Before recitation, for your own good

- Review lecture summary
- Read problems and think about it
- Problems in parenthesis are R exercise from hws

After recitation

- Memorize key concepts first (what)
- Solve problems second (how)
- ? in recitation slides are further steps; should be done ONLY after the first two steps (why)

Summary: strive for two solutions

What:

- Random 101: **condition** and weighted average (**expectation)**
- Markov and Martingale
- first uses fundamental calculus (dx integ) second is sugar syntax

How:

- memorize, understand, memorize
- 1.geometric and exponential weights
- 2.condition-joint-marginalize structure
- 3.Binomial, poisson, normal pdf, mean, variance, laplace

More questions to practice from (R)

sol in the end; if there are any questions you want me to attend, send an email and i might be able to help

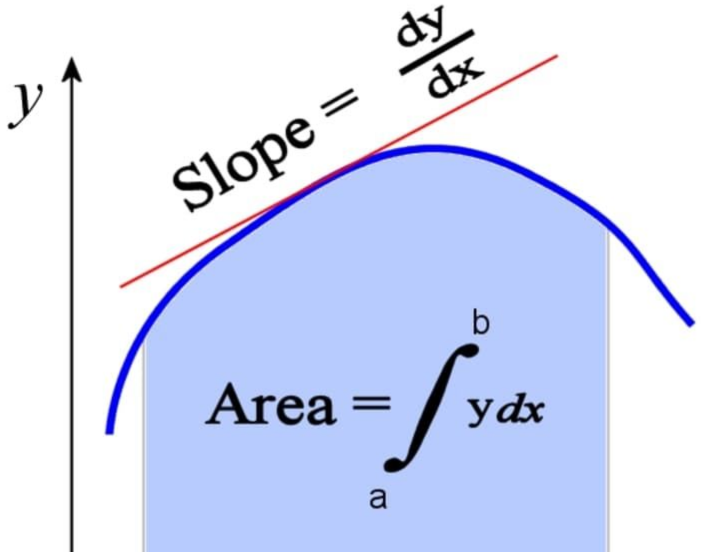
Do way

d:

- lambda in inventory demand model
- mu in drifted BM

Integ: Every E[??]

- total amount of inventory needed
- total waiting time



General tips.

1. Memorize - the more the better.
2. Understand the trend for 1 (make stories)
3. Repeat 1, 2.

E.g.

Story of a gambler who ended up winning with p 1-rho^(-a+b)/ 1- rho^()(binary,) for better memorization

- Trend, probability, expectation
- Stick to Boundary then to hierarchy
- Boundary are t^-2,
- gives sense on global when having to work on local (concession for linearity)

Graph of integ -> inf near zero for running max dist

Tip1: multiplicative weights

X: red vs +: green throughout the slide

Log, exp, sin, ax^b, constant,

E[e^cX] for summary, **c, I(w), Bin(n, p), Pois(r), Z, X_T** to generate

1. Bernoulli: 0, 1= I — p(A)

2. Geometric: **p^x** — 1/ (1-p)

   1/(1-x) = 1+x+x^2+..., f(x+1)= xf(x), M/M/1

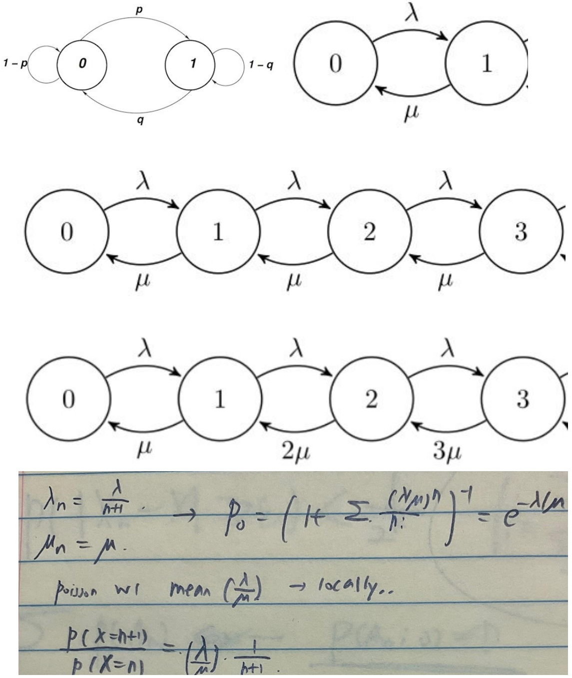
3. Poisson: **c^x/x!** — e^-c

   1 + 2/x + 4/x^2 +..., f(x+1) = (c/x)f(x), M/M/∞

4. X_T: **generating ftn e^{c*(e^t-1)}**

   e^{e^t-1} = 1 + (e^t-1) + (e^t-1)^2/2 + (e^t-1)^3/3!

   geom sum of exp = exp



Tip2: conversion “=”

sum one vector = probability

- bounded function = indicator function

- $Z \sim N(0,1)$ as RV basis (CLT, BPN)

Conversion table:

B_t^2-t, $\exp(cB_t - c^2t)$,

W12 topic of exponential weight

EW? Exponential weighting?

- mastery of probability lies in recognizing / memorizing conversion
- simplex normalization -1/lambda-t, e^-lambda(e^t), e^(t^2/2)
- independence and wlog as probabilitist’s tool
- tpe: conversion between time and prob

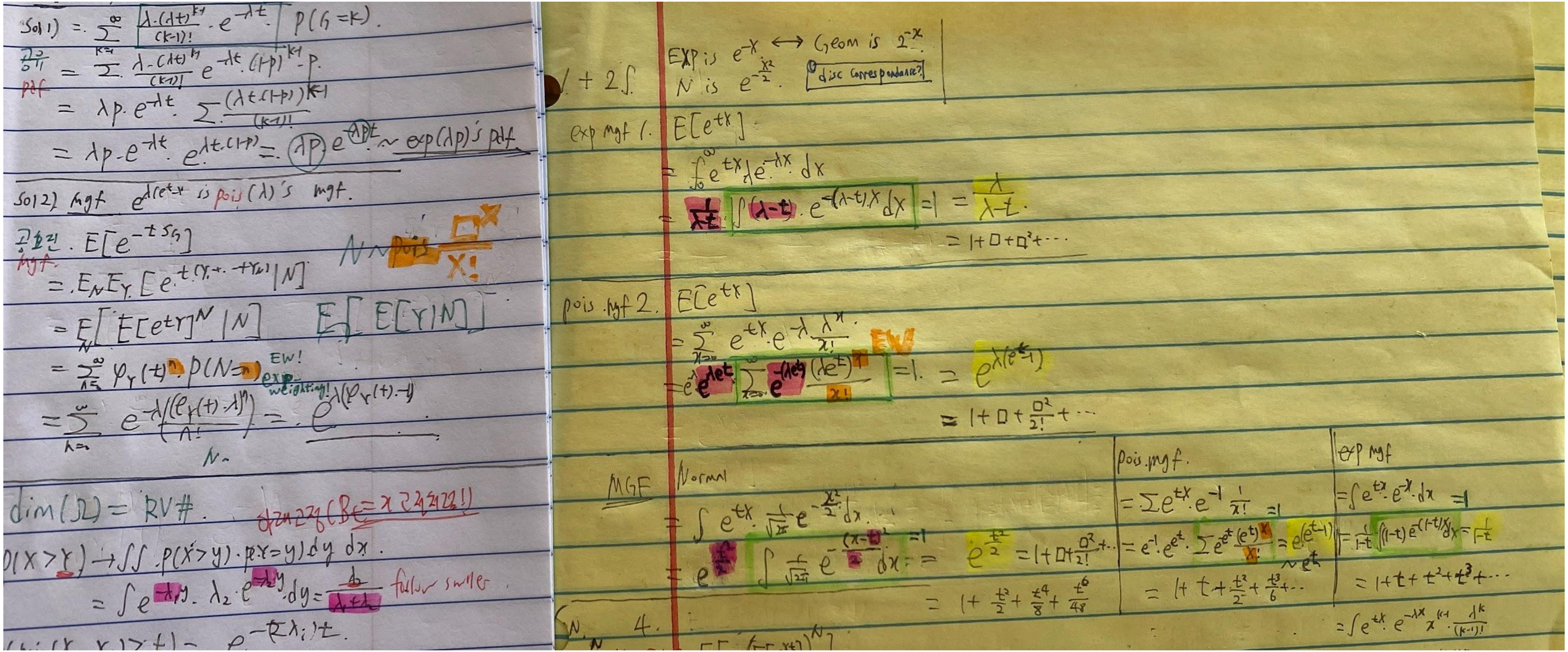
EW!

- drifted bm and gamble ruin TPE

EW~ Expected Winning time

- two proof for infinite expected winning time (contradiction vs direct pf)

Universal Exp



Sol 1) $= \sum_{k=1}^{\infty} \boxed{\frac{\lambda (\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t}} \, P(G=k)$

공리 pdf $= \sum \frac{\lambda (\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t} (1-p)^{k-1} p$

$= \lambda p \, e^{-\lambda t} \sum \frac{(\lambda t (1-p))^{k-1}}{(k-1)!}$

$= \lambda p \, e^{-\lambda t} e^{\lambda t (1-p)} = \lambda p \, e^{-\lambda p t} \sim$ exp(λp)'s pdf

Sol 2) mgf $e^{\lambda(e^t-1)}$ is pois(λ)'s mgf.

공도긴 mgf. $E[e^{-t S_N}]$ $N \sim$ pois

$= E_N E_Y [e^{t(Y_1 + \cdots + Y_N)} | N]$

$= E_N [E[e^{tY}]^N | N]$ $E_N[E[Y|N]]$

$= \sum_{n=0}^{\infty} \varphi_Y(t)^n \, P(N=n)$ EW! exp weighting!

$= \sum_{n=0}^{\infty} e^{-\lambda} \frac{(\varphi_Y(t) \lambda)^n}{n!} = e^{\lambda(\varphi_Y(t)-1)}$

$\dim(\Omega) = $ RV#

$P(X > Y) \to \iint P(X > y) \cdot P(Y=y) \, dy \, dx$

$= \int e^{-\lambda_1 y} \lambda_2 e^{-\lambda_2 y} dy = \frac{\lambda_2}{\lambda_1 + \lambda_2}$ follow smaller

$e^{-(\sum \lambda_i) t}$

Exp is $e^{-x} \longleftrightarrow$ Geom is 2^{-x}

N is $e^{-\frac{x^2}{2}}$ disc correspondance?

1 + 2 ∫

exp mgf 1. $E[e^{tX}]$

$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx$

$= \frac{\lambda}{\lambda - t} \boxed{\int (\lambda - t) e^{-(\lambda-t)x} dx} = 1 = \frac{\lambda}{\lambda - t}$

$= 1 + \square + \square^2 + \cdots$

pois mgf 2. $E[e^{tX}]$

$= \sum_{x=0}^{\infty} e^{tx} e^{-\lambda} \frac{\lambda^x}{x!}$ EW

$= e^{-\lambda} e^{\lambda e^t} \boxed{\sum_{x=0}^{\infty} e^{-\lambda e^t} \frac{(\lambda e^t)^x}{x!}} = 1. = e^{\lambda(e^t-1)}$

$= 1 + \square + \frac{\square^2}{2!} + \cdots$

MGF

Normal

$\int e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$= e^{\frac{t^2}{2}} \boxed{\int \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}} dx} = 1 = e^{\frac{t^2}{2}} = 1 + \square + \frac{\square^2}{2!} + \cdots$

$= 1 + \frac{t^2}{2} + \frac{t^4}{8} + \frac{t^6}{48}$

Pois mgf.

$= \sum e^{tx} e^{-1} \frac{1}{x!}$ =1

$= e^{-1} e^{e^t} \boxed{\sum e^{-e^t} \frac{(e^t)^x}{x!}} = e^{(e^t - 1)}$

$= 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots$

exp mgf

$= \int e^{tx} e^{-x} dx$ =1

$= \frac{1}{1-t} \boxed{\int (1-t) e^{-(1-t)x} dx} = \frac{1}{1-t}$

$= 1 + t + t^2 + t^3 + \cdots$

$= \int e^{tx} e^{-\lambda x} x^{k-1} \frac{\lambda^k}{(k-1)!}$

4. S_N

$E[e^{tX}] = e^{\lambda(e^{t}-1)}$

N condition

$N \sim \text{Geo}(p)$ $X_i \sim \text{Exp}(\lambda) \to S_N = \sum \lambda e^{-\lambda x}$ $p^{x}(1-p)^{n} = (\lambda p) e^{-(\lambda p-1)x}$. $\text{Exp}(\lambda p)$ | pois. split.

$N \sim \text{pois}$, $X_i \sim \text{pois}(\lambda) \to N_T = \int_{t=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \mu e^{-\mu t} \, dt = (1-p)^k p$.

$\boxed{T \text{ condition}}$

(= interarrival exp)

$N \sim \text{Pois}$ $X_i \sim \text{Bern}(p) \to S_N = \sum p^{x} \cdot (1-p)^{1-x}$ don't know. help. $\text{pois}(\lambda p)$. | Poiss split.

1.3. $\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}$ diffusion eq. unique sol.

1.4. $p(x\,t \mid x_0) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-x_0)^2}{2t}\right).$

1.5 $p_k(n+1) - p_k(n) = \frac{1}{2}\left(2p_{k+1}(n) - 2p_k(n) + p_{k-1}(n)\right).$

$p_k(n)$: prob. RW, move k left from start at n^{th} step.

$p_k(n+1) = \frac{1}{2}p_{k+1}(n) + \frac{1}{2}p_{k-1}(n)$

② $\Delta, \eta \to 0$
$n, k \to \infty$
$\Delta = \eta^2$
$k\eta = x$
$n\Delta = t$

1.6 $\frac{p_{k\eta}((n+1)\Delta) - p_{k\eta}(n\Delta)}{\Delta} = \frac{\frac{1}{2}\left(p_{(k+1)\eta}(n\Delta) - 2p_{k\eta}(n\Delta) + p_{(k-1)\eta}(n\Delta)\right)}{\Delta}$

invariance principle

1.7. $\lim_{n\to\infty} \sum_{k=-\infty}^{\sqrt{n}x} p_k(n) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}}du$ (CT from $p_k(n) = P(X_1 + \cdots + X_n = k)$)

2.1 $P[B_t \le x \mid B_{t_0} = x_0 \cdots B_{t_n} = x_n] = P(B_t \le x \mid B_{t_0} = x_0]$. MK

2.2 $P(B_t \le x \mid B_{t_0} = x_0) = P(B_t - B_{t_0} \le x - x_0) = \frac{1}{\sqrt{2\pi(t-t_0)}} \int_{-\infty}^{x-x_0} \exp\left(-\frac{u^2}{2(t-t_0)}\right)d$

$\frac{\partial \varphi}{\partial t} = -\frac{1}{2} \cdot t^{-3/2}$
$+ \frac{1}{2} t^{-1/2}$
$= +\frac{1}{2} t^{-1/2} \cdot ($

$\frac{\partial \varphi}{\partial x} = t^{-1/2} \cdot -\frac{x}{}$

$\frac{\partial^2 \varphi}{\partial x^2} = \exp(-$

$= \exp($

$e($

Adherent OST's: Mg portraits tension (equilibrium) of Time, Space using "Osum in Space and Time" (OS_T) 1. Adherency. 2. Bounded. Check!

Alg: [return $E[e^{\lambda T}]$ if : $EX_T = EX \cdot ET$ else : alarm nonzero system [3. transient or null recur.]]

1. $S_{N_T} \not\perp T$: $ES_{N(t)} \neq ES \cdot EN_{(t)}$, $E[e^{\beta X_T - \lambda T}] \neq E[e^{\beta X_T}] \cdot E[e^{-\lambda T}]$ (other source?) | $ET = \infty$ certificate 2. $T_1 := \inf\{t : B_t = 1\}$ as $EB_1 = 0 \neq EB_T \cdot ET \to ET = \infty$.

3. Knowing $ET_a = \infty$, $EX \neq EX_T$ $1 \leq E[e^{\sup_{0 \leq u \leq t} X(u)}] = E[e^{\sup_{0 \leq u \leq t} X(u)}] \to E[e^{-\infty}] =$

$Var(T) = \frac{b \cdot \delta^2}{\mu^3}$ $\left(\frac{b}{\mu}\right) \cdot \left(\frac{\delta}{\mu}\right)^2$

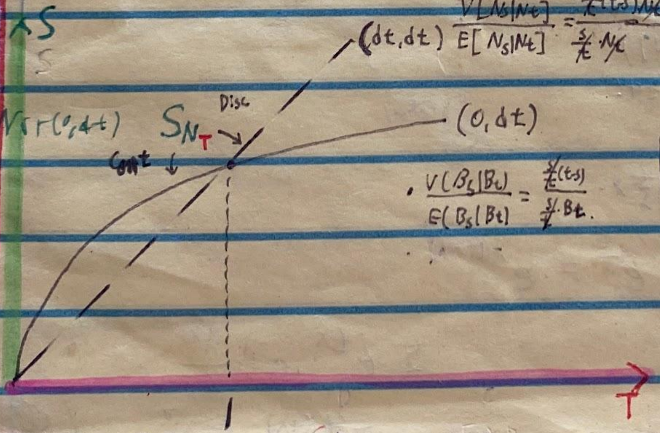
$E[B_T] = E[B]ET$
$EX_T = \frac{ET}{\mu}$

TPE = C J M on BPN

Binary Discrete Cont.

#/s $I_{Bern(p)}$ Unif. $N \sim r(0, dt)$

$Bin(n,p) \xrightarrow{a.} Pois(\lambda) \xrightarrow{b.} N(\mu, \delta^2)$



$\frac{V[N_s|N_t]}{E[N_s|N_t]} = \frac{\frac{s}{t}(t-s) N_t}{\frac{s}{t} \cdot N_t} = t-s$

$\frac{V(B_s|B_t)}{E(B_s|B_t)} = \frac{\frac{s}{t}(t-s)}{\frac{s}{t} \cdot B_t}$

$E\left[\sum^{T} S\right] = \left[\sum \quad \right]$

sum RV.

$\sum^{E} B = G$. $\sum$.

$\sum^{G} E = E$ PP split. $\sum \lambda e^{-\lambda t}$.

$\sum$

$\frac{1}{\sqrt{n}} \sum^{nt} N = N$. BM. $\left(\lim_{n \to \infty} \sum_{k=-\infty}^{\sqrt{n}x} P_k(n) = \lim_{n \to \infty} \sum_{k=-\infty}^{\sqrt{n}x} \underbrace{P\left(\sum_{i=1}^{n} X_i = k\right) = \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2}\right) du}_{CLT} \right)$

M engine: design $\int$ = 1

($\frac{1}{n}$ in MC) Conditioner weighting

- type follows
- Rev. Eng S_T for $E[e^{\gamma T}]$
- $Var(T_b) = \frac{b}{\mu} \cdot \left(\frac{\delta}{\mu}\right)^2 =$
- PP is time-trend-Var tied. BM is time-Var tied.

REVISIT $\frac{1}{\sqrt{n}}$ scaling?

Where should neg binom be placed?

B Geom(p) D Exp(λ) C Gamma weibull

(b) Mbl. (a)

Indicator → Simple → Signed

B D C

6. Suppose you own one share of a stock whose price changes according to a standard Brownian motion process. Suppose that you purchased the stock at a price $b + c$, $c > 0$, and the present price is b. You have decided to sell the stock either when it reaches the price $b + c$ or when an additional time t goes by (whichever occurs first). What is the probability that you do not recover your purchase price?

7. Compute an expression for

$$P\left\{\max_{t_1 \leqslant s \leqslant t_2} B(s) > x\right\}$$

*10. Let $\{X(t),\ t \geqslant 0\}$ be a Brownian motion process with drift coefficient μ and variance parameter σ^2. What is the conditional distribution of $X(t)$ given that $X(s) = c$ when

(a) $s < t$?

(b) $t < s$?

$\mu t + \sigma B_t = -2t + 2B_t$.

BM $e^{-\frac{2\mu}{\sigma^2}} = e^{-\frac{2}{2}} = e^{-1} =$

DTMC $= \frac{.25}{.75}$ q

CTMC $= \frac{\lambda}{\mu}$ μ.

$e^{-\frac{2\mu}{\sigma^2}} = \rho$

P_a : gambler ends loser $= \frac{\rho^{-a} - \rho^{-a+b}}{1-\rho^{a+b}}$

P_b : " Winner $= \frac{1-\rho^{a}}{1-\rho^{a+b}}$

$P_b =$

*19. Show that $\{Y(t), t \geqslant 0\}$ is a Martingale when

$$Y(t) = \exp\{cB(t) - c^2t/2\}$$

where c is an arbitrary constant. What is $E[Y(t)]$? $= E[Y_0] = 1$.

An important property of a Martingale is that if you continually observe the process and then stop at some time T, then, subject to some technical conditions (which will hold in the problems to be considered),

$$E[Y(T)] = E[Y(0)]$$

The time T usually depends on the values of the process and is known as a *stopping time* for the Martingale. This result, that the expected value of the stopped Martingale is equal to its fixed time expectation, is known as the *Martingale stopping theorem*.

*20. Let

$$T = \text{Min}\{t : B(t) = 2 - 4t\}$$

$E[X_T] = \mu \cdot E[T]$

That is, T is the first time that standard Brownian motion hits the line $2 - 4t$. Use the Martingale stopping theorem to find $E[T]$. $B_t + 4t$ hit 2.

22. Let $X(t) = \sigma B(t) + \mu t$, and for given positive constants A and B, let p denote the probability that $\{X(t),\ t \geqslant 0\}$ hits A before it hits $-B$.

(a) Define the stopping time T to be the first time the process hits either A or $-B$. Use this stopping time and the Martingale defined in Exercise 19 to show that

$$E[\exp\{c(X(T) - \mu T)/\sigma - c^2T/2\}] = 1$$

$p = p(T_A < T_{-B})$

gamble winner quit $\frac{1-\rho^a}{1-\rho^{a+b}}$

(b) Let $c = -2\mu/\sigma$, and show that

$$E[\exp\{-2\mu X(T)/\sigma\}] = 1$$

① $P_a + P_b = 1$

(c) Use part (b) and the definition of T to find p.

② $\left[p_a \cdot \frac{-2\mu A}{\sigma^2} + p_b \cdot \frac{+2\mu B}{\sigma^2}\right] = 1$

Hint: What are the possible values of $\exp\{-2\mu X(T)/\sigma^2\}$?

23. Let $X(t) = \sigma B(t) + \mu t$, and define T to be the first time the process $\{X(t),\ t \geqslant 0\}$ hits either A or $-B$, where A and B are given positive numbers. Use the Martingale

$E[tB_{\frac{1}{t}}] = \ldots = 0$

$Cov(s \cdot B_{\frac{1}{s}},\ t B_{\frac{1}{t}})$

$= s \cdot t \cdot Cov(B_{\frac{1}{s}}, B_{\frac{1}{t}}) = s \cdot t (\frac{1}{t}) = s$ w.l.o.g. $s<t$.

$B_1 B_2 | B_4$: $\frac{2}{16}B_4^2 + \frac{1 \cdot 2}{4}$?

B_4

$\forall t \in (0,\infty)$

③ $Y_t := \sqrt{c} \cdot B_{c \cdot t}$ $\quad B_{ct} = \sqrt{c} \cdot Z$

i) $B_t = \sqrt{t} \cdot Z \to \frac{1}{\sqrt{c}} B_{c \cdot t} = Z$

$B_1 = Z$ $\quad \frac{1}{\sqrt{c}} B_{2c} = \sqrt{2} Z$ $\quad \frac{1}{\sqrt{c}} B_{3c} = \sqrt{3} Z$ $\quad \frac{1}{\sqrt{c}} B_{4c} = 2Z$

$N(0,2c)$ $\quad N(0,3c)$ $\quad N(0,4c)$

$\Delta t = \sqrt{\Delta x}$ $\quad \frac{1}{2}B_4, \frac{1}{4}T_2$

$(\Delta t)^2 = \Delta x$ $\quad \boxed{\frac{1}{c} T_{\sqrt{c} y}}$

FIXED "T" $=4$

④ $Z_t := B_{T-t} = B_{4-t}$

$Z_t - Z_s = B_{T-t} - B_{T-s} = B$

$B_{4-3} = Z$ $\quad B_{4-2} = \sqrt{2} Z$ $\quad B_{4-1} = \sqrt{3} Z$ $\quad B_{4-0} = 2Z$

D-R-M ed by Yao.

① DBM

Drifted BM.

$\mu = -2,\ \sigma = 2$, $p = e^{-1} \sim \frac{1}{3}$

② RRW.

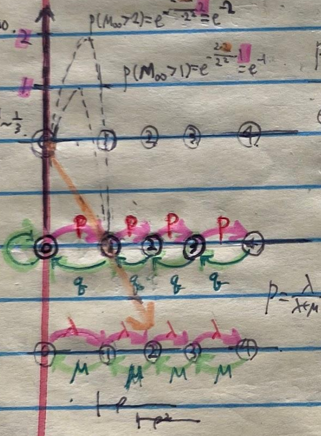
Reflected RW.

$p = .25,\ q = .75$

③ MM1

M/M/1 queue.

$\lambda = 1,\ \mu = 3$.



$P(M_\infty > 2) = e^{-\frac{2 \cdot 2 \cdot 2}{2^2}} = e^{-2}$

$P(M_\infty > 1) = e^{-\frac{2 \cdot 2 \cdot 1}{2^2}} = e^{-1}$

$p = \frac{\lambda}{\lambda + \mu}$

$X_t = \mu t + \sigma B_t$

$P(\hat{M}_\infty > b) = e^{-2ab}$ SBM line crossing = DBM M_∞

$P(X>x) = \sup_t X_t > x$ dist.		1	2	3	4
$e^{-\frac{2\mu x}{\sigma^2}}$	$\exp(\lambda)$, $\lambda = \frac{-2\mu}{\sigma^2} = 1$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$
$(\frac{p}{q})^x$	$geom(\tilde{p})$, $M_\infty(x)$, $\tilde{p}^{-1} = \frac{p}{q} = \frac{.25}{.75}$	4	16	52	160
p^{x+1}, $p^x(1-p)$	$geom(p)$, $p = \frac{\lambda}{\mu} = \frac{1}{3}$	ET_0: 1	ET_1: $\frac{1-3^2}{1-3} = 4$	ET_2: $\frac{1-3^3}{1-3} = 13$	ET_3: $\frac{1-3^4}{1-3} = 40$

$\times \frac{1}{\lambda + \mu}$ $\quad \times \frac{1}{\lambda + \mu} = \frac{1}{4}$.

$\boxed{\frac{P(X=x+1)}{P(X=x)} = e^{-\lambda} = \tilde{p}^{-1} = p = \frac{1}{3}}$

$\frac{q}{p} = \hat{p} > 1 > p = \frac{\lambda}{\mu}$

Time to reach 0.

$E[T] = \frac{1}{\mu}(bP_b - aP_a) = \frac{a \to -\infty}{} \frac{b}{\mu}$

$= (\#\text{steps from } i \text{ to } 0) X$

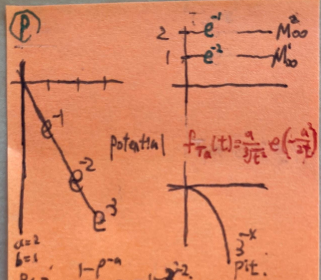
T_i := Time from i to $i+1$

$ET_i := \frac{1}{\lambda}(1 + 3ET_{i+1})$ $\quad 1-p^{-(i+1)}$

$= \frac{1}{\lambda}(1 + p^{-1} + p^{-2} + \ldots + p^{-i})$

$V(T_i) =$

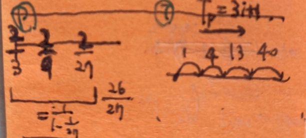
$ET_i = \frac{i}{q-p} - \frac{N}{q-p}\left(\frac{1-\hat{p}^i}{1-\hat{p}^N}\right) = qET_{i-1} + pET_{i+1}$



$f_{T_a}(t) = \frac{a}{\sqrt{2\pi t^3}} e^{(-\frac{a^2}{2t})}$

$a=2,\ b=1$

$P_b = \frac{1 - \hat{p}^{-a}}{1 - \hat{p}^{a+b}}$ $\quad 1 - 3^{-2}$



$\boxed{\text{Normalizing: } 1 - p^x}$ ⓪①② $\frac{1}{1-p^3} x$.

Pb. PP. EXP.

Why p_a + p_b = 1?
[return e^-cT: ES = EX*ET?: alarm system]

need independence and boundedness
to use OST

Adherent OST's: Mg portraits tension (equilibrium) of Time, Space using "Osum in Space and Time" (OS_T) 1. Adherency. 2. Bounded. Check!

Alg: [return $E[e^{\lambda T}]$ if : $EX_T = EX \cdot ET$ else : alarm nonzero system (3. transient or null recur.)]

1. $S_{N_T} \not\perp T$: $ES_{N(t)} \neq ES \cdot EN_{(t)}$, $E[e^{\beta X_T - \lambda T}] \neq E[e^{\beta X_T}] \cdot E[e^{-\lambda T}]$ 1. other source?

$ET = \infty$ certificate 2. $T_1 := \inf\{t : B_t = 1\}$ as $EB_1 = 0 \neq EB_T \cdot ET \to ET = \infty$.

3. Knowing $ET_a = \infty$, $EX \neq EX_T$

$1 \leq E[e^{\sup_{0 \leq u \leq t} X(u)}] = E[e^{\sup_{0 \leq u \leq t} X(u)}] \to E[e^{-\infty}] = 0$

EW~ Bill gates survives war

$$E[\tau_a] = \underline{2a \cdot \int \cdot \frac{1}{\sqrt{t}} \cdot \varphi\left(\frac{a}{\sqrt{t}}\right) . dt}$$

$$= \frac{2a}{\sqrt{2\pi}} \int_0^{\infty} \frac{1}{\sqrt{t}} \cdot e^{-\frac{a^2}{2t}} dt$$

$$\geq \frac{2a}{\sqrt{2\pi}} \cdot \int_1^{\infty} \frac{1}{\sqrt{t}} e^{-\frac{a^2}{2t}} dt$$

$$\geq \frac{2a}{\sqrt{2\pi}} \cdot e^{-\frac{a^2}{2}} \boxed{\int_{\cancel{1}\,0}^{\infty} \frac{1}{\sqrt{t}} dt}$$

$$= \left[2 t^{1/2}\right]_0^{\infty} \to \infty$$

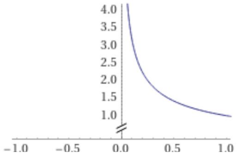
icate 2. $T_1 := \inf\{t : B_t = 1\}$ as $E B_1 = 0 \neq E B_T \cdot E T \to E T = \infty$.

$1 \leq E\left[e^{\sup_{0 \leq u \leq t} X(u)}\right] = E\left[e^{\sup_{u \leq t} X(u)}\right] \to E\left[e^{-u}\right] =$

Almost sure convergence does not imply L1 convergence

$\frac{1}{\sqrt{t}}$

Plots



W10, Create joint nature with artificial conditionals

- joint
- conditional
- interpolation

1. Distributional properties. of BM. Hitting time.

A. Joint. $f(x_1 \cdots x_n) = f_{t_1}(x_1) \cdot f_{t_2-t_1}(x_2) \cdots f_{t_n-t_{n-1}}(x_n) = \dfrac{\exp\left(-\frac{1}{2}\left(\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{(t_2-t_1)} + \cdots + \frac{(x_n-x_{n-1})^2}{t_n-t_{n-1}}\right)\right)}{(2\pi)^{n/2} \cdot (t_1 (t_2-t_1) \cdots (t_n-t_{n-1}))^{1/2}}$

B. Cond. $f_{s|t}(x_1|B) = \dfrac{f_s(x_1) \cdot f_{t-s}(B-x_1)}{f_t(B)} = K_2 \cdot \exp\left(-\frac{x_1^2}{2s} - \frac{(B-x_1)^2}{2(t-s)}\right)$

$= K_2 \cdot \exp\left(-\frac{(t-s)+s}{2s(t-s)} x_1^2 + \frac{B}{t-s} x_1\right)$

don't care constant $\frac{B}{2(t-s)}$

$-\dfrac{b'}{2a} = -\dfrac{1}{2} \dfrac{\frac{B}{t-s}}{\frac{t}{2s(t-s)}} = \dfrac{s}{t}B \quad = K_3 \cdot \exp\left(-\frac{t}{2s(t-s)}\left(x_1 - \frac{s}{t}B\right)^2\right)$

$\to \sim N\left(\frac{s}{t}B, \frac{s}{t}(t-s)\right)$

C. Cond 2. $x_3 = B$

$f_{s,t|u}(x_1, x_2, B)$

$\boxed{B_t | B_u \sim N\left(\frac{t}{u}, \frac{t}{u}(u-t)\right)}$

i) $E[B_s B_t | B_u] = E[E[B_s B_t | B_t] | B_u] = E\left[\frac{s}{t} B_t^2 | B_u\right] = \frac{s}{t}\left[\frac{t^2}{u^2} + \frac{t}{u}(u-t)\right] = \frac{st}{u^2} + \frac{s}{u}(u-t)$

ii) $f(x_1, x_2, x_3) = \exp\left(-\frac{1}{2}\left(\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{t_2-t_1} + \frac{(x_3-x_2)^2}{t_3-t_2}\right)\right)$ $= (*)$

$(*) = \dfrac{x_1^2}{t_1} + \dfrac{(x_2-x_1)^2}{t_2-t_1} + \dfrac{(B-x_2)^2}{t_3-t_2}$

$(*) = \left(\dfrac{1}{t_1} + \dfrac{1}{t_2-t_1}\right)x_1^2 + \left(\dfrac{1}{t_2-t_1} + \dfrac{1}{t_3-t_2}\right)x_2^2$

$- \dfrac{2}{t_2-t_1} x_1 x_2 - \dfrac{2B}{t_3-t_2} x_2$

1. $E[B_t^n$ ...

$= \left[\begin{array}{l} (2n)! \ldots \\ 0. \end{array}\right.$

- $E[B_t^2]$

$E[B_t^4]$

$Var(B_t^2$

1. $E[B_t^n]$

$= \begin{cases} (2n)!! & (n \equiv 1 (2)) \\ 0. & \text{o.w} \end{cases}$

- $E[B_t^2] = t.$

$E[B_t^4] = 3t^2.$

$Var(B_t^2) = 2t^2.$

2. $B_t = \sqrt{t} Z.$

$E[B_t^2] = E[tZ^2] = t$

3. Mgf Normal

$E[e^X] = e^{\mu + \frac{\sigma^2}{2}}$

when $X \sim N(\mu, \sigma^2).$

Pf. $\downarrow$

$E[e^{tX}].$

$E[e^{\lambda X - \frac{\lambda^2}{2} t}]$

holds $\forall$ dist info.
(moment)

$\frac{d}{dt} E[e^{tX}]\Big|_{t=0} = E[X]$

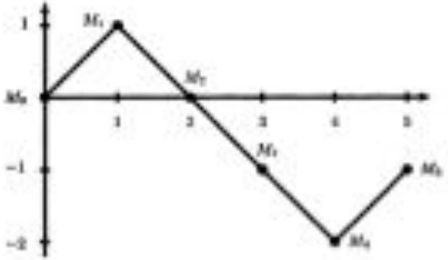
(dynamic)
weight e^{t-1} e^{t-2}

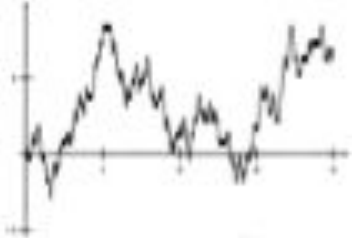
$$A = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \mu_N & -\mu_N \end{pmatrix}$$

y solution $y = (y_0, y_1, \dots)$ of the linear system

$$Ay = 0.$$

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$$





trend, probability, expectation on TS

		—	$\times$ ($p = 1/4$)
Prob — reach	+	1	<1
reach	−	1	1
return	+, −	1	<1
!! quiz $T_a > T_b$		$\frac{a}{a+b}$	$\frac{\rho^{-a}-1}{\rho^{-a}-\rho^{b}} = \frac{1-\rho^{a}}{1-\rho^{a+b}}$
!! quiz		$\frac{b}{a+b}$	$\frac{1-\rho^{b}}{\rho^{-a}-\rho^{b}} = \frac{\rho^{a}-\rho^{a+b}}{1-\rho^{a+b}}$
Time — reach	+	∞	∞
	−	∞	finite
return	+	∞	∞
	−	∞	∞

12

- $\rho = e^{-\frac{\mu}{2\sigma^2}}$; $P(\sup X_t > b)$
- $P(\sup X_t > b) = e^{-\frac{2\mu}{\sigma^2}\cdot b} = \rho^b$.

$$\frac{a\cdot\rho^{b} + b\cdot\rho^{-a} - a - b}{\mu(\rho^{-a}-\rho^{b})} = \frac{E[X_T]}{\mu}$$

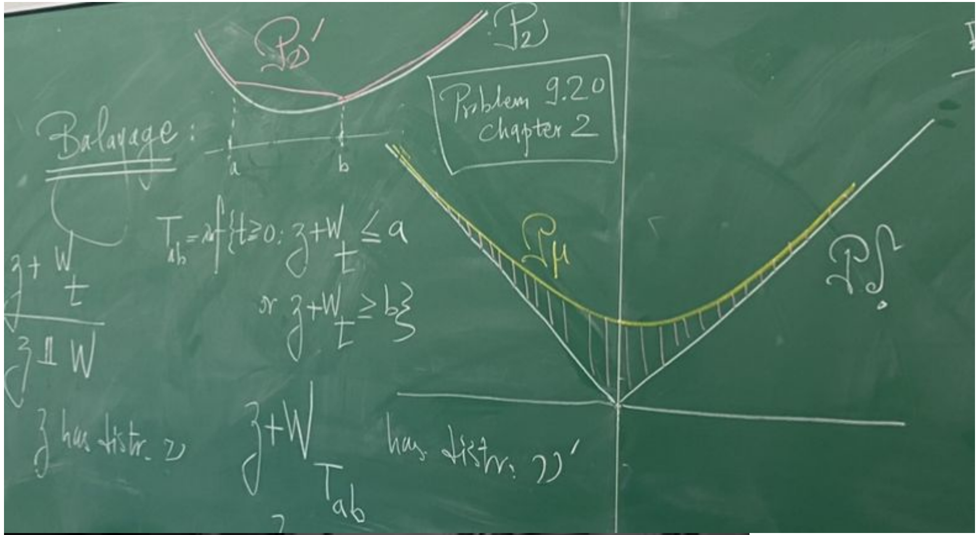
$ET = \frac{1}{\mu}(bp_b - a(1-p_b))$ $\left(= \frac{b}{\mu}\right)$. $Var(T) =$

Point transient

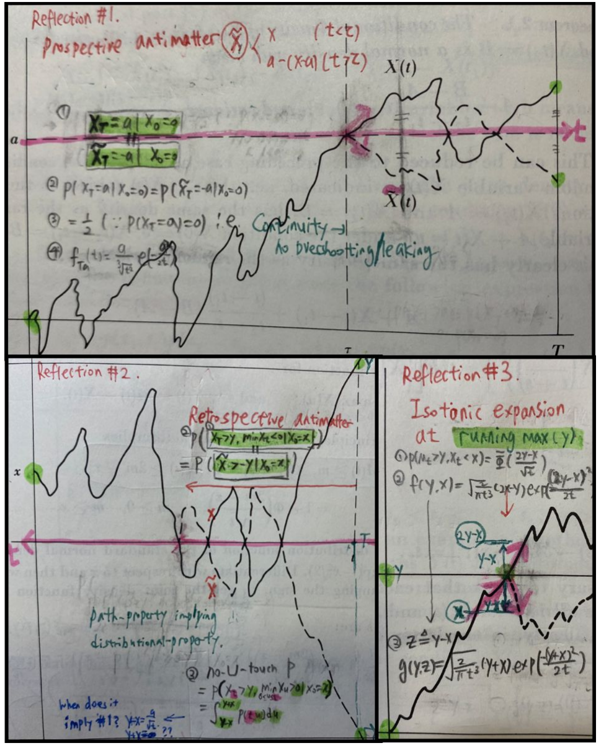
1? finitness hierarchy in PE

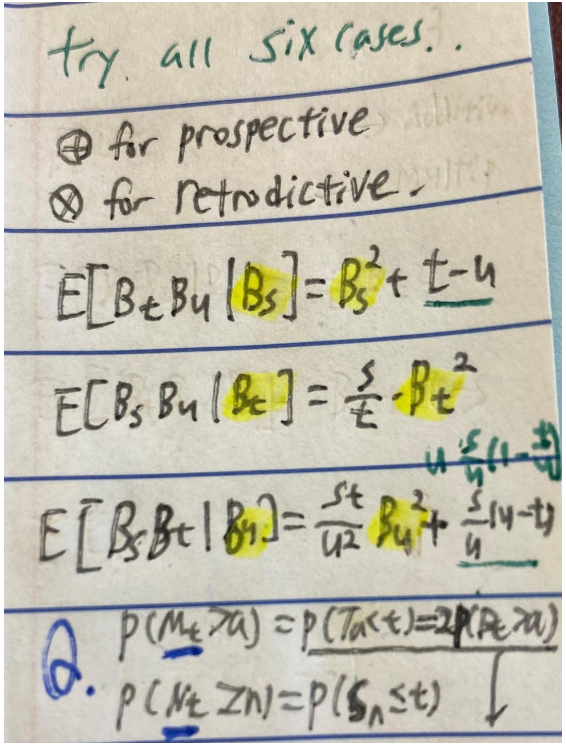
T,S / S|T

- mutual characterization of T, S
- linear interpolation of S|T
- T|S?



Markov reflection

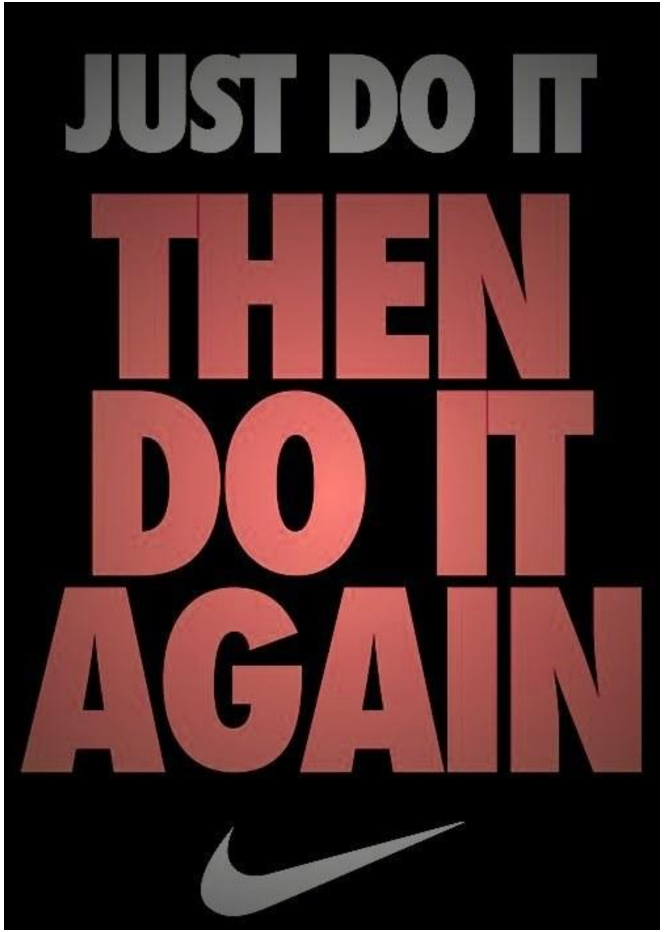




Just do it

- just normal
- just independent
- just identically distributed
- just finite p-th moment

then induction extend



W8 recitation: 3Es

memoryless after midterm!

1. Exponential
2. Ergodicity
3. Extension

1. Exponential

Where is exponential in queueing.

Queueing: eQuilibrium, using, Ergodicity, under the hood lies, Exponential jump chain

Memoryless from exponential jump chain allows recurrence relation leading to next E: ergodicity

2. Ergodicity: Two ways of solving Midterm #1-(b)

Global view: geometric mean

(ii) When we replace after each draw and inspection then it turns into event follow geometric distribution

$X_i \sim \text{Bernoulli}\left(\frac{g}{g+b}\right)$ draw good unit instead of bad

$E(X) = \frac{g+b}{g} = 1 + \frac{b}{g}$

EX = 1/p - 1

Local view: recurrence relation

ii) $E[X] = E[X \mid \text{First unit is good}] \cdot P(\text{First unit is good}) + \underbrace{E[X \mid \text{First unit is bad}]}_{=0} \cdot P(\text{First unit is bad})$.

$= (1 + E[X]) \frac{g}{g+b}$

$E[X] = \frac{g}{g+b} + \frac{g}{g+b} E[X]$

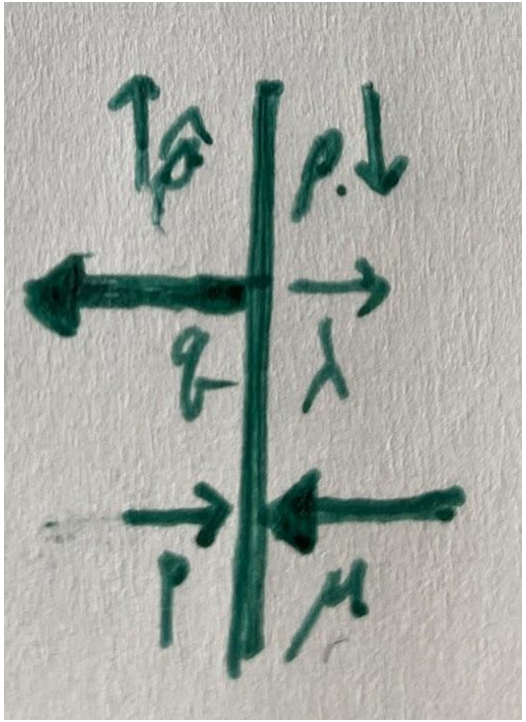
$E[X]\left(1 - \frac{g}{g+b}\right) = \frac{g}{g+b} \rightarrow E[X]\left(\frac{b}{g+b}\right) = \frac{g}{g+b}$

$E[X] = \frac{\frac{g}{g+b}}{\frac{b}{g+b}} = \frac{g}{g+b} \cdot \frac{g+b}{b} = \frac{g}{b}$

Shuqi and Cindy (Hon)'s sol for 1-i and 1-ii each. For recurrent structure, **conditioning on the next step** characterize the entire distribution (stationary distribution via detailed balance).

- Average over time = average over space (ergodicity)
- recurrence relation at certain time gave the same answer ignorant of the global view (geometric distribution) because of homogeneity from ergodicity
- Geometric "mean" can be computed from recurrence relation (could other statistics e.g. variance be computed? -> Yes. Martingale)

2. Ergodicity: two different rho notations



rw

- bigger rhohat is ergodic
- strong negative drift

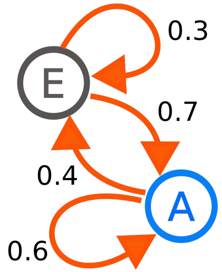
ctmc

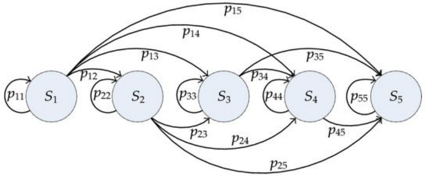
- lower rho is ergodic
- low server utilization

3. Extension: from discrete to continuous

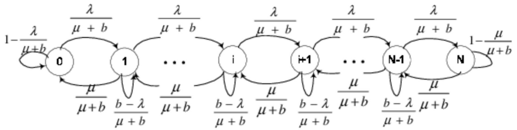
Time extension happened in DTMC to CTMC

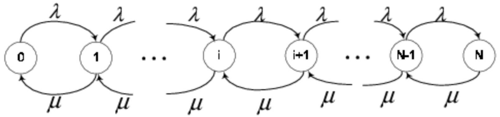
- exponential sojourn time (mean : 1/ v_i)
- jump chain like DTMC
- from prob -> rate: q_ij = v_i * p_ij (how are the units? rate: #/s)



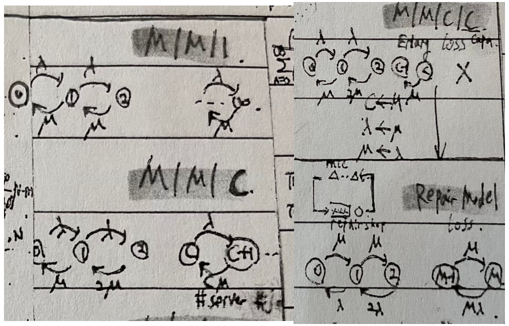


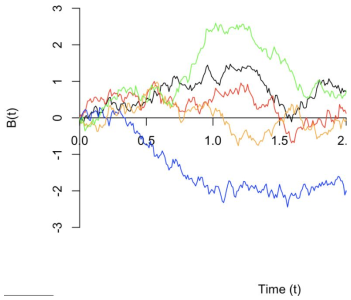
$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ 0 & p_{22} & p_{23} & p_{24} & p_{25} \\ 0 & 0 & p_{33} & p_{34} & p_{35} \\ 0 & 0 & 0 & p_{44} & p_{45} \end{bmatrix} \text{ where } p_{ij} \geq 0 \text{ and } \sum_j p_{ij} = 1$$





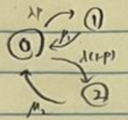
Space extension will happen in Brownian motion (finite, countable, infinite states)





Do not copy.

17. Expectation & probability. (W6. keyword 3)



$\lambda P_0 = \mu_1 P_1 + \mu_2 P_2$

$\mu_1 P_1 = \lambda p P_0$

$\mu_2 P_2 = \lambda(1-p) P_0$.

$P_0 = \left(1 + \left(\frac{\lambda p}{\mu_1}\right) + \left(\frac{\lambda(1-p)}{\mu_2}\right)\right)^{-1}$

larger $\lambda \to$ lower P_0. (outflo

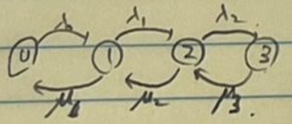
22. $\lambda_n = \frac{\lambda}{n+1}$. $\to P_0 = \left(1 + \sum \frac{(\lambda/\mu)^n}{n!}\right)^{-1} = e^{-\lambda/\mu}$

$\mu_n = \mu$.

poisson w/ mean $\left(\frac{\lambda}{\mu}\right)$ → locally..

$\frac{P(X=n+1)}{P(X=n)} = \left(\frac{\lambda}{\mu}\right) \cdot \frac{1}{n+1}$.

23.



$= \frac{3}{10}, \frac{2}{10}, \frac{1}{10}$

$= \frac{1}{8}, \frac{2}{8}, \frac{2}{8}$

$P_1 = \frac{3/10}{1/8} P_0$

$P_2 = \frac{2/10}{2/8} P_1$

$P_3 = \frac{1/10}{2/8} P_1$.

(a) $P_1 + 2P_2 + 3P_3$

(b) $P_2 + P_3$.

8.1 (b) P(X > Y)

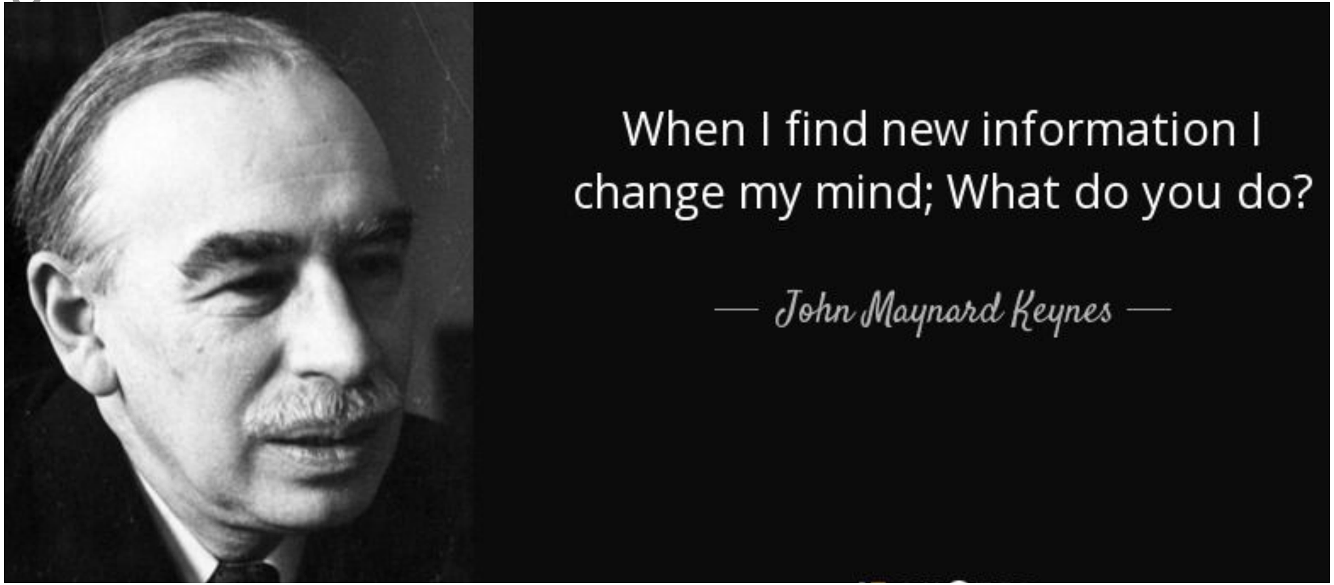
3. total cost consists of fixed + variable cost. variable cost need L (# in system) * cost per unit

W6. recitation

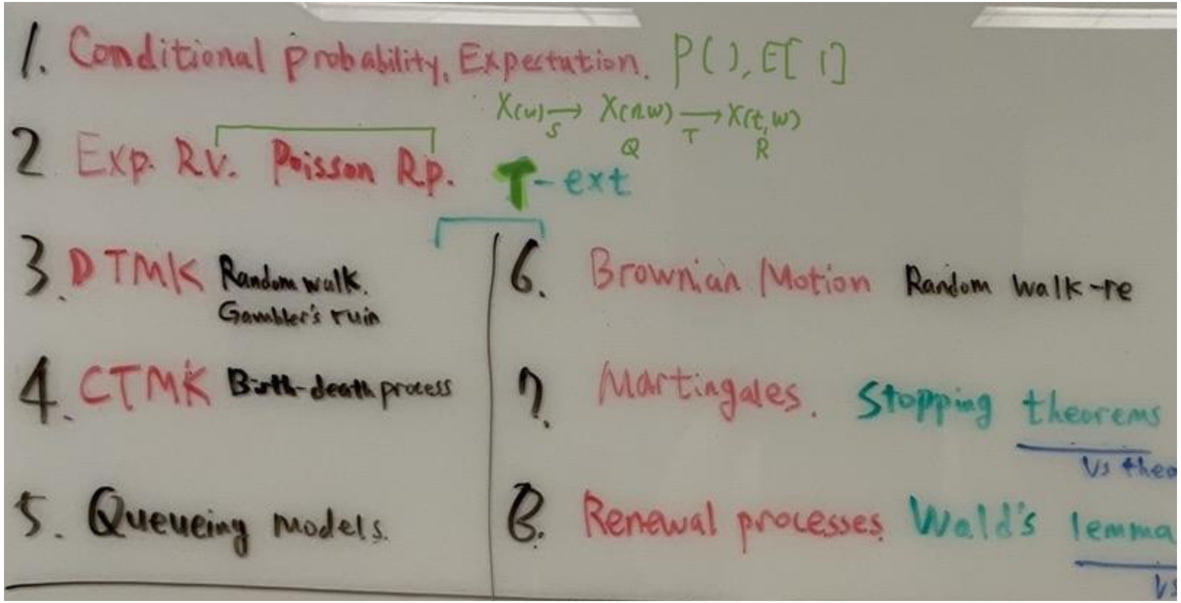
Review for midterm with subjective commentary on key concepts

1. random, conditional, expectation, Markov
2. Binomial, Poisson, Normal
3. Trend, probability, expectation

1. Keynes' random, conditional, expectation, Markov, Martingale



Our random, conditional, expectation, Markov, Martingale



random is a function of w (scenario)

c: new info can change how we measure randomness

e: measured randomness is what we expect

mk: but let’s give a structure* for generative model

mg: randomness measuring tool on the new structure is needed

* memoryless (constant “probability”) is the simplest structure; next step is constant “probabilities”

Gulliver (Poisson)'s travel to Lilliput (Binomial) and Brobdingnag (Normal)

Mural depicting Gulliver surrounded by citizens of Lilliput.

Gulliver exhibited to the Brobdingnag Farmer (painting by

Our Binomial-Poisson-Normal (BPN)

3) Poisson(λ)

norm constant

$$P(N=k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad k=0,1,2,\dots \quad \lambda \geq 0$$

$$\left(\text{recall: } \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}\right)$$

verify) $E(N) = \lambda = Var(N)$

$$\text{Binom}(n,p) \xrightarrow[np \to \lambda \geq 0]{n\to\infty,\ p \to 0} \text{Poisson}(\lambda)$$

verify) $$\binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!\,n^k} \cdot \frac{(np)^k}{(1-p)^k}(1-p)^n$$

if n large, p small

$$= \frac{n!}{k!(n-k)!\,n^k} \cdot \frac{\lambda^k}{(1-p)^k}\left(1-\frac{\lambda}{n}\right)^n + \varepsilon$$

$\left(1-\frac{\lambda}{n}\right)^n \to e^{-\lambda}$, $\quad (1-p)^k \to 1$

$$\frac{n!}{(n-k)!\,n^k} = \frac{n(n-1)\cdots(n-k+1)}{\underbrace{n \cdot n \cdots n}_{k}} \to 1$$

$$= \frac{\lambda^k}{k!} e^{-\lambda}$$

Binomial-Poisson (BP), Poisson-Normal (PN) approximation are two end of the spectrum

1. BP: given poisson, you add time axis to make it binomial (= lift then chop up)

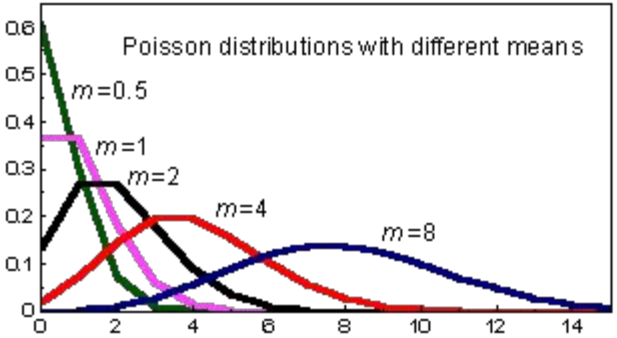
Our BPN

Show $\lim_{n\to\infty} e^{-n} \sum_{k=1}^{n} n^k \,/\, k! = 1/2$

This is $P[N_n \leqslant n]$ where N_n is a random variable with Poisson distribution of parameter n. Hence each N_n is distributed like $X_1 + \cdots + X_n$ where the random variables (X_k) are independent and identically distributed with Poisson distribution of parameter 1.

By the central limit theorem, $Y_n = \frac{1}{\sqrt{n}}(X_1 + \cdots + X_n - n)$ converges in distribution to a standard normal random variable Z, in particular, $P[Y_n \leqslant 0] \to P[Z \leqslant 0]$.

Finally, $P[Z \leqslant 0] = \frac{1}{2}$ and $[N_n \leqslant n] = [Y_n \leqslant 0]$ hence $P[N_n \leqslant n] \to \frac{1}{2}$, QED.

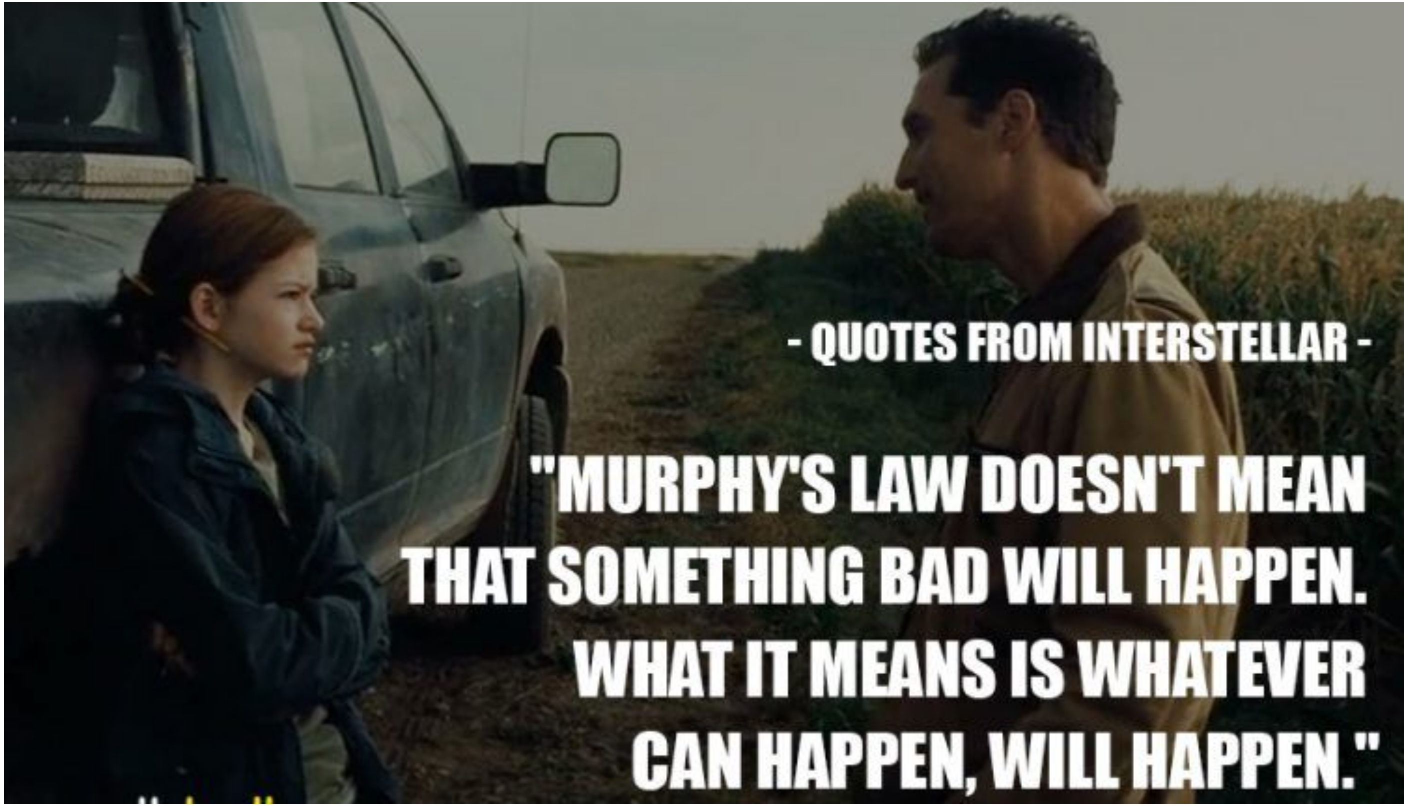


2. PN

if you stack up (X1 + …) it becomes normal distribution.

i.e. given normal, you add t axis and assume Poisson (= lift then chop up)

Murphy’s trend, probability, expectation (TPE)



Our TPE: hierarchy of happeness

1. P(X < inf) = 1
2. E(X) < inf
3. P(X < M) = 1 for some M < inf
4. X in [-a,b]

Consider

- 1 vs 2 for null vs positive recurrence

Preview:

importance of X and T for measuring: due date (T) with minimum quality (X)

Our TPE: importance of due date (T bound for measure)

(i-a) positive drift (p>q): reach any positive state w.p. 1, and any negative state w.p. \in (0,1)
expected time (# of steps) to reach any positive state is finite, to reach any negative state is infinite
(i-b) negative drift (p<q): reach any negative state w.p. 1, and any positive state w.p. \in (0,1)
expected time (# of steps) to reach any negative state is finite, to reach any positive state is infinite

(ii) 0 drift/symmetric (p=q): reach any positive or negative state w.p. 1;
expected time (# of steps) to reach any positive or negative state is infinite

consequences:
(a) positive/negative drift: return ("round trip") to any state w.p. f_i <1;
hence such a state i will be visited a finite number of times:
the number of visits follow a geometric distribution (p=1-f_i)
such a state is called "transient"
(b) 0 drift/symmetric: return to any state i w.p. f_i=1; hence such a state will be visited
an infinite number of times — such a state is called "recurrent" (null)

Preview:

importance of X and
T for measuring:
due date (T) with
minimum quality (X)

56. Suppose that on each play of the game a gambler either wins 1 with probability p or loses 1 with probability $1 - p$. The gambler continues betting until she or he is either up n or down m. What is the probability that the gambler quits a winner?

tpe: [-n, m] in asymmetric RW

$$\frac{1-(q/p)^m}{1-(q/p)^{n+m}}$$

58. In the gambler's ruin problem of Section 4.5.1, suppose the gambler's fortune is presently i, and suppose that we know that the gambler's fortune will eventually reach N (before it goes to 0). Given this information, show that the probability he wins the next gamble is

$$\frac{p[1-(q/p)^{i+1}]}{1-(q/p)^i}, \qquad \text{if } p \neq \tfrac{1}{2}$$

$$\frac{i+1}{2i},$$

$$\begin{aligned}
&\mathsf{P}\left(X_{n+1}=i+1 \mid X_n=i, \lim X_m=N\right) \\
&= \frac{\mathsf{P}\left(X_{n+1}=i+1, \lim X_m=N \mid X_n=i\right)}{\mathsf{P}\left(\lim X_m=N \mid X_n=i\right)} \\
&= \frac{\mathsf{P}\left(\lim X_m=N \mid X_{n+1}=i+1, X_n=i\right)\mathsf{P}\left(X_{n+1}=i+1 \mid X_n=i\right)}{\mathsf{P}\left(\lim X_m=N \mid X_n=i\right)} \\
&= \frac{P_{i+1}p}{P_i} = \begin{cases} \frac{[1-(q/p)^{i+1}]p}{1-(q/p)^i}, & \text{if } p \neq \frac{1}{2} \\ \frac{i+1}{2i}, & \text{if } p = \frac{1}{2}. \end{cases}
\end{aligned}$$

59. For the gambler's ruin model of Section 4.5.1, let M_i denote the mean number of games that must be played until the gambler either goes broke or reaches a fortune of N, given that he starts with i, $i = 0, 1, \ldots, N$. Show that M_i satisfies

$$M_0 = M_N = 0; \quad M_i = 1 + pM_{i+1} + qM_{i-1}, \quad i = 1, \ldots, N-1$$

Solve these equations to obtain

$$\begin{aligned} M_i &= i(N-i), & \text{if } p = \tfrac{1}{2} \\ &= \frac{i}{q-p} - \frac{N}{q-p}\frac{1-(q/p)^i}{1-(q/p)^N}, & \text{if } p \neq \tfrac{1}{2} \end{aligned}$$

It is clear that $M_0 = M_N = 0$. Conditioning on the first play, for $i = 1, \ldots, N-1$,

$$\begin{aligned} M_i &= \mathsf{E}[\#\text{games from } i|\text{win first}]\,\mathsf{P}(\text{win first}) + \mathsf{E}[\#\text{games from } i|\text{lose first}]\,\mathsf{P}(\text{lose first}) \\ &= (1 + M_{i+1})p + (1 + M_{i-1})q \\ &= 1 + pM_{i+1} + qM_{i-1}. \end{aligned}$$

W4. recitation

Distributions in Poisson process

Compound Poisson process (5.50, 5.85)

Independent Stationary increments (5.85)

- + Memoryless wlog fixing (5.44)

Distributions in Poisson process

Poisson counts: N_(t +s) - N_t ~ Pois(lambda * s)

Exponential interarrival times: S_2 - S_1 ~ exp(lambda)

P(Sn < t) = P(N_t >= n)

Uniform: P(S_i | N_t)

Binomial:

- one PP: P(N_u = k | N_t = n)
- two PP: P(N^1_1 = k | N^1_t + N^2_t = n)

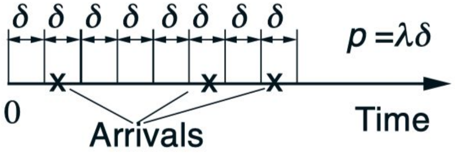
Geometric: geometric sum of exp is exp

Negative binomial: P(N_u | N_t)

Times of Arrival	Continuous	Discrete
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time CDF	Exponential	Geometric
Arrival Rate	λ/unit time	p/per trial

$$p = \lambda \frac{t}{N}$$

Law of conservation



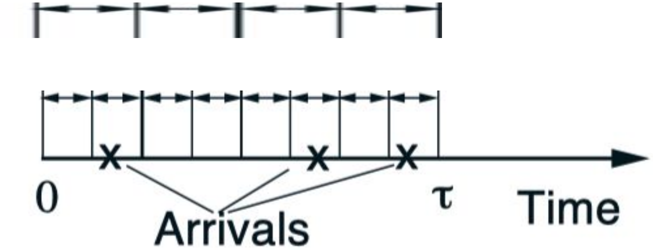
	POISSON	BERNOULLI
Times of Arrival	Continuous	Discrete
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time CDF	Exponential	Geometric
Arrival Rate	λ/unit time	p/per trial

Conserved: lambda

- what can happen will happen <*Interstellar*>

$$p_1 = \lambda * \frac{t}{N}$$

$$p_2 = \lambda * \frac{t}{2N}$$



5.44

W: waiting time, T: time until the first car

(a) $\mathsf{P}(W=0)=\mathsf{P}(X\geq T)=e^{-\lambda T}$.

(b)

$$\begin{aligned}
\mathsf{E}[W] &= \int_0^\infty \mathsf{E}[W|X=x]\,\lambda e^{-\lambda x}dx \\
&= \int_0^T \mathsf{E}[W|X=x]\,\lambda e^{-\lambda x}dx + \int_T^\infty \mathsf{E}[W|X=x]\,\lambda e^{-\lambda x}dx \\
&= \int_0^T (x+\mathsf{E}[W])\lambda e^{-\lambda x}dx + 0\cdot e^{-\lambda T} \\
&= \int_0^T \lambda x e^{-\lambda x}dx + \mathsf{E}[W]\,(1-e^{-\lambda T})
\end{aligned}$$

5.50, 85

Let T denote the time until the next train arrives; and so T is uniform on $(0,1)$. Note that, conditional on T, X is Poisson with mean $7T$.

(a) $\mathsf{E}[X] = \mathsf{E}[\mathsf{E}[X|T]] = \mathsf{E}[7T] = 7/2$.

(b) Note that $\mathsf{E}[X|T] = 7T$ and $\mathsf{Var}(X|T) = 7T$. By the conditional variance formula

$$\mathsf{Var}(X) = \mathsf{E}[7T] + \mathsf{Var}(7T) = 7\mathsf{E}[T] + 49\mathsf{Var}(T) = \frac{7}{2} + \frac{49}{12} = \frac{91}{12}.$$

The amount of money $X(T)$ paid by the insurance company in T weeks is a Compound Poisson Process, therefore

$$\mathsf{E}[X(4)] = 4\lambda\mathsf{E}[Y] = 4 \cdot 5 \cdot 2{,}000 = 40{,}000,$$

$$\mathsf{Var}(X(4)) = 4\lambda\mathsf{E}\left[Y^2\right] = 4\lambda(\mathsf{Var}(Y) + \mathsf{E}[Y]^2) = 4 \cdot 5 \cdot (2{,}000^2 + 2{,}000^2) = 1.6 \times 10^8,$$

where $Y \sim \mathrm{Exp}(1/2{,}000)$.

Problems were solved but refer to the solution set

5.86

(a) $\mathsf{P}(N(t) = n) = 0.3e^{-3t}(3t)^n/n! + 0.7e^{-5t}(5t)^n/n!$.

(b) No!

(c) Yes! The probability of n events in any interval of length t will, by conditioning on the type of year, be as given in (a).

(d) No! Knowing how many storms occur in an interval changes the probability that it is a good year and this affects the probability distribution of the number of storms in other intervals.

(e)

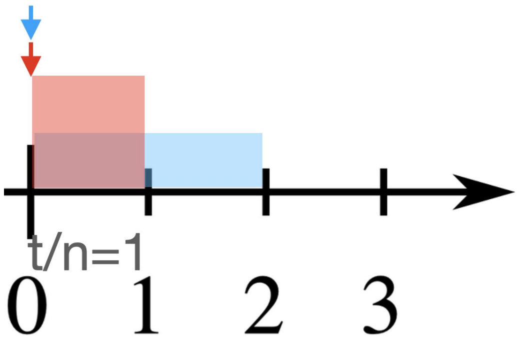
$$\begin{aligned}\mathsf{P}(\text{good}|N(1) = 3) &= \frac{\mathsf{P}(N(1) = 3|\text{good})\,\mathsf{P}(\text{good})}{\mathsf{P}(N(1) = 3|\text{good})\,\mathsf{P}(\text{good}) + \mathsf{P}(N(1) = 3|\text{bad})\,\mathsf{P}(\text{bad})} \\ &= \frac{(e^{-3}3^3/3!)\cdot 0.3}{(e^{-3}3^3/3!)\cdot 0.3 + (e^{-5}5^3/3!)\cdot 0.7}.\end{aligned}$$

Symmetry-based wlog fixing

Symmetry lead to reflection/shift/rotation invariant

Fix start time

- P(X > Y) X, Y ~ exp(1) should be .5
- P(X > Y) X ~ exp(1), Y ~ exp(2) should be .5 / (1+.5)



Fix one point

- [Circle Triangle Picking]

W4. re recitation

Continuous and Discrete (inverse)

- How can times (cont) and #s (disc) characterize each other? (dt, forall in disc.)
- When to use: superimpose for counts (discrete), independence for time (cont)

Random sum of random variables (nested)

- How can it be solved with generating function? Could this explain geometric sum of exp is exp?
- When not to use: nonlinear

Conservation Law

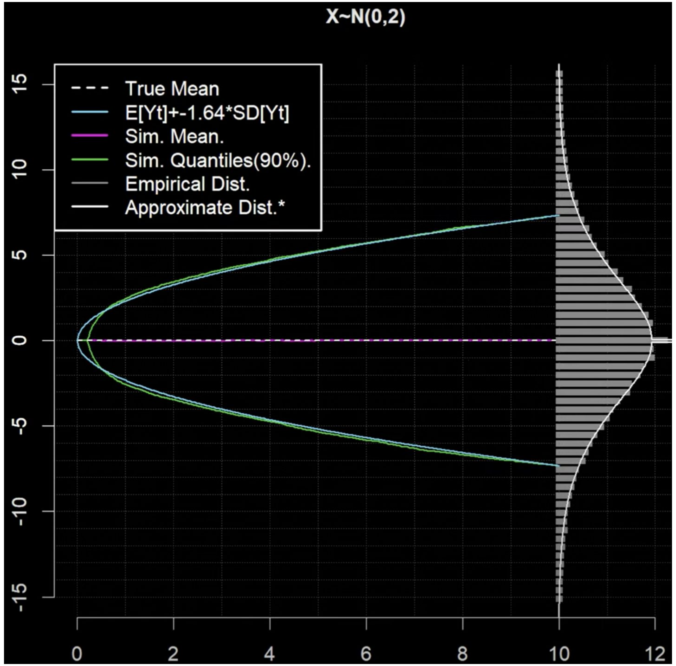
- How can this be formularized otherwise?
- When to use:
  - exit distributions from domains,
  - expected occupation times of subsets prior to exiting a domain,
  - trajectories

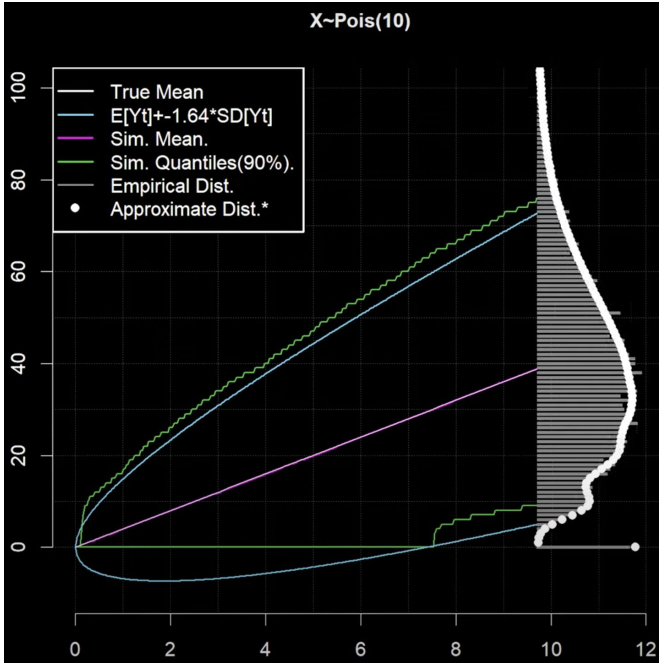
+ When is General (= / !=) Special and what are other examples?

Conservation

- Law of conservation of energy $K1 + U1 = K2 + U2$
- Wald’s lemma $ES_N = ES * EN$
- Optional stopping theorem $EX_T = EX_0$
- Ergodicity (time-averaged equals space-averaged if well-mixed)

Linearity in compound Poisson process





W3 recap

- Exponential: unique distribution with memoryless property (used as good as new)
- Poisson process: **counting process** of events (“arrivals”) with **i.i.d. exponential** inter-arrival times
- Imagine a i) non iid ii) without exponential arrival time (5.4) process and how it affects memoryless property
- N(t), for any given t, can be expressed as a sum of iid Bernoulli(p) rv's with p=\lambda*h and h=t/n; hence, letting n to infinity leads to the Poisson distribution of N(t), with mean=\lambda*t

W4 recap

- Distributions in Poisson process
    - Poisson - Exponential
    - Binomial - Geometric
    - Negative Binomial
    - Uniform
    - Gamma (Erlang)

- Compound Poisson process
    - (5.50, 5.85)
    - vs conditional poisson (5.86)

5.4 post office with 2 servers, 3 customers

P(A + B > C) = P(A_renew > C | A > B) * P(A > B)

a. 0
b. A should take very long. If A is 1, 2, 3. 1/27
c. P(A_renew > C ~~| A > B~~) * P(A > B) = P(A_renew > C) * P(A > B) = 1/4

intuition: increasing randomness, symmetry from memoryless of exponential dist.

5.6 post office

- Decomposing merged process using memoryless
- min(X1..Xn) = X1 can be factorized into pairwise comparison:
- p(C not last) = p(A > B+C|A > C) * p(A > C) + p(B > A+C|B > A) * p(B > A)= (lambda1 / lambda1 + lambda2)^2 + (lambda2 / lambda1 + lambda2)^2

p(X > Y) = mu / (lambda + mu)

= lambda^-1 / (lambda^-1 + mu^-1)

= EX / (EX + EY)

mnemonics: olympic (4Y) vs sunrise (1D); sunrise likely happens first

5.22 total time = wait + service

$$\mathsf{E}\left[\text{time}\right] = \mathsf{E}\left[\text{time waiting for server 1}\right] + \frac{1}{\mu_1} + \mathsf{E}\left[\text{time waiting for server 2}\right] + \frac{1}{\mu_2}.$$

$$\mathsf{E}\left[\text{time waiting for server 1}\right] = \frac{1}{\mu_1} + \mathsf{E}\left[\text{Additional}\right]$$

$$= \frac{1}{\mu_1} + \sum_{i=1}^{2} \mathsf{E}\left[\text{time waiting server 2} \mid \text{server } i \text{ finishes first}\right] \mathsf{P}\left(\text{server } i \text{ finishes first}\right)$$

$$= \frac{1}{\mu_1} + \frac{1}{\mu_2} \cdot \frac{\mu_1}{\mu_1 + \mu_2} + 0 \cdot \frac{\mu_2}{\mu_1 + \mu_2}.$$

$$\mathsf{E}\left[\text{time waiting for server 2}\right]$$

$$= \sum_{i=1}^{2} \mathsf{E}\left[\text{time waiting server 2} \mid \text{server } i \text{ finishes first}\right] \mathsf{P}\left(\text{server } i \text{ finishes first}\right)$$

$$= \frac{1}{\mu_2} \cdot \frac{\mu_1}{\mu_1 + \mu_2} + 0 \cdot \frac{\mu_2}{\mu_1 + \mu_2}.$$

$$\mathsf{E}\left[\text{time}\right] = \frac{2}{\mu_1} + \frac{2}{\mu_2} \cdot \frac{\mu_1}{\mu_1 + \mu_2} + \frac{1}{\mu_2}$$

5.43 hint: (1) define RVs

S_i: service time at each server
X: time until new arrival
p: proportion of customers that are served by both servers

5.43 hint: (1) define RVs (2) express answer with them

$$p = \mathsf{P}\left(X > S_1 + S_2\right)$$

5.47 (a) three clocks - merging strategy

$$
\begin{aligned}
&\mathsf{E}\left[\text{time until next customer enters the system}\right] \\
&= \mathsf{E}\left[\text{time until one of the servers gets free}\right] + \mathsf{E}\left[\text{time for a new arrival}\right] = \frac{1}{2\mu} + \frac{1}{\lambda}.
\end{aligned}
$$

5.47 (b) recurrence relation with supplementary RVs

T_i denote the time until both servers are busy when you start with i busy servers

$$\mathsf{E}[T_0] = \frac{1}{\lambda} + \mathsf{E}[T_1]$$

link: recurrence relation with supplementary RVs (3.40, 41)

5.47 (b) recurrence relation with supplementary RVs

starting with 1 server busy, let T be the time until the first event (arrival or departure)

$$X = \begin{cases} 1, & \text{if the first event is an arrival,} \\ 0, & \text{if it is a departure;} \end{cases}$$

Y be the additional time after the first event until both servers are busy.

$$\begin{aligned} \mathsf{E}[T_1] &= \mathsf{E}[T] + \mathsf{E}[Y] \\ &= \frac{1}{\lambda+\mu} + \mathsf{E}[Y|X=1]\frac{\lambda}{\lambda+\mu} + \mathsf{E}[Y|X=0]\frac{\mu}{\lambda+\mu} \\ &= \frac{1}{\lambda+\mu} + \mathsf{E}[T_0]\frac{\mu}{\lambda+\mu}, \end{aligned}$$

$\mathsf{E}[Y|X=1] = 0$ and $\mathsf{E}[Y|X=0] = \mathsf{E}[T_0]$.

$$\mathsf{E}[T_0] = \frac{1}{\lambda} + \frac{1}{\lambda+\mu} + \mathsf{E}[T_0]\frac{\mu}{\lambda+\mu}$$

Appendix.

hopefully helpful diagrams
from textbooks
or by Angie Moon

chopping up [0,t] into n, n->infty

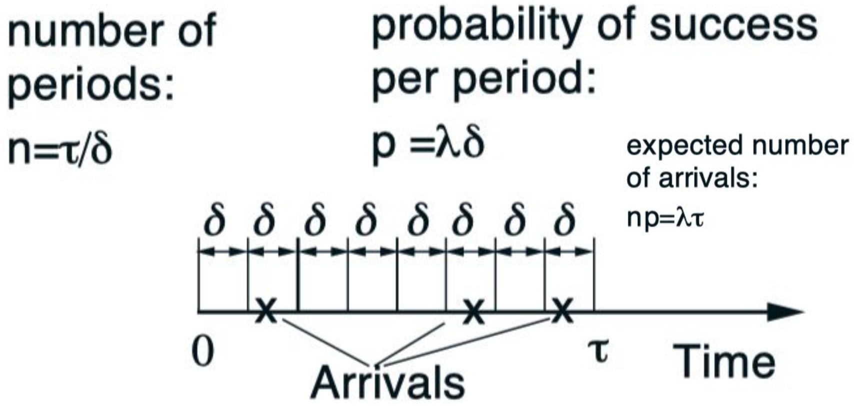
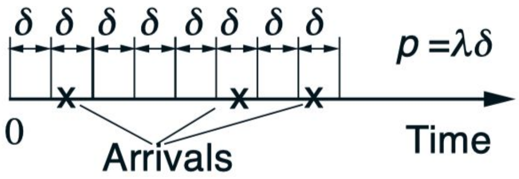


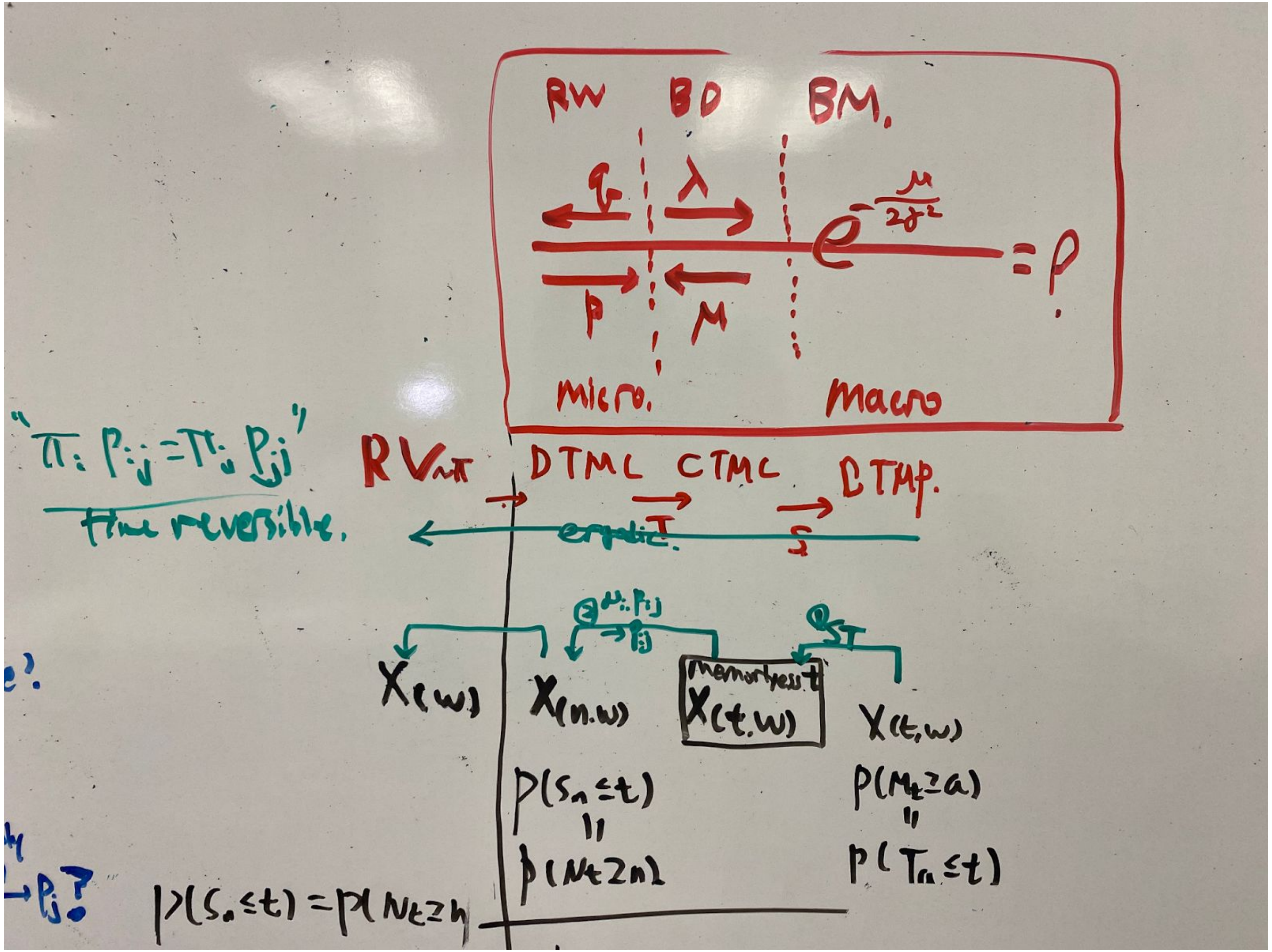
Figure 5.5: Bernoulli approximation of the Poisson process.

DTMC - T_ext - CTMC - S_ext - CTMP



	POISSON	**BERNOULLI**
Times of Arrival	Continuous	Discrete
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time CDF	Exponential	Geometric
Arrival Rate	λ/unit time	p/per trial

Figure 5.6: View of the Bernoulli process as the discrete-time version of the Poisson. We discretize time in small intervals δ and associate each interval with a Bernoulli trial whose parameter is $p = \lambda\delta$. The table summarizes some of the basic correspondences.



liable to chg.
with new info input.

Ross exercise

outside hw

33. Let $X(t) = N(t+1) - N(t)$ where $\{N(t),\ t \geq 0\}$ is a Poisson process with rate λ. Compute

$$\text{Cov}[X(t), X(t+s)]$$

34. Let $\{N(t), t \geq 0\}$ denote a Poisson process with rate λ and define $Y(t)$ to be the time from t until the next Poisson event.

(a) Argue that $\{Y(t),\ t \geq 0\}$ is a stationary process.

(b) Compute $\text{Cov}[Y(t), Y(t+s)]$.

var(T),

sigma = 1

- p = .5(1+mu*sqrt(dt))

sigma

- p = .5(1+mu * sqrt(dt)/sigma)

24. Let $\{X(t),\ t \geq 0\}$ be Brownian motion with drift coefficient μ and variance parameter σ^2. Suppose that $\mu > 0$. Let $x > 0$ and define the stopping time T (as in Exercise 21) by

$$T = \text{Min}\{t : X(t) = x\}$$

Use the Martingale defined in Exercise 18, along with the result of Exercise 21, to show that

$$\text{Var}(T) = x\sigma^2/\mu^3$$

In Exercises 25 to 27, $\{X(t), t \geq 0\}$ is a Brownian motion process with drift parameter μ and variance parameter σ^2.

25. Suppose every Δ time units a process either increases by the amount $\sigma\sqrt{\Delta}$ with probability p or decreases by the amount $\sigma\sqrt{\Delta}$ with probability $1 - p$ where

$$p = \frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{\Delta}).$$

Show that as Δ goes to 0, this process converges to a Brownian motion process with drift parameter μ and variance parameter σ^2.

28. Compute the mean and variance of

(a) $\int_0^1 t\, dB(t)$

(b) $\int_0^1 t^2\, dB(t)$

35. Let $\{X(t), -\infty < t < \infty\}$ be a weakly stationary process having covariance function $R_X(s) = \text{Cov}[X(t), X(t+s)]$.

(a) Show that

$$\text{Var}(X(t+s) - X(t)) = 2R_X(0) - 2R_X(t)$$

(b) If $Y(t) = X(t+1) - X(t)$ show that $\{Y(t), -\infty < t < \infty\}$ is also weakly stationary having a covariance function $R_Y(s) = \text{Cov}[Y(t), Y(t+s)]$ that satisfies

$$R_Y(s) = 2R_X(s) - R_X(s-1) - R_X(s+1)$$