EE092IU
DIGITAL SIGNAL PROCESSING LABORATORY

Lab 4

Z TRANSFORM

Full name:………………………………..

Student number:………………………...

Class:……………………………………..

Date:……………………………………...

I. OBJECTIVES

Students know how to write m-file in Matlab to illustrate the properties of Z transform.

II. REQUIRED EQUIPMENT

1. Computer
2. Matlab software

III. INTRODUCTION

Just as with the Laplace transform for continuous-time signals and systems, the Z-transform provides a way to represent discrete-time signals and systems, and to process discrete-time signals. Although the Z-transform can be related to the Laplace transform, the relation is operationally not very useful. However, it can be used to show that the complex z-plane is in a polar form where the radius is a damping factor and the angle corresponds to the discrete frequency w in radians. Thus, the unit circle in the z-plane is analogous to the jw axis in the Laplace plane, and the inside of the unit circle is analogous to the left-hand s-plane. We will see that once the connection between the Laplace plane and the z-plane is established, the significance of poles and zeros in the z-plane can be obtained like in the Laplace plane.

The representation of discrete-time signals by the Z-transform is very intuitive—it converts a sequence of samples into a polynomial. The inverse Z-transform can be achieved by many more methods than the inverse Laplace transform, but the partial fraction expansion is still the most commonly used method. Using the one-sided Z-transform, for solving difference equations that could result from the discretization of differential equations, but not exclusively, is an important application of the Z-transform.

3.1. Mathematical Definition

The z transform of a discrete time signal is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{1}$$

Example 1: Compute the z-transform of the sequence $x(n) = [3, 5, 4, 3], 0 \le n \le 3$

Commands	Results	Comments

`syms z`
`x0=3;`
`x1=5;`
`x2=4;`
`x3=3;`
`Xz=x0*(z^0)+x1*(z^-1)+x2*(z^2)+x3*(z^-3)`
`pretty(Xz)`
	$3+\frac{5}{z}+\frac{4}{z^2}+\frac{3}{z^3}$	z-Transform of the sequence $x[n]$.

An alternative and more elegant computation of the z-transform

Commands	Results	Comments

`syms z`
`x=[3 5 4 3];`
`n=[0 1 2 3];`
`X=sum(x.*(z.^-n))`
`pretty(X)`
	$3+\frac{5}{z}+\frac{4}{z^2}+\frac{3}{z^3}$	Alternative way of computing the z-transform of the sequence $x[n]$.

Example 2: Compute the z-transform of the sequence $x(n)=0.9^n u(n)$

Commands	Results	Comments

`syms n z`
`x=0.9^n;`
`X=symsum(x.*(z.^-n),n,0,inf)`
	`X=10*z/(10*z-9)`	The z-transform of the sequence $x[n]=0.9^n\,u[n]$ is $X(z)=\frac{z}{z-0.9}$.

In MATLAB, the z-transform $F(z)$ of a sequence $f(n)$ is computed easily by using the command *ztrans*. Moreover, the inverse z-transform of a function $F(z)$ is computed by using the command *iztrans*.

Before using these two commands, the declaration of the complex variable z and of the discrete time n as symbolic variables is necessary. Recall that in order to define a symbolic variable, the command syms is used.

Example 3: Compute the z-transform of the sequence $f(n)=2^n$

Commands	Results	Comments

`syms n z`
`f= 2^n;`
`ztrans(f)`
`simplify(ans)`
	`ans=z/(z-2)`	The z-transform of the sequence $f[n]=2^n$.

Example 4: Compute the inverse z-transform of the function $F(z)=z/(z-2)$

Commands	Results	Comments

`syms n z`
`F=z/(z-2);`
`iztrans(F)`
	`ans=2^n`	Inverse z-transform of the function $F(z)=\dfrac{z}{z-2}$.

Example 5: Using Partial Fraction Expansion of a Rational Function, express in the partial fraction form the signal which in the z-domain is given by

$$X(z)=\frac{z^2+3z+1}{z^3+5z^2+2z-8}$$

Commands	Results	Comments
`A=[1 5 2 -8];` `ro=roots(A)`	`ro=-4.0000 -2.0000` `1.0000`	The vector containing the coefficients of the denominator polynomial is defined and its roots are computed.

`syms z`
`X=(z^2+3*z+1)/(z^3+5*z^2+2*z-8);`
`c1=limit((z-ro(1))*X,z,ro(1))`
`c2=limit((z-ro(2))*X,z,ro(2))`
`c3=limit((z-ro(3))*X,z,ro(3))`
	`c1=1/2`
	`c2=1/6`
	`c3=1/3`
		The coefficients c_i are calculated according to Equation 10.27.

Example 6: Using Partial Fraction Expansion of a Rational Function, express in the partial fraction form the signal which in the z-domain is given by

$$X(z)=\frac{z^2+3z+1}{z^3-3z+2}$$

Commands	Results	Comments
`A= [1 0 −3 2];` `rt=roots(A)`	`rt=−2.0000 1.0000 1.0000`	First, the roots of the denominator are calculated. The root $\lambda=1$ is repeated two times. Notice that the coefficient of z^2 is zero and must be taken into account when formulating matrix A.

`syms z`
`X= (z^2+3*z+1)/(z^3−3*z+2);`
`c1=limit((z-rt(1))*X,z,rt(1))`
	$c_1=-1/9$	The coefficient c_1 is computed according to the lower part of (10.29).

`r=2`		In order to compute the coefficients c_2 and c_3 (that correspond to $i=1$ and $i=2$, respectively), we set $r=2$ as there are two repeated roots. First, we compute coefficient c_2.
`f= ((z−1)^r )*X;`		Calculation of $(z-\lambda_1)^r X(z)$.
`di=diff(f,z,r−1);`		Calculation of $\frac{d^{r-i}[(z-\lambda_1)^r X(z)]}{dz^{r-i}}$.
`fact=1/factorial(r−1);`		Calculation of $\frac{1}{(r-i)!}$. Notice that $i=1$ when calculating c_2.
`c2=limit(fact*di,z,1)`	$c_2=10/9$	Coefficient c_2 is computed according to the upper part of (10.29).

`di=diff(f,z,r−2);`
`fact=1/factorial(r−2);`
`c3=limit(fact*di,z,1)`
	$c_3=5/3$	Coefficient c_3 is computed in the same way to c_2, but here we set $i=2$.

We also can use *residue* command

Commands	Results	Comments
`num= [ 1 3 1];` `den= [ 1 0 −3 2]`		The coefficients of numerator and denominator polynomials are defined as usual.

`[R,P,K] =residue(num,den)`	`R=−0.1111 1.1111 1.6667`
	`P=−2.0000 1.0000 1.0000`
	`K= []`
		$X(z)$ is expressed in partial fraction form as $X(z)=\frac{-0.1111}{z+2}+\frac{1.1111}{z-1}+\frac{1.6667}{(z-1)^2}$, which is the same result as the one obtained by the analytical way.

3.2. Basic Properties of the Z Transform

Properties of the z-Transforms

Property	$x[n]$	$X(z)$
1. Linearity	$ax[n] + by[n]$	$aX(z) + bY(z)$
2. Time-shifting	$x[n-m]$	$z^{-m}X(z) + z^{-m+1}x[-1] + \cdots + z^{-1}x[-m+1] + x[-m]$
3. Frequency scaling	$a^n x[n]$	$X\left(\frac{z}{a}\right)$
4. Time reversal	$x[-n]$	$X\left(\frac{1}{z}\right)$
5. Multiplication by n	$nx[n]$	$-z\frac{d}{dz}X(z)$
6. Multiplication by n^2	$n^n x[n]$	$z\frac{d}{dz}X(z) + z^2\frac{d^2}{dz^2}X(z)$
7. Modulation		
Multiplication by $e^{j\Omega_0 n}$	$e^{j\Omega_0 n}x[n]$	$X(e^{-j\Omega_0}z)$
Multiplication by $\cos\Omega n$	$(\cos\Omega n)x[n]$	$\frac{1}{2}\left[X(e^{j\Omega}z) + X(e^{-j\Omega}z)\right]$
Multiplication by $\sin\Omega n$	$(\sin\Omega n)x[n]$	$\frac{j}{2}\left[X(e^{j\Omega}z) - X(e^{-j\Omega}z)\right]$
8. Accumulation	$\sum_{k=0}^{n} x[k]$	$\frac{z}{z-1}X(z)$
9. Convolution	$x[n] * h[n]$	$X(z)H(z)$
10. Initial value	$x[0] = \lim_{z\to\infty} X(z)$	
11. Final value	$x[\infty] = \lim_{z\to 1}(1-z^{-1})X(z)$	

Table 1: Z-transform property

3.3. Z Transform Pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

Table 2: Z-transform pairs

3.4 Using the z-Transform to Solve Difference Equations

Example 7: Find the solutions of the difference equation y[n] +0.5y[n-1] + 2y[n-2] = 0.9^n u[n], where y[n] = 0, n < 0

Solution:

We have:

$$Z\{y[n]\} = Y(z).$$

$$Z\{y[n-1]\} = z^{-1}Z\{y[n]\} = z^{-1}Y(z).$$

$$Z\{y[n-2]\} = z^{-2}Z\{y[n]\} = z^{-2}Y(z).$$

The computational procedure as follow:

1. Applying z-transform to both parts of the differential equation yields $Z\{y[n] + 0.5y[n-1] + 2y[n-2]\} = Z\{0.9^n u[n]\}$.
2. Due to the linearity property, we get

$$Z\{y[n]\} + 0.5Z\{y[n-1]\} + 2Z\{y[n-2]\} = \frac{z}{z-0.9},$$

   where the z-transform pair $a^n u[n] \leftrightarrow \dfrac{z}{z-a}$ is used.

3. The z-transforms of $y[n]$, $y[n-1]$, and $y[n-2]$ are substituted according to Equations 10.35 through 10.37, and the difference equation is converted to the algebraic equation

$$Y(z) = 0.5z^{-1}Y(z) + 2z^{-2}Y(z) = \frac{z}{z-0.9}.$$

4. We solve the equation for $Y(z)$ and get

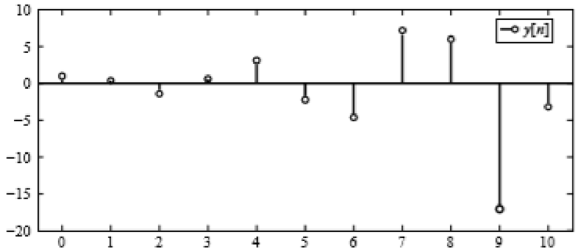
$$Y(z) = \frac{z}{(z-0.9)(1+0.5z^{-1}+2z^{-2})} = \frac{z^3}{(z-0.9)(z^2+0.5z+2)}.$$

5. The solution $y[n]$ of the difference equation is obtained by applying inverse z-transform to $Y(z)$, i.e., $y[n] = Z^{-1}\{Y(z)\}$.

In Matlab:

Commands	Results	Comments
`syms n z Y`		$Y(z)$ is denoted by Y.
`X=ztrans(0.9^n,z)`		The z-transform of the right side of the difference equation is computed.
`Y1=z^(-1)*Y;`		The z-transform of $y[n-1]$, that is, $Z\{y[n-1]\}$ is defined according to (10.36). The result is assigned to variable $Y1$.
`Y2=z^(-2)*Y;`		$Z\{y[n-2]\}$ is denoted by $Y2$ and is defined according to (10.37).
This is the crucial point of the computational procedure. The term X is moved to the left side of the difference equation and the whole left side is assigned to a term G		
`G=Y+0.5*Y1+2*Y2-X;`		The term G is a function of Y and z.
`SOL=solve(G,Y);` `pretty(SOL);`	$\frac{z^3}{(z-0.9)(z^2+0.5z+2)}$.	Using the command `solve` allows us to solve for Y. This is the solution of the differential equation expressed in the z-domain. The obtained solution is same as the analytically computed solution.
`y=iztrans(SOL,n);`		Applying inverse z-transform yields the solution $y[n]$ of the difference equation.

Commands	Results	Comments

`n1=0:10;`
`y_n=subs(y,n,n1);`
`stem(n1,y_n)`
`legend('y[n]')`
`xlim([-.5 10.5])`
		The sequence $y[n]$ is defined as a symbolic expression; thus in order to implement its graph, the symbolic variable n is substituted by a vector.

In order to confirm that y[n] is in fact the solution of the difference equation, y[n] is inserted to the difference equation, if it satisfies the difference equation, then it is indeed its solution

Commands	Results	Comments
`yn1=subs(y,n,n-1);` `yn2=subs(y,n,n-2);`		The terms $y[n-1]$ and $y[n-2]$ are computed from the derived sequence $y[n]$.
`test=y+0.5*yn1+2* yn2-0.9^n;` `test=simplify(test)`	`test=0`	The obtained sequence $y[n]$ is indeed the solution of the difference equation $y[n]+0.5y[n-1]+2y[n-2]=0.9^n$.

PROCEDURE

Problem 1:

Express in partial fraction form the signal and determine $x(n)$

$$X(z)=\frac{12-38z^{-1}+11z^{-2}+3z^{-3}+54z^{-4}}{1-5z^{-1}+6z^{-2}}$$

$$X(z)=\frac{2z^{2}+z-1}{z^{3}-3z+2}$$

$$X(z)=\frac{5-11z^{-1}}{1-5z^{-1}+6z^{-2}}$$

Problem 2:

Compute of the Z-transform of the following sequence using the symbolic toolbox of MATLAB

$$h_1[n]=0.8u(n)$$
$$h_2[n]=u[n]-u[n-10]$$
$$h_3[n]=cos(\omega_0 n)u[n]$$
$$h_4[n]=h_1[n]h_3[n]$$

Problem 3:

Consider an FIR filter with impulse response

$$h(n)=\delta(n)+\delta(n-1)+\delta(n-2)$$

Find the filter output for an input

$$x(n) = \cos\left(2\pi n/3\right)\left(u(n) - u(n-14)\right)$$

Use the convolution sum to find the output, and verify your results with MATLAB.

Use z-transform to find the output

Problem 4:

Use z-transform to find the solution of the difference equations:

a/ y[n] + 1.5 y[n-1] + 0.5y[n-2] = x[n] + x[n-1], where x[n] = 0.8^n u[n]

b/ y[n] – y[n-1] = x[n] + x[n-1], where x[n] = 0.8^n u[n]

Plot the solution for 0 <n < 20.
Confirm your result by inserting the obtained solution in the difference equation.