STAT443 Forecasting - Assignment 3

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1 Theory

Problem 1

Consider the AR(1) model: $Y_t = \phi Y_{t-1} + a_t$, where $a_t \sim i.i.N.[0, \sigma^2]$. Suppose you have the following data:

$$\sum Y_t Y_{t-1} = 800, \quad \sum Y_{t-1}^2 = 1000 \;\; and \;\; \sum \hat{a}_t = 60$$

Show that the estimated model resulting from applying the least squares method is stationary. Use the asymptotic distributino of the least squares estimator

$$\hat{\phi}_{OLS} \overset{asy}{\sim} N\Big[\phi, \hat{\sigma}\frac{1}{\Sigma Y_{t-1}^2}\Big] \tag{1}$$

to construct an approximate 95% confidence interval for the true population coefficient ϕ and explain its meaning. You can assume an effective sample size of 100 observations.

Consider immediately that we are solving for $Y_t = \phi Y_{t-1} + a_t$, so if we are solving for the standard OLS regression of the form $\hat{Y}_t = \hat{\beta}_1 + \hat{\beta}_2 X_t + \hat{a}_t$ we know that $\hat{\beta}_1 = 0$ and $\hat{\beta}_2 = \phi$.

Therefore we have that:

$$\begin{aligned} \beta_1 = 0 &= \bar{Y}_t - \hat{\beta}_2 \bar{Y}_{t-1} \\ \bar{Y}_t &= \hat{\beta}_2 \bar{Y}_{t-1} \end{aligned} \tag{2}$$

and

$$\begin{aligned} \phi = \hat{\beta}_2 &= \frac{\sum (Y_{t-1} - \bar{Y}_{t-1})(Y_t - \bar{Y}_t)}{\sum (Y_{t-1} - \bar{Y}_{t-1})^2} \\ \hat{\beta}_2 &= \frac{\sum Y_{t-1} Y_t - 100\bar{Y}_t \bar{Y}_{t-1}}{\sum Y_{t-1}^2 - 100\bar{Y}_{t-1}^2} \\ \hat{\beta}_2 &= \frac{800 - 100\bar{Y}_t \bar{Y}_{t-1}}{1000 - 100\bar{Y}_{t-1}^2} \\ \hat{\beta}_2 &= \frac{800 - 100(\hat{\beta}_2 \bar{Y}_{t-1})\bar{Y}_{t-1}}{1000 - 100\bar{Y}_{t-1}^2} \\ \hat{\beta}_2(1000 - 100\bar{Y}_{t-1}^2) &= 800 - 100\hat{\beta}_2 \bar{Y}_{t-1}^2 \\ 1000\hat{\beta}_2 &= 800 \\ \hat{\phi} = \hat{\beta}_2 &= 0.8 \end{aligned} \tag{3}$$

Following the above derived information we also have that:

$$var(\hat{\phi}_{OLS}) = \hat{\sigma}^2 \frac{1}{\sum y_{t-1}^2}$$

where

$$\hat{\sigma}^2 = RSS/n = 60/100 = 0.6$$

so

$$var(\hat{\phi}_{OLS}) = \frac{0.6}{1000} = 0.0006$$

Finally:

$$\begin{aligned} 0.95 \approx Pr\Big(\hat{\phi} - 1.96\sqrt{var(\hat{\phi}_{OLS})} < \phi < \hat{\phi} + 1.96\sqrt{var(\hat{\phi}_{OLS})}\Big) \\ 0.95 \approx Pr\Big(0.8 - 1.96\sqrt{0.0006} < \phi < 0.8 + 1.96\sqrt{0.0006}\Big) \\ 0.95 \approx Pr\Big(0.75199 < \phi < 0.84801\Big) \end{aligned} \tag{4}$$

We are 95% certain that the interval $(0.75199, 0.84801)$ contains the true unknown ϕ.

Problem 2

Consider the estimated AR(2) model

$$Y_t = 0.049Y_{t-1} + 0.22Y_{t-2} + \hat{a}_t$$

.

Perform a simple t-test to test the significance of Y_{t-2}. Assume the effective sample size is 164.

Using the following formulas:

$$\sqrt{n}\left(\begin{bmatrix}\hat{\phi}_1 \\ \hat{\phi}_2\end{bmatrix} - \begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix}\right) \overset{asy}{\sim} N\left(\begin{bmatrix}0 \\ 0\end{bmatrix}, \begin{bmatrix}1-\phi_2^2 & -\phi_1(1+\phi_2) \\ -\phi_1(1+\phi_2) & 1-\phi_2^2\end{bmatrix}\right)$$
$$\left(\begin{bmatrix}\hat{\phi}_1 \\ \hat{\phi}_2\end{bmatrix}\right) \overset{asy}{\sim} N\left(\begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix}, \begin{bmatrix}\frac{1-\phi_2^2}{n} & \frac{-\phi_1(1+\phi_2)}{n} \\ \frac{-\phi_1(1+\phi_2)}{n} & \frac{1-\phi_2^2}{n}\end{bmatrix}\right)$$

Using the above formulas we need to perform a t-test, which tests the following hypothesis:

Test $H_0: \ \phi_2 = 0$ versus $H_1: \ \phi_2 \neq 0$

Our t-statistic is defined as $t_{stat} = \frac{\phi_2 \hat{-} \phi_2}{SE[\hat{\phi}_2]}$ and our critical t-value is $t_{crit}(\alpha = 5\%, 2 - sided, df = 164) = 1.974535 \approx 2$.

$$t_{stat} = \frac{0.22 - 0}{\sqrt{(1 - 0.22^2)/164}} = \frac{0.22}{0.07617374} = 2.888134$$

Since $t_{stat} > t_{crit} => 2.888134 > 1.974535$ we reject H_0 at the 5% significance level.

Problem 3

Consider the quartery seasonally adjusted Canadian Personal Expenditure on Consumer Goods and Services raw time series that you used in Assignment 1. The data range covers teh first quarter of the year 1961 till the first quarter of 2007, i.e., 185 observations. Define $X_t = lnW_t$. Extract the cycle Y_t using the TS approach and fit the following AR(2) model to it:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + a_t, \qquad a_t \sim i.i.N[0, \sigma^2]$$

Suppose/verify that you obtained the following estimated model

$$\hat{Y}_t = \underset{(0.0741)}{0.98}\, Y_{t-1} - \underset{(0.0731)}{0.006}\, Y_{t-2} + \hat{a}_t; \qquad \sum \hat{a}_t^2 = 0.0136100383$$

where the figures in the brackets are the corresponding standard errors. Verify that the asymptotic least squares estimator of σ^2 is 0.000074. Construct an approximate 95% confidence interval for ϕ_1. Perform a test of significance for each individual coefficient. Test the restriction that $\phi_1 + \phi_2 = 1$.

Verify that the asymptotic least squares estimator of_ σ^2 *is* 0.000074.

$$\begin{aligned} \hat{\sigma}^2 &= RSS/n \\ \hat{\sigma}^2 &= \frac{\sum \hat{a}_t^2}{n} = \frac{0.0136100383}{185} = 0.000074 \end{aligned} \tag{5}$$

Construct an approximate 95% confidence interval for ϕ_1

$$\begin{aligned} \hat{\phi}_1 &= 0.98 \\ SE[\hat{\phi}_1] &= 0.0741 \end{aligned}$$

$$\begin{aligned} 0.95 &\approx Pr\Big(\hat{\phi} - 1.96SE[\hat{\phi}_1] < \phi_1 < \hat{\phi} + 1.96SE[\hat{\phi}_1]\Big) \\ 0.95 &\approx Pr\Big(0.98 - 1.96(0.0741) < \phi < 0.98 + 1.96(0.0741)\Big) \\ 0.95 &\approx Pr\Big(0.834764 < \phi < 1.125236\Big) \end{aligned} \tag{6}$$

We are 95% certain that the interval $(0.834764, 1.125236)$ contains the true unknown ϕ_1.

Perform a test of significance for each individual coefficient

ϕ_1

Test $H_0: \ \phi_1 = 0$ versus $H_1: \ \phi_1 \neq 0$

Our t-statistic is defined as $t_{stat} = \frac{\phi_1 \hat{-} \phi_1}{SE[\hat{\phi}_1]}$ and our critical t-value is $t_{crit}(\alpha = 5\%, 2-sided, df = 185) = 1.97287 \approx 2$.

$$t_{stat} = \frac{0.98 - 0}{0.0741} = \frac{0.98}{0.0741} = 13.22537$$

Since $t_{stat} > t_{crit} => |13.22537| > 1.97287$ we reject H_0 at the 5% significance level.

ϕ_2

Test $H_0: \ \phi_2 = 0$ versus $H_1: \ \phi_2 \neq 0$

Our t-statistic is defined as $t_{stat} = \frac{\hat{\phi_2} - \phi_2}{SE[\hat{\phi_2}]}$ and our critical t-value is $t_{crit}(\alpha = 5\%, 2 - sided, df = 185) = 1.97287 \approx 2$.

$$t_{stat} = \frac{-0.006 - 0}{0.0731} = \frac{-0.006}{0.0731} = -0.08207934$$

Since $t_{stat} < t_{crit} => |-0.08207934| < 1.97287$ we do not reject H_0 at the 5% significance level.

Test the restriction that $\phi_1 + \phi_2 = 1$

Test $H_0: \ \phi_1 + \phi_2 = 1$ versus $H_1: \ \phi_1 + \phi_2 \neq 1$

Our t-statistic is defined as $t_{stat} = \frac{\phi_1 \hat{+} \phi_2 - (\phi_1 + \phi_2)}{SE[\phi_1 \hat{+} \phi_2]}$ and our critical t-value is $t_{crit}(\alpha = 5\%, 2 - sided, df = 185) = 1.97287 \approx 2$.

$$t_{stat} = \frac{0.974 - 1}{\sqrt{2 * 0.0054056 + 2 * -0.005265514}} = \frac{-0.026}{0.01673834} = -1.55332$$

Since $t_{stat} < t_{crit} => |-1.55332| < 1.97287$ we do not reject H_0 at the 5% significance level.

Problem 4

Consider the AR(2) model:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + a_t, \qquad a_t \sim i.i.N[0, \sigma^2]$$

where the parameter vector $\theta' = (\phi_1, \phi_2, \sigma^2) \in \Theta$, and Θ is the parameter space. Based on an effective sample S_n of size n given as

$$S_n = \{Y_n, Y_{n-1}, ..., Y_1, Y_0, Y_{-1}\}$$

where Y_0 and Y_{-1} *are the starting values, i.e., the information set $I_0 = \{Y_0, Y_{-1}\}$. Show that the approximate log-likelihood function is*

$$l(\theta) = -\frac{n}{2} ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^{n} (Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2})^2$$

Use $l(\theta)$ to derive expressions for the asymptotic maximum likelihood estimators $(\hat{\phi}_1)_{ML}$, $(\hat{\phi}_2)_{ML}$, and $(\hat{\sigma}^2)_{ML}$. *Show that*

$$\sqrt{n}\left(\begin{bmatrix} (\hat{\phi}_1)_{ML} \\ (\hat{\phi}_2)_{ML} \end{bmatrix} - \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \right) \overset{asy}{\sim} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 - \phi_2^2 & -\phi_1(1 + \phi_2) \\ -\phi_1(1 + \phi_2) & 1 - \phi_2^2 \end{bmatrix} \right)$$

Our objective is to find the joint density of the sequence $\{Y_t\}_{t=1}^n$.

We cannot use the product of a marginal distribution though because the lags of an AR model are correlated. Instead we will use conditional probability from Bayes Theorem. I.e. $P(A, B) = P(A|B)P(B)$.

If our sample set S_n is given as:

$$S_n = \{Y_n, Y_{n-1}, ..., Y_1, Y_0, Y_{-1}\}$$

Then define $S_{n-1} = \{Y_{n-1}, ..., Y_1, Y_0, Y_{-1}\}$, and this pattern should hold for all n until S_0.

Combining this with Bayes Theorem and Recursive Substitution we have that:

$$\begin{aligned}
f(Y_n, S_{n-1}) &= f(Y_n|S_{n-1})f(S_{n-1}) \\
f(Y_n, S_{n-1}) &= f(Y_n|S_{n-1})f(Y_{n-1}|S_{n-2})f(S_{n-2}) \\
f(Y_n, S_{n-1}) &= f(Y_n|S_{n-1})f(Y_{n-1}|S_{n-2})f(Y_{n-2}|S_{n-3})f(S_{n-3}) \\
&... \\
f(Y_n, S_{n-1}) &= f(Y_n|S_{n-1})f(Y_{n-1}|S_{n-2})f(Y_{n-2}|S_{n-3}) \cdot ... \cdot f(Y_1|S_0)f(S_0)
\end{aligned} \tag{7}$$

Where $f(S_0)$ is the density of the starting values.

Since $a_t \sim N[0, \sigma^2]$, then by *Gaussian Law* and since the first 2 values of AR(2) are known, $Y_t \overset{asy}{\sim} N$.

-> $Y_t|_{S_{t-1}=\{Y_{t-1},Y_{t-2}\}} \sim N[\phi_1 Y_{t-1} + \phi_2 Y_{t-2}, \sigma^2]$

and

-> $f(Y_t|S_{t-1}) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\Big\{ - \frac{1}{2\sigma^2} a_t^2 \Big\}$ for any t.

We now have that:

$f(Y_n, Y_{n-1}, ..., Y_1; Y_0, Y_{-1}|\theta) = (2\pi\sigma^2)^{-n/2} \cdot exp\Big\{ - \frac{1}{2\sigma^2} \sum a_t^2 \Big\} \cdot f(S_0)$ is the joint density function of Y_t

Define $L(\theta|Y_n, Y_{n-1}, ..., Y_1; Y_0, Y_{-1}) = (2\pi\sigma^2)^{-n/2} \cdot exp\Big\{ - \frac{1}{2\sigma^2} \sum a_t^2 \Big\} \cdot f(S_0)$ as the likelihood function.

Since $ln(x)$ is a globally increasing function of x, we can:

$$arg\ max\ l(\cdot) = logL(\cdot)$$

<–>

$$arg\ max\ l(\cdot) = \frac{n}{2}log(2\pi) - \frac{n}{2}log(\sigma^2) - \frac{1}{2\sigma^2}\sum a_t^2 + log(f(S_0))$$

- $\frac{n}{2}log(2\pi)$ is a constant
- $log(f(S_0))$ is asymptotically negligible
- and by definition $\hat{a}_t = Y_t - \hat{Y}_t = Y_t - \hat{\phi}_1 Y_{t-1} - \hat{\phi}_2 Y_{t-2}$

Therefore we have that:

$$l(\theta) = -\frac{n}{2}ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}(Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2})^2$$

We need to set the First Order Equations to zero and solve for our parameters i.e. $\frac{\delta l}{\delta \phi_1} = 0$, $\frac{\delta l}{\delta \phi_1} = 0$, and $\frac{\delta l}{\delta \sigma} = 0$ these equations become quite complicated however. A simpler method is to notice that maximizing $l(\theta)$ is the same as minimizing $\sum a_t^2$

Because Y_t is Gaussian we know that MLE and OLS are asymptotically equivalent therefore:

$$\begin{bmatrix}(\hat{\phi}_1)_{ML} \\ (\hat{\phi}_2)_{ML}\end{bmatrix} = \begin{bmatrix}(\hat{\phi}_1)_{OLS} \\ (\hat{\phi}_2)_{OLS}\end{bmatrix}$$

So we have that:

$$\sqrt{n}\left(\begin{bmatrix}(\hat{\phi}_1)_{ML} \\ (\hat{\phi}_2)_{ML}\end{bmatrix} - \begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} \right) \overset{asy}{\sim} N\left(\begin{bmatrix}0 \\ 0\end{bmatrix}, \begin{bmatrix} 1-\phi_2^2 & -\phi_1(1+\phi_2) \\ -\phi_1(1+\phi_2) & 1-\phi_2^2 \end{bmatrix} \right)$$

Problem 5

Suppose you estimated the following AR(2) model:

$$\hat{Y}_t = \underset{(0.0774)}{0.049}\, Y_{t-1} + \underset{(0.0770)}{0.23}\, Y_{t-2} + \hat{a}_t, \qquad n = 164, \qquad RSS = 0.0000617$$

where the figures between brackets are the corresponding standard errors. Suppose, however, that you are not sure whether the AR(2) model is adequate. You decided to overfit an AR(4) over the same sample and test the null-hypothesis that Y_t follows an AR(2). Let the estimated regression results for fitting the AR(4) model be

$$\hat{Y}_t = \underset{(0.029)}{0.044} Y_{t-1} + \underset{(0.037)}{0.25}\, Y_{t-2} + \underset{(0.04)}{0.035} Y_{t-3} + \underset{(0.045)}{0.094} Y_{t-4} + \hat{a}_t, \qquad n = 164, \qquad RSS = 0.0000611$$

Use the LRT to test, at the 5% significance, $H_0 : Y_t \sim AR(2)$ versus $H_1 : Y_t \sim AR(4)$, and make a decision by either comparing the statistic to the critical value or by computing the p-value of the test.

We need to test that $H_0 : AR(2)$ or $H_0 : \phi_3 = \phi_4 = 0$ versus $H_1 : AR(4)$.

We would like to test against the $\chi^2_{q,test}$ using the Likelihood Ratio Test:

$$LRT_{stat} = n \cdot ln\left(\frac{\hat{\sigma}^2_R}{\hat{\sigma}^2_{MLE}}\right)$$
$$q = the\ number\ of\ restrictions = 2$$

We also have that $R_{q\times p}\Phi_{p\times 1} = r_{q\times 1}$.

Under our H_0, the quantity $LRT_{stat} = -2ln(\Lambda) \overset{asy}{\sim} \chi^2(2)$

Where $\Lambda = \frac{L(\theta_R)|_{\theta_R=\hat{\theta}_R}}{L(\theta)|_{\theta=\hat{\theta}}}$, $\hat{\theta} = \begin{bmatrix}\hat{\Phi}\\ \hat{\sigma}^2\end{bmatrix}$, and $\hat{\theta}_R = \begin{bmatrix}\hat{\Phi}_R\\ \hat{\sigma}^2_R\end{bmatrix}$

$$\begin{aligned}
\Lambda &= \frac{(2\pi\hat{\sigma}^2_R)^{-n/2} exp\{-\frac{1}{2\hat{\sigma}^2_R}\sum a^2_{R_t}\}}{(2\pi\hat{\sigma}^2)^{-n/2} exp\{-\frac{1}{2\hat{\sigma}^2}\sum a^2_t\}},\ \hat{\sigma}^2_R = \sum a^2_{R_t}/n,\ \hat{\sigma}^2 = \sum a^2_t/n \\
\Lambda &= \left(\frac{\hat{\sigma}^2_R}{\hat{\sigma}^2}\right)^{-n/2} \\
\Lambda &= \left(\frac{RSS_R/n}{RSS/n}\right)^{-n/2} \\
\Lambda &= \left(\frac{RSS_R}{RSS}\right)^{-n/2}
\end{aligned} \tag{8}$$

Therefore:

$$\begin{aligned}
LRT_{stat} &= -2(-n/2)ln(RSS_R/RSS) \sim \chi^2(2) \\
LRT_{stat} &= nln(RSS_R/RSS) \sim \chi^2(2) \\
LRT_{stat} &= 164ln(0.0000617/0.0000611) \sim \chi^2(2) \\
LRT_{stat} &= 1.602619 \sim \chi^2(2)
\end{aligned} \tag{9}$$

(Our $\chi^2(2)_{crit} \approx 5.991465$ therefore we *do not* reject our $H_0 : \phi_3 = \phi_4 = 0$ because $\chi^2(2)_{stat} < \chi^2(2)_{crit}$ -> $1.602619 < 5.991465$.

Problem 6

Suppose you have the following data pertaining to an AR(3) representation:

$$\sum Y_t^2 = 108, \quad \sum Y_{t-1}^2 = 5, \quad \sum Y_{t-2}^2 = 55, \quad \sum Y_{t-3}^2 = 129,$$
$$\sum Y_{t-1}Y_{t-2} = 15, \quad \sum Y_{t-1}Y_{t-3} = 25, \quad \sum Y_{t-2}Y_{t-3} = 81,$$
$$\sum Y_{t-1}Y_t = 20, \quad \sum Y_{t-2}Y_t = 76, \quad \sum Y_{t-3}Y_t = 109, \quad n = 30$$

Test asumptotically the claim that the true model is AR(1) using the LRT test.

Begin by calculating $\Phi = (\sum \bar{Y}^t \bar{Y})^{-1} \sum \bar{Y} Y_t$:

$$\Phi = \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \hat{\phi}_3 \end{bmatrix} = \begin{bmatrix} 5 & 15 & 25 \\ 15 & 55 & 81 \\ 25 & 81 & 129 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 76 \\ 109 \end{bmatrix} = \begin{bmatrix} 4 \\ 2.5 \\ -1.5 \end{bmatrix}$$

Then we can calculate the unrestricted RSS given by:

$$RSS_U = \sum \hat{a}_t^2 = \sum Y_t^2 - \hat{\Phi}^T (\sum \bar{Y} Y_t) = 108 - \begin{bmatrix} 4 & 2.5 & -1.5 \end{bmatrix} \begin{bmatrix} 20 \\ 76 \\ 109 \end{bmatrix} = 108 - 106.5 = 1.5$$

Next using $R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $r = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we can calculate our restricted model's Phi's using:

$$\hat{\Phi}_R = \hat{\Phi} + (\sum \bar{Y}^T \bar{Y})^{-1} R^T (R(\sum \bar{Y}^T \bar{Y})^{-1} R^T)^{-1} (r - R\Phi) = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

I calculated $\hat{\phi}_R$ in R using `Phi + solve(X) %*% t(R) %*% solve(R %*% solve(X) %*% t(R)) %*% (r - R %*% Phi)` to get approximately $\hat{\phi}_R = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$

Next we calculate (in R) our restricted model's RSS using the following:

$$RSS_R = RSS + (\hat{\Phi} - \hat{\Phi}_R)^T (\sum \bar{Y} \bar{Y}^T)(\hat{\Phi} - \hat{\Phi}_R) = 28$$

Finally,

$$LRT_{stat} = n \cdot ln(\frac{\sigma_R^2}{\sigma^2}) = 30ln(\frac{RSS_R}{RSS}) = 30ln(\frac{28}{1.5}) \approx 87.80218 \sim \chi^2(2)$$

Since $LRT_{stat} > LRT_{crit}$ we reject the null hypothesis that $H_0 : \phi_2 = \phi_3 = 0$

2 Practicum

Use the seasonally adjusted series you worked with on Assignment 1 to perform the following tasks: Construct a measure of the business cycle Y_t using the Trend Stationary approach and for an AR(p) process using both the BIC(k) and AIC(k) to estimate p. Include standard errors in brackets under you estimates of $\hat{\phi}_j$, and provide an astimate of σ. Perform a likelihood ratio test at the 5% level of

$H_0 : Y_t \sim AR(p)$ versus $H_1 : Y_t \sim AR(p+2)$

where p is chosen from the BIC(k). Calculate the p-value of the test statistic. From the estimated AR(p) model calculate $\gamma(0)^{1/2}$. Also calculate 1) the infinite moving average weights ψ_k, 2) the autocorrelations $\rho(k)$, 3) the forecasts $E_T[Y_{T+k}]$, and 4) the confidence intervals for the forecasts for $k = 0, 1, 2, 3, 4, 5, 6$. Do not just give numbers but explain how you made your calculations. Put your calculations in one easy-to-read table. (This is a good practice on writing your report.)

When running our AR(p) model fits for p={0,..,9} we find that the optimal model according to both AIC and BIC, is an AR(4) model:

```
## [1] 4
```

```
## [1] 4
```

Our lag 4 model results including standard errors:

```
##
## Call:
## arima(x = Y_t, order = c(which(aic.array == min(aic.array)) - 1, 0, 0), include.mean = FALSE)
##
## Coefficients:
##          ar1     ar2     ar3      ar4
##       0.9669  0.1874  0.0925  -0.2563
## s.e.  0.0711  0.1006  0.1009   0.0720
##
## sigma^2 estimated as 6.961e-05:  log likelihood = 620.6,  aic = -1231.2
```

$$\underset{(s.e.)}{\hat{Y}_t} = \underset{(0.0711)}{\hat{\phi}_1} Y_{t-1} + \underset{(0.1006)}{\hat{\phi}_2} Y_{t-2} + \underset{(0.1009)}{\hat{\phi}_3} Y_{t-3} + \underset{(0.0720)}{\hat{\phi}_4} Y_{t-4} = 0.9669Y_{t-1} + 0.1874Y_{t-2} + 0.0925Y_{t-3} - 0.2563Y_{t-4}$$

An estimate for $\hat{\sigma}$ is given by summing the squared residuals and dividing by the length of our data:

```
## [1] "sigma^2:"
```

```
## [1] 6.961005e-05
```

```
## [1] "sigma:"
```

```
## [1] 0.008343264
```

Likelihood Ratio Test

We next want to do a comparison using the Likelihood Ratio Test for an $AR(4)$ versus $AR(6)$.

Therefore $H_0 : AR(2)$ or $H_0 : \phi_5 = \phi_6 = 0$ and $H_1 : AR(6)$

```
## [1] "chi-statistic"
```

```
## [1] 2.230767
```

```
## [1] "p-value for test"
```

```
## [1] 0.3277896
```

with a p-value = 0.333717, we *do not* reject the null hypothesis that $H_0 : \phi_5 = \phi_6 = 0$

Summary AIC & BIC Table

```
##     p   AIC   BIC
## 1  0 -5.57 -5.57
## 2  1 -9.47 -9.45
## 3  2 -9.46 -9.42
## 4  3 -9.47 -9.42
## 5  4 -9.53 -9.46
## 6  5 -9.52 -9.43
## 7  6 -9.52 -9.42
## 8  7 -9.51 -9.39
## 9  8 -9.51 -9.37
## 10 9 -9.51 -9.35
```

The value of $\gamma(0)^{1/2}$

```
##             [,1]
## [1,] 0.07774875
```

1) Calculate the Infinite Moving Average Weights

```
## [1] 4 3 2 1
```

The ψ_k's are calculated using the following formulas: $\psi_0 = 1$ $\psi_1 = \phi_1\psi_0$ $\psi_2 = \phi_1\psi_1 + \phi_2\psi_0$ $\psi_3 = \phi_1\psi_2 + \phi_2\psi_1 + \phi_3\psi_0$ $\psi_4 = \phi_1\psi_3 + \phi_2\psi_2 + \phi_3\psi_1 + \phi_4\psi_0$ $\psi_5 = \phi_1\psi_4 + \phi_2\psi_3 + \phi_3\psi_2 + \phi_4\psi_1$ $\psi_6 = \phi_1\psi_5 + \phi_2\psi_4 + \phi_3\psi_3 + \phi_4\psi_2$

2) Calculate the Autocorrelations $\rho(k)$

```
##         0         1         2         3         4         5         6
## 1.0000000 0.9936398 0.9870772 0.9784463 0.9666531 0.9546559 0.9417295
```

Our ρ_k are taken directly from the `ARMAacf(estimates,lag.max= 6)` function output.

3) Calculate the forecasts $E_T[Y_{T+k}]$

```
##         185
## -0.01524047
```

We have an $AR(4)$ model, so to calculate $E_T[Y_{T+k}]$ we used the following formula $E_T[Y_{T+k}] = \phi_1 Y_{T+k-1} + \phi_2 Y_{T+k-2} + \phi_3 Y_{T+k-3} + \phi_4 Y_{T+k-4}$ for $Y_T = Y_{186:191}$ and we already know $Y_{T=185}$ so we did not need to calculate for $k = 0$, because $E_T[Y_{185}]$ is given.

4) Calculate the confidence intervals for the forecasts for $k = 0, 1, 2, 3, 4, 5, 6$

To calculate the Confidence Intervals for the forecasts $k = 0, ..., 6$ we need to:

1) Calculate $var(E_T[Y_{T+k}]) = \hat{\sigma}^2 \cdot \sum_{j=1}^{k} \psi_j^2$ for each k
2) Calculate the lower bound using $E_T[Y_{T+k}] - 2\sqrt{var(E_T[Y_{T+k}])}$
3) Calculate the upper bound using $E_T[Y_{T+k}] + 2\sqrt{var(E_T[Y_{T+k}])}$

Summary Data

All of the above four sets of calculations are summarized in the table below:

	k	Infinite_Moving_Average_Weights	Autocorrelation	Forecasts	Lower_CI	Upper_CI
0	0	1.000	1.000	-0.015	-0.032	0.001

	k	Infinite_Moving_Average_Weights	Autocorrelation	Forecasts	Lower_CI	Upper_CI
1	1	0.967	0.994	-0.014	-0.037	0.010
2	2	1.122	0.987	-0.013	-0.043	0.017
3	3	1.359	0.978	-0.012	-0.049	0.026
4	4	1.357	0.967	-0.011	-0.055	0.032
5	5	1.423	0.955	-0.011	-0.061	0.039
6	6	1.468	0.942	-0.010	-0.066	0.045

Appendix Code

```r
library(xlsx)
library(dplyr)
CPE_Cons_Goods  <- read.xlsx("CONS_Canada.xls", sheetName = "Sheet3")
CPE_Cons_Goods <- CPE_Cons_Goods %>% mutate(X_t = log(Seasonally.Adjusted))
CPE_Cons_Goods <- CPE_Cons_Goods[complete.cases(CPE_Cons_Goods),]
model.CPE <- lm(X_t ~ seq_along(CPE_Cons_Goods$X_t) , data = CPE_Cons_Goods)

# Yt is the residuals of regression
Y_t <- residuals(model.CPE)

## Defining your own AIC + BIC function rather than using R's.
AIC <- function(res, k, N){
  aic <-log(sum(res^2) / N)
  aic <- aic + 2 * k /N
  aic
}

BIC <- function(res, k, N){
  bic <-log(sum(res^2) / N)
  bic <- bic +  k *log(N)/N
  bic
}


#obtaining the aic and bic
aic.array = rep(NA, 10)
bic.array = rep(NA, 10)
N = length(Y_t)
for(ii in 0:9) {
  model.arima = arima(Y_t, order = c(ii, 0, 0), include.mean=FALSE) # AR(ii) model
  res.arima = model.arima$residuals
  aic.array[ii + 1] = AIC(res.arima, ii, N)
  bic.array[ii + 1] = BIC(res.arima, ii, N)
}

which(aic.array == min(aic.array))-1
which(bic.array == min(bic.array))-1


model.arima = arima(Y_t, order = c(which(aic.array == min(aic.array))-1, 0, 0), include.mean=FALSE)
model.arima
```

```r
sigma.hat = sum(model.arima$residuals^2)/N
print("sigma^2:")
sigma.hat
print("sigma:")
sqrt(sigma.hat)


model.arima.a <- arima(Y_t, order = c(6, 0, 0), include.mean=FALSE)
sigma.hat.a <- sum(model.arima.a$residuals^2)/N
print("chi-statistic")
Chi_2 = N * log(sigma.hat / sigma.hat.a)
Chi_2
print("p-value for test")
pval_2 = 1- pchisq(Chi_2, 2)
pval_2


# Puttng AIC/BIC into a table
AIC_table <- data.frame(cbind(c(0:9),aic.array,bic.array))
rownames(AIC_table) <- c()
colnames(AIC_table) <- c("p", "AIC","BIC")
round(AIC_table, digits=2)


estimates <-  model.arima$coef # Estimates from AR(X) model. Find X with AIC/BIC
rhos <- ARMAacf(estimates,lag.max= 4) ### REPLACE 0 WITH X
## Extracting the theoretical acf values for k=0,..,X.
phi <- model.arima$coef ### equal to AR(X) model coefficients -> phi hats
gamma0 <- (sigma.hat^2 / (1 - phi %*% rhos[-1])) ### gamma0, fill in '...'
sqrt(gamma0)

#### 1) Calculate the Infinite Moving Average Weights

# calucluating psi's, the infinite MA weights
psis <- rep(NA,7)
psis[1] <- 1
psis[2] <- phi[1]
psis[3] <- phi[1] * psis[2] + phi[2]
psis[4] <- phi[1] * psis[3] + phi[2] * psis[2] + psis[1] * phi[3]
j = 5
(j-1):-1:(j-4)
for( j in 5:7) {
  psis[j] <- psis[(j-1):-1:(j-4)] %*% phi
}

#### 2) Calculate the Autocorrelations $\rho(k)$
rhos

#### 3) Calculate the forecasts $E_T[Y_{T+k}]$
lastYt <- Y_t[length(Y_t)]
lastYt
Etk <- rep(NA,7)
```

```
Etk[1] <- lastYt
Etk[2] <- phi[1] * Y_t[length(Y_t)] + phi[2] * Y_t[length(Y_t)-1] + phi[3] * Y_t[length(Y_t)-2] + phi[4
Etk[3] <- phi[1] * Etk[2] + phi[2] * Etk[1] + phi[3] * Y_t[length(Y_t)-1] + phi[4]*Y_t[length(Y_t)-2]
Etk[4] <- phi[1] * Etk[3] + phi[2] * Etk[2] + phi[3] * Etk[1] + phi[4]*Y_t[length(Y_t)-1]
Etk[5] <- phi[1] * Etk[4] + phi[2] * Etk[3] + phi[3] * Etk[2] + phi[4]*Etk[1]
Etk[6] <- phi[1] * Etk[5] + phi[2] * Etk[4] + phi[3] * Etk[3] + phi[4]*Etk[2]
Etk[7] <- phi[1] * Etk[6] + phi[2] * Etk[5] + phi[3] * Etk[4] + phi[4]*Etk[3]

#### 4) Calculate the confidence intervals for the forecasts for $k = 0, 1, 2, 3, 4, 5, 6$
varEtk <- rep(NA,7)
psis
for(j in 1:7){
## varEtk[j] = sigma.hat * sum(psis^2 to j)
  varEtk[j] = sigma.hat * sum(psis[1:j]^2)
  }

EtkCIlow <- rep(NA,7)
EtkCIhigh <- rep(NA,7)
### fill in (using a for loop recommended)
for(j in 1:7){
  EtkCIlow[j] <- Etk[j] - 2 * sqrt(varEtk[j])
  EtkCIhigh[j] <- Etk[j] + 2 * sqrt(varEtk[j])
}

#### Summary Data
summaryData <- data.frame(k=0:6, Infinite_Moving_Average_Weights= psis, Autocorrelation = c(rhos, NA, N
summaryData
```