5.

A.

5.

5.

a) Since dice is unbaised the probablity for occurance of every face is same.
Hence for k faced dice, probablity of occurance of every face is $\frac{1}{k}$ $\therefore p = \frac{1}{k}$ —— (1)

We need to find the number of trials for this we will use Geometric distribution.

3

A Geometric random variable X with parameter p takes positive integer values. The probability distribution on $i=1,2,\ldots$ is,

$$P(X==i) = (1-p)^{i-1} \cdot p \qquad \text{where } X = \lfloor \sqrt{k} \rfloor$$

$$E[X] = 1p + 2(1-p)p + 3(1-p)^2 p \ldots$$

$$-(1-p)E[X] = (1-p)p + 2(1-p)^2 p + 3(1-p)^3 p \ldots$$

$$pE[x] = p + (1-p)p + (1-p)^2 p + \ldots$$

$$E[x] = 1 + (1-p) + (1-p)^2 + \ldots$$

$$= \frac{1}{1-(1-p)} = \frac{1}{p}$$

from (1) $p = \frac{1}{k}$

$$\therefore E[x] = \frac{1}{p} = \frac{1}{1/k} = k$$

Ans : $\boxed{E[\lfloor \sqrt{k} \rfloor] = k}$

Hence,
we need to roll a dice k number of times (Expectation)
untill we see $\lfloor \sqrt{k} \rfloor$ on upward face

References:
Slides 116, and 117 from sirs ppt

B.

5.

b. This problem is an example of Coupon Collector Problem.

So, no. of times we need to roll a dice untill we see every number from 1 to k atleast once on upward face,

$$E\left[\sum_{i=1}^{k} X_i\right] = \sum_{i=1}^{k} E[X_i]$$

X_i is ~~the event~~ random variable for event where we see number 'i' on upward face

$$\therefore \sum_{i=1}^{k} E[X_i] = \sum_{i=1}^{k} \frac{1}{p_i}$$

Now we rolled the die the probablity of getting a unique number (which we havent seen) $= \frac{k}{k} = 1$

If we roll again, (2nd time)
the probablity of seeing a unique number $= \frac{(k-1)}{k}$

If we roll again, (3rd time)
the probablity of seeing a unique number $= \frac{(k-2)}{k}$

⋮

$$\therefore \sum_{i=1}^{k} \frac{1}{p_i} = 1 + \frac{k}{k-1} + \frac{k}{k-2} + \frac{k}{k-3} + \dots + k$$

$$= \sum_{i=1}^{k} \frac{k}{k-i-1}$$

$$= \boxed{k\log(k)}$$ Ans.

References:
Slide 118 , 119 from sirs ppt
https://www.quora.com/How-many-times-do-I-need-to-throw-a-die-to-get-all-the-sides-at-least-once-at-average

C.

5.c.

$P(1) = 1/4$
$P(3) = 1/4$
$P(2) = 1/2$

from reference attached

$P(1) = 1$

$$P(2) = P(1)\left[P(2)+P(3)\right] + P(2)\left[P(1)+P(3)\right] + P(3)\left[P(1)+P(2)\right]$$

$$= \frac{1}{4}\left(\frac{1}{4}+\frac{1}{2}\right) + \frac{1}{2}\left[\frac{1}{4}+\frac{1}{4}\right] + \frac{1}{4}\left(\frac{1}{4}+\frac{1}{2}\right)$$

$$= \frac{1}{4}\left[\frac{3}{4}\right] + \frac{1}{2}\left[\frac{1}{2}\right] + \frac{1}{4}\left[\frac{3}{4}\right]$$

$$= \frac{10}{16} = \frac{5}{8}$$

$$P(3) = 3 * P(1) * P(2) * P(3)$$

$$= 3 * \frac{1}{4} * \frac{1}{2} * \frac{1}{4} = \frac{3}{32}$$

$$E = \sum_{i=1}^{3} \frac{1}{P_i} - \sum_{i,j} \frac{1}{P_i+P_j} + \frac{1}{P_i+P_j+P_k}$$

$$= \left[1 + \frac{8}{5} + \frac{32}{3}\right] - \left[\left(\frac{1}{1+\frac{5}{8}}\right) + \left(\frac{1}{1+\frac{3}{32}}\right) + \left(\frac{1}{\frac{5}{8}+\frac{3}{32}}\right)\right] + \left[\frac{1}{1+\frac{5}{8}+\frac{3}{32}}\right]$$

$$= 1 + \frac{8}{5} + \frac{32}{3} - \frac{8}{13} - \frac{32}{35} - \frac{32}{23} + \frac{32}{55}$$

$$= 10.83 \approx \boxed{11}$$

References:
https://math.stackexchange.com/questions/600012/coupon-collectors-problem-with-unequal-probabilities?rq=1

3.

A.

3.

a. Given sample space Ω, powerset $P(\Omega)$

where $|P(\Omega)| = 2^{|\Omega|}$

Prove or disprove $P(A) = \frac{|A|}{|\Omega|}$ for every event $A \in P(\Omega)$.

For a measure to be a valid probability measure it should follow 3 axioms

Axiom 1: for any event A, $P(A) \geq 0$

$$\left.\begin{array}{ll} \text{if } |A| = 0 & P(A) = 0 \\ \text{if } |A| > 0 & P(A) > 0 \end{array}\right\} \text{Axiom 1 follows}$$

Axiom 2: $P(\Omega) = 1$

if $|A| = |\Omega|$

$P(A) = \frac{|\Omega|}{|\Omega|} = 1$ $\therefore$ Axiom 2 follows

Axiom 3: If $A_1, A_2, A_3 \ldots$ are disjoint events, then

$P(A_1 \cup A_2 \cup A_3 \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$

Let ~~P(A)~~ =

$A = A_1 \cup A_2 \cup A_3$ $\therefore |A| = |A_1| + |A_2| + |A_3|$

$P(A_1) = \frac{|A_1|}{|\Omega|}$ $P(A_2) = \frac{|A_2|}{|\Omega|}$ $P(A_3) = \frac{|A_3|}{|\Omega|}$

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3)$$
$$= \frac{|A_1|}{|\Omega|} + \frac{|A_2|}{|\Omega|} + \frac{|A_3|}{|\Omega|}$$
$$P(A) = \frac{|A|}{|\Omega|}$$

$\therefore$ Axiom 3 holds true

Hence $P(A) = \frac{|A|}{|\Omega|}$ for every event $A \in P(\Omega)$ is true

3.

b.

References:

https://www.probabilitycourse.com/chapter1/1_3_2_probability.php#:~:text=Axioms%20of%20Probability%3A,P(A3)%2B%E2%8B%AF

B.

3b.

$$\sum P(A_i) - \sum_{i \le j \le n} P(A_i \cap A_j) \le P(\bigcup_{i \le n} A_i)$$

from inclusion exclusion

$$\left|\bigcup_{i \ge 1}^{n} A_i\right| = \sum |A| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| + (-1)^{n+1} |A_1 \cap A_2 \cdots A_n|$$

i.e.

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum |A_i \cap \dots A_k| \right)$$

lets prove by induction
if it holds for n, it should hold for $n+1$

$$\therefore P(A_1 \cup A_2 \cdots \cup A_{n+1})$$
$$= P(A_1 \cup A_2 \cdots A_n) + P(A_{n+1}) - \sum P(A_1 \cap A_{n+1}) \cdots \cup P(A_n \cap A_{n+1})]$$

$$= \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots A_n) + P(A_{n+1}) -$$

$$= \sum P(A_i) - \sum P(A_i \cap A_j) + (-1)^{n+1} \sum P(A_i) + (-1)^{n+2} P(A_1 \cap A_2 \cdots A_{n+1})$$

$\therefore$ It holds true for $n+1$ hence (1) is true

since it holds for $n+1$

we can say that after adding one more term one additional term is introduced in the relation

for even terms $\longrightarrow$ we subtract

coz $(-1)^{n+1} \Rightarrow -ve$

and if we subtract a term then the RHS will become smaller than LHS.

Hence

$$P(\cup A_i) \geq \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \sum P(A_i \cap A_j \cap A_k \cap A_l)$$