Algorithms
ShanghaiTech University
Rui Fan

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CS 140
Problem Set 5

Problem Set 5

Due date: Email your solutions to cs140_17fall@163.com or submit a written copy to the TA before 7:30 PM, November 30, 2017.

Problem 5-1. Save the Bridges! [30 points]

Ben Bitdiddle is the mayor of GatesTown, a city that has been at war for a long time with DreyfoosCity. The good people of GatesTown believe that roads are immoral and only use bridges or tunnels instead. Thus, GatesTown has n houses, and no roads. Instead of roads, there are m bridges going from house to house. Bridges are made of stones and the bridge going from house u to house v has $s_{u,v}$ stones. GatesTown is a connected city: from any house, it is possible to reach any other house by following some sequence of bridges.

Ben is worried that Alyssa P. Hacker from DreyfoosCity will try to blow up the bridges in GatesTown, so he asks his engineers to make a plan to protect the bridges. The engineers come back to Ben with a set containing k different plans. Each bridge protection plan i has a "power" p_i and a cost c_i. If the i^{th} plan is enacted, all the bridges with $s_{u,v} \leq p_i$ will be protected, while the others will be destroyed in case of attack. Notice that the number of bridges that are saved does not impact the cost of the plan. If a plan with sufficient power is enacted, there is no limit to how many bridges can be saved.

(a) [12 points] Ben would like to choose a plan that keeps the entire city connected, while spending as little as possible. Develop an efficient algorithm to determine the best plan for this purpose, or to establish that no plan will keep the entire city connected.

(b) [18 points] It turns out that the city council finds this plan too expensive. Ben decides that the best way to convince the inhabitants is to concretely tell them what their money can achieve. He asks each household u to tell him what maximum cost c_u^{max} they deem appropriate for the city to spend on bridge protection. He then replies back to each household with the maximum number of houses they would be connected with, if the city adopts one of the k plans whose cost is constrained to their suggested budget (each household only gets a reply about their own suggested budget). Design an algorithm to efficiently compute all these numbers. By efficiently, we mean an algorithm that runs in $O(m \log m + k \log k)$ time.

Problem 5-2. Parity Problems [30 points] Your friend has discovered a black magic algorithm to solve the all-pairs shortest paths problem on a weighted undirected graph $G = (V, E)$ in $O(V^2)$ time. However, due to some technical details, it only works on graphs with odd edge weights. You are tasked with designing a way to make the algorithm work on more general graphs. You may assume that all edge weights are natural numbers.

(a) [12 points] Describe an algorithm to produce a function $h : V \to \{0, 1\}$ with the property that $h(u) - h(v) + w(u, v)$ is odd for every edge $(u, v) \in E$, where $w(u, v)$ is the weight of the edge (u, v). You may assume that there exists at least one function h satisfying the above property.

(b) [12 points] Prove that a such a function $h : V \to \{0, 1\}$ exists if G contains no cycles with an odd number of even-weight edges.

(c) [6 points] Design an algorithm to solve the all-pairs shortest paths problem on an undirected graph that does not contain any cycles with an odd number of even-weight edges in $O(V^2)$ time. You may assume that your friend's algorithm works as described.

Problem 5-3. Largest Weight Cycle [40 points]

You are given a weighted tree on n vertices with possibly negative weights. You are allowed to add a single edge of weight 0 to this tree. Your goal is to maximize the weight of the cycle thus formed. Find an efficient way to do this.

(a) [10 points] Give an algorithm with running time $O(n^3)$.

(b) [10 points] Give an algorithm with running time $O(n^2)$.

(c) [20 points] Give an algorithm with running time $O(n)$.