CS244 Theory of Computation
Homework 1 Solution

Let $\Sigma = \{0, 1\}$ if not otherwise specified.

Problem 1

Let A be the set of all strings where there are no consecutive 0's. Show that A is regular in the following ways:

(a) by giving an NFA that recognizes A,

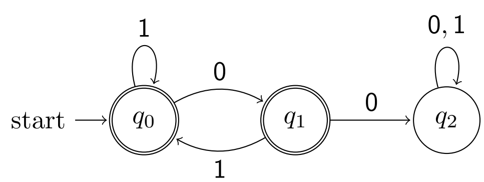
(b) by giving a DFA that recognizes A,

(c) by giving a regular expression that describes A, and

(d) by giving a right linear grammar that describes A.

Solution

(a) An NFA which is also a DFA that recognizes A:



(b) Same as in part (a).

(c) $(1 \cup 01)^*(0 \cup 1 \cup \varepsilon)$

(d) $S \to 1S \mid 0T \mid \varepsilon$
$T \to 1S \mid \varepsilon$

Problem 2

Let B be the set of all strings with even length that contain at least one 1 in their first half.

(a) Show that B is not regular.

(b) Show that B is context-free.

Solution

(a) *Proof.* Assume for contradiction that B is regular and let p be the pumping length given by the pumping lemma. Let $s = 0^p 1 0^{p+1}$, which is in B. If $s = xyz$ with $|xy| \leq p$ and $|y| > 0$, then y can only consist of 0s, and the string $xyyz$ is of the form $0^k 1 0^{p+1}$ where $k \geq p + 1$. However, $xyyz \notin B$ contradicting the pumping lemma. Thus B is not regular. □

(b) *Proof.* Construct a PDA that recognizes B:

1) Read the input until a 1 appears, while at the same time pushing 0s.
2) Continue reading the input while pushing 0s and nondeterministically switch to reading the input while popping the stack.
3) *Accept* if a 1 was found while pushing and the stack becomes empty at the end of the input.

Or consider the following CFG:

$$
\begin{aligned}
S &\to XSX \mid 1XT \\
T &\to XXT \mid \varepsilon \\
X &\to 0 \mid 1
\end{aligned}
$$

Here T is any string of even length, and S is any string of even length with $1XT$ in the middle (so the 1 falls at least one character to the left of the middle). □

Problem 3

Show that the following languages are not regular. B refers to the language in Problem 2.

(a) $C = \{w \mid w \in B \text{ or } w \text{ has odd length}\}$.

(b) $D = \{w \mid w \text{ has even length but } w \notin B\}$.

Solution

(a) *Proof.* Formally, $C = B \cup \Sigma(\Sigma\Sigma)^*$. Assume C is regular, then $B = C \cap (\Sigma\Sigma)^*$ would also be regular, which contradicts part (b). Thus C is not regular. □

(b) *Proof.* Formally, $D = (\Sigma\Sigma)^* - B$. $L_1 - L_2 = L_1 \cap \overline{L_2}$, so regular languages are closed under difference. Assume D is regular, then $B = (\Sigma\Sigma)^* - D$ would also be regular, which contradicts part (b). Thus D is not regular. □

Problem 4

For any string $w = w_1 w_2 \dots w_n$, the *reverse* of w, written as $w^{\mathcal{R}}$, is the string w in reverse order, $w_n \dots w_2 w_1$. For any language A, let $A^{\mathcal{R}} = \{w^{\mathcal{R}} \mid w \in A\}$. Show that if A is regular, so is $A^{\mathcal{R}}$, i.e., regular languages are closed under reversal.

Solution

Proof. For any regular language A, let M be the DFA that recognizes it. Construct an NFA N that recognizes $A^{\mathcal{R}}$. Keep all the states in M and reverse the direction of all the arrows in M. Let the accept state of N be the start state of M. Since there should be only one start state in an FA, we introduce a new state as the start state of N which goes to every accept state of M by an ε-transition. □

Problem 5

Consider the following CFG G:

$$S \to 0S1 \mid 00S1 \mid \varepsilon$$

Describe $L(G)$ and show that G is ambiguous. Give an unambiguous grammar H where $L(H) = L(G)$ and sketch a proof that H is unambiguous.

Solution

$L(G) = \{0^m 1^n \mid 0 \le n \le m \le 2n\}$. Grammar G generates $w = 00011$ in two different ways. Here are two leftmost derivations:

$$S \to 0S1 \to 000S11 \to 00011 \text{ and } S \to 00S1 \to 000S11 \to 00011$$

Thus G is ambiguous.
Consider the CFG H:

$$T \to 00T1 \mid U$$
$$U \to 0U1 \mid \varepsilon$$

H and G both generate one or two 0s with each 1, so $L(H) = L(G)$. However, H forces all cases of generating two 0s to precede the cases of generating one 0. If $w = 0^m 1^n \in L(G)$, then H generates w using $(m - n)$ applications of the $T \to 00T1$ rule, then an application of the $T \to U$ rule, then $(2n - m)$ applications of the $U \to 0U1$ rule, and finally an application of the $U \to \varepsilon$ rule. The leftmost derivation of every string is unique, thus H is unambiguous.

Problem 6

Let $E = \{rst \mid r, t \in \mathsf{0}^* \text{ and } s \in \mathsf{0}^*\mathsf{1}\mathsf{0}^* \text{ where } |r| = |s| = |t|\}$. Show that E is context-free in two ways:

(a) by giving a CFG that generates E, and

(b) by giving a PDA that recognizes E.

Solution

(a) $S \to \mathsf{0}S\mathsf{00} \mid \mathsf{00}S\mathsf{0} \mid \mathsf{010}$

(b) Construct a PDA that recognizes E:

1) Read 0s and pushing them on the stack.
2) At any point nondeterministically read 010.
3) Repeatedly, either read 0 and pop 00, or read 00 and pop 0.
4) *Accept* if the stack is empty at the end of the input.

A formal definition: $M = (\{q\}, \{\mathsf{0}, \mathsf{1}\}, \{\mathsf{0}, \mathsf{1}, S\}, \delta, q, S, \emptyset)$ accepting by empty stack, where

$$\begin{aligned}
\delta(q, \varepsilon, S) &= \{(q, \mathsf{0}S\mathsf{00}), (q, \mathsf{00}S\mathsf{0}), (q, \mathsf{010})\} \\
\delta(q, \mathsf{0}, \mathsf{0}) &= \{(q, \varepsilon)\} \\
\delta(q, \mathsf{1}, \mathsf{1}) &= \{(q, \varepsilon)\}
\end{aligned}$$

We accept the shorthand in $\delta(q, \varepsilon, S)$, though it is better to expand those terms with extra states.

Problem 7

Let $F = \{rst \mid r, s, t \in \mathtt{0}^*\mathtt{1}\mathtt{0}^* \text{ where } |r| = |s| = |t|\}$.

(a) Show that F is not context-free.

(b) Is $F \cup (\Sigma\Sigma\Sigma)^*$ context-free? Prove your answer.

(c) Is $F \cup \Sigma(\Sigma\Sigma\Sigma)^*$ context-free? Prove your answer.

Solution

(a) *Proof.* Assume for contradiction that F is context-free and let p be the pumping length given by the pumping lemma for CFLs. Let $w = \mathtt{0}^p\mathtt{1}\mathtt{0}^p\mathtt{1}\mathtt{1}\mathtt{0}^p$. Because $w \in F$ and $|w| = 3p + 3 \geq p$, we can write $w = uvxyz$ satisfying the three conditions of the pumping lemma. Neither v nor y can include a 1. Otherwise, the string uxz would contain fewer than three 1s and so $uxz \notin F$, contradicting the pumping lemma (pumping down). $|vxy| \leq p$, therefore vxy cannot include 0s from both the first block of 0s in w and from the last block of 0s in w. If vxy doesn't include 0s from the first block, then both v and y must include 0s only from the second and third blocks of 0s. Then the first 1 in uxz would not occur within its first third and so $uxz \notin F$. Similarly, if vxy doesn't include 0s from the last block, then both v and y must include 0s only from the first and second blocks of 0s. So the last 1 in uxz would not occur within its last third and so $uxz \notin F$. Both cases give a contradiction, so F cannot be context-free. ☐

(b) All strings in F have length divisible by 3 and $(\Sigma\Sigma\Sigma)^*$ is the language of all strings over Σ that have length divisible by 3. $F \cup (\Sigma\Sigma\Sigma)^* = (\Sigma\Sigma\Sigma)^*$, which is regular thus context-free.

(c) Let $L = F \cup \Sigma(\Sigma\Sigma\Sigma)^*$. Then $F = L \cap (\Sigma\Sigma\Sigma)^*$. If F is context-free, then F would also be context-free because the intersection of a CFL and a regular language is a CFL. We know from part (a) that F is not context-free. Thus L is not context-free.