

Continuous Random Variables: Quantiles, Expected Value, and Variance

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Outline

Quantiles

Expected Value

Variance

Functions of random variables

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Random Variables:
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Quantiles of continuous distributions

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- ▶ The p -quantile of a random variable, X , is the number, $Q(p)$, such that:

$$P(X \leq Q(p)) = p$$

- ▶ In terms of the cumulative distribution function (cdf):

$$\begin{aligned} F(Q(p)) &= p \\ Q(p) &= F^{-1}(p) \end{aligned}$$

Example

- ▶ Let Y be the time delay (s) between a 60 Hz AC circuit and the movement of a motor on a different circuit.

$$f(y) = \begin{cases} 60 & 0 < y < \frac{1}{60} \\ 0 & \text{otherwise} \end{cases}$$

- ▶ $Q(0.95)$:

$$\begin{aligned} 0.95 &= P(Y \leq Q(0.95)) = \int_{-\infty}^{Q(0.95)} f(y) dy \\ &= \int_{-\infty}^0 0 dx + \int_0^{Q(0.95)} 60 dy = 0 + (60|_0^{Q(0.95)}) \\ &= 60Q(0.95) \\ Q(0.95) &= \frac{0.95}{60} = \frac{19}{1200} \approx 0.0158 \end{aligned}$$

Interpretation: on average, 95% of the time delays will be below 0.0158 seconds.

Example

- ▶ You can also calculate quantiles directly from the cdf:

$$F(y) = \begin{cases} 0 & y \leq 0 \\ 60y & 0 < y \leq \frac{1}{60} \\ 1 & y > \frac{1}{60} \end{cases}$$

- ▶ $Q(0.25)$:

$$\begin{aligned} 0.25 &= P(Y \leq Q(0.25)) = F(Q(0.25)) \\ &= 60 \cdot Q(0.25) \end{aligned}$$

Hence:

$$Q(0.25) = \frac{0.25}{60} = \frac{1}{240} \approx 0.00417$$

Interpretation: on average, 25% of the time delays will be below 0.00417 seconds.

Your turn: calculating quantiles

- $T \sim \text{Exp}(\alpha = 1/2)$:

$$f(t) = \begin{cases} 0 & t \leq 0 \\ 2e^{-2t} & t \geq 0 \end{cases} \quad F(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-2t} & t \geq 0 \end{cases}$$

- Find:

1. $Q(0.05)$
2. $Q(0.5)$
3. $Q(p)$ for some p with $0 \leq p \leq 1$

Answers: calculating quantiles

1. $Q(0.05)$:

$$0.05 = P(T \leq Q(0.05)) = F(Q(0.05)) = 1 - e^{-2Q(0.05)}$$

$$0.95 = e^{-2Q(0.05)}$$

$$\log(0.95) = -2Q(0.05)$$

$$Q(0.05) = \frac{\log(0.95)}{-2} \approx 0.0256$$

2. $Q(0.5)$:

$$0.5 = P(T \leq Q(0.5)) = F(Q(0.5)) = 1 - e^{-2Q(0.5)}$$

$$0.5 = e^{-2Q(0.5)}$$

$$\log(0.5) = -2Q(0.5)$$

$$Q(0.5) = \frac{\log(0.5)}{-2} \approx 0.347$$

Answers: calculating quantiles

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3. $Q(p)$

$$p = P(T \leq Q(p)) = F(Q(p)) = 1 - e^{-2Q(p)}$$

$$1 - p = e^{-2Q(p)}$$

$$\log(1 - p) = -2Q(p)$$

$$Q(p) = \frac{\log(1 - p)}{-2}$$

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Expected value

- ▶ The expected value of a continuous random variable is:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

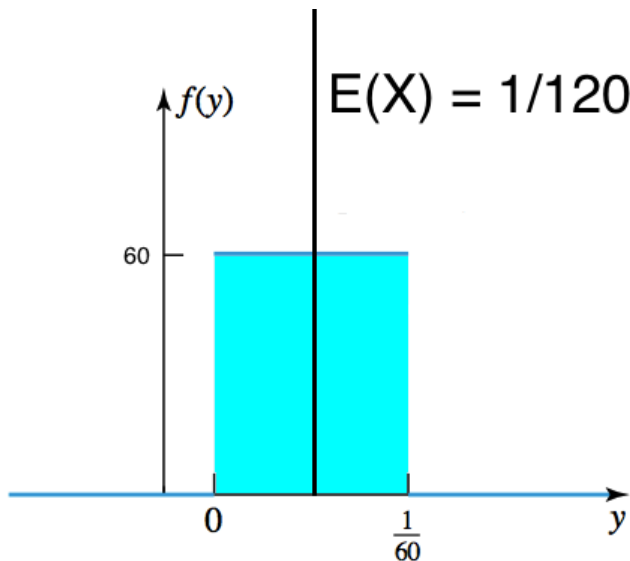
- ▶ As with continuous random variables, $E(X)$ (often denoted by μ) is the mean of X , a measure of center.

Example: time delay, Y

$$f(y) = \begin{cases} 60 & 0 \leq y \leq \frac{1}{60} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} y \cdot f(y) dy \\ &= \int_{-\infty}^0 y \cdot 0 dy + \int_0^{1/60} y \cdot 60 dy + \int_{1/60}^{\infty} y \cdot 0 dy \\ &= 0 + \left(\frac{y^2}{2} \cdot 60 \right)_0^{1/60} + 0 \\ &= \frac{1}{2} \left(\frac{1}{60} \right)^2 \cdot 60 = \frac{1}{120} \end{aligned}$$

$E(X)$ is the “center of mass” of a distribution



Your turn: calculate $E(X)$

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{\alpha} e^{-x/\alpha} & x \geq 0 \end{cases}$$

1. $X \sim \text{Exp}(3)$
2. $X \sim \text{Exp}(\alpha)$

Answers: Calculate $E(X)$

1. $X \sim \text{Exp}(3)$:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x \cdot f(x) dx \\ &= \int_{-\infty}^0 x \cdot 0 dx + \int_0^{\infty} x \cdot \frac{1}{3} e^{-x/3} dx \end{aligned}$$

integration by parts:

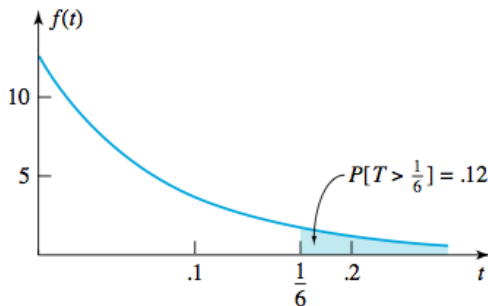
$$\begin{aligned} &= 0 + \left(x(-e^{-x/3}) \right)_0^{\infty} - \int_0^{\infty} (-e^{-x/3}) dx \\ &= \left(-\infty e^{-\infty/3} + 0e^{-0/3} \right) + \int_0^{\infty} e^{-x/3} dx \\ &= 0 + \left(-3e^{-x/3} \right)_0^{\infty} \\ &= \left(-3e^{-\infty/3} + 3e^{-0/3} \right) \\ &= 3 \end{aligned}$$

2. Similarly, $E(X) = \alpha$ when $X \sim \text{Exp}(\alpha)$.

Example: waiting time for the next student to arrive at the library

- ▶ From 12:00 to 12:10 PM, about 12.5 students per minute enter on average.
- ▶ Hence, the average waiting time for the next student is $\frac{1}{12.5} = 0.08$ minutes for the next student.
- ▶ Let $T \sim \text{Exp}(0.08)$ be the time until the next student arrives.
- ▶ $P(\text{wait is more than 10 seconds}) =$

$$P(T > 1/6) = 1 - F(1/6) = 1 - (1 - e^{(-0.08 \cdot 1/6)}) = 0.12$$



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- ▶ The variance of a continuous random variable X is:

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x - E(X))^2 \cdot f(x) dx$$

Shortcut formulas:

$$\begin{aligned}\text{Var}(X) &= \int_{-\infty}^{\infty} x^2 f(x) dx - E^2(X) \\ &= E(X^2) - E^2(X)\end{aligned}$$

- ▶ The standard deviation is $\text{SD}(X) = \sqrt{\text{Var}(X)}$

Your turn: checkout time

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Let X denote the amount of time for which a book on 2-hour reserve at a college library is checked out by a randomly selected student and suppose that X has density function

$$f(x) = \begin{cases} .5x & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Calculate:

1. $E(X)$
2. $\text{Var}(X)$

Answers: checkout time

1.

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x \cdot f(x) dx = \int_0^2 x \cdot \frac{1}{2} x dx \\ &= \frac{1}{2} \int_0^2 x^2 dx = \left(\frac{x^3}{6} \right)_0^2 = \frac{8}{6} \approx 1.333 \end{aligned}$$

2.

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^2 x^2 \frac{1}{2} x dx = \frac{1}{2} \int_0^2 x^3 dx = \left(\frac{x^4}{8} \right)_0^2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E(X^2) - E^2(X) = 2 - \left(\frac{8}{6} \right)^2 \\ &= \frac{2}{9} \end{aligned}$$

Your turn: ecology

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- ▶ An ecologist wishes to mark off a circular sampling region having radius 10 m. However, the radius of the resulting region is actually a random variable R with pdf:

$$f(r) = \begin{cases} \frac{3}{2}(10 - r)^2 & 9 \leq r \leq 11 \\ 0 & \text{otherwise} \end{cases}$$

- ▶ Calculate:

1. $E(R)$
2. $SD(R)$

Answers: ecology

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1.

$$\begin{aligned} E(R) &= \int_{-\infty}^{\infty} r \cdot f(r) dr \\ &= \int_9^{11} r \cdot \frac{3}{2}(10-r)^2 dr \\ &= \int_9^{11} \left(\frac{3}{2}r^3 - 30r^2 + 150r \right) dr \\ &= \left(\frac{3}{8}r^3 - 10r^2 + 75r^2 \right)_9^{11} \\ &= \left(\frac{3}{8}(11)^3 - 10(11)^3 + 75(11)^2 \right) - \left(\frac{3}{8}9^3 - 10(9)^3 + 75(9)^2 \right) \\ &= 10 \end{aligned}$$

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Answers: ecology

2.

$$\begin{aligned} E(R^2) &= \int_{-\infty}^{\infty} r^2 \cdot f(r) dr \\ &= \int_9^{11} r^2 \cdot \frac{3}{2}(10-r)^2 dr \\ &= \int_9^{11} \left(\frac{3}{2}r^4 - 30r^3 + 150r^2 \right) dr \\ &= \left(\frac{3}{10}r^5 - \frac{15}{2}r^4 + 50r^3 \right)_9^{11} \\ &= \left(\frac{3}{10}(11)^5 - \frac{15}{2}(11)^4 + 50(11)^3 \right) - \left(\frac{3}{10}(9)^5 - \frac{15}{2}(9)^4 + 50(9)^3 \right) \\ &= \frac{503}{5} = 100.6 \end{aligned}$$

$$\text{Var}(R) = E(R^2) - E^2(R) = \frac{503}{5} - 10^2 = \frac{3}{5} = 0.6$$

$$\text{SD}(R) = \sqrt{\text{Var}(R)} = \sqrt{0.6} \approx .77$$

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Expectation of a function of a random variable

- ▶ Why does $E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) dx$?
- ▶ It turns out that for any function g of a random variable:

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x) dx$$

- ▶ Hence:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) dx$$

if we take $g(X) = X^2$.

- ▶ In the ecology example, the expected *area* of the circular sampling region is:

$$E(\pi R^2) = \int_{-\infty}^{\infty} \pi r^2 \cdot f(r) dr$$

where $\pi R^2 = g(R)$ is the sampling area.

Expectation of a linear function of X

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- For constants a and b :

$$\begin{aligned} E(aX + b) &= \int_{-\infty}^{\infty} (ax + b) \cdot f(x) dx \\ &= a \underbrace{\int_{-\infty}^{\infty} x \cdot f(x) dx}_{E(X)} + b \underbrace{\int_{-\infty}^{\infty} f(x) dx}_1 \\ &= \boxed{aE(X) + b} \end{aligned}$$

- Example: the expected *diameter* of the ecologist's sampling region is:

$$E(2 \cdot R + 0) = 2 \cdot E(R) + 0 = 2 \cdot 10 = 20$$

Variance of a linear function of X

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- For constants a and b :

$$\begin{aligned}\text{Var}(aX + b) &= E((aX + b)^2) - E^2(aX + b) \\&= E(a^2X^2 + abX + b^2) - (aE(X) + b)^2 \\&= (a^2E(X^2) + abE(X) + b^2) \\&\quad - (a^2E^2(X) + abE(X) + b^2) \\&= a^2(E(X^2) - E^2(X)) \\&= \boxed{a^2\text{Var}(X)}\end{aligned}$$

- Example: the variance of the *diameter* of the ecologist's sampling region is:

$$\text{Var}(2 \cdot R + 0) = 4\text{Var}(R) = 4 \cdot 0.6 = 2.4$$

Standardization

- **Standardization:** converting a random variable X into another random variable Z by subtracting the mean and dividing by the standard deviation:

$$Z = \frac{X - E(X)}{SD(X)}$$

- Z has mean 0:

$$\begin{aligned} E(Z) &= E\left(\frac{X - E(X)}{SD(X)}\right) = E\left(\frac{1}{SD(X)} \cdot X - \frac{E(X)}{SD(X)}\right) \\ &= \frac{1}{SD(X)} \cdot E(X) - \frac{E(X)}{SD(X)} = 0 \end{aligned}$$

- Z has variance (and standard deviation) 1:

$$\begin{aligned} \text{Var}(Z) &= \text{Var}\left(\frac{X - E(X)}{SD(X)}\right) = \text{Var}\left(\frac{1}{SD(X)} \cdot X - \frac{E(X)}{SD(X)}\right) \\ &= \frac{1}{SD^2(X)} \text{Var}(X) = \text{Var}(X) \frac{1}{\text{Var}(X)} = 1 \end{aligned}$$