gradient_descent

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1 Gradient Descent Method

- **Gradient descent** is an optimization algorithm used to minimize a function by iteratively moving towards the steepest direction.
- It is widely used in machine learning and deep learning for training models.

1.1 Optimization Algorithm We Learn't so far

- **Ordinary Least Squares (OLS)**

Question: 1. What is the main goal of OLS?

2. What does it minimize?
3. How does it work?

1.2 Question:

How many iterations does OLS require to converge?

1.3 Gradient Descent Algorithm

1.3.1 Why use Gradient Descent if we have OLS?

- Remember if you put in the values of X and y in the normal equation and perform some finite number of operations, you will get the optimal values of the parameters **in a single iteration**.
- **OLS is an non-iterative** approach for optimization
- The inverse operation, $(X^T X)^{-1}$ is computationally very expensive
- The time complexity of this operation increases significantly with the increase in number of features.
- So for a large dataset with a large number of features, OLS becomes computationally expensive and infeasible.

1.3.2 But how does Gradient Descent Work?

Step 1: Initialize the value of x randomly

Step 2: Calculate the gradient of $f(x)$ with respect to x ie. $\frac{\partial\ f(x)}{\partial\ x}$

Step 3: Update x as:

$$x := x - \alpha \frac{\partial\ f(x)}{\partial\ x}$$

where, α is the **learning rate** and ':=' is assignment operator

Step 4: Repeat steps 1, 2 and 3 until the value of $f(x)$ converges to the minimum value.

As the function approaches the minimum point, its gradient approaches zero and so the updates don't change x much. At the minimum point of the function, $\frac{\partial\ f(x)}{\partial\ x} \cong 0$ and the solution converges at that point after certain number of iterations.

[174]:
```
import matplotlib.pyplot as plt
import plotly.express as px

plt.style.use('seaborn-v0_8')  # Use a valid style
```

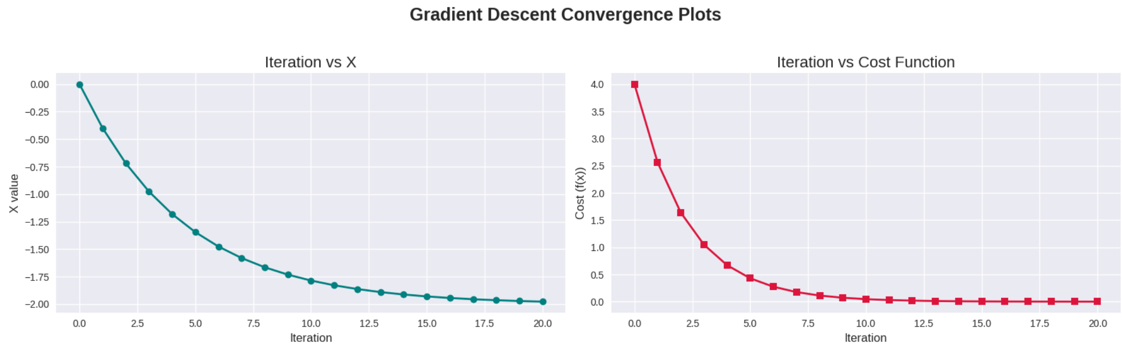
[175]:
```
history = dict()


def gradient_descent(
    f, df, x0, learning_rate=0.01, tolerance=1e-6, max_iterations=1000
):
    """
    Perform gradient descent to minimize the function f.

    Parameters:
    - f: The function to minimize.
    - df: The derivative of the function f.
    - x0: Initial guess for the minimum.
    - learning_rate: Step size for each iteration.
    - tolerance: Threshold for convergence.
    - max_iterations: Maximum number of iterations.

    Returns:
    - x: The estimated minimum point.
    """
    x = x0
    history["x"] = [x0]
    history["iterations"] = [0]
    history["cost"] = [f(x0)]
    for i in range(max_iterations):
        grad = df(x)
```

```
        x_new = x - learning_rate * grad
        history["x"].append(x_new)
        history["iterations"].append(i+1)
        history["cost"].append(f(x_new))
        if abs(x_new - x) < tolerance:
            break

        # print(f"Iteration {i}: (Initial) X = {x:.6f}")
        x = x_new

    return x, i
```

[176]:
```
def f(x):
    return x**2 + 4*x + 4  # Example function
def df(x):
    return 2*x + 4  # Derivative of the function
```

[177]:
```
x0 = 0  # Initial guess
minimum, iteration = gradient_descent(f, df, x0, learning_rate=0.1,␣
 ↪max_iterations=20)
print(f"The minimum point is at Iteration {iteration}: x = {minimum}")


fig, axes = plt.subplots(1, 2, figsize=(16, 5))

# Plot: Iteration vs X
axes[0].plot(history['iterations'], history['x'], color='teal', marker='o',␣
 ↪linewidth=2)
axes[0].set_title("Iteration vs X", fontsize=16)
axes[0].set_xlabel("Iteration", fontsize=12)
axes[0].set_ylabel("X value", fontsize=12)
axes[0].grid(True)

# Plot: Iteration vs Cost
axes[1].plot(history['iterations'], history['cost'], color='crimson',␣
 ↪marker='s', linewidth=2)
axes[1].set_title("Iteration vs Cost Function", fontsize=16)
axes[1].set_xlabel("Iteration", fontsize=12)
axes[1].set_ylabel("Cost (f(x))", fontsize=12)
axes[1].grid(True)

plt.suptitle("Gradient Descent Convergence Plots", fontsize=18,␣
 ↪fontweight='bold')
plt.tight_layout(rect=[0, 0, 1, 0.95])
plt.show()
```

```
The minimum point is at Iteration 19: x = -1.976941569907863
```



[178]:
```
from plotly.subplots import make_subplots
import plotly.graph_objects as go

fig = make_subplots(
    rows=1, cols=2, subplot_titles=("Iteration vs X", "Iteration vs Cost")
)

fig.add_trace(
    go.Scatter(x=history["iterations"], y=history["x"], mode="lines+markers",␣
 ↪name="X"),
    row=1,
    col=1,
)

fig.add_trace(
    go.Scatter(
        x=history["iterations"], y=history["cost"], mode="lines+markers",␣
 ↪name="Cost"
    ),
    row=1,
    col=2,
)

fig.update_layout(title_text="Gradient Descent Progress", width=1400,␣
 ↪height=600)
fig.show()
```

1.4 Key Takeaways

- Gradient of the cost function w.r.t each of the parameters can be derived easily using calculus.
- The parameters are updated iteratively using their corresponding gradients.
- When the dataset is large with a large number of features, Gradient descent is preferred instead of OLS of the function.