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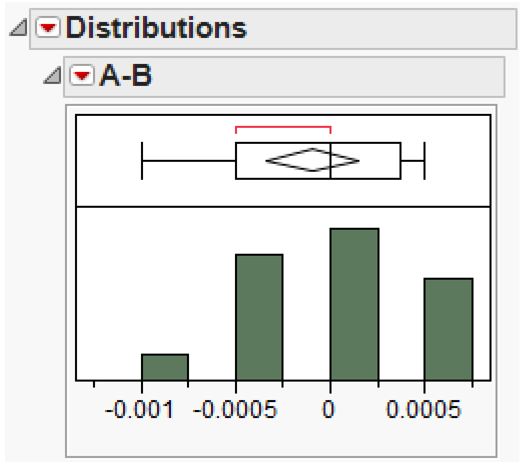
1. Chapter 6 - End of chapter exercises: Problem 6

   (a) *(2 pts)* The methods mentioned are specifically for two *independent* samples. The data provided isn't really independent since each bushing is measured by both of the students. We would assume that the repeated measurements on each of the bushings would be correlated so we would instead want to use the methods talked about in section 6.3.2 on "inference for the mean of paired differences".

   (b) *(5 pts)* We take student's A measurement and subtract student B's measurement for each bushing. This gives us 16 differences.



Quantiles

Summary Statistics

Mean	-9.375e-5
Std Dev	0.0004553
Std Err Mean	0.0001138
Upper 95% Mean	0.0001489
Lower 95% Mean	-0.000336
N	16

The formula for our confidence interval using t-based distributions is of the form

$$\bar{d} \pm t_{1-\alpha/2,n-1}\frac{s_d}{\sqrt{n}}$$

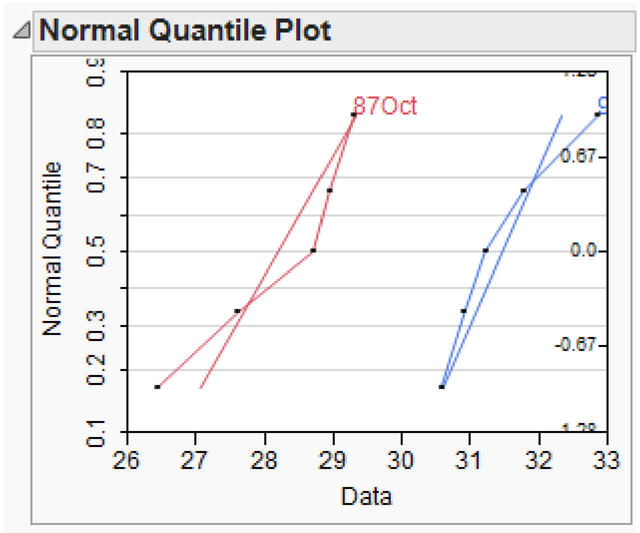
where $\bar{d}$ is the mean of the differences, n is the number of differences and s_d is the sample standard deviation of the differences. We want to use $t_{.975,15}$ as our quantile and from the back of the book we get that this is equal to 2.131. Our confidence interval is then

$$-0.00009375 \pm 2.131\frac{.0004553}{\sqrt{16}}$$

which gives a 95% confidence interval of (-0.0003363111, 0.0001488111) which matches the JMP output.

2. Chapter 6 - End of chapter exercises: Problem 13 (a-d)

   (a) *(3 pts)* Here is a normal quantile plot for both data sets.



The normality assumption seems to be fine in this case. We will compare the sample variances to assess if the equal variance assumption is reasonable in this case

Level	Number	Mean	Std Dev
87Oct	5	28.1980	1.17246
90Oct	5	31.4640	0.89626

So the ratio of our variances is $\frac{1.17^2}{0.89^2} = 1.72$. Since this is less than 2 we'll say that the equal variance assumption is fine. As long as there was some assessment of the equal variance assumption that would be fine for the sake of this problem - it didn't need to be through the 'ratio ¡ 2' criteria.

So it looks like both assumptions seem reasonable for this data.

(b) *(3 pts)*

$$\begin{aligned} s_p^2 &= \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} \\ &= \frac{4 * 1.17246^2 + 4 * 0.89626^2}{5+5-2} \\ &= \frac{8.711778}{8} \\ &= 1.088972 \end{aligned}$$

So $s_p = \sqrt{s_p^2} = 1.043538$. In this context we are assuming that the mileage for tankfuls of fuel are both normally distributed with the same standard deviation. This value s_p represents our estimate of the shared standard deviation in the mileage of tankfuls of fuel for these two Octanes in this particular car.

(c) *(4 pts)* Let $t^* = t_{1-\alpha/2, n_1+n_2-2}$ then our confidence interval will have the form

$$(\bar{x_1} - \bar{x_2}) \pm t^* s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Plugging in our data we have

$$(28.198 - 31.464) \pm 2.306 * 1.043538 * \sqrt{1/5 + 1/5}$$

which gives a confidence interval of (-4.78794, -1.74406). I used 87Oct - 90Oct so if you calculated the difference the other way around the confidence interval would be (1.74406, 4.78794).

From the nature of the confidence interval I would expect a small p-value. Since the confidence interval doesn't contain 0 we know that the p-value would be less than .05 but the confidence interval isn't even that close to 0 so I would expect a small p-value.

(d) *(5 pts)* We will do a one sided hypothesis test with $\alpha = .05$ assuming equal variances. We will use the critical value method.

i.

$$H_o : \mu_{87} = \mu_{90}$$
$$H_a : \mu_{87} < \mu_{90}$$

ii. We will use $\alpha = 0.05$

iii. Our test statistic will be

$$T = \frac{(\bar{x_{87}} - \bar{x_{90}}) - 0}{s_p\sqrt{1/n_1 + 1/n_2}}$$

$T \sim T_{n_1+n_2-2}$ assuming 1) H_o is true and 2) the data in both groups are independent and come from normal distributions with a common variance.
Reject H_o if $T < t_{.05,8} = -1.86$

iv. We have

$$T = \frac{28.198 - 31.464}{1.043538\sqrt{1/5 + 1/5}} = -4.948549$$

v. With $T = -4.94 < -1.86$ we reject H_o in favor of H_a.

vi. There is enough evidence to conclude that higher-octane gasoline provides a higher average mileage than the lower octane gasoline. (Note: we can only really make a conclusion about M. Murphy in this particular 9-year-old economy car).

3. Chapter 6 - End of chapter exercises: Problem 27

(a) *(2 pts)* The researchers wanted to make inferences about zircaloy-4. However they only have a single zircaloy-4 specimen in the experiment. If the population of interest is zircaloy-4 specimens then the actual sample size here is just 1. They wouldn't necessarily be able to generalize their results to the population of interest. On the other hand with 821 measurements on this particular specimen they can be really confident in their findings for this particular specimen.

(b) *(4 pts)* With a sample this large we can use normal-distribution based methods and the results really won't differ from if we used t-based methods. The form for our confidence interval will be

$$\bar{x} \pm z_{.99}\frac{s}{\sqrt{n}}$$

which gives

$$.055 \pm 2.33\frac{.028}{\sqrt{821}}$$

so our confidence interval is (0.05272311, 0.05727689). So we can be 98% confident that the true mean diameter of the particles in this particular Zircaloy-4 specimen is between .0527 and .0572 μm.

(c) *(5 pts)* You could do this test with any of the testing methods you prefer. If you used a $\alpha = .01$ and used the confidence interval method then you could just use the previous results. I will use the p-value method to do the test.

i.

$$H_o : \mu = .057$$
$$H_a : \mu \neq .057$$

ii. We will use $\alpha = 0.05$

iii. Our test statistic will be

$$Z = \frac{(\bar{x} - .057}{\sigma/\sqrt{n}}$$

$Z \sim N(0,1)$ assuming 1) H_o is true and 2) the data is normally distributed with a variance of σ^2. Since we have a very large sample size we can treat s^2 from the data as if it were the true variance.

Reject H_o if $|Z| > z_{.975} = 1.96$

iv. We have

$$Z = \frac{.055 - .057}{.028/\sqrt{821}} = -2.04665$$

v. With $|Z| = 2.04665 > 1.96$ we reject H_o in favor of H_a.

vi. There is enough evidence to conclude that the true mean particle diameter for this specimen is different from the standard of $.057\mu m$.

(d) *(2 pts)* The observed difference here is $.002\mu m$. This was statistically significant at the .05 level. However, this difference might not matter at all in a practical sense. The statistical significance tells us that we wouldn't see this difference entirely due just to random chance. Practical significance isn't something statistics can tell us - we need to rely on our knowledge about the data/problem at hand to tell us if the difference actually means anything.

4. You have a coin that has a probability of p of landing on heads. Your friend told you that they didn't think it was a fair coin so you decide you want to run a test. Your testing procedure to test $H_0 : p = 0.5$ against $H_a : p \neq 0.5$ is as follows:

- Flip the coin 7 times and count how many heads there are (call this number X)
- If $X = 0$ or if $X = 7$ reject H_0, otherwise fail to reject H_0.

Answer the following questions:

(a) *(2 pts)* What is the type I error rate for this testing procedure?
The type I error rate is the probability that we reject the null hypothesis conditioned on the null being true. If the null is true then $X \sim Bin(7, 0.5)$ and the probability we reject the null is just

$$\begin{aligned} P(X = 0 or X = 7) &= P(X = 0) + P(X = 7) \\ &= \binom{7}{0}.5^0(1 - .5)^{7-0} + \binom{7}{7}.5^7(1 - .5)^{7-7} \\ &= 0.015625 \end{aligned}$$

So our type I error rate for this testing procedure is .015625.

(b) *(2 pts)* If in actuality the coin is such that $p = .75$ what is the type II error rate for this testing procedure? The type II error rate is the probability that we fail to reject the null conditioned on the alternative hypothesis being true. To do a probability calculation we need all of the parameters in our model specified so this is why I specified that "in actuality $p = .75$". So we just need to find the probability that we don't reject the null when $p = .75$. If $p = .75$ then $X \sim Bin(7, 0.75)$

$$\begin{aligned} P(\text{don't reject null}) &= P(X = 1, 2, 3, 4, 5, or 6) \\ &= 1 - P(X = 0 or X = 7) \\ &= 1 - (P(X = 0) + P(X = 7)) \\ &= 1 - (\binom{7}{0}.75^0(1 - .75)^{7-0} + \binom{7}{7}.75^7(1 - .75)^{7-7}) \\ &= 0.8664551 \end{aligned}$$

(c) If you change the rejection criteria to: If X is either 0, 1, 6, or 7 then reject H_0, otherwise fail to reject

i. *(2 pts)* What is the type I error rate now?

Once again if the null is true then $X \sim Bin(7, 0.5)$ and the probability we reject the null is just

$$\begin{aligned} P(X = 0,1,6,7) &= P(X = 0)P(X = 1) + P(X = 6) + P(X = 7) \\ &= \binom{7}{0}.5^7(1-.5)^{7-0} + \binom{7}{1}.5^1(1-.5)^{7-1} + \binom{7}{7}.5^7(1-.5)^{7-7} \\ &= 0.125 \end{aligned}$$

So our type I error rate for this testing procedure is 0.125.

ii. *(2 pts)* What is the type II error rate (assuming $p = .75$)?

Once again we just need to find the probability that we don't reject the null when $p = .75$. If $p = .75$ then $X \sim Bin(7, 0.75)$

$$\begin{aligned} P(\text{don't reject null}) &= P(X = 2,3,4,5) \\ &= P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) \\ &= \binom{7}{2}.75^2(1-.75)^{7-2} + \binom{7}{3}.75^3(1-.75)^{7-3} + \binom{7}{4}.75^4(1-.75)^{7-4} + \binom{7}{5}.75^5(1-.75)^7 \end{aligned}$$

One thing we notice is that by making the rejection region 'bigger' we have increased our type I error rate but decreased our type II error rate.

(d) *(4 pts)* By changing the value for n and the rejection region find a test such that the type I error rate is less than .1 and the type II error rate is less than .5 (assume $p = .75$ when calculating the type II error rate)

Answers will vary. One solution that works is using $n = 14$ with rejecting the null if $X = 0, 1, 2, 3, 11, 12, 13, 14$. In this case our type I error rate is 0.05737305 and the type II error rate is 0.4786202.

(e) *(4 bonus pts)* Bonus: You have a biased coin such that p is not equal to 0, 0.5, or 1. Can you come up with a procedure for using that coin to generate outcomes from a $Binomial(n = 1, p = .5)$ random variable? (Essentially can you find a way to remove the bias from a weighted coin)

i. Flip the coin two times

ii. If the result was Heads then Tails call the outcome 'heads'. If the result was Tails and then Heads call the result 'tails'.

iii. If you got either heads both times or tails both times start over.

If the probability of getting a heads is p then the probability that you get a heads then tails is $p(1-p)$. The probability that you get a tails and then heads is $(1-p)p$. These are the same probabilities so the overall probability that the result is called 'heads' must be 0.50.