STAT443 Forecasting - Assignment 2

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1 Theory

Problem 1

Suppose that Y_t is stationary, Gaussian, with zero mean. Let $I_t = \{Y_t, Y_{t-1}\}$. Show that the optimal linear forecast of Y_{t+2} is

$$\hat{Y}_{t+2} = \left(\frac{\rho(2) - \rho(3)\rho(1)}{1 - \rho^2(1)}\right) Y_t + \left(\frac{\rho(3) - \rho(2)\rho(1)}{1 - \rho^2(1)}\right) Y_{t-1}$$

Begin with:

$$\hat{Y}_{t+2} = \alpha + \phi_0 Y_t + \phi_1 Y_{t-1}$$

but $\alpha = 0$ to ensure that $\hat{Y}_{t+2}$ is unbiased.

The forecasting error e_{t+2} is:

$$e_{t+2} = Y_{t+2} - \hat{Y}_{t+2} = Y_{t+2} - \phi_0 Y_t - \phi_1 Y_{t-1}$$

Our goal is to minimize $var(e_{t+2})$:

$$\begin{aligned}
S(\phi_0, \phi_1) &= var(e_{t+2}) = E\Big[\Big((Y_{t+2} - \phi_0 Y_t - \phi_1 Y_{t-1}) - E[Y_{t+2} - \phi_0 Y_t - \phi_1 Y_{t-1}]\Big)^2\Big] \\
S(\phi_0, \phi_1) &= E\Big[(Y_{t+2} - \phi_0 Y_t - \phi_1 Y_{t-1})^2\Big] \\
S(\phi_0, \phi_1) &= E\big[Y_{t+2}^2\big] - 2\phi_0 E\big[Y_{t+2}Y_t\big] - 2\phi_1 E\big[Y_{t+2}Y_{t-1}\big] + \phi_0^2 E\big[Y_t^2\big] + 2\phi_0\phi_1 E\big[Y_t Y_{t-1}\big] + \phi_1^2 E\big[Y_{t-1}^2\big] \\
S(\phi_0, \phi_1) &= \gamma(0) - 2\phi_0\gamma(2) - 2\phi_1\gamma(3) + \phi_0^2\gamma(0) + 2\phi_0\phi_1\gamma(1) + \phi_1^2\gamma(0)
\end{aligned} \tag{1}$$

Next we take the partial derivatives with respect to ϕ_0 and ϕ_1 and set $\frac{\delta S}{\delta \phi_0}$ and $\frac{\delta S}{\delta \phi_1}$ equal to 0.

$$\begin{aligned}
\frac{\delta S}{\delta \phi_0} &= -2\gamma(2) + 2\phi_0\gamma(0) + 2\phi_1\gamma(1) = 0 \\
\phi_0 &= \frac{\gamma(2) - \phi_1\gamma(1)}{\gamma(0)}
\end{aligned} \tag{2}$$

and

$$\begin{aligned}
\frac{\delta S}{\delta \phi_1} &= -2\gamma(3) + 2\phi_0\gamma(1) + 2\phi_1\gamma(0) = 0 \\
\phi_1 &= \frac{\gamma(3) - \phi_0\gamma(1)}{\gamma(0)}
\end{aligned} \tag{3}$$

Using substitution:

$$\begin{aligned}
\phi_0 &= \frac{\gamma(2) - \frac{\gamma(3)-\phi_0\gamma(1)}{\gamma(0)}\gamma(1)}{\gamma(0)} \\
\phi_0 &= \rho(2) - (\rho(3) - \phi_0\rho(1))\rho(1) \\
\phi_0 &= \rho(2) - \rho(3)\rho(1) + \phi_0\rho(1)^2 \\
\phi_0 - \phi_0\rho(1)^2 &= \rho(2) - \rho(3)\rho(1) \\
\phi_0(1-\rho(1)^2) &= \rho(2) - \rho(3)\rho(1) \\
\phi_0 &= \frac{\rho(2)-\rho(3)\rho(1)}{1-\rho(1)^2}
\end{aligned} \tag{4}$$

and then

$$\begin{aligned}
\phi_1 &= \frac{\gamma(3)-\phi_0\gamma(1)}{\gamma(0)} \\
\phi_1 &= \rho(3) - \phi_0\rho(1) \\
\phi_1 &= \rho(3) - \frac{\rho(2)-\rho(3)\rho(1)}{1-\rho(1)^2}\rho(1) \\
\phi_1 &= \frac{\rho(3)(1-\rho(1)^2)}{1-\rho(1)^2} - \frac{\rho(2)\rho(1)-\rho(3)\rho(1)^2}{1-\rho(1)^2} \\
\phi_1 &= \frac{\rho(3)(1-\rho(1)^2) - \rho(2)\rho(1) + \rho(3)\rho(1)^2}{1-\rho(1)^2} \\
\phi_1 &= \frac{\rho(3)-\rho(2)\rho(1)}{1-\rho(1)^2}
\end{aligned} \tag{5}$$

We now need to check that what we have found is in fact a minimum, which can be shown if the determinant of the Hessian Matrix is greater than zero, defined by:

$$\begin{aligned}
det\left(\begin{bmatrix} \frac{\delta^2 S}{\delta\phi_0^2} & \frac{\delta^2 S}{\delta\phi_0\delta\phi_1} \\ \frac{\delta^2 S}{\delta\phi_1\delta\phi_0} & \frac{\delta^2 S}{\delta\phi_1^2} \end{bmatrix}\right) &= \frac{\delta^2 S}{\delta\phi_0^2}\frac{\delta^2 S}{\delta\phi_1^2} - \frac{\delta^2 S}{\delta\phi_0\delta\phi_1}\frac{\delta^2 S}{\delta\phi_1\delta\phi_0} \\
det\left(\begin{bmatrix} 2\gamma(0) & 2\gamma(1) \\ 2\gamma(1) & 2\gamma(0) \end{bmatrix}\right) &= 4\gamma(0)^2 - 4\gamma(1)^2
\end{aligned} \tag{6}$$

and we know that $\gamma(k) < \gamma(0)$ because of $var(Y_t - Y_{t+k}) = \gamma(0) + \gamma(0) - 2\gamma(k) \geq 0 \quad \forall k > 0$ and we can drop the equality because of the Law of Imperfect Prediction, which implies that:

$$\begin{aligned}
\gamma(0) &> \gamma(k) \\
\gamma(0)^2 &> \gamma(k)^2 \\
4\gamma(0)^2 &> 4\gamma(k)^2 \\
4\gamma(0)^2 - 4\gamma(k)^2 &> 0
\end{aligned} \tag{7}$$

Finally we need to show that e_{t+2} is independent of $I_t = \{Y_t, Y_{t-1}\}$ which we can do by checking that the covariance is zero. This is sufficient because e_{t+2}, Y_t, Y_{t-1} are Gaussian:

$$
\begin{aligned}
cov(e_{t+2}, Y_t) &= E[(e_{t+2} - E[e_{t+2}])(Y_t - E[Y_t])] \\
cov(e_{t+2}, Y_t) &= E[(Y_{t+2} - \phi_0 Y_t - \phi_1 Y_{t-1})(Y_t)] \\
cov(e_{t+2}, Y_t) &= E[Y_{t+2}Y_t - \phi_0 Y_t^2 - \phi_1 Y_{t-1}Y_t] \\
cov(e_{t+2}, Y_t) &= E[Y_{t+2}Y_t] - \phi_0 E[Y_t^2] - \phi_1 E[Y_{t-1}Y_t] \\
cov(e_{t+2}, Y_t) &= \gamma(2) - \phi_0\gamma(0) - \phi_1\gamma(1) \\
cov(e_{t+2}, Y_t) &= \gamma(2) - (\frac{\gamma(2) - \phi_1\gamma(1)}{\gamma(0)})\gamma(0) - \phi_1\gamma(1) \\
cov(e_{t+2}, Y_t) &= \gamma(2) - \gamma(2) + \phi_1\gamma(1) - \phi_1\gamma(1) = 0
\end{aligned}
\tag{8}
$$

and

$$
\begin{aligned}
cov(e_{t+2}, Y_{t-1}) &= E[(e_{t+2} - E[e_{t+2}])(Y_{t-1} - E[Y_{t-1}])] \\
cov(e_{t+2}, Y_{t-1}) &= E[(Y_{t+2} - \phi_0 Y_t - \phi_1 Y_{t-1})(Y_{t-1})] \\
cov(e_{t+2}, Y_{t-1}) &= E[Y_{t+2}Y_{t-1} - \phi_0 Y_t Y_{t-1} - \phi_1 Y_{t-1}^2] \\
cov(e_{t+2}, Y_{t-1}) &= E[Y_{t+2}Y_{t-1}] - \phi_0 E[Y_t Y_{t-1}] - \phi_1 E[Y_{t-1}^2] \\
cov(e_{t+2}, Y_{t-1}) &= \gamma(3) - \phi_0\gamma(1) - \phi_1\gamma(0) \\
cov(e_{t+2}, Y_{t-1}) &= \gamma(3) - \phi_0\gamma(1) - (\frac{\gamma(3) - \phi_0\gamma(1)}{\gamma(0)})\gamma(0) \\
cov(e_{t+2}, Y_{t-1}) &= \gamma(3) - \phi_0\gamma(1) - \gamma(3) + \phi_0\gamma(1) = 0
\end{aligned}
\tag{9}
$$

Both covariances equal 0, therefore e_{t+2} is independent of $I_t = \{Y_t, Y_{t-1}\}$.

Problem 2

Suppose Y_t is Gausssian. Let $I_t = \{Y_{t-1}, Y_{t-2}\} = \{0.02, 0.01\}$, $\gamma(0) = 0.0016$, $\rho(1) = \frac{2}{3}$, and $\rho(2) = \frac{1}{3}$. Construct a 95% confidence interval for Y_t and explain its meaning.

From Theorem 44 (Sampson 2001), if the stationary stochastic process is Gaussian (normally distributed) a 95% confidence interval for Y_{t+k} is:

$$E_t\big[Y_{t+k}\big] \pm 1.96\sqrt{var_t(Y_{t+k})}$$

Also from tutorial

$$E_t\big[Y_t\big] = \phi_{21}Y_{t-1} + \phi_{22}Y_{t-2}$$

To solve for ϕ_{21} and ϕ_{22} we minimize $vare_t$ and solve:

$$\begin{aligned}
&var(e_t) = var(Y_t - \phi_{21}Y_{t-1} - \phi_{22}Y_{t-2})\\
&var(e_t) = E\bigg[\Big((Y_t - \phi_{21}Y_{t-1} - \phi_{22}Y_{t-2}) - E\big[(Y_t - \phi_{21}Y_{t-1} - \phi_{22}Y_{t-2})\big]\Big)^2\bigg]\\
&var(e_t) = E\bigg[\Big((Y_t - \phi_{21}Y_{t-1} - \phi_{22}Y_{t-2})\Big)^2\bigg]\\
&var(e_t) = E\big[Y_t^2\big] - 2\phi_{21}E\big[Y_tY_{t-1}\big] - 2\phi_{22}E\big[Y_tY_{t-2}\big] + \phi_{21}^2E\big[Y_{t-1}^2\big] - 2\phi_{21}\phi_{22}E\big[Y_{t-1}Y_{t-2}\big] + \phi_{22}^2E\big[Y_{t-2}^2\big]\\
&var(e_t) = \gamma(0) - 2\phi_{21}\gamma(1) - 2\phi_{22}\gamma(2) + \phi_{21}^2\gamma(0) - 2\phi_{21}\phi_{22}\gamma(1) + \phi_{22}^2\gamma(0)
\end{aligned} \tag{10}$$

The we differentiate with respect to ϕ.

$$\begin{aligned}
\frac{\delta S}{\delta\phi_{21}} &= -2\gamma(1) + 2\phi_{21}\gamma(0) + 2\phi_{22}\gamma(1) = 0\\
\phi_{21} &= \frac{\gamma(1) - \phi_{22}\gamma(1)}{\gamma(0)} = \rho(1) - \phi_{22}\rho(1)
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
\frac{\delta S}{\delta\phi_{22}} &= -2\gamma(2) + 2\phi_{21}\gamma(1) + 2\phi_{22}\gamma(0) = 0\\
\phi_{21} &= \frac{\gamma(2) - \phi_{21}\gamma(1)}{\gamma(0)} = \rho(2) - \phi_{21}\rho(1)
\end{aligned} \tag{12}$$

Next we use substitution:

$$\begin{aligned}
\phi_{21} &= \rho(1) - (\rho(2) - \phi_{21}\rho(1))\rho(1)\\
\phi_{21} &= \frac{\rho(1) - \rho(1)\rho(2)}{1 - \rho(1)^2}\\
\phi_{21} &= \frac{\frac{2}{3} - \frac{2}{3}\frac{1}{3}}{1 - (\frac{2}{3})^2}\\
\phi_{21} &= \frac{\frac{6}{9} - \frac{2}{9}}{\frac{5}{9}} = \frac{4}{5}
\end{aligned} \tag{13}$$

and

$$\phi_{21} = \frac{1}{3} - \frac{4}{5}\frac{2}{3} = -\frac{3}{15} \tag{14}$$

$$\therefore E_t\big[Y_t\big] = \phi_{21}Y_{t-1} + \phi_{22}Y_{t-2} = (0.8)(0.02) + (-0.2)(0.01) = 0.014$$

and

$$\begin{aligned}
var(Y_t|I_t) &= \gamma(0)(1 - r_2^T R_2^{-1} r_2) \\
var(Y_t|I_t) &= 0.0016(1 - \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{3} \\ \frac{2}{3} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}) \\
var(Y_t|I_t) &= 0.0016(1 - \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1.8 & -1.2 \\ -1.2 & 1.8 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}) \\
var(Y_t|I_t) &= 0.0016(1 - \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0.8 \\ -0.2 \end{bmatrix}) \\
var(Y_t|I_t) &= 0.0016(1 - \frac{7}{15}) \\
var(Y_t|I_t) &= 0.0008533333
\end{aligned} \tag{15}$$

Finally:

$$\begin{aligned}
0.95 &\approx Pr\Big(E[Y_t] - 1.96\sqrt{var(Y_t)} < Y_t < E[Y_t] + 1.96\sqrt{var(Y_t)}\Big) \\
0.95 &\approx Pr\Big(0.0141.96\sqrt{0.0008533333} < Y_t < 0.014 + 1.96\sqrt{0.0008533333}\Big) \\
0.95 &\approx Pr\Big(-0.04326 < Y_t < 0.07126\Big)
\end{aligned} \tag{16}$$

We are 95% certain that the interval $(-0.04326, 0.07126)$ contains the true unknown $Y_t|Y_{t-1}, Y_{t-2}$.

2 Practicum

Problem 1

Use a random number generator to generate θ from a uniform distribution with lower bound 0.5 and upper bound 0.9. Generate σ from a uniform distribution with lower bound 0.02 and upper bound 0.08. Suppose that Y_t follows the $MA(1)$ representation:

$$Y_t = \epsilon_t + \theta\epsilon_{t-1}, \qquad \epsilon_t \sim i.i.d.N[0, \sigma^2]$$

Calculate $\rho(k)$, ϕ_{kk} for $k = 1, 2, 3$. For $k = 3$, you can use the matrix inversion and multiplication commands in your computer package. Calculate the optimal forecast rule

$$E[Y_t|Y_{t-1}, Y_{t-2}, Y_{t-3}]$$

and

$$var[Y_t|Y_{t-1}, Y_{t-2}, Y_{t-3}]$$

Using a random number generator simulate $T = 200$ observations of $Y_t = \epsilon_t + \theta\epsilon_{t-1}$ and present a graph of Y_t. Using your simulated data find $\hat{\rho}(k)$, $\hat{\phi}_{kk}$ for $k = 1, 2, 3$ by running the appropriate regressions. Why do $\rho(k)$, ϕ_{kk} and $\hat{\rho}(k)$, and $\hat{\phi}_{kk}$ differ?

Generate θ and σ

```
## [1] 0.6017529
```

```
## [1] 0.05238621
```

Calculate $\rho(k)$, ϕ_{kk} for $k = 1, 2, 3$

$$\rho(k) = \begin{cases} \frac{\theta}{1+\theta^2} & k = 1 \\ 0 & k > 1 \end{cases}$$

$$\rho(k) = \begin{cases} \frac{0.6017529}{1+0.6017529^2} & = 0.4417811 & k = 1 \\ 0 & & k > 1 \end{cases}$$

k = 1

$$\begin{gathered} \hat{Y}_t = \phi_{11} Y_{t-1} \\ \phi_{11} = R_1 r_1 = [1]\rho(1) = 0.4417811 \end{gathered} \tag{17}$$

```
## [1] 0.4417811
```

k = 2

$$\hat{Y}_t = \phi_{21}Y_{t-1} + \phi_{22}Y_{t-2}$$

$$\hat{\Phi} = \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = R_2^{-1} r_2 = \begin{bmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho(1) \\ \rho(2) \end{bmatrix} \tag{18}$$

$$\hat{\Phi} = \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0.4417811 \\ 0.4417811 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.4417811 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5489126 \\ -0.2424992 \end{bmatrix}$$

```
##           [,1]      [,2]
## [1,] 1.0000000 0.4417811
## [2,] 0.4417811 1.0000000
```

```
##           [,1]
## [1,] 0.4417811
## [2,] 0.0000000
```

```
##            [,1]
## [1,]  0.5489126
## [2,] -0.2424992
```

k = 3

$$\hat{Y}_t = \phi_{31}Y_{t-1} + \phi_{32}Y_{t-2} + \phi_{33}Y_{t-3}$$

$$\hat{\Phi} = \begin{bmatrix} \phi_{31} \\ \phi_{32} \\ \phi_{33} \end{bmatrix} = R_3^{-1} r_3 = \begin{bmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho(1) \\ \rho(2) \\ \rho(3) \end{bmatrix} \tag{19}$$

$$\hat{\Phi} = \begin{bmatrix} \phi_{31} \\ \phi_{32} \\ \phi_{33} \end{bmatrix} = R_3^{-1} r_3 = \begin{bmatrix} 1 & 0.4417811 & 0 \\ 0.4417811 & 1 & 0.4417811 \\ 0 & 0.4417811 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.4417811 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5832087 \\ -0.3201306 \\ 0.1414277 \end{bmatrix}$$

```
##           [,1]      [,2]      [,3]
## [1,] 1.0000000 0.4417811 0.0000000
## [2,] 0.4417811 1.0000000 0.4417811
## [3,] 0.0000000 0.4417811 1.0000000
```

```
##           [,1]
## [1,] 0.4417811
## [2,] 0.0000000
## [3,] 0.0000000
```

```
##            [,1]
## [1,]  0.5832087
## [2,] -0.3201306
## [3,]  0.1414277
```

So $\phi_{11} = 0.4417811, \phi_{22} = -0.2424992, \phi_{33} = 0.1414277$

Calculate the optimal forecast rule:

$$E[Y_t|Y_{t-1},Y_{t-2},Y_{t-3}] = \hat{Y}_t = \phi_{31}Y_{t-1} + \phi_{32}Y_{t-2} + \phi_{33}Y_{t-3}$$
$$E[Y_t|Y_{t-1},Y_{t-2},Y_{t-3}] = 0.583Y_{t-1} - 0.320Y_{t-2} + 0.141Y_{t-3}$$

and

$$\begin{aligned}
var(Y_t|Y_{t-1},Y_{t-2},Y_{t-3}) &= \gamma(0)(1 - r_3^T R_3^{-1} r_3) \\
var(Y_t|Y_{t-1},Y_{t-2},Y_{t-3}) &= \sigma^2(1+\theta^2)(1 - r_3^T R_3^{-1} r_3) \\
var(Y_t|Y_{t-1},Y_{t-2},Y_{t-3}) &= \sigma^2(1+\theta^2)(1 - \begin{bmatrix}0.4417811 & 0 & 0\end{bmatrix}\begin{bmatrix}0.5832087 \\ -0.3201306 \\ 0.1414277\end{bmatrix}) \\
var(Y_t|Y_{t-1},Y_{t-2},Y_{t-3}) &= (0.05238621^2)(1+0.6017529^2)(1-0.2576506) = 0.002774939
\end{aligned} \tag{20}$$

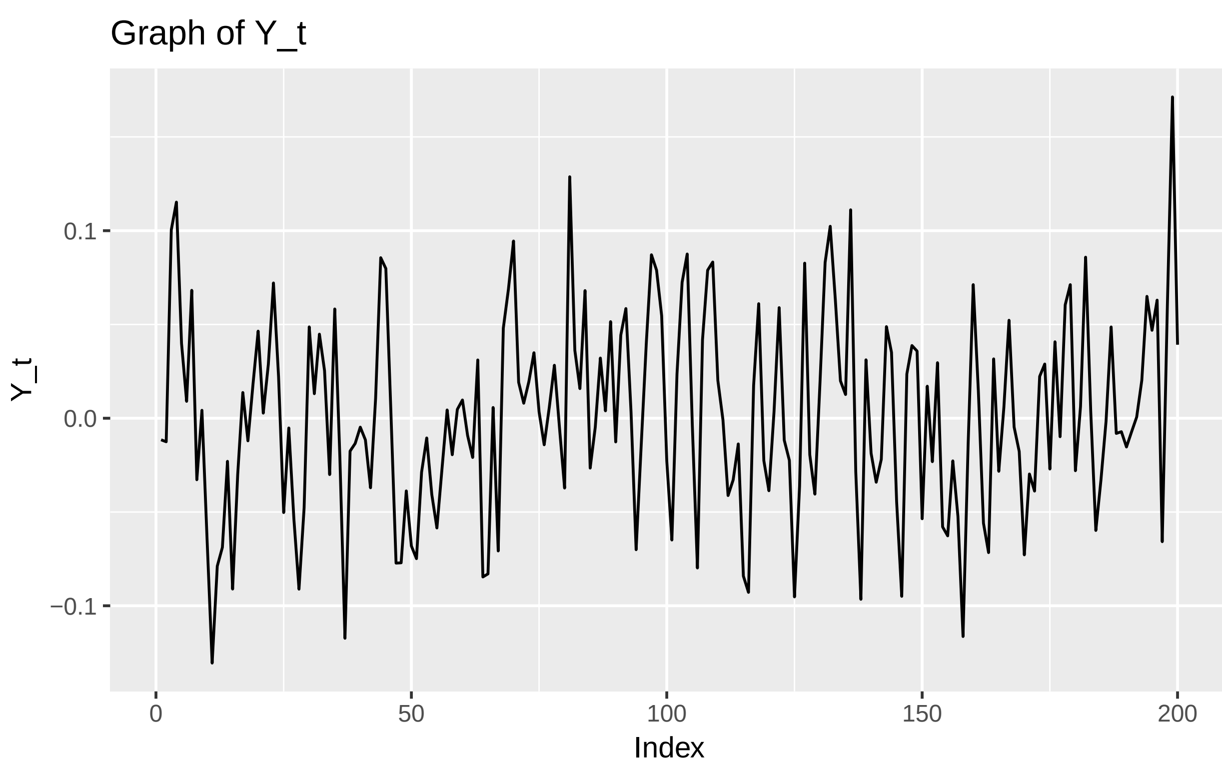
```
##             [,1]
## [1,] 0.002774939
```

Simulate $T = 200$ Observations of $Y_t = \epsilon_t + \theta\epsilon_{t-1}$

```
## [1] -0.016044802 -0.001847028 -0.011439050  0.107337597  0.050641679
## [6]  0.009471836
```

```
## [1] -0.011502034 -0.012550504  0.100454116  0.115232389  0.039945613
## [6]  0.009000256
```

Present a graph of Y_t



5 different models using two equations the $\hat{Y}_t = \rho(k)Y_{t-k}$ and $\hat{Y}_t = \phi_{k1}Y_{t-1} + \phi_{k2}Y_{t-2} + \phi_{k3}Y_{t-3} + ...$

Estimation of $\hat{\rho}(1)$ and $\hat{\phi}_{11}$ using formula $\hat{Y}_t = \rho(1)Y_{t-1}$

```
##
## Call:
## lm(formula = Y_t[-1] ~ Y_t[-length(Y_t)] - 1)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.110037 -0.034005 -0.002416  0.038093  0.150410
##
## Coefficients:
##                   Estimate Std. Error t value Pr(>|t|)
## Y_t[-length(Y_t)]  0.35170    0.06662   5.279  3.4e-07 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.04946 on 198 degrees of freedom
## Multiple R-squared:  0.1234, Adjusted R-squared:  0.119
## F-statistic: 27.87 on 1 and 198 DF,  p-value: 3.4e-07

## [1] 0.3517026
```

Our estimate $\hat{\rho}(1) = \hat{\phi}_{11} = 0.35170$ is fairly close to our theoretical value of $\rho(1) = \phi_{11} = 0.4417811$

Estimation of $\hat{\rho}(2)$ using formula $\hat{Y}_t = \rho(2)Y_{t-2}$

```
##
## Call:
## lm(formula = Y_t[-c(1, 2)] ~ Y_t[-((length(Y_t) - 1):length(Y_t))] -
##     1)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.130362 -0.032254  0.001836  0.036471  0.168348
##
## Coefficients:
##                                        Estimate Std. Error t value Pr(>|t|)
## Y_t[-((length(Y_t) - 1):length(Y_t))] -0.04525    0.07324  -0.618    0.537
##
## Residual standard error: 0.0529 on 197 degrees of freedom
## Multiple R-squared:  0.001934,   Adjusted R-squared:  -0.003132
## F-statistic: 0.3818 on 1 and 197 DF,  p-value: 0.5374

## [1] -0.04524958
```

This estimate $\hat{\rho}(2)$ is close to zero which makes sense for an MA(1).

Estimation of $\hat{\rho}(3)$ using formula $\hat{Y}_t = \rho(3)Y_{t-3}$

```
##
## Call:
## lm(formula = Y_t[-c(1, 2, 3)] ~ Y_t[-((length(Y_t) - 2):length(Y_t))] -
```

```
##     1)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.129255 -0.030552  0.000934  0.038425  0.168833
##
## Coefficients:
##                                        Estimate Std. Error t value Pr(>|t|)
## Y_t[-((length(Y_t) - 2):length(Y_t))]  0.03960    0.07301   0.542    0.588
##
## Residual standard error: 0.05256 on 196 degrees of freedom
## Multiple R-squared:  0.001498,   Adjusted R-squared:  -0.003596
## F-statistic: 0.2941 on 1 and 196 DF,  p-value: 0.5882
```

```
## [1] 0.03959823
```

This estimate $\hat{\rho}(3)$ is close to zero which makes sense for an MA(1).

Estimation of $\hat{\phi}_{21}$ and $\hat{\phi}_{22}$ using formula $\hat{Y}_t = \phi_{21}Y_{t-1} + \phi_{22}Y_{t-2}$

```
##
## Call:
## lm(formula = Y_t[-c(1, 2)] ~ Y_t[-c(1, length(Y_t))] + Y_t[-((length(Y_t) -
##     1):length(Y_t))] - 1)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.102756 -0.034455 -0.002293  0.036688  0.143283
##
## Coefficients:
##                                        Estimate Std. Error t value Pr(>|t|)
## Y_t[-c(1, length(Y_t))]                 0.41777    0.07016   5.955 1.19e-08
## Y_t[-((length(Y_t) - 1):length(Y_t))] -0.19507    0.07210  -2.706  0.00742
##
## Y_t[-c(1, length(Y_t))]               ***
## Y_t[-((length(Y_t) - 1):length(Y_t))] **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.0488 on 196 degrees of freedom
## Multiple R-squared:  0.1548, Adjusted R-squared:  0.1462
## F-statistic: 17.95 on 2 and 196 DF,  p-value: 6.929e-08
```

```
## [1] 0.417769
```

```
## [1] -0.1950678
```

Our estimates are fairly close to the theoretical values of 0.5489 and -0.2425.

Estimation of $\hat{\phi}_{31}$, $\hat{\phi}_{31}$, and $\hat{\phi}_{33}$ using formula $\hat{Y}_t = \phi_{31}Y_{t-1} + \phi_{32}Y_{t-2} + \phi_{33}Y_{t-3}$

```
##
## Call:
## lm(formula = Y_t[-c(1, 2, 3)] ~ Y_t[-c(1, 2, length(Y_t))] +
##     Y_t[-c(1, (length(Y_t) - 1):length(Y_t))] + Y_t[-((length(Y_t) -
##     2):length(Y_t))] - 1)
```

```
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.102473 -0.032289 -0.000975  0.036659  0.139871
##
## Coefficients:
##                                          Estimate Std. Error t value
## Y_t[-c(1, 2, length(Y_t))]                0.44701    0.07024   6.364
## Y_t[-c(1, (length(Y_t) - 1):length(Y_t))] -0.25366    0.07667  -3.309
## Y_t[-((length(Y_t) - 2):length(Y_t))]      0.14857    0.07217   2.059
##                                          Pr(>|t|)
## Y_t[-c(1, 2, length(Y_t))]               1.38e-09 ***
## Y_t[-c(1, (length(Y_t) - 1):length(Y_t))]  0.00112 **
## Y_t[-((length(Y_t) - 2):length(Y_t))]      0.04087 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.04797 on 194 degrees of freedom
## Multiple R-squared:  0.1768, Adjusted R-squared:  0.1641
## F-statistic: 13.89 on 3 and 194 DF,  p-value: 3.048e-08
```

```
## [1] 0.4470114
```

```
## [1] -0.2536598
```

```
## [1] 0.1485742
```

Our estimates are fairly close to the theoretical values of 0.583, -0.320, and 0.141.

Obviously all of our values are just estimates and will not be expected to match the theoretical values exactly. As T increases we would expect our estimate to become more accurate.

In summary:

- $\hat{\rho}(1) = 0.3517026$
- $\hat{\rho}(2) = -0.04524958$
- $\hat{\rho}(3) = 0.03959823$
- $\hat{\phi}_{11} = 0.3517026$
- $\hat{\phi}_{22} = -0.1950678$
- $\hat{\phi}_{33} = 0.1485742$

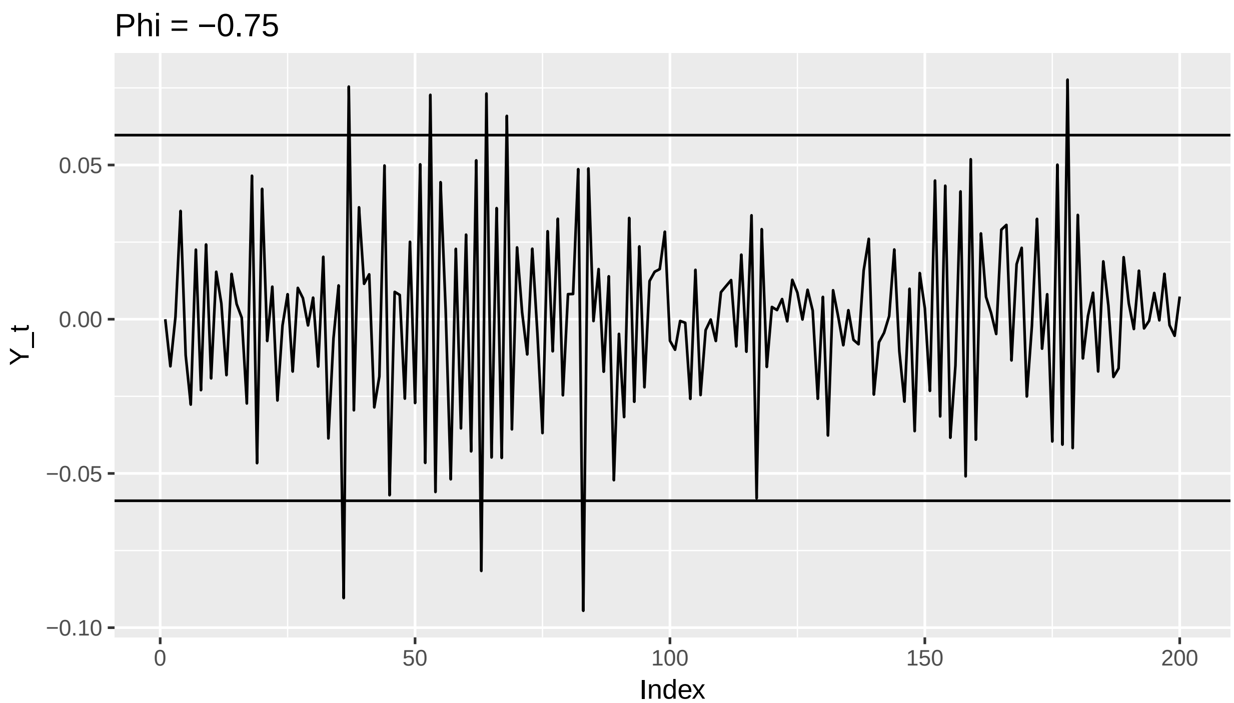
Problem 2

Using a standard normal random number generator, simulate $T = 200$ observations of the $AR(1)$ representation for Y_t:

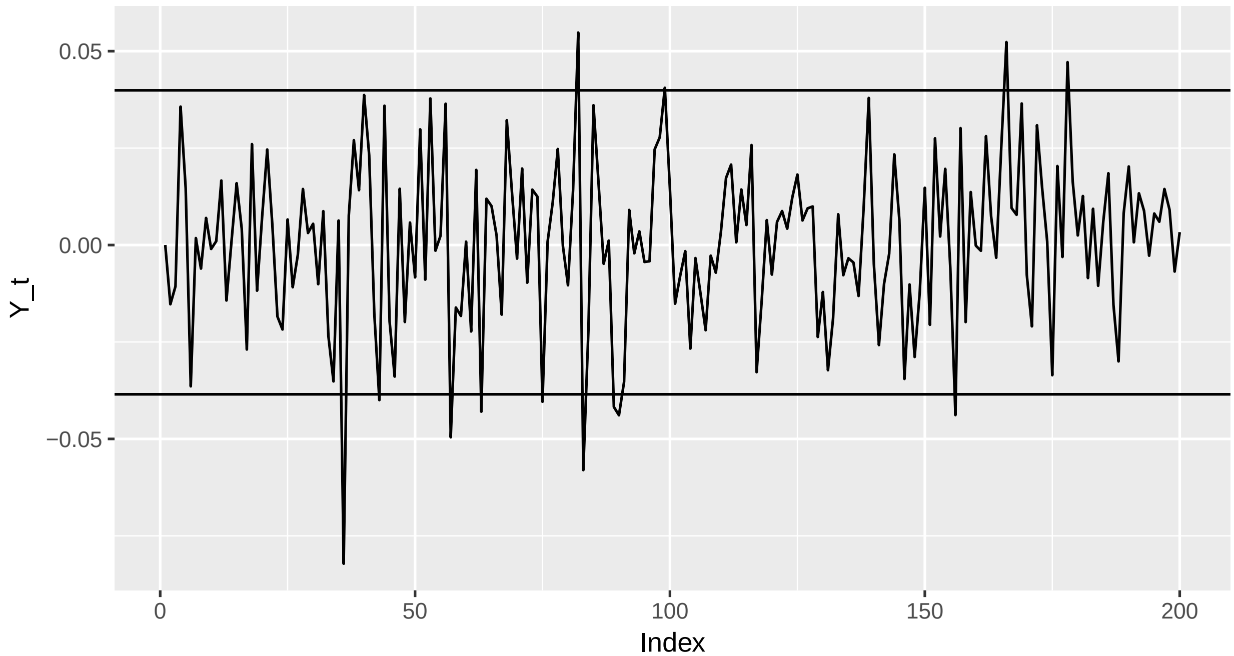
$$Y_t = \phi Y_{t-1} + \epsilon_t, \quad \epsilon_t \sim i.i.dN[0, \sigma^2],$$

where $\sigma = 0.02$ for $\phi = -0.75$, $\phi = 0$, $\phi = 0.75$, $\phi = 0.95$, and $\phi = 1$. Plot all five graphs with 95% confidence bands.

```
## [1] 0.05407414
```

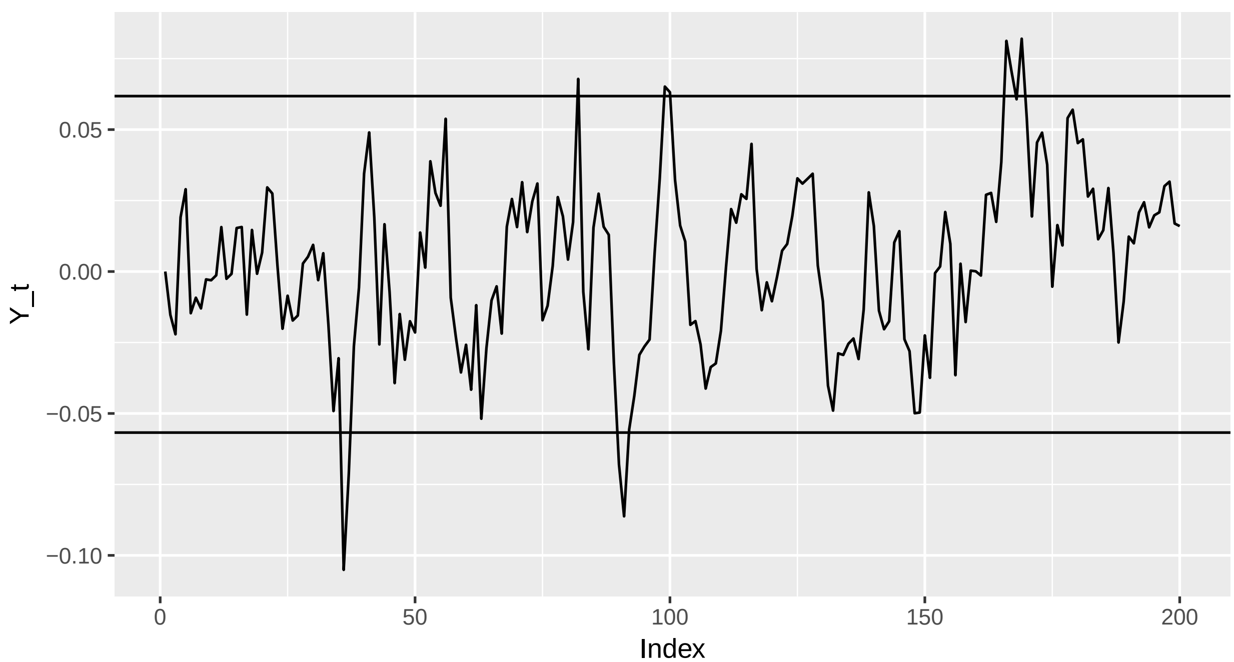


Phi = 0

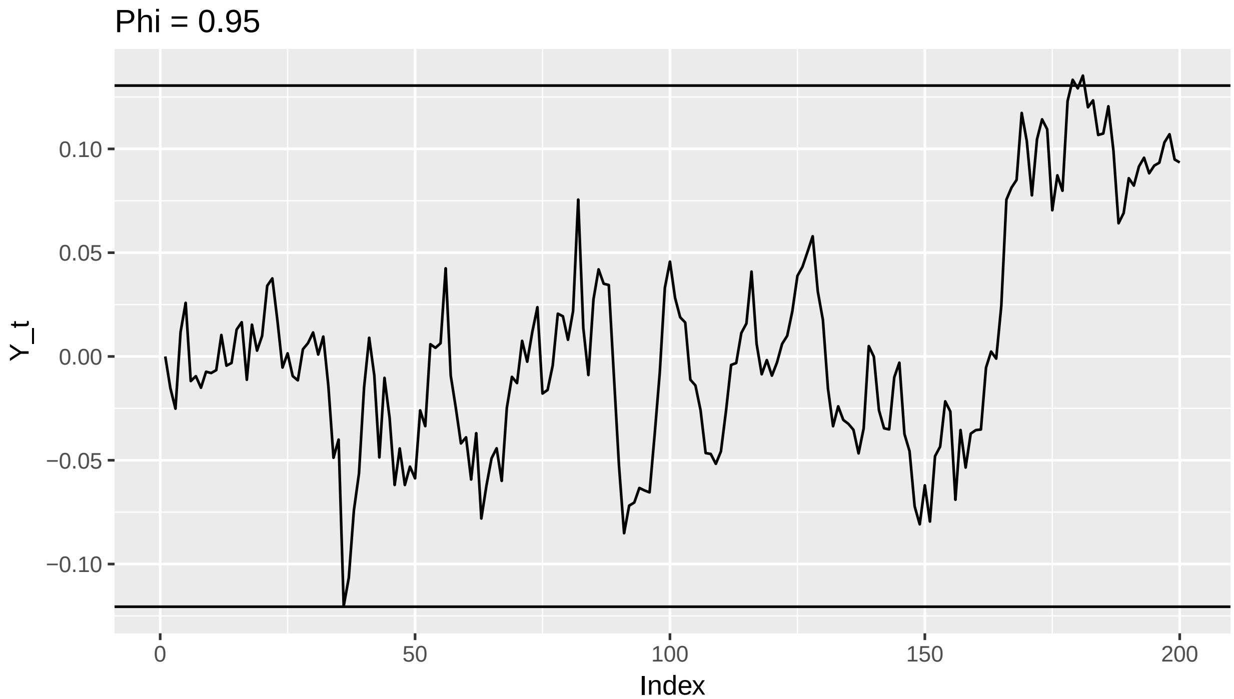


Y_t
0.05
0.00
−0.05
0
50
100
150
200
Index

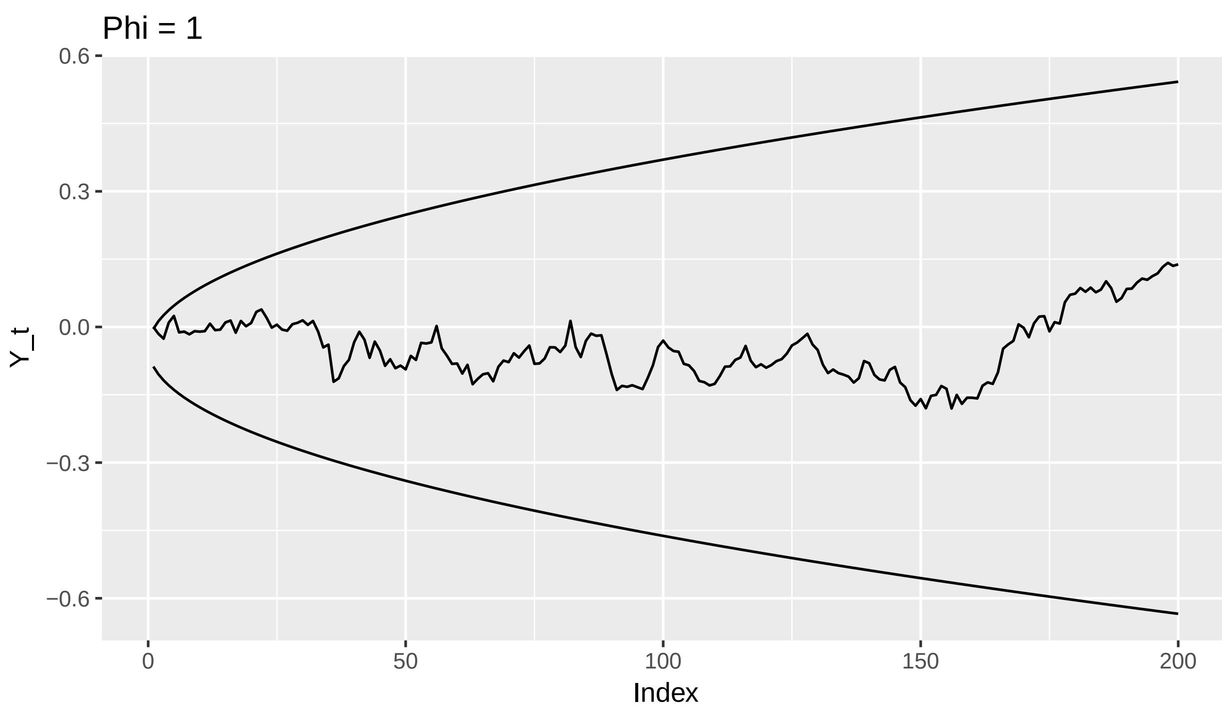
Phi = 0.75



Y_t
0.05
0.00
−0.05
−0.10
0
50
100
150
200
Index



Phi = 0.95
0.10
0.05
0.00
−0.05
−0.10
Y_t
0
50
100
150
200
Index



Phi = 1
0.6
0.3
0.0
−0.3
−0.6
Y_t
0
50
100
150
200
Index

Appendix Code

```
#### Generate $\theta$ and $\sigma$
set.seed(271)
theta <- runif(1, 0.5, 0.9)
sigma <- runif(1, 0.02, 0.08)
theta
sigma
#### Calculate $\rho(k)$, $\phi_{kk}$ for $k = 1,2,3$

##### k = 1

rho1 <- theta/(1 + theta^2)
rho1

##### k = 2

library(matlib)
R_2 <- matrix(c(1, rho1,rho1, 1), ncol = 2, nrow = 2)
r_2 <- matrix(c(rho1, 0), ncol = 1, nrow = 2)
R_2
r_2
Phi_2 <- inv(R_2) %*% r_2
Phi_2


##### k = 3
R_3<- matrix(c(1, rho1, 0,  rho1, 1, rho1, 0, rho1, 1), ncol = 3, nrow = 3)
r_3 <- matrix(c(rho1, 0, 0), ncol = 1, nrow = 3)
R_3
r_3
Phi_3 <- inv(R_3) %*% r_3
Phi_3

#### Calculate the optimal forecast rule:
sigma
theta
(sigma^2)*(1+theta^2)*(t(r_3) %*% Phi_3)

#### Simulate $T = 200$ Observations of $Y_t = \epsilon_t + \theta \epsilon_{t-1}$
epsilon_t <- rnorm(201, mean = 0, sd = sigma)
Y_t <- epsilon_t[-1] + theta*epsilon_t[-length(epsilon_t)]

#### Present a graph of $Y_t$
library(ggplot2)
qplot(seq_along(Y_t), Y_t, geom="line") + xlab("Index") + ggtitle("Graph of Y_t")

##### Estimation of $\hat{\rho}(1)$ and $\hat{\phi}_{11}$ using formula $\hat{Y}_t = \rho(1) Y_{t-1}$
model_rho1 <- lm(Y_t[-1] ~ Y_t[-length(Y_t)] -1)
summary(model_rho1)
rho_1 <- as.numeric(model_rho1$coef)
rho_1
```

```r
##### Estimation of $\hat{\rho}(2)$ using formula $\hat{Y}_t = \rho(2) Y_{t-2}$
model_rho2 <- lm(Y_t[-c(1,2)] ~ Y_t[-((length(Y_t)-1):length(Y_t))] -1)
summary(model_rho2)
rho_2 <- as.numeric(model_rho2$coef)
rho_2

##### Estimation of $\hat{\rho}(3)$ using formula $\hat{Y}_t = \rho(3) Y_{t-3}$
model_rho3 <- lm(Y_t[-c(1,2,3)] ~ Y_t[-((length(Y_t)-2):length(Y_t))] -1)
summary(model_rho3)
rho_3 <- as.numeric(model_rho3$coef)
rho_3

##### Estimation of $\hat{\phi}_{21}$ and $\hat{\phi}_{22}$ using formula $\hat{Y}_t = \phi_{21} Y_{t-1
model_Phi_2 <- lm(Y_t[-c(1,2)] ~ Y_t[-c(1,length(Y_t))] + Y_t[-((length(Y_t)-1):length(Y_t))] -1)
summary(model_Phi_2)
phi_21 <- as.numeric(model_Phi_2$coef)[1]
phi_22 <- as.numeric(model_Phi_2$coef)[2]
phi_21
phi_22

##### Estimation of $\hat{\phi}_{31}$, $\hat{\phi}_{31}$, and  $\hat{\phi}_{33}$ using formula $\hat{Y}
model_Phi_3 <- lm(Y_t[-c(1,2,3)] ~ Y_t[-c(1,2,length(Y_t))] + Y_t[-c(1,(length(Y_t)-1):length(Y_t))] + Y
summary(model_Phi_3)
phi_31 <- as.numeric(model_Phi_3$coef)[1]
phi_32 <- as.numeric(model_Phi_3$coef)[2]
phi_33 <- as.numeric(model_Phi_3$coef)[3]
phi_31
phi_32
phi_33

### Problem 2
library(ggplot2)
sigma <- 0.02
epsilon_t <- rnorm(200, mean = 0, sd = 0.02)

### For Phi = -0.75
Y_n075 <- rep(0, 200)
for(i in 1:199){
  Y_n075[i+1] <- Y_n075[i]*(-0.75) + epsilon_t[i]
  i = i + 1
}
sqrt(var(diff(Y_n075)))
lwd_Yt = mean(Y_n075) - 1.96*sigma/sqrt(1-(-0.75)^2)
upr_Yt = mean(Y_n075) + 1.96*sigma/sqrt(1-(-0.75)^2)
qplot(seq_along(Y_n075), Y_n075, geom = "line") +
  geom_hline(yintercept=lwd_Yt) +
  geom_hline(yintercept=upr_Yt) + ggtitle("Phi = -0.75") + xlab("Index") + ylab("Y_t")

### For Phi = 0
Y_0 <- rep(0, 200)
for(i in 1:199){
  Y_0[i+1] <- Y_0[i]*(0) +  epsilon_t[i]
  i = i + 1
```

```
}
lwd_Yt = mean(Y_0) - 1.96*sigma/sqrt(1-(0)^2)
upr_Yt = mean(Y_0) + 1.96*sigma/sqrt(1-(0)^2)
qplot(seq_along(Y_0), Y_0, geom = "line") +
  geom_hline(yintercept=lwd_Yt) +
  geom_hline(yintercept=upr_Yt) + ggtitle("Phi = 0") + xlab("Index") + ylab("Y_t")

### For Phi = 0.75
Y_075 <- rep(0, 200)
for(i in 1:199){
  Y_075[i+1] <- Y_075[i]*(0.75) + epsilon_t[i]
  i = i + 1
}
lwd_Yt = mean(Y_075) - 1.96*sigma/sqrt(1-(0.75)^2)
upr_Yt = mean(Y_075) + 1.96*sigma/sqrt(1-(0.75)^2)
qplot(seq_along(Y_075), Y_075, geom = "line") +
  geom_hline(yintercept=lwd_Yt) +
  geom_hline(yintercept=upr_Yt) + ggtitle("Phi = 0.75") + xlab("Index") + ylab("Y_t")

### For Phi = 0.95
Y_095 <- rep(0, 200)
for(i in 1:199){
  Y_095[i+1] <- Y_095[i]*(0.95) + epsilon_t[i]
  i = i + 1
}
lwd_Yt = mean(Y_095) - 1.96*sigma/sqrt(1-(0.95)^2)
upr_Yt = mean(Y_095) + 1.96*sigma/sqrt(1-(0.95)^2)
qplot(seq_along(Y_095), Y_095, geom = "line") +
  geom_hline(yintercept=lwd_Yt) +
  geom_hline(yintercept=upr_Yt) + ggtitle("Phi = 0.95") + xlab("Index") + ylab("Y_t")

### For Phi = 1
Y_1 <- rep(0, 200)
tSeq <- 1:200
for(i in 1:199){
  Y_1[i+1] <- Y_1[i]*(1) + epsilon_t[i]
  i = i + 1
}
lwd_Yt = data.frame(lwd_Yt = mean(Y_1) - 1.96*sqrt(var(diff(Y_1)))*sqrt(tSeq))
upr_Yt = data.frame(upr_Yt = mean(Y_1) + 1.96*sqrt(var(diff(Y_1)))*sqrt(tSeq))
Y_1 <- data.frame(Y_1)
ggplot(Y_1, aes(x = seq_along(Y_1), y = Y_1)) +
  geom_line() +
  geom_line(aes(x = 1:200 , y = lwd_Yt)) +
  geom_line(aes(x = 1:200 , y = upr_Yt)) + ggtitle("Phi = 1") + xlab("Index") + ylab("Y_t")
```