Problem Set 7

Due date: Email your solutions to cs140_17fall@163.com or submit a written copy to a TA before 7:30 PM, January 4, 2018.

Problem 7-1. Matrix Fixing [60 points]

You are given an $n \times n$ matrix M of integers. You can modify the matrix by repeated performing the following operation: pick an integer $1 \leq i \leq n$ and set all the entries from the i'th row and the i'th column to zero (a total of $2n - 1$ entries). Your goal is to use the minimum number of these operations to turn M into a matrix of all zeros. We'll call this the *matrix-fixing problem*.

(a) [5 points] State the matrix-fixing problem as a decision problem.

(b) [15 points] Show that the decision problem is in NP.

(c) [25 points] Show that the decision problem is NP-complete.

Hint: You may use the fact that the decision variant of the clique problem is NP-complete.

Now, we'll focus on the optimization variant of the matrix-fixing problem. Consider the following greedy approximation algorithm: as long as there are still nonzero entries in M, perform the operation that converts the maximum number of nonzero entries to zero. In the remaining parts of this problem, you will prove that this greedy algorithm always gives an approximation ratio of $2 \log n$. That is, if the optimal solution uses k operations, then the greedy algorithm will use no more than $2k \log n$ operations.

(d) [5 points] Let k be the number of operations used by the optimal solution to a matrix-fixing problem. Show that there must exist an operation on M that changes at least a $1/k$ fraction of the nonzero entries in M to zero.

Hint: Prove this by contradiction.

(e) [10 points] Using what you just proved, show that the greedy approximation algorithm will use no more than $2k \log n$ operations.

Hint: You may find the following fact useful: $1 - m < e^{-m}$ for all positive m.

Problem 7-2. Proving NP-Completeness [40 points]

(a) [15 points] Let TRIPLE-SAT denote the set of Boolean formulas that have at least three distinct satisfying assignments. In other words, TRIPLE-SAT is the problem: given a Boolean formula, decide whether it has at least three distinct satisfying assignments.

Prove that TRIPLE-SAT is NP-complete.

(b) [25 points] Define BAGEL to be the decision problem where given (G, p, k), where $G = (V, E)$ is an undirected graph, p maps each vertex $u \in V$ to a nonnegative integer $p(u)$, and k is a nonnegative integer, the following holds: there exists a subset $U \subseteq V$ such that no two vertices in U are neighbors in G, and $\sum_{u \in U} p(u) \geq k$.

Prove that BAGEL is NP-Hard by reducing 3SAT to BAGEL. Note that 3SAT is the satisfiability problem for 3-CNF formulas, and is different from TRIPLE-SAT.