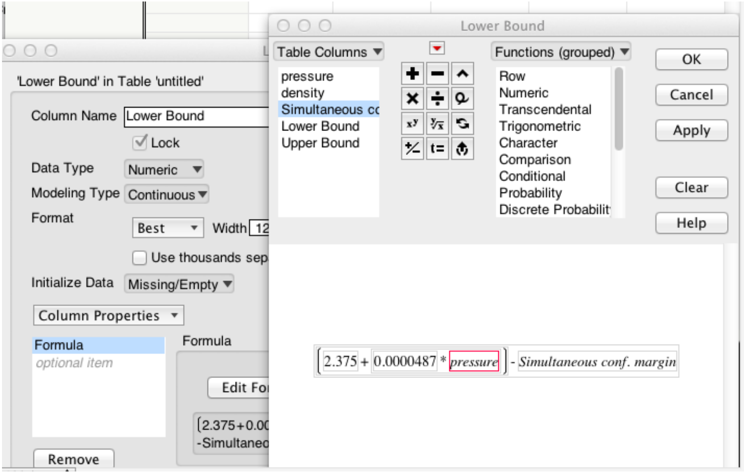
ceramics.jmp
Source

Columns (3/1)
pressure
density
Simultaneous conf

Rows	
All rows	15
Selected	1
Excluded	0
Hidden	0
Labelled	0

	pressure	density	Simultaneous conf. margin
1	2000	2.486	0.0245077929
2	2000	2.479	0.0245077929
3	2000	2.472	0.0245077929
4	4000	2.558	0.0173017766
5	4000	2.57	0.0173017766
6	4000	2.58	0.0173017766
7	6000	2.646	0.0141040654
8	6000	2.657	0.0141040654
9	6000	2.653	0.0141040654
10	8000	2.724	0.0173017766
11	8000	2.774	0.0173017766
12	8000	2.808	0.0173017766
13	10000	2.861	0.0245077929
14	10000	2.879	0.0245077929
15	10000	2.858	0.0245077929

Ceramics: calculating the margins of error in JMP



Ceramics: calculating the margins of error in JMP

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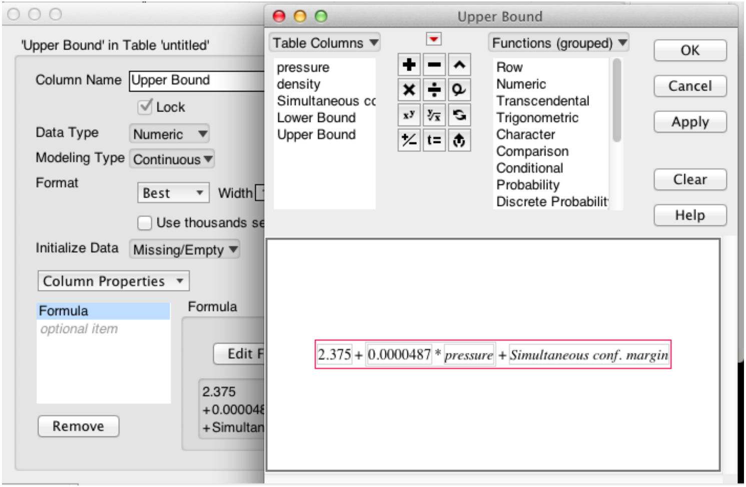
Columns (5/0)
- pressure
- density
- Simultaneous conf
- Lower Bound
- Upper Bound

	pressure	density	Simultaneous conf. margin	Lower Bound	Upper Bound
1	2000	2.486	0.0245077929	2.4478922071	•
2	2000	2.479	0.0245077929	2.4478922071	•
3	2000	2.472	0.0245077929	2.4478922071	•
4	4000	2.558	0.0173017766	2.5524982234	•
5	4000	2.57	0.0173017766	2.5524982234	•
6	4000	2.58	0.0173017766	2.5524982234	•
7	6000	2.646	0.0141040654	2.6530959346	•
8	6000	2.657	0.0141040654	2.6530959346	•
9	6000	2.653	0.0141040654	2.6530959346	•
10	8000	2.724	0.0173017766	2.7472982234	•
11	8000	2.774	0.0173017766	2.7472982234	•
12	8000	2.808	0.0173017766	2.7472982234	•
13	10000	2.861	0.0245077929	2.8374922071	•
14	10000	2.879	0.0245077929	2.8374922071	•
15	10000	2.858	0.0245077929	2.8374922071	•

Ceramics: calculating the margins of error in JMP



Ceramics: calculating the margins of error in JMP

	pressure	density	Simultaneous conf. margin	Lower Bound	Upper Bound
1	2000	2.486	0.0245077929	2.4478922071	2.4969077929
2	2000	2.479	0.0245077929	2.4478922071	2.4969077929
3	2000	2.472	0.0245077929	2.4478922071	2.4969077929
4	4000	2.558	0.0173017766	2.5524982234	2.5871017766
5	4000	2.57	0.0173017766	2.5524982234	2.5871017766
6	4000	2.58	0.0173017766	2.5524982234	2.5871017766
7	6000	2.646	0.0141040654	2.6530959346	2.6813040654
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14	10000	2.879	0.0245077929	2.8374922071	2.8865077929
15	10000	2.858	0.0245077929	2.8374922071	2.8865077929

untitled

Columns (5/1)
- pressure
- density
- Simultaneous conf
- Lower Bound +
- Upper Bound +