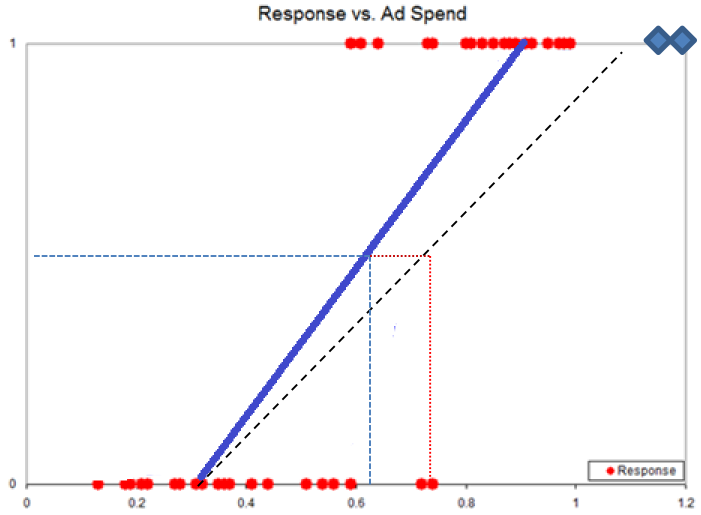
Logistic Regression

Classification Types

- Two-class
  - $y \in \{0,1\}$
    - 0: “negative” class
    - 1: “positive” class
  - Yes/No; Benign/Malignant; Click/No click
- Multi-class
  - $y \in \{0,1,2,\ldots,n\}$
  - Grades; Colors; Item Categories

Two Class via Linear Regression



- Goal: Assign objects to y = 0 or y = 1
- Decision rule:
  - $h_\theta(x) \geqslant 0.5$; y = 1
  - $h_\theta(x) < 0.5$; y = 0
- Issues:
  - Hard to fit properly
  - $h_\theta(x)$ can be > 1 or < 0
    - Can't extract probabilities

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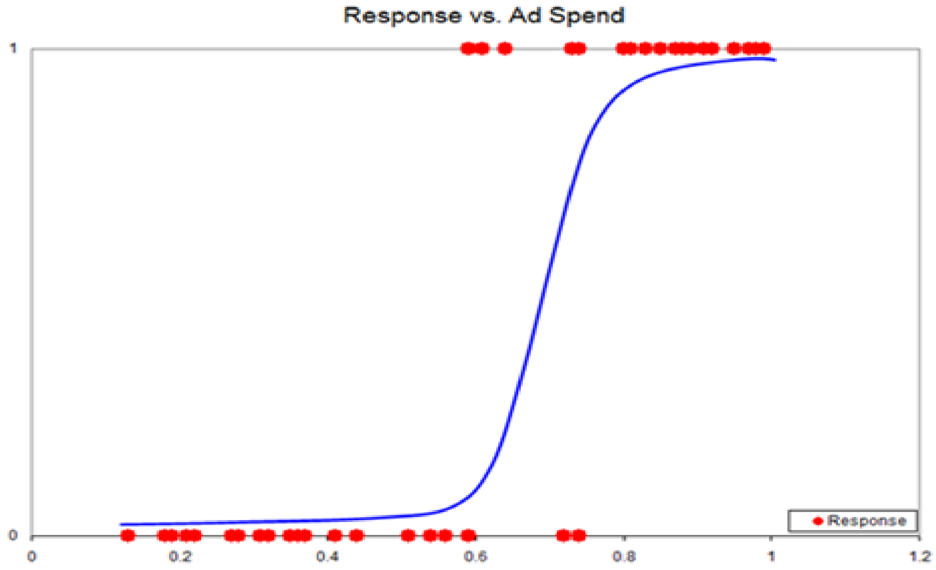
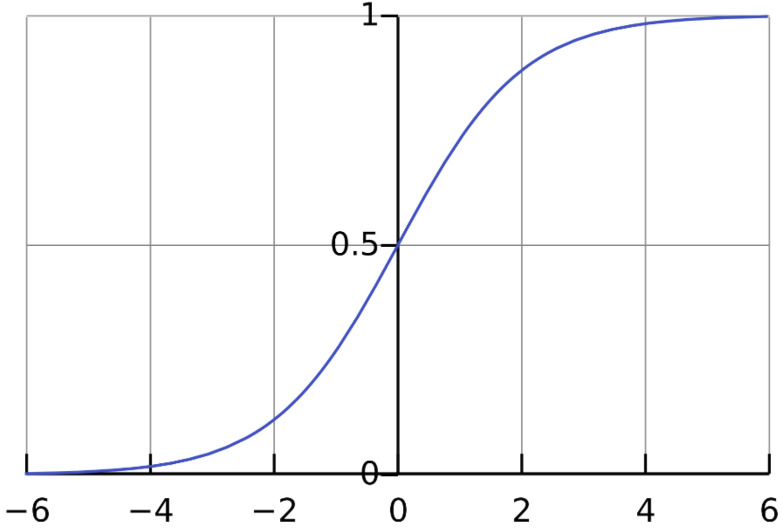
Logistic Regression



- Use “Logistic Function”
- Shape better suited to two-class problem

Logistic Function

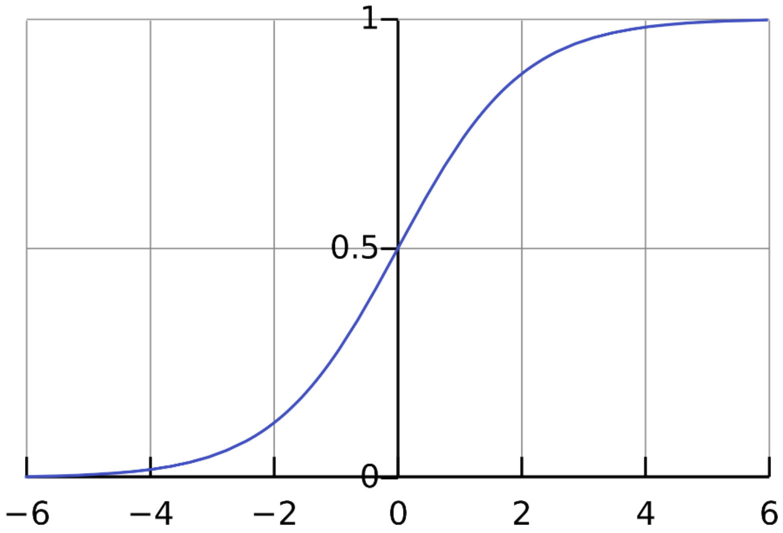


$$h_\theta(x) = \frac{1}{1 + e^{-\theta^T x}}$$

- Class Assignment:
  - y = 1 if $h_\theta(x) \geqslant 0.5$
  - y = 0 if $h_\theta(x) < 0.5$

- Asymptotic
  - $h_\theta(x) \to 1$ as $x \to \infty$
  - $h_\theta(x) \to 0$ as $x \to -\infty$

Logistic Function



$$h_\theta(x) = \frac{1}{1 + e^{-\theta^T x}}$$

Probabilistic interpretation

- $P_\theta(y = 1|x) = h_\theta(x)$
- $P_\theta(y = 0|x) = 1 - h_\theta(x)$

- Always between 0 and 1!

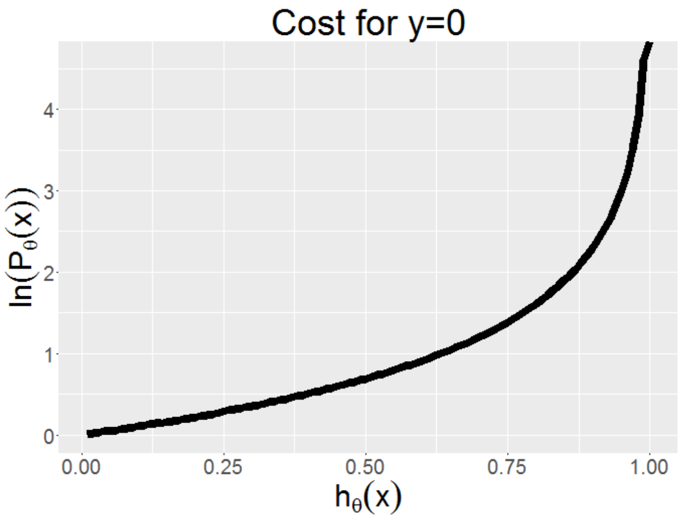
Reminder: Gradient Descent

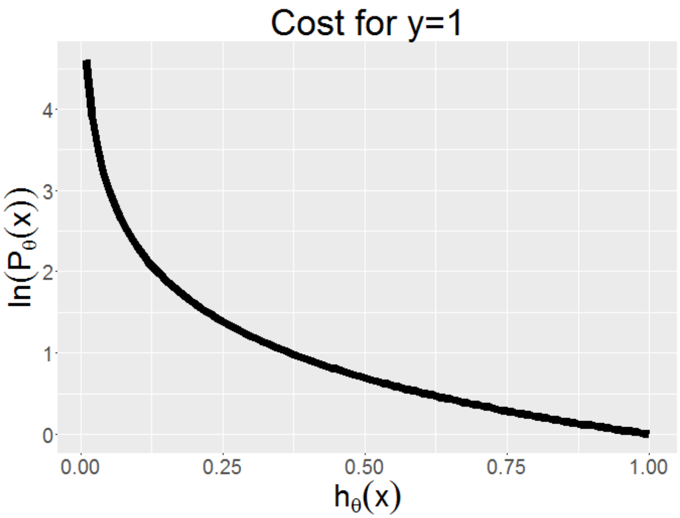
$$\theta_{j+1} = \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta)$$

- α is the "learning rate"
- Each time, the algorithm takes a step in the direction of the steepest downward slope, so $J(\theta)$ decreases.
- α determines how quickly or slowly the algorithm will converge to a solution

Cost Function: Log-Likelihood

$$J(\theta) = \frac{1}{m}\sum_{i=1}^{m} -\ln\left(P_\theta\left(y = y_i | x^{(i)}\right)\right)$$





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Cost Function: Log-Likelihood

Need to find $\frac{\partial}{\partial \theta_j} J(\theta)$ for log-likelihood

$$J(\theta) = \frac{1}{m} \sum_{i=1}^{m} -\ln\left(P_\theta\left(y = y_i | x^{(i)}\right)\right)$$

Since y is only 1 or 0, rewrite this as:

$$J(\theta) = -\frac{1}{m}\left[\sum_{i=1}^{m} y_i \ln\left(h_\theta\left(x^{(i)}\right)\right) + (1 - y_i) \ln\left(1 - h_\theta\left(x^{(i)}\right)\right)\right]$$

Quick Recap

Hypothesis: $$h_\theta(x) = \frac{1}{1 + e^{-\theta^T x}}$$

Cost Function: $$J(\theta) = -\frac{1}{m}\left[\sum_{i=1}^{m} y_i \ln\left(h_\theta\left(x^{(i)}\right)\right) + (1 - y_i)\ln\left(1 - h_\theta\left(x^{(i)}\right)\right)\right]$$

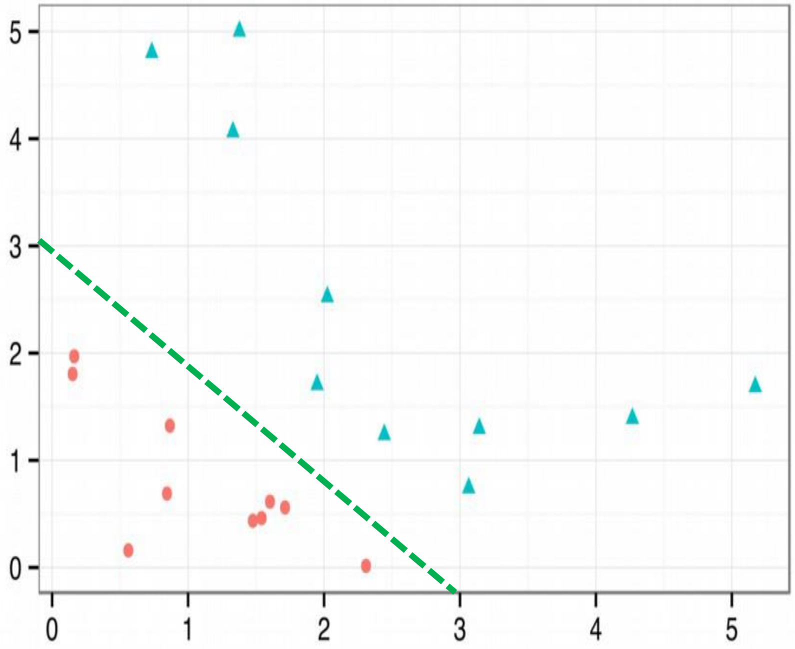
Parameters: $$\theta^T = [\theta_1, \dots, \theta_n]$$

Decision Boundary

$$h_\theta(x) = \frac{1}{1 + e^{-\theta^T x}}$$

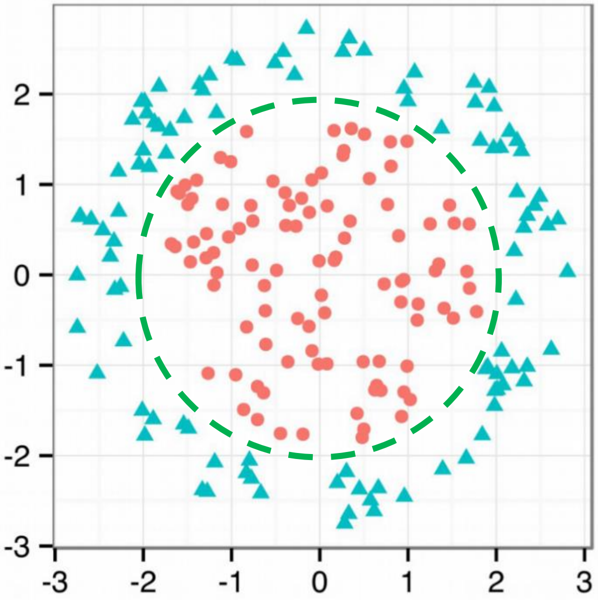
- Consider the exponent $\theta^T x$
  - Assume threshold of 0.5
  - $0.5 = \frac{1}{1+e^{-\theta^T x}} \rightarrow e^{-\theta^T x} = 1 \rightarrow \theta^T x = 0$
  - $D(\theta, x) = \theta^T x$ is the "decision boundary"
  - Visualize class separation

Decision Boundary: Linear



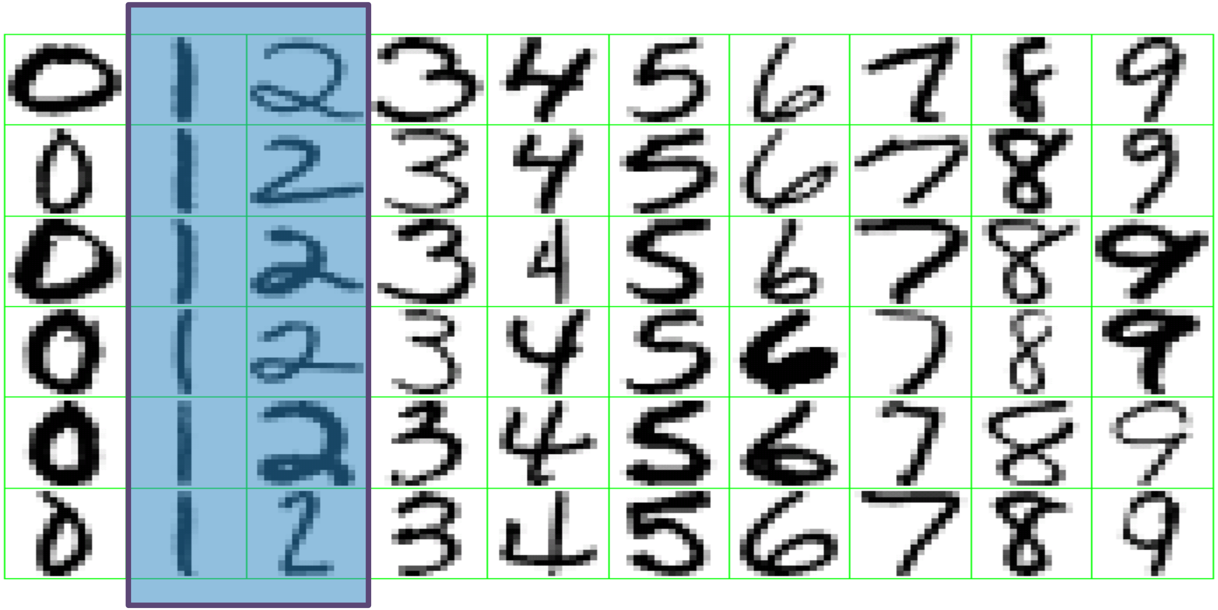
- Logistic regression is a "linear classifier"
- Example in 2-D
  - $D(\theta, x) = (\theta_0 + \theta_1 x_1 + \theta_2 x_2)$
  - $\theta^T = [-3, 1, 1]$
  - $\theta^T x = -3 + x_1 + x_2 \geq 0$
  - $\rightarrow x_1 + x_2 \geq 3$

Decision Boundary: Non-Linear

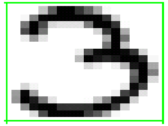


- "Non-linear" algorithms have more complicated decision boundaries
- $D(\theta, x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1^2 + \theta_4 x_2^2$
- $\theta^T = [-2, 0, 0, 1, 1]$
- $\rightarrow x_1^2 + x_2^2 \geq 2$

Example: Handwritten Digit Recognition



Extracting Features For Learning



$\{x_1, x_2, x_3 \ldots\ldots\ldots X_{256}, y = \text{‘three’}\}$

- Each xi corresponds to a feature value in the image
- y is a label of the training data; can be numeric or categorical, ‘3’ or ‘three’
- Each image is converted to row vectors and the appropriate learning algorithm is used
- Convention
  - x_i represents the ith feature in a training sample
  - y represents the label for the training sample

QUESTIONS