26. Let W be the line with parametric equations

$$x = 2t, \quad y = -t, \quad z = 4t$$

(a) Find a basis for W.

(b) Find the standard matrix for the orthogonal projection on W.

27. Find the orthogonal projection of $\mathbf{u} = (5, 6, 7, 2)$ on the solution space of the homogeneous linear system

$$\begin{aligned} x_1 + \quad x_2 + x_3 \quad\quad &= 0 \\ 2x_2 + x_3 + x_4 &= 0 \end{aligned}$$

28. Show that if $\mathbf{w} = (a, b, c)$ is a nonzero vector, then the standard matrix for the orthogonal projection of R^3 onto the line span$\{\mathbf{w}\}$ is

$$P = \frac{1}{a^2 + b^2 + c^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

29. Let A be an $m \times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of R^n onto the row space of A.

Working with Proofs

30. Prove: If A has linearly independent column vectors, and if $A\mathbf{x} = \mathbf{b}$ is consistent, then the least squares solution of $A\mathbf{x} = \mathbf{b}$ and the exact solution of $A\mathbf{x} = \mathbf{b}$ are the same.

31. Prove: If A has linearly independent column vectors, and if $\mathbf{b}$ is orthogonal to the column space of A, then the least squares solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{0}$.

32. Prove the implication $(b) \Rightarrow (a)$ of Theorem 6.4.3.

True-False Exercises

TF. In parts (a)–(h) determine whether the statement is true or false, and justify your answer.

(a) If A is an $m \times n$ matrix, then A^TA is a square matrix.

(b) If A^TA is invertible, then A is invertible.

(c) If A is invertible, then A^TA is invertible.

(d) If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system, then $A^TA\mathbf{x} = A^T\mathbf{b}$ is also consistent.

(e) If $A\mathbf{x} = \mathbf{b}$ is an inconsistent linear system, then $A^TA\mathbf{x} = A^T\mathbf{b}$ is also inconsistent.

(f) Every linear system has a least squares solution.

(g) Every linear system has a unique least squares solution.

(h) If A is an $m \times n$ matrix with linearly independent columns and $\mathbf{b}$ is in R^m, then $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution.

Working with Technology

T1. (a) Use Theorem 6.4.4 to show that the following linear system has a unique least squares solution, and use the method of Example 1 to find it.

$$\begin{aligned} x_1 + \quad x_2 + x_3 &= 1 \\ 4x_1 + 2x_2 + x_3 &= 10 \\ 9x_1 + 3x_2 + x_3 &= 9 \\ 16x_1 + 4x_2 + x_3 &= 16 \end{aligned}$$

(b) Check your result in part (a) using Formula (9).

T2. Use your technology utility to perform the computations and confirm the results obtained in Example 2.

6.5 Mathematical Modeling Using Least Squares

In this section we will use results about orthogonal projections in inner product spaces to obtain a technique for fitting a line or other polynomial curve to a set of experimentally determined points in the plane.

Fitting a Curve to Data

A common problem in experimental work is to obtain a mathematical relationship $y = f(x)$ between two variables x and y by "fitting" a curve to points in the plane corresponding to various experimentally determined values of x and y, say

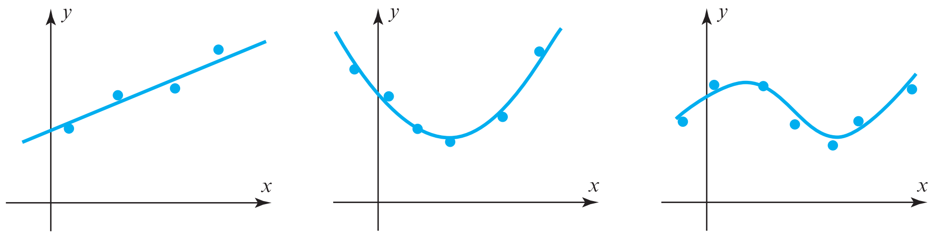
$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

On the basis of theoretical considerations or simply by observing the pattern of the points, the experimenter decides on the general form of the curve $y = f(x)$ to be fitted. This curve is called a ***mathematical model*** of the data. Some examples are (Figure 6.5.1):

(a) A straight line: $y = a + bx$

(b) A quadratic polynomial: $y = a + bx + cx^2$

(c) A cubic polynomial: $y = a + bx + cx^2 + dx^3$



(a) $y = a + bx$ (b) $y = a + bx + cx^2$ (c) $y = a + bx + cx^2 + dx^3$

▶ Figure 6.5.1

Least Squares Fit of a Straight Line

When data points are obtained experimentally, there is generally some measurement "error," making it impossible to find a curve of the desired form that passes through all the points. Thus, the idea is to choose the curve (by determining its coefficients) that "best fits" the data. We begin with the simplest case: fitting a straight line to data points.

Suppose we want to fit a straight line $y = a + bx$ to the experimentally determined points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

If the data points were collinear, the line would pass through all n points, and the unknown coefficients a and b would satisfy the equations

$$\begin{aligned} y_1 &= a + bx_1 \\ y_2 &= a + bx_2 \\ &\vdots \\ y_n &= a + bx_n \end{aligned} \tag{1}$$

We can write this system in matrix form as

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

or more compactly as

$$M\mathbf{v} = \mathbf{y} \tag{2}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad M = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{3}$$

If there are measurement errors in the data, then the data points will typically not lie on a line, and (1) will be inconsistent. In this case we look for a least squares approximation to the values of a and b by solving the normal system

$$M^T M\mathbf{v} = M^T\mathbf{y}$$

For simplicity, let us assume that the x-coordinates of the data points are not all the same, so M has linearly independent column vectors (Exericse 14) and the normal system has the unique solution

$$\mathbf{v}^* = \begin{bmatrix} a^* \\ b^* \end{bmatrix} = (M^TM)^{-1}M^T\mathbf{y}$$

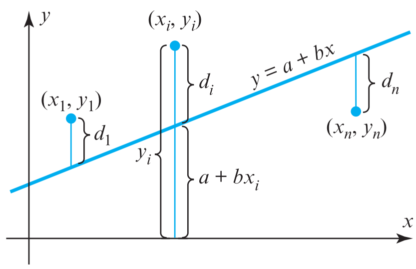
[see Formula (9) of Theorem 6.4.4]. The line $y = a^* + b^*x$ that results from this solution is called the ***least squares line of best fit*** or the ***regression line***. It follows from (2) and (3) that this line minimizes

$$\|\mathbf{y} - M\mathbf{v}\|^2 = [y_1 - (a + bx_1)]^2 + [y_2 - (a + bx_2)]^2 + \cdots + [y_n - (a + bx_n)]^2$$

The quantities

$$d_1 = |y_1 - (a + bx_1)|, \quad d_2 = |y_2 - (a + bx_2)|, \ldots, \quad d_n = |y_n - (a + bx_n)|$$

are called ***residuals***. Since the residual d_i is the distance between the data point (x_i, y_i) and the regression line (Figure 6.5.2), we can interpret its value as the "error" in y_i at the point x_i. If we assume that the value of each x_i is exact, then all the errors are in the y_i so the regression line can be described as *the line that minimizes the sum of the squares of the data errors*—hence the name, "least squares line of best fit." In summary, we have the following theorem.



▶ Figure 6.5.2 d_i measures the vertical error.

THEOREM 6.5.1 Uniqueness of the Least Squares Solution

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be a set of two or more data points, not all lying on a vertical line, and let

$$M = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \quad \textit{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{4}$$

Then there is a unique least squares straight line fit

$$y = a^* + b^*x \tag{5}$$

to the data points. Moreover,

$$\mathbf{v}^* = \begin{bmatrix} a^* \\ b^* \end{bmatrix} \tag{6}$$

is given by the formula

$$\mathbf{v}^* = (M^TM)^{-1}M^T\mathbf{y} \tag{7}$$

which expresses the fact that $\mathbf{v} = \mathbf{v}^*$ is the unique solution of the normal equation

$$M^TM\mathbf{v} = M^T\mathbf{y} \tag{8}$$

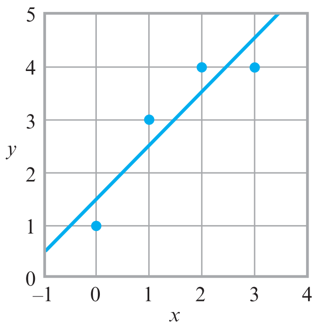


Figure 6.5.3

▶ EXAMPLE 1 Least Squares Straight Line Fit

Find the least squares straight line fit to the four points (0, 1), (1, 3), (2, 4), and (3, 4). (See Figure 6.5.3.)

Solution We have

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad M^TM = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}, \quad \text{and} \quad (M^TM)^{-1} = \frac{1}{10}\begin{bmatrix} 7 & -3 \\ -3 & 2 \end{bmatrix}$$

$$\mathbf{v}^* = (M^TM)^{-1}M^T\mathbf{y} = \frac{1}{10}\begin{bmatrix} 7 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

so the desired line is $y = 1.5 + x$.

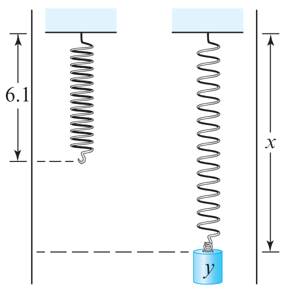


Figure 6.5.4

▶ EXAMPLE 2 Spring Constant

Hooke's law in physics states that the length x of a uniform spring is a linear function of the force y applied to it. If we express this relationship as $y = a + bx$, then the coefficient b is called the ***spring constant***. Suppose a particular unstretched spring has a measured length of 6.1 inches (i.e., $x = 6.1$ when $y = 0$). Suppose further that, as illustrated in Figure 6.5.4, various weights are attached to the end of the spring and the following table of resulting spring lengths is recorded. Find the least squares straight line fit to the data and use it to approximate the spring constant.

Weight y (lb)	0	2	4	6
Length x (in)	6.1	7.6	8.7	10.4

Solution The mathematical problem is to fit a line $y = a + bx$ to the four data points

$$(6.1, 0), \quad (7.6, 2), \quad (8.7, 4), \quad (10.4, 6)$$

For these data the matrices M and $\mathbf{y}$ in (4) are

$$M = \begin{bmatrix} 1 & 6.1 \\ 1 & 7.6 \\ 1 & 8.7 \\ 1 & 10.4 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix}$$

so

$$\mathbf{v}^* = \begin{bmatrix} a^* \\ b^* \end{bmatrix} = (M^TM)^{-1}M^T\mathbf{y} \approx \begin{bmatrix} -8.6 \\ 1.4 \end{bmatrix}$$

where the numerical values have been rounded to one decimal place. Thus, the estimated value of the spring constant is $b^* \approx 1.4$ pounds/inch. ◀

Least Squares Fit of a Polynomial

The technique described for fitting a straight line to data points can be generalized to fitting a polynomial of specified degree to data points. Let us attempt to fit a polynomial of fixed degree m

$$y = a_0 + a_1x + \cdots + a_mx^m \tag{9}$$

to n points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

Substituting these n values of x and y into (9) yields the n equations

$$\begin{aligned} y_1 &= a_0 + a_1x_1 + \cdots + a_m x_1^m \\ y_2 &= a_0 + a_1x_2 + \cdots + a_m x_2^m \\ &\vdots \\ y_n &= a_0 + a_1x_n + \cdots + a_m x_n^m \end{aligned}$$

or in matrix form,

$$\mathbf{y} = M\mathbf{v} \tag{10}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad M = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^m \\ 1 & x_2 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^m \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{bmatrix} \tag{11}$$

As before, the solutions of the normal equations

$$M^TM\mathbf{v} = M^T\mathbf{y}$$

determine the coefficients of the polynomial, and the vector $\mathbf{v}$ minimizes

$$\|\mathbf{y} - M\mathbf{v}\|$$

Conditions that guarantee the invertibility of M^TM are discussed in the exercises (Exercise 16). If M^TM is invertible, then the normal equations have a unique solution $\mathbf{v} = \mathbf{v}^*$, which is given by

$$\mathbf{v}^* = (M^TM)^{-1}M^T\mathbf{y} \tag{12}$$

EXAMPLE 3 Fitting a Quadratic Curve to Data

According to Newton's second law of motion, a body near the Earth's surface falls vertically downward in accordance with the equation

$$s = s_0 + v_0t + \tfrac{1}{2}gt^2 \tag{13}$$

where

$$\begin{aligned} s &= \text{vertical displacement downward relative to some reference point} \\ s_0 &= \text{displacement from the reference point at time } t = 0 \\ v_0 &= \text{velocity at time } t = 0 \\ g &= \text{acceleration of gravity at the Earth's surface} \end{aligned}$$

Suppose that a laboratory experiment is performed to approximate g by measuring the displacement s relative to a fixed reference point of a falling weight at various times. Use the experimental results shown in the following table to approximate g.

Time t (sec)	.1	.2	.3	.4	.5
Displacement s (ft)	−0.18	0.31	1.03	2.48	3.73

Solution For notational simplicity, let $a_0 = s_0$, $a_1 = v_0$, and $a_2 = \frac{1}{2}g$ in (13), so our mathematical problem is to fit a quadratic curve

$$s = a_0 + a_1 t + a_2 t^2 \tag{14}$$

to the five data points:

$$(.1, -0.18), \quad (.2, 0.31), \quad (.3, 1.03), \quad (.4, 2.48), \quad (.5, 3.73)$$

With the appropriate adjustments in notation, the matrices M and $\mathbf{y}$ in (11) are

$$M = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \\ 1 & t_4 & t_4^2 \\ 1 & t_5 & t_5^2 \end{bmatrix} = \begin{bmatrix} 1 & .1 & .01 \\ 1 & .2 & .04 \\ 1 & .3 & .09 \\ 1 & .4 & .16 \\ 1 & .5 & .25 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{bmatrix} = \begin{bmatrix} -0.18 \\ 0.31 \\ 1.03 \\ 2.48 \\ 3.73 \end{bmatrix}$$

Thus, from (12),

$$\mathbf{v}^* = \begin{bmatrix} a_0^* \\ a_1^* \\ a_2^* \end{bmatrix} = (M^T M)^{-1} M^T \mathbf{y} \approx \begin{bmatrix} -0.40 \\ 0.35 \\ 16.1 \end{bmatrix}$$

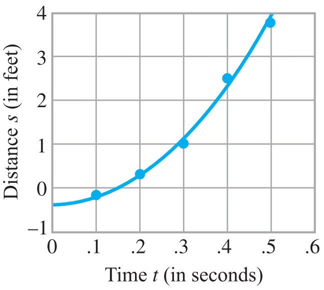
so the least squares quadratic fit is

$$s = -0.40 + 0.35t + 16.1t^2$$

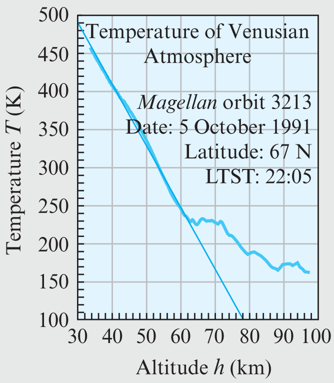
From this equation we estimate that $\frac{1}{2}g = 16.1$ and hence that $g = 32.2$ ft/sec^2. Note that this equation also provides the following estimates of the initial displacement and velocity of the weight:

$$s_0 = a_0^* = -0.40 \text{ ft}$$
$$v_0 = a_1^* = 0.35 \text{ ft/sec}$$

In Figure 6.5.5 we have plotted the data points and the approximating polynomial. ◀



▲ Figure 6.5.5



Source: NASA

Historical Note On October 5, 1991 the *Magellan* spacecraft entered the atmosphere of Venus and transmitted the temperature T in kelvins (K) versus the altitude h in kilometers (km) until its signal was lost at an altitude of about 34 km. Discounting the initial erratic signal, the data strongly suggested a linear relationship, so a least squares straight line fit was used on the linear part of the data to obtain the equation

$$T = 737.5 - 8.125h$$

By setting $h = 0$ in this equation, the surface temperature of Venus was estimated at $T \approx 737.5$ K. The accuracy of this result has been confirmed by more recent flybys of Venus.