Time Series Analysis - Assignment 1

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1 Theory

Problem 1

Consider the model:

$$Y_t = \epsilon_t + \theta\epsilon_{t-1}^2, \quad where \quad \epsilon_t \ \sim i.i.d.N(0, \sigma^2)$$

Compute the autocovariances $\gamma(k)$ of Y_t for $k \geq 0$ and show whether the process is stationary.

In order for a process to be stationary we need its mean, variance, and covariances to **not** depend on time:

I.e.

1) $E[Y_t] = E[Y_s] = \mu \quad \forall s, t$
2) $var(Y_t) = var(Y_s) = \sigma^2 \quad \forall s, t$
3) $cov(Y_t, Y_{t\pm k}) = cov(Y_s, Y_{s\pm k})$ for any k.

Proof of 1):

$$\begin{aligned} &E[Y_t] = E[\epsilon_t] + \theta E[\epsilon_{t-1}^2] = 0 + \theta(Var(\epsilon_{t-1}) + E[\epsilon_{t-1}]^2) \\ &E[Y_t] = \theta\sigma^2 + 0 = \theta\sigma^2 \end{aligned} \tag{1}$$

$\therefore E[Y_t]$ does not depend on t.

Proof of 2):

$$\begin{aligned} &var(Y_t) = var(\epsilon_t + \theta\epsilon_{t-1}^2) \\ &var(Y_t) = var(\epsilon_t) + \theta^2 var(\epsilon_{t-1}^2) + 2cov(\epsilon_t, \theta^2\epsilon_{t-1}^2) \\ &var(Y_t) = \sigma^2 + \theta^2 var(\epsilon_{t-1}^2) + 0 \\ &var(Y_t) = \sigma^2 + \theta^2 E[(\epsilon_{t-1}^2 - E[\epsilon_{t-1}^2])^2] \\ &var(Y_t) = \sigma^2 + \theta^2 E[\epsilon_{t-1}^4 - 2\epsilon_{t-1}^2 E[\epsilon_{t-1}^2] + E[\epsilon_{t-1}^2]^2] \\ &var(Y_t) = \sigma^2 + \theta^2 E[\epsilon_{t-1}^4 - 2\epsilon_{t-1}^2\sigma^2 + \sigma^4] \\ &var(Y_t) = \sigma^2 + \theta^2(E[\epsilon_{t-1}^4] - 2\sigma^2 E[\epsilon_{t-1}^2] + \sigma^4) \end{aligned} \tag{2}$$

$E[\epsilon_{t-1}^4]$ is the 4th central moment of the Normal Distribution which is equal to $3\sigma^4$.

$$\begin{aligned} &var(Y_t) = \sigma^2 + \theta^2(3\sigma^4 - 2\sigma^4 + \sigma^4) \\ &var(Y_t) = \sigma^2 + 2\theta^2\sigma^4 \end{aligned} \tag{3}$$

$\therefore var[Y_t]$ does not depend on t.

Proof of 3):

$$\gamma(k) := cov(Y_t, Y_{t\pm k}) \tag{4}$$

For k = 0

$$\gamma(0) = cov(Y_t, Y_t) = var(Y_t) = \sigma^2 + 2\theta^2\sigma^4 \tag{5}$$

For k = 1

$$\begin{aligned}
\gamma(1) &= E[(Y_t - E[Y_t])(Y_{t-1} - E[Y_{t-1}])] \\
\gamma(1) &= E[(Y_t - \theta\sigma^2)(Y_{t-1} - \theta\sigma^2)] \\
\gamma(1) &= E[Y_tY_{t-1} - \theta\sigma^2 Y_t - \theta\sigma^2 Y_{t-1} + \theta^2\sigma^4] \\
\gamma(1) &= E[Y_tY_{t-1}] - \theta\sigma^2 E[Y_t] - \theta\sigma^2 E[Y_{t-1}] + \theta^2\sigma^4 \\
\gamma(1) &= E[Y_tY_{t-1}] - \theta\sigma^2(\theta\sigma^2) - \theta\sigma^2(\theta\sigma^2) + \theta^2\sigma^4 \\
\gamma(1) &= E[Y_tY_{t-1}] - \theta^2\sigma^4 \\
\gamma(1) &= E[(\epsilon_t + \theta\epsilon_{t-1}^2)(\epsilon_{t-1} + \theta\epsilon_{t-2}^2)] - \theta^2\sigma^4 \\
\gamma(1) &= E[\epsilon_t\epsilon_{t-1}] + \theta E[\epsilon_t\epsilon_{t-2}^2] + \theta E[\epsilon_{t-1}^3] + \theta^2 E[\epsilon_{t-1}^2\epsilon_{t-2}^2] - \theta^2\sigma^4
\end{aligned} \tag{6}$$

Note that the third central moment of a Normal Distribution is $E[\epsilon_{t-1}^3] = 0$

$$\begin{aligned}
\gamma(1) &= 0 + \theta(0) + \theta(0) + \theta^2\sigma^4 - \theta^2\sigma^4 \\
\gamma(1) &= 0
\end{aligned} \tag{7}$$

For k = 2 & k > 2

Because ϵ_t is i.i.d. we will always have $E[\epsilon_t\epsilon_s] = 0 \;\; \forall s, t.$

Following the pattern using brute force we will never have it so that $s = t$ for any further lags. So we know:

$$\gamma(k) = -\theta^2\sigma^4 \;\; \forall\, k \geq 2$$

To be explicit

$$\begin{aligned}
\gamma(2) &= E[(Y_t - E[Y_t])(Y_{t-2} - E[Y_{t-2}])] \\
\gamma(2) &= E[(Y_t - \theta\sigma^2)(Y_{t-2} - \theta\sigma^2)] \\
\gamma(2) &= E[Y_tY_{t-2} - \theta\sigma^2 Y_t - \theta\sigma^2 Y_{t-2} + \theta^2\sigma^4] \\
\gamma(2) &= E[Y_tY_{t-2}] - \theta\sigma^2 E[Y_t] - \theta\sigma^2 E[Y_{t-2}] + \theta^2\sigma^4 \\
\gamma(2) &= E[Y_tY_{t-2}] - \theta\sigma^2(\theta\sigma^2) - \theta\sigma^2(\theta\sigma^2) + \theta^2\sigma^4 \\
\gamma(2) &= E[Y_tY_{t-2}] - \theta^2\sigma^4 \\
\gamma(2) &= E[(\epsilon_t + \theta\epsilon_{t-1}^2)(\epsilon_{t-2} + \theta\epsilon_{t-3}^2)] - \theta^2\sigma^4 \\
\gamma(2) &= E[\epsilon_t\epsilon_{t-2}] + \theta E[\epsilon_t\epsilon_{t-3}^2] + \theta E[\epsilon_{t-1}^2\epsilon_{t-2}] + \theta^2 E[\epsilon_{t-1}^2\epsilon_{t-3}^2] - \theta^2\sigma^4
\end{aligned} \tag{8}$$

For all pairs of ϵ inside of $E[.]$ we will never have $\epsilon_t\epsilon_s$ where $s = t$ so these terms are always 0 for all $k \geq 2$.

$$\therefore \gamma(o) = \sigma^2 + 2\theta^2\sigma^4 \quad \gamma(k) = -\theta^2\sigma^4 \;\; \forall\, k \geq 1 \tag{9}$$

Our autocovariance function does not depend on t for all k. This means our final condition for stationarity is satisfied.

Problem 2

Consider the stationary $AR(1)$ representation:

$$Y_t = \phi Y_{t-1} + a_t, \quad a_t \sim i.i.d\ N(0, \sigma^2)$$

Show that

$$Y_t \sim N(0, \frac{\sigma^2}{1-\phi^2})$$

To start:

$$\begin{aligned} E[Y_t] &= E[\phi Y_{t-1} + a_t] \\ E[Y_t] &= \phi E[Y_{t-1}] + E[a_t] \\ E[Y_t] &= \phi E[Y_{t-1}] + 0 \end{aligned} \tag{10}$$

Y_t is stationary so $E[Y_t] = E[Y_{t-1}]$

$$\begin{aligned} E[Y_t] &= \phi E[Y_t] \\ (1-\phi)E[Y_t] &= 0 \end{aligned} \tag{11}$$

$\therefore$ as long as $\phi \neq 0$ then we know $E[Y_t] = 0$.

Next:

$$\begin{aligned} var(Y_t) &= var(\phi Y_{t-1} + a_t) \\ var(Y_t) &= \phi^2 var(Y_{t-1}) + var(a_t) + zero\ cross\ cov. \end{aligned} \tag{12}$$

Again Y_t is stationary so $var(Y_{t-1}) = var(Y_t)$ so,

$$\begin{aligned} var(Y_t) &= \phi^2 var[Y_t] + \sigma^2 \\ (1-\phi^2)var(Y_t) &= \sigma^2 \\ var(Y_t) &= \sigma^2/(1-\phi^2) \end{aligned} \tag{13}$$

Also note that $var(Y_t) \neq \infty \therefore \phi \neq 1$ so $E[Y] = 0$

$$\therefore Y_t \sim N(0, \frac{\sigma^2}{1-\phi^2})$$

Problem 3

Consider a random walk model with a drift parameter m expressed as

$$Y_t = m + \phi Y_{t-1} + \epsilon_t$$

such that $E[\epsilon_t] = 0 \ \forall \ t$, $var[\epsilon_t] = \sigma^2 \ \forall \ t$, $cov[\epsilon_t, \epsilon_s] = 0 \ for \ t \neq s$ and $cov[\epsilon_t, Y_t] = 0 \forall t$. Assume further that $|\phi| < 1$. Derive an expression for (1) $E[Y_t]$, (2) $var[Y_t]$, (3) the autocovariance function $\gamma(k)$, and (4) the autocorrelation function $\rho(k)$ of Y_t.

For $E[Y_t]$

$$\begin{aligned} Y_t &= m + \phi Y_{t-1} + \epsilon_t \\ Y_t &= m + \phi(m + \phi Y_{t-2} + \epsilon_{t-1}) + \epsilon_t \end{aligned} \tag{14}$$

Continue substituting and expand:

$$\begin{aligned} Y_t &= m + \phi m + \phi^2 m + \phi^3 m + ... + \epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + ... \\ Y_t &= \frac{m}{1-\phi} + \epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + ... \end{aligned} \tag{15}$$

We can now take the expectations:

$$\begin{aligned} E[Y_t] &= E[\frac{m}{1-\phi} + \epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + ...] \\ E[Y_t] &= E[\frac{m}{1-\phi}] + E[\epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + ...] \\ E[Y_t] &= E[\frac{m}{1-\phi}] + E[\epsilon_t] + E[\phi\epsilon_{t-1}] + E[\phi^2\epsilon_{t-2}] + ... \\ E[Y_t] &= \frac{m}{1-\phi} + 0 + 0 + 0 + ... \\ E[Y_t] &= \frac{m}{1-\phi} \end{aligned} \tag{16}$$

For $var[Y_t]$

From (15) above:

$$\begin{aligned} Y_t &= \frac{m}{1-\phi} + \epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + ... \\ var(Y_t) &= var(\frac{m}{1-\phi} + \epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + ...) \\ var(Y_t) &= var(\frac{m}{1-\phi}) + var(\epsilon_t) + var(\phi\epsilon_{t-1}) + var(\phi^2\epsilon_{t-2}) + ... + zero \ cross \ cov. \\ var(Y_t) &= 0 + var(\epsilon_t) + \phi^2 var(\epsilon_{t-1}) + \phi^4 var(\epsilon_{t-2}) + ... \\ var(Y_t) &= \frac{\sigma^2}{1-\phi^2} \end{aligned} \tag{17}$$

For $\gamma(k)$

$$
\begin{aligned}
\gamma(k) &= E[(Y_t - \frac{m}{1-\phi})(Y_{t-k} - \frac{m}{1-\phi})] \\
\gamma(k) &= E[(Y_tY_{t-k} - \frac{m}{1-\phi}Y_{t-k} - \frac{m}{1-\phi}Y_t + \frac{m^2}{(1-\phi)^2})] \\
\gamma(k) &= E[Y_tY_{t-k}] - E[\frac{m}{1-\phi}Y_{t-k}] - E[\frac{m}{1-\phi}Y_t] + E[\frac{m^2}{(1-\phi)^2}] \\
\gamma(k) &= E[Y_tY_{t-k}] - \frac{m}{1-\phi}E[Y_{t-k}] - \frac{m}{1-\phi}E[Y_t] + \frac{m^2}{(1-\phi)^2} \\
\gamma(k) &= E[Y_tY_{t-k}] - \frac{m}{1-\phi}\frac{m}{1-\phi} - \frac{m}{1-\phi}\frac{m}{1-\phi} + \frac{m^2}{(1-\phi)^2} \\
\gamma(k) &= E[Y_tY_{t-k}] - \frac{m^2}{(1-\phi)^2} \\
\gamma(k) &= E[(m + \phi Y_{t-1} + \epsilon_t)Y_{t-k}] - \frac{m^2}{(1-\phi)^2} \\
\gamma(k) &= E[mY_{t-k} + \phi Y_{t-1}Y_{t-k} + \epsilon_t Y_{t-k}] - \frac{m^2}{(1-\phi)^2} \\
\gamma(k) &= mE[Y_{t-k}] + E[\phi Y_{t-1}Y_{t-k}] + E[\epsilon_t Y_{t-k}] - \frac{m^2}{(1-\phi)^2} \\
\gamma(k) &= \frac{m^2}{1-\phi} + \phi E[Y_{t-1}Y_{t-k}] + 0 - \frac{m^2}{(1-\phi)^2} \\
\gamma(k) &= \frac{m^2(1-\phi)}{(1-\phi)^2} - \frac{m^2}{(1-\phi)^2} + \phi E[Y_{t-1}Y_{t-k}] \\
\gamma(k) &= \frac{-m^2\phi}{(1-\phi)^2} + \phi E[Y_{t-1}Y_{t-k}] \\
\gamma(k) &= \phi(-\frac{m^2}{(1-\phi)^2} + E[Y_{t-1}Y_{t-k}]) \\
\gamma(k) &= \phi(-\frac{m^2}{(1-\phi)^2} + E[Y_{t-1}Y_{t-k}]) \\
\gamma(k) &= \phi\gamma(k-1) = \phi^2\gamma(k-2) = \phi^3\gamma(k-3) = \phi^4\gamma(k-4) = ... = \phi^k\gamma(0) \\
\gamma(k) &= \phi^k\frac{\sigma^2}{(1-\phi^2)}
\end{aligned} \tag{18}
$$

For $\rho(k)$ By definition:

$$\rho(k) = \gamma(k)/\gamma(0) \tag{19}$$

But we already know that:

$$\gamma(k) = \phi^k \gamma(0) \tag{20}$$

Therefore:

$$\begin{aligned} &\rho(k) = \gamma(k)/\gamma(0) = \phi^k \gamma(0)/\gamma(0) = \phi^k \\ &\rho(k) = \phi^k \end{aligned} \tag{21}$$

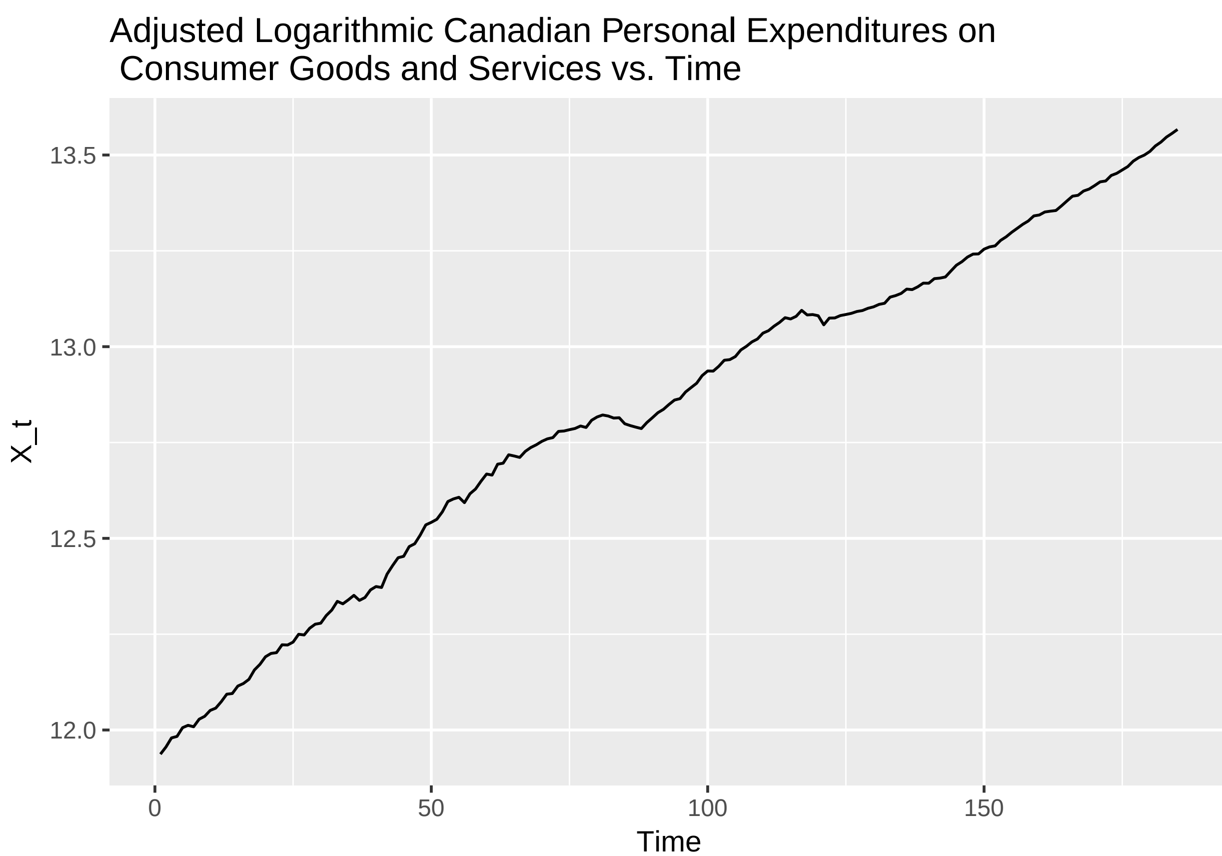
2 Practicum

Problem 1 - For the *seasonally adjusted* version do the following:

(a) Linearize the raw time series; that is, obtain $X_t = ln(W_t)$ and present a graph of the resulting series X_t.

```
library(xlsx)
library(dplyr)
library(ggplot2)
CPE_Cons_Goods  <- read.xlsx("CONS_Canada.xls", sheetName = "Sheet3")
CPE_Cons_Goods <- CPE_Cons_Goods %>% mutate(X_t = log(Seasonally.Adjusted))
CPE_Cons_Goods <- CPE_Cons_Goods[complete.cases(CPE_Cons_Goods),]

qplot(seq_along(CPE_Cons_Goods$X_t), CPE_Cons_Goods$X_t, geom = "line") +
  xlab("Time") +
  ylab("X_t") +
  ggtitle("Adjusted Logarithmic Canadian Personal Expenditures on \n Consumer Goods and Services vs. Ti
```



(b) TS approach:

Regress X_t on a constant and a time trend as

$$X_t = \alpha + \mu * t + Y_t \quad with \;\; Y_t \;\; \sim \;\; i.i.d. \;\; N.(0, \sigma^2)$$

```r
model_b <- lm(X_t ~ seq_along(CPE_Cons_Goods$X_t) , data = CPE_Cons_Goods)
summary(model_b)
```

```
##
## Call:
## lm(formula = X_t ~ seq_along(CPE_Cons_Goods$X_t), data = CPE_Cons_Goods)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.14397 -0.04027 -0.02086  0.05518  0.12311
##
## Coefficients:
##                                Estimate Std. Error t value Pr(>|t|)
## (Intercept)                   1.207e+01  9.159e-03 1318.15   <2e-16 ***
## seq_along(CPE_Cons_Goods$X_t) 8.158e-03  8.540e-05   95.52   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.06203 on 183 degrees of freedom
## Multiple R-squared:  0.9803, Adjusted R-squared:  0.9802
## F-statistic:  9124 on 1 and 183 DF,  p-value: < 2.2e-16
```

```r
sum(resid(model_b)^2)
```

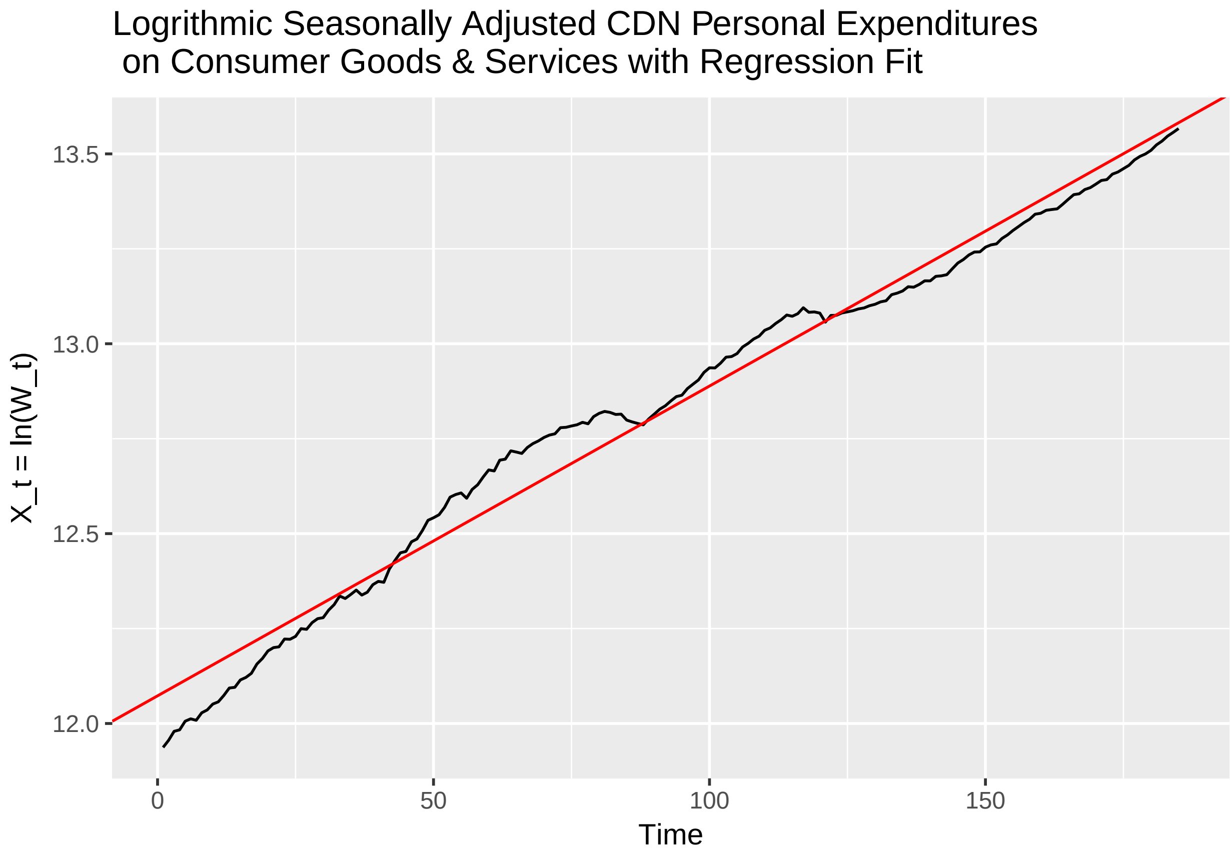
```
## [1] 0.704245
```

```r
summary(model_b)$r.squared
```

```
## [1] 0.9803367
```

$$\underset{t}{\hat{X}_t} = \underset{1318.15}{\hat{\alpha}} + \underset{95.52}{\hat{\mu}} \; t = 12.07 + 0.0082t$$

$n = 185 \qquad F - ratio = 9124 \qquad RSS = 0.704245 \qquad R^2 = 0.9803367$



What is the annual growth rate of W_t?

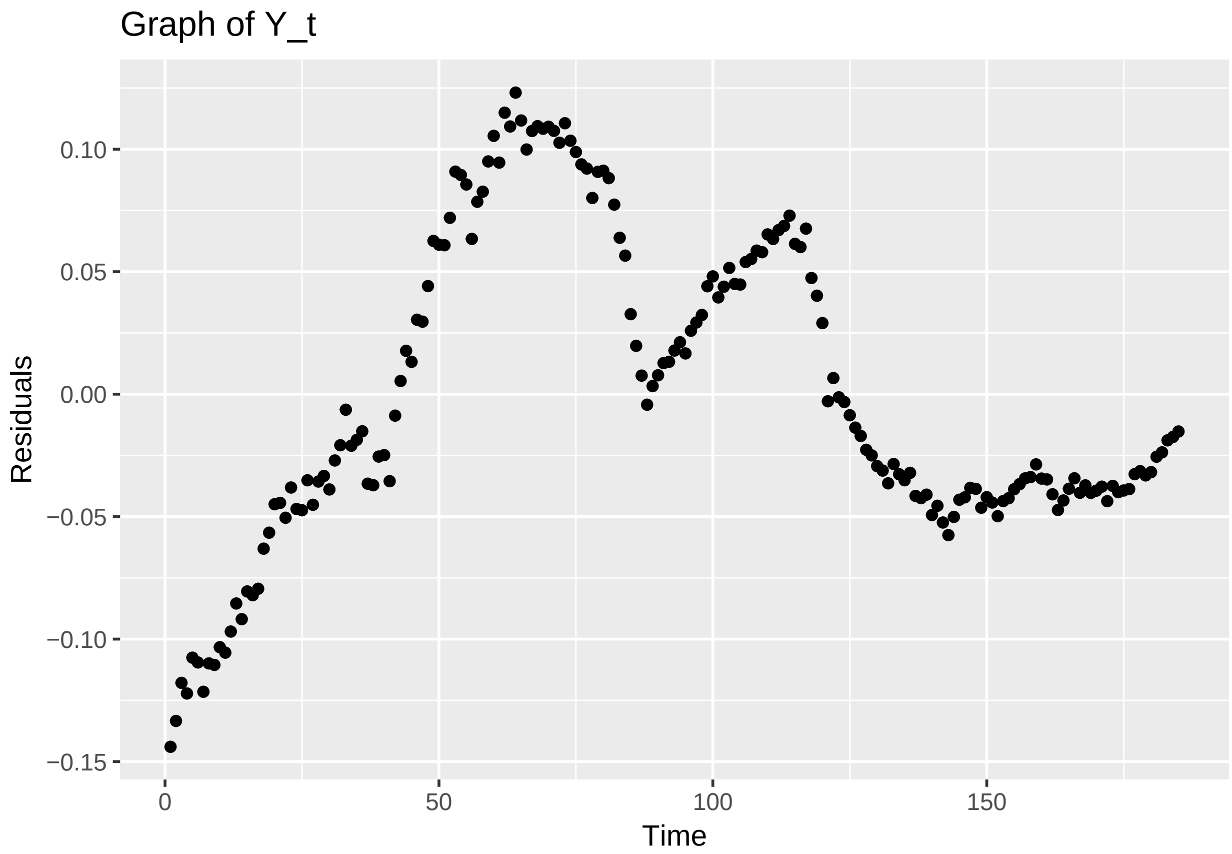
The annual growth rate of $\hat{W}_t$ is equal to $\hat{\mu}_t * 4 \times 100\%$. So growth annually is 3.263%.

Use the Rule of 72 to determine how long it takes for W_t to double

Using the *Rule of 72* it will take W_t, approximately 22.07 years to double.

Present a graph of Y_t

Below we have various plots assessing the model fit. The first plot shows the graph of Y_t



What is the sample mean of Y_t?

The sample mean of Y_t is given by:

```
mean(model_b$residuals)
```

```
## [1] 7.488247e-18
```

How large is the largest percentage deviation of W_t from its long-run trend? When does this occur?

```
CPE_Cons_Goods <- cbind(CPE_Cons_Goods[1:185,], model_b$residuals)
CPE_Cons_Goods$W_t_fitted_values <- exp(model_b$fitted.values)
CPE_Cons_Goods$W_t_Percent_Deviation <- abs(CPE_Cons_Goods$Seasonally.Adjusted - CPE_Cons_Goods$W_t_fit

max(CPE_Cons_Goods$W_t_Percent_Deviation)
```

```
## [1] 0.1340848
```

```
max_dev_index <- which(CPE_Cons_Goods$W_t_Percent_Deviation == max(CPE_Cons_Goods$W_t_Percent_Deviation)
max_dev_index
```

```
## [1] 1
```

```r
CPE_Cons_Goods[max_dev_index,]
```

```
##       NA. Seasonally.Unadjusted Seasonally.Adjusted      X_t
## 1 1961:01                 36070              152821 11.93702
##   model_b$residuals W_t_fitted_values W_t_Percent_Deviation
## 1        -0.1439683          176484.9             0.1340848
```

The largest percentage deviation from W_t from its long-run trend is 14.397% and this occurs in our first observation which is the first quarter (Q1) of 1961.

Obtain the cycle from the previous regression and fit the following AR(1) model and estimate ϕ using ordinary least squares:

$$Y_t = \phi Y_{t-1} + a_t$$

```r
Y_t <- model_b$residuals
Y_t_minus1 <- lag(Y_t)
resid <- data.frame(cbind(Y_t, Y_t_minus1))
resid <- resid[-1,]
model_AR_1 <- lm(Y_t ~ Y_t_minus1 - 1 , data = resid)
summary(model_AR_1)
```

```
##
## Call:
## lm(formula = Y_t ~ Y_t_minus1 - 1, data = resid)
##
## Residuals:
##        Min         1Q     Median         3Q        Max
## -0.0312062 -0.0048960  0.0007353  0.0053835  0.0258891
##
## Coefficients:
##            Estimate Std. Error t value Pr(>|t|)
## Y_t_minus1   0.9754     0.0103   94.73   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.00864 on 183 degrees of freedom
## Multiple R-squared:   0.98,  Adjusted R-squared:  0.9799
## F-statistic:  8973 on 1 and 183 DF,  p-value: < 2.2e-16
```

```r
sum(resid(model_AR_1)^2)
```

```
## [1] 0.01366125
```

```r
summary(model_AR_1)$r.squared
```

```
## [1] 0.9800133
```

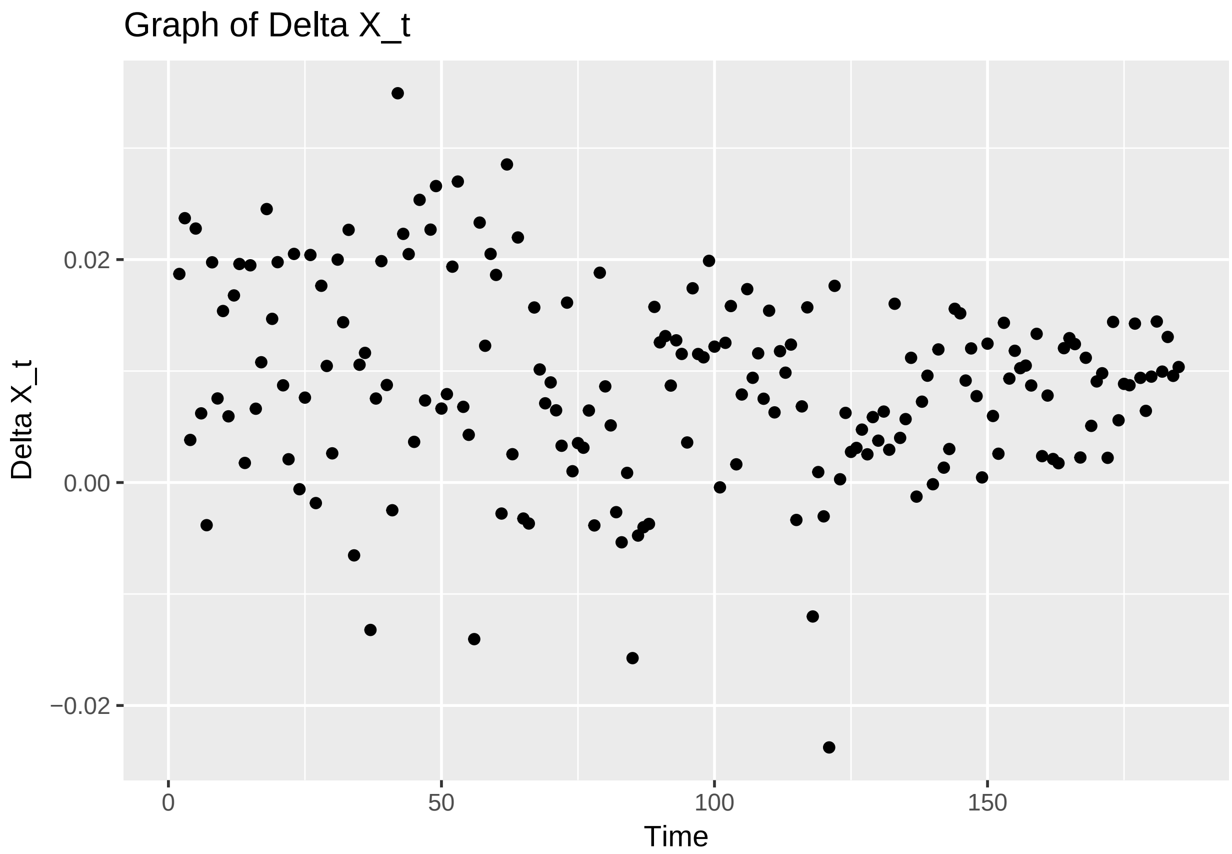
$$\underset{t}{\hat{Y}_t} = \underset{94.73}{\hat{\phi}} Y_{t-1} = 0.9754 Y_{t-1}$$

$n = 184 \qquad F - ratio = 8973 \qquad RSS = 0.01366125 \qquad R^2 = 0.9800133$

$\hat{Y}_t$ is stationary *iff* $-1 < \phi < 1$, therefore we can assume our model is stationary because our estimate of $\hat{\phi} = 0.9754$ satisfies this condition.

(c) DS Approach:

Present a graph of ΔX_t.



Run the regression to obtain the Difference Stationary cycle as in $\Delta X_t = \mu + Y_t$

In an intercept only regression model, the intercept can be found by taking the mean of the response variable

```r
model_c <- lm(DeltaX_t ~ 1, data = CPE_Cons_Goods)
summary(model_c)
```

```
##
## Call:
## lm(formula = DeltaX_t ~ 1, data = CPE_Cons_Goods)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.032618 -0.005742  0.000056  0.005527  0.026062
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 0.0088572  0.0006447   13.74   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.008745 on 183 degrees of freedom
```

```
##   (1 observation deleted due to missingness)
```

What is the annual growth rate of W_t?

The constant growth model is based on a quarterly differences so we need mu*4:

```
mu <- model_c$coefficients[1]
mu*4
```

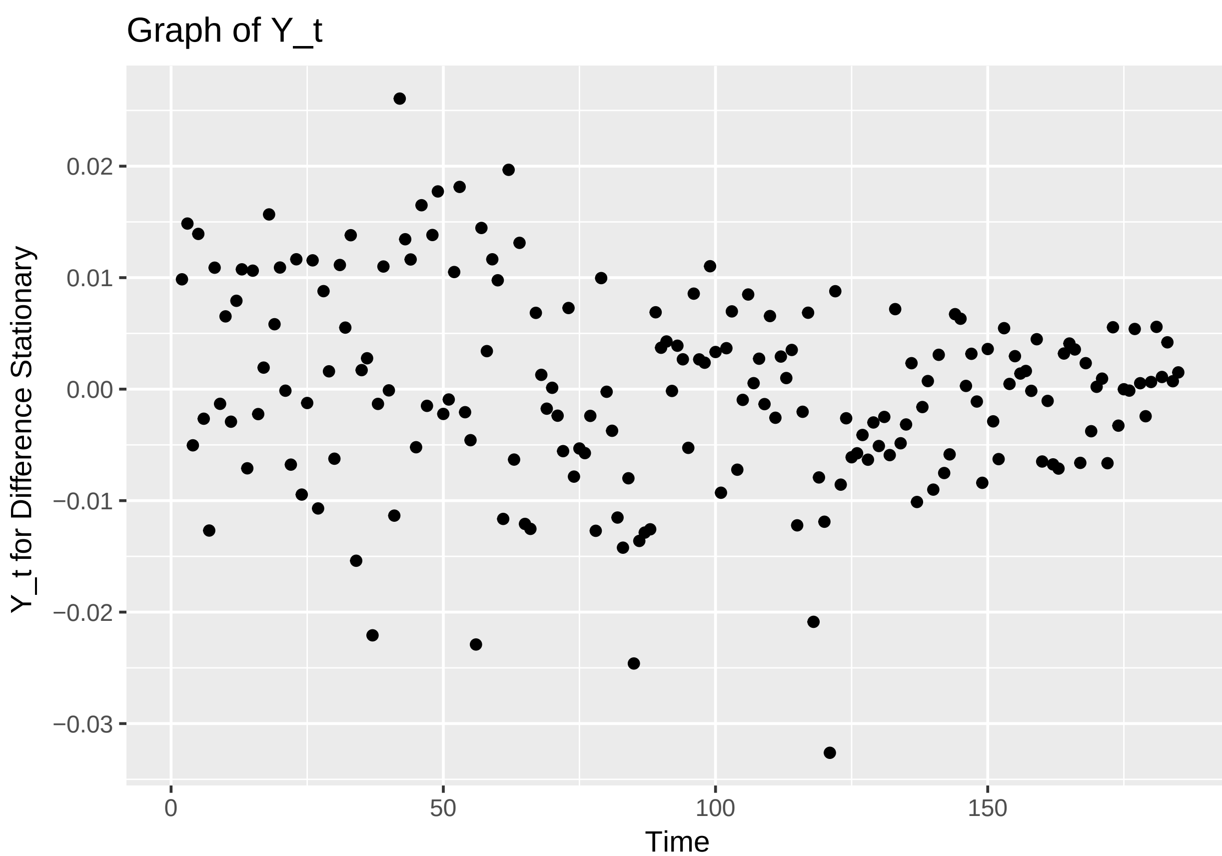
```
## (Intercept)
##  0.03542867
```

Therefore the annual growth rate is 3.54%

Use the Rule of 72 to determine how long it takes W_t to double.

Using the *Rule of 72* it will take W_t, approximately 20.32 years to double.

Present a graph of Y_t



What is the sample mean of Y_t?

```
mean(CPE_Cons_Goods$Y_t_DS, na.rm=TRUE)
```

```
## [1] -7.668755e-20
```

What is the largest value of Y_t? When did it occur?

```
max_abs_Y_t <- max(abs(CPE_Cons_Goods$Y_t_DS),  na.rm=TRUE)
max_abs_Y_t
```

```
## [1] 0.03261828
```

```
max_dev_index <- which((CPE_Cons_Goods$Y_t_DS == max_abs_Y_t) | (CPE_Cons_Goods$Y_t_DS == -max_abs_Y_t))
max_dev_index
```

```
## [1] 121
```

```
CPE_Cons_Goods[max_dev_index,]
```

```
##          NA. Seasonally.Unadjusted Seasonally.Adjusted      X_t    DeltaX_t
## 121 1991:01                112096              468359 13.05699 -0.02376111
##           Y_t_DS
## 121 -0.03261828
```

The largest value of Y_t (assuming absolute difference from 0) is 0.03261828, this value occurs between the final quarter of 1990 and the first quarter of 1991.

Estimate ϕ in the AR(1) model $Y_t = \phi Y_{t-1} + a_t$ using ordinary least squares:

```
Y_t <- model_c$residuals
Y_t_minus1 <- lag(Y_t)
resid <- data.frame(cbind(Y_t, Y_t_minus1))
resid <- resid[-1,]
model_AR_1 <- lm(Y_t ~ Y_t_minus1 - 1 , data = resid)
summary(model_AR_1)
```

```
##
## Call:
## lm(formula = Y_t ~ Y_t_minus1 - 1, data = resid)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.032554 -0.005715  0.000008  0.005481  0.026123
##
## Coefficients:
##            Estimate Std. Error t value Pr(>|t|)
## Y_t_minus1 0.005395   0.073872   0.073    0.942
##
## Residual standard error: 0.008739 on 182 degrees of freedom
## Multiple R-squared:  2.93e-05,   Adjusted R-squared:  -0.005465
## F-statistic: 0.005333 on 1 and 182 DF,  p-value: 0.9419
```

```
sum(resid(model_AR_1)^2)
```

```
## [1] 0.01389837
```

```
summary(model_AR_1)$r.squared
```

```
## [1] 2.930044e-05
```

$\hat{Y}_t$ is stationary *iff* $-1 < \phi < 1$, therefore we can assume our model is stationary because our estimate of $\hat{\phi} = 0.005395$ satisfies this condition.

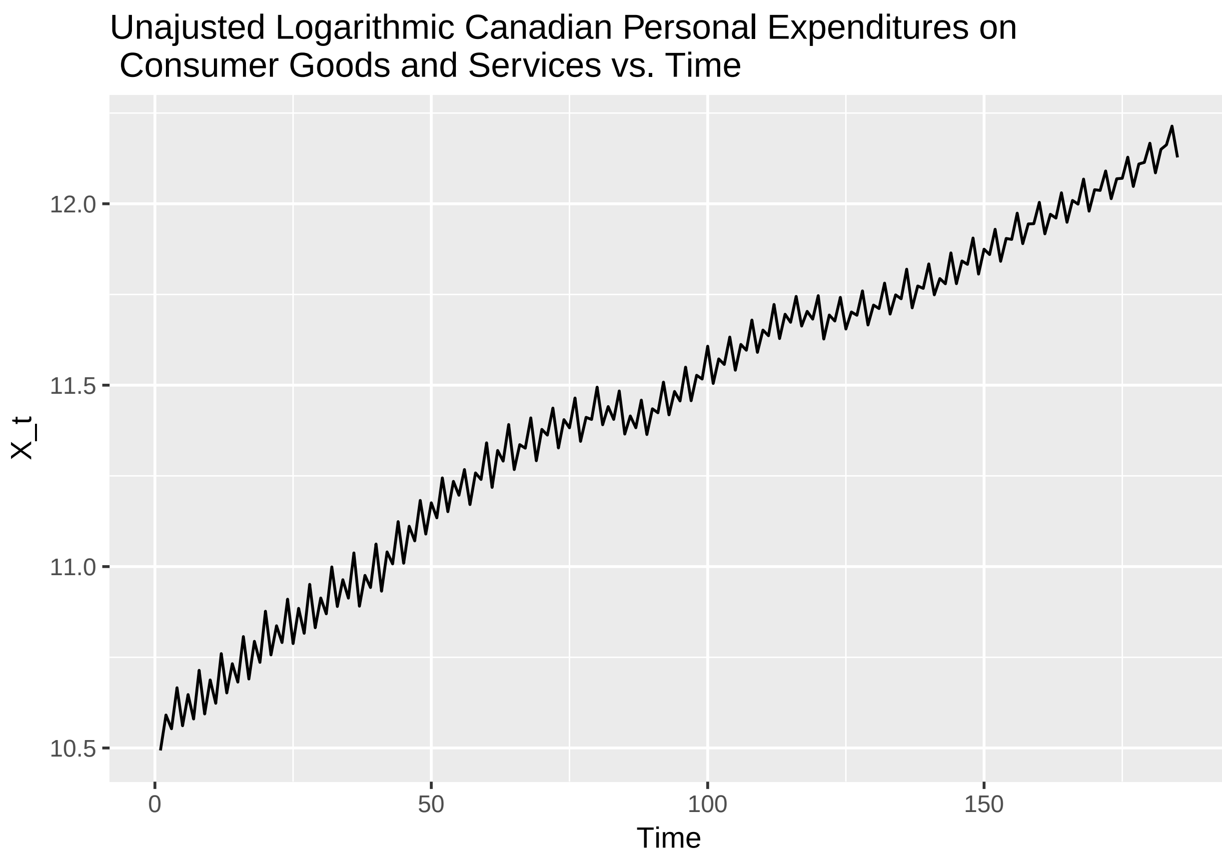
Problem 2 - For the *seasonally unadjusted* version, do the following:

a)

Present a graph of $X_t = ln(W_t^S)$:

```
CPE_Cons_Goods <- CPE_Cons_Goods %>% mutate(X_t_S = log(Seasonally.Unadjusted))

qplot(seq_along(CPE_Cons_Goods$X_t_S), CPE_Cons_Goods$X_t_S, geom = "line") +
  xlab("Time") +
  ylab("X_t") +
  ggtitle("Unajusted Logarithmic Canadian Personal Expenditures on \n Consumer Goods and Services vs. Ti
```



Run the regression to obtain the Trend Stationary Cycle Y_t:

```
CPE_Cons_Goods$d1 <- c(rep(c(1,0,0,0), times = length(CPE_Cons_Goods$X_t_S)/4),1)
CPE_Cons_Goods$d2 <- c(rep(c(0,1,0,0), times = length(CPE_Cons_Goods$X_t_S)/4),0)
CPE_Cons_Goods$d3 <- c(rep(c(0,0,1,0), times = length(CPE_Cons_Goods$X_t_S)/4),0)
CPE_Cons_Goods$d4 <- c(rep(c(0,0,0,1), times = length(CPE_Cons_Goods$X_t_S)/4),0)

model_2_a <- lm(X_t_S ~ d1 + d2 + d3 + d4 + seq_along(X_t_S) - 1, data = CPE_Cons_Goods)
summary(model_2_a)
```

```
##
## Call:
## lm(formula = X_t_S ~ d1 + d2 + d3 + d4 + seq_along(X_t_S) - 1,
```

```
##     data = CPE_Cons_Goods)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.14748 -0.04227 -0.01573  0.05222  0.12369
##
## Coefficients:
##                   Estimate Std. Error t value Pr(>|t|)
## d1               1.063e+01  1.231e-02  863.97   <2e-16 ***
## d2               1.070e+01  1.232e-02  867.86   <2e-16 ***
## d3               1.066e+01  1.238e-02  861.30   <2e-16 ***
## d4               1.075e+01  1.244e-02  863.82   <2e-16 ***
## seq_along(X_t_S) 8.164e-03  8.730e-05   93.52   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.06341 on 180 degrees of freedom
## Multiple R-squared:      1,  Adjusted R-squared:      1
## F-statistic: 1.207e+06 on 5 and 180 DF,  p-value: < 2.2e-16
```

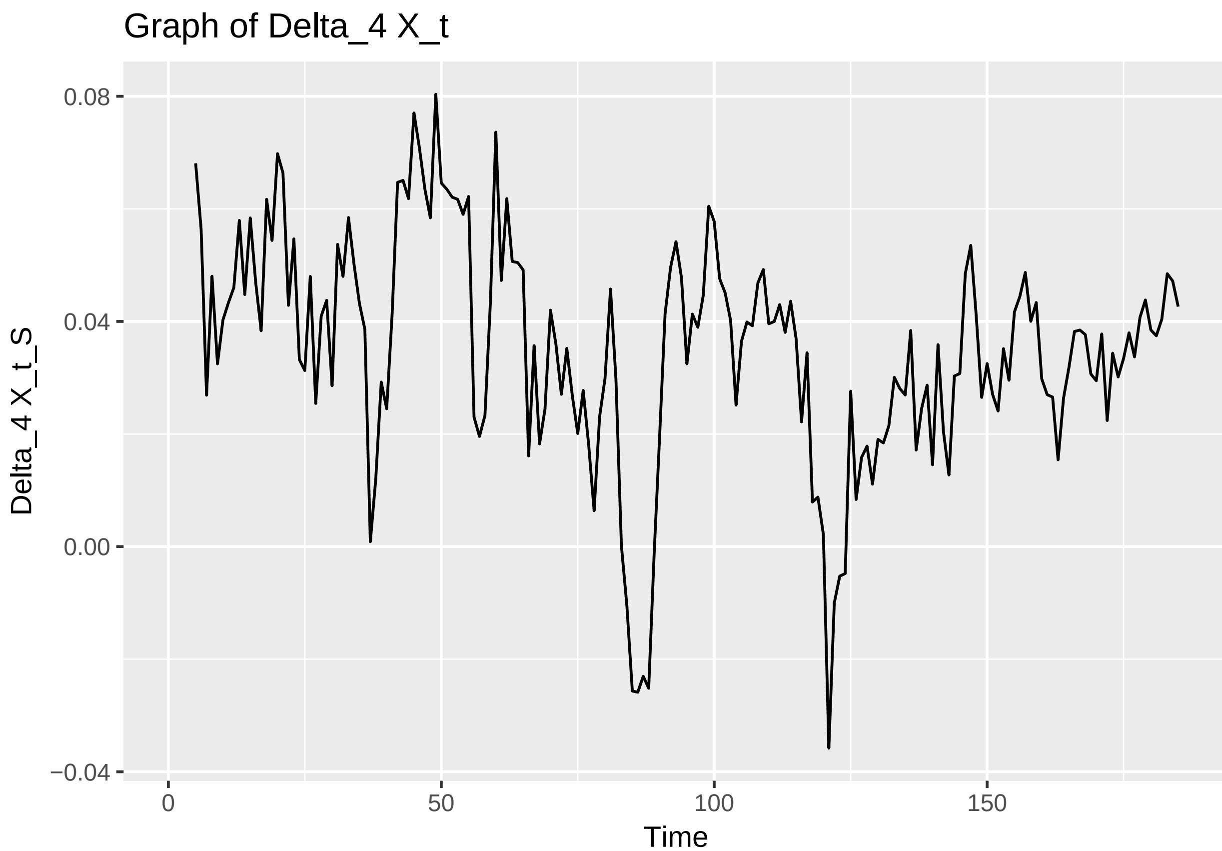
Present a graph of Y_t:

```r
qplot(seq_along(model_2_a$residuals), model_2_a$residuals) +
    ylab("Residuals") +
    xlab("Time") +
    ggtitle("Graph of Y_t")
```

b)

Present a graph of $\Delta_4 X_t$:

```r
CPE_Cons_Goods$XDelta4X_t_S <- CPE_Cons_Goods$X_t_S - lag(CPE_Cons_Goods$X_t_S, 4)
qplot(seq_along(CPE_Cons_Goods$XDelta4X_t_S), CPE_Cons_Goods$XDelta4X_t_S, geom ="line") +
    ylab("Delta_4 X_t_S") +
    xlab("Time") +
    ggtitle("Graph of Delta_4 X_t")
```



Run the regression to obtain the Difference Stationary cycle as in $\Delta_4 X_t = \mu + Y_t$

```r
model_2b <- lm(XDelta4X_t_S ~ 1, data = CPE_Cons_Goods)
summary(model_2b)
```

```
##
## Call:
## lm(formula = XDelta4X_t_S ~ 1, data = CPE_Cons_Goods)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.070883 -0.009656  0.002691  0.012190  0.045276
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 0.035090   0.001495   23.47   <2e-16 ***
```

```
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.02012 on 180 degrees of freedom
##   (4 observations deleted due to missingness)
```

What is the annual growth rate of W_t?

```
model_2b$coefficients
```

```
## (Intercept)
##  0.03508967
```

The annual growth rate of W_T is 3.509%.

Use the Rule of 72 to determine how long it takes W_t to double.

Using the *Rule of 72* it will take W_t, approximately 20.52 years to double.

Present a graph of Y_t

```
qplot(seq_along(model_2b$residuals), model_2b$residuals) +
    ylab("Residuals") +
    xlab("Time") +
    ggtitle("Graph of Y_t")
```