9. ARCH/GARCH

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So far we've dealt with time series that have trends, wander around, have persistent shocks and exhibit seasonality.

What about series that have periods of volatility clustered together, while the rest of the time the series is relatively tranquil?

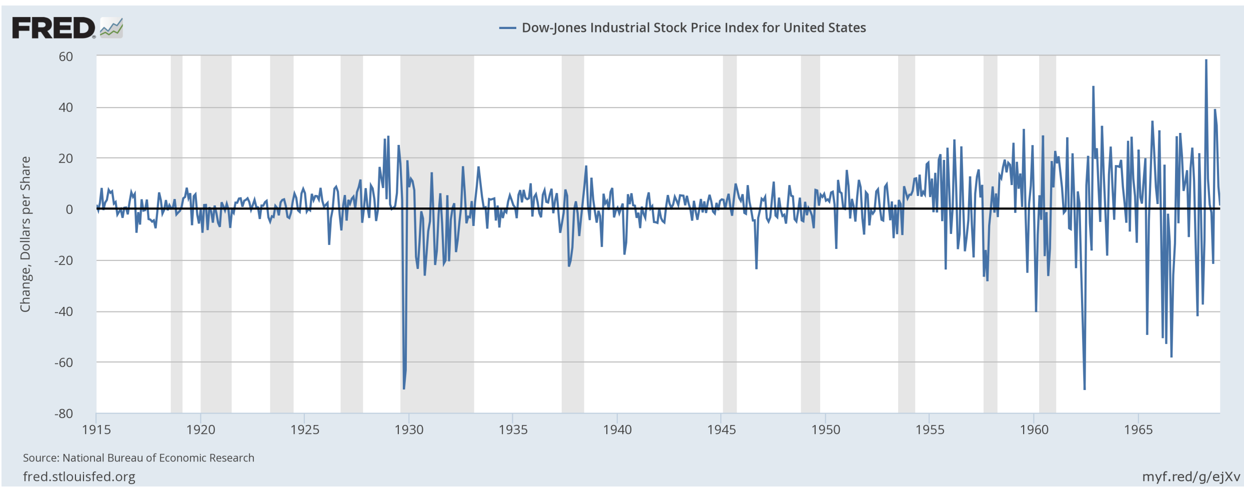

Figure 1:

ACF of Differenced DJI

PACF of Differenced DJI

Try Squaring…

Try Squaring…

ARCH

ARCH is an acronym for: **Autoregressive Regressive Conditional Heteroskedasticity**

We'll begin with a multivariate model… - Recall, inference from OLS assumes that (and is only valid if) the series is homoskedastic, i.e. - uniform variance: $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma^2$

== Our interest is in trying to model a series that has **conditional** heteroskedasticity

We'll start with

$$y_{t+1} = \epsilon_{t+1} x_t$$

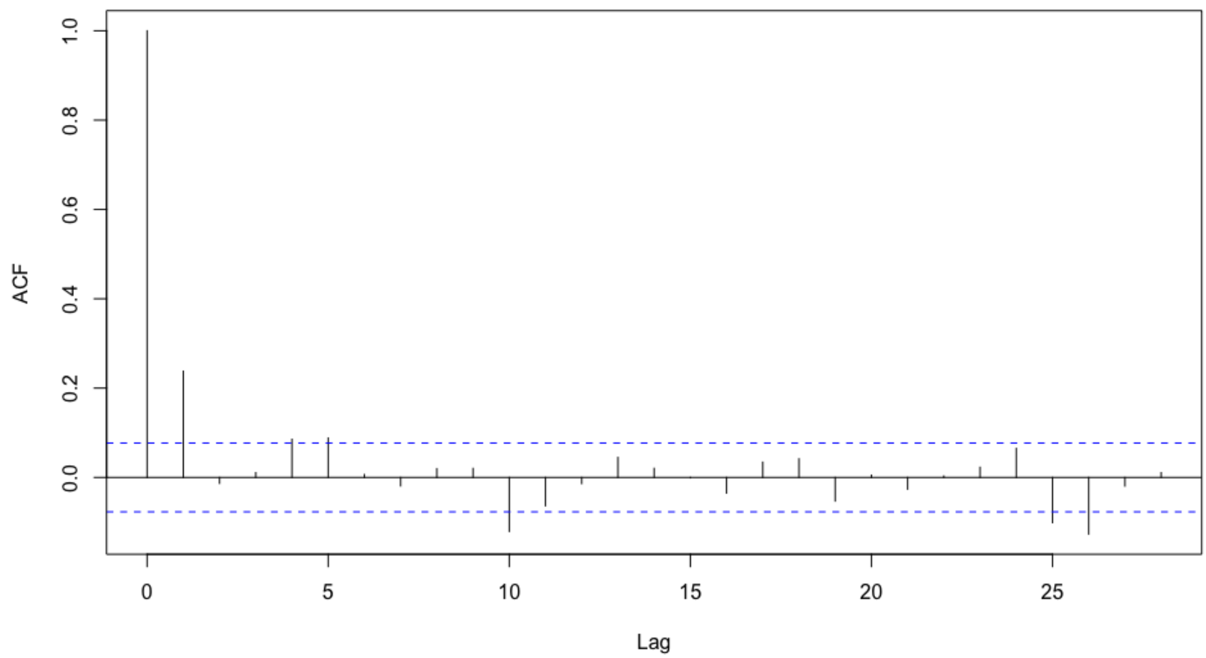

Figure 2:

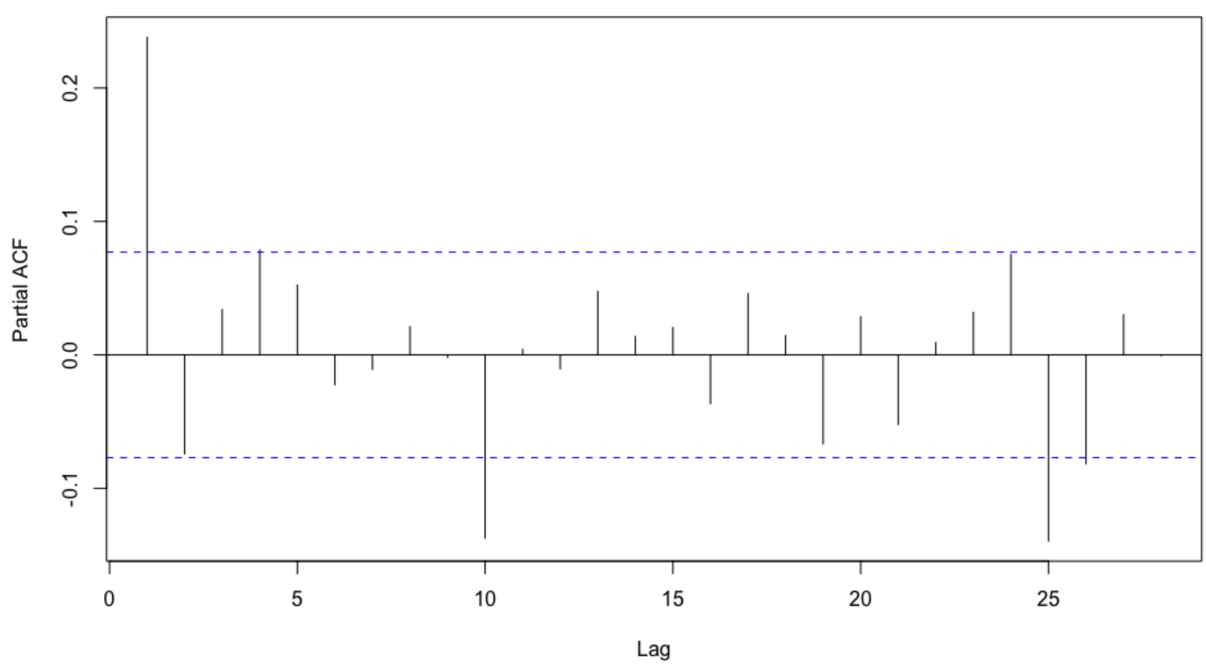

Figure 3:

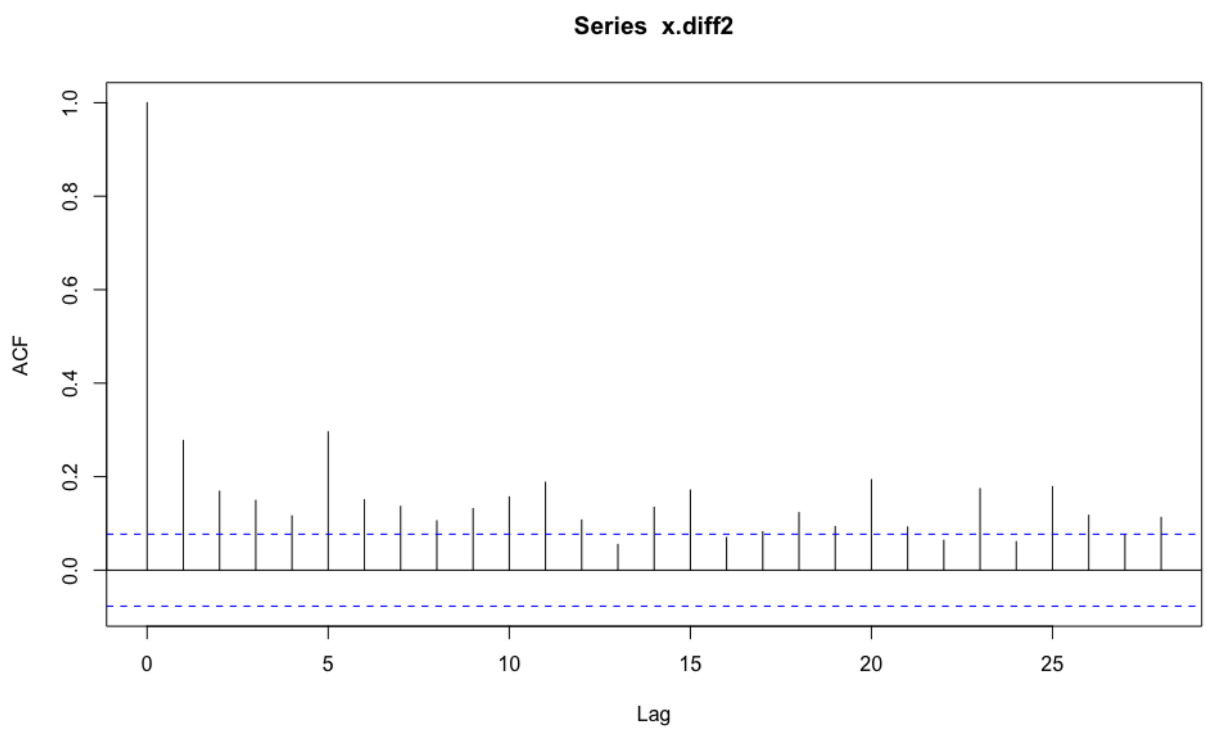

Figure 4:

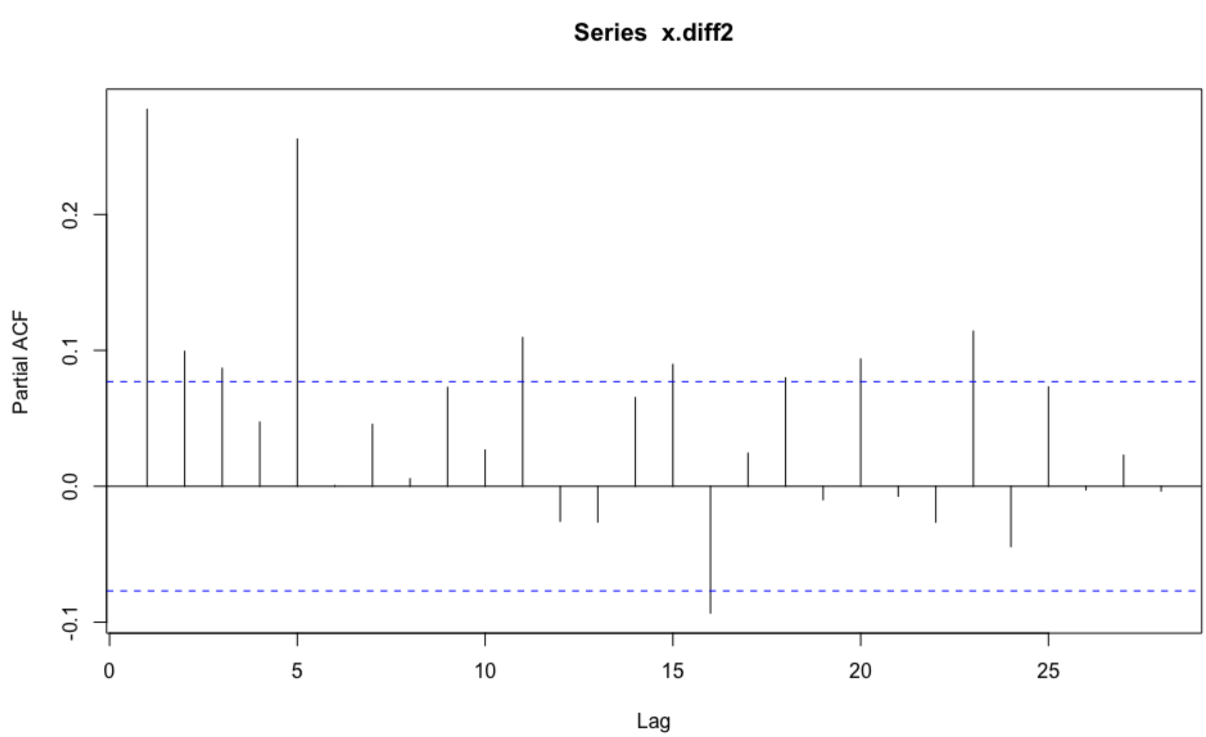

Figure 5:

If $x_t = x_{t-1} = x_{t-2} = \cdots = c$, where c is some constant, then the sequence $\{y_t\}$ is a white noise process.

However, when the the elements of the series $\{x_t\}$ are not equal, then the variance of $\{y_t\}$ is **conditional** on the observed value of $\{x_t\}$

== Attempting to estimate

$$y_{t+1} = \epsilon_{t+1} x_t$$

One possible solution:

$$\ln(y_t) = a_0 + a_1 \ln(x_{t-1}) + e_t$$

Problems with this method include: - Choice of $\{x_t\}$ - Necessary transformation of the data - Assumes $\{e_t\}$ has constant variance

== Instead, we can model the mean and variance simultaneously - For this example, we'll use a simple ARMA model:

$$y_t = a_0 + a_1 y_{t-1} + e_t$$

such that

$$\begin{aligned}\mathbf{E_t}(y_{t+1}|y_t) &= \mathbf{E_t}\left[(y_{t+1} - a_0 - a_1 y_t)^2\right] \\ &= \mathbf{E_t}\left[(e_{t+1})^2\right]\end{aligned}$$

Of course, this assumes $\mathbf{E_t}\left[(e_{t+1})^2\right] = \sigma^2$

== But what if the conditional variance is not a constant?

Then we can forecast the variance as an AR(q) model using the squares of the residuals:

$$\hat{e_t}^2 = a_0 + a_1 \hat{e_{t-1}}^2 + a_2 \hat{e_{t-2}}^2 + \cdots + a_q \hat{e_{t-q}}^2 + w_t$$

where $w_t \sim N(0,1)$

This is an **ARCH** model, for which there are many applications. - residuals can come from AR, ARMA, or standard regression models

== Although the previous model is usable and useful for the idea behind conditional heteroskedasticity, a better method uses maximum likelihood techniques to estimate the model and the variance simultaneously

- Multiplicative ARCH(1)

$$\epsilon_t = w_t \sqrt{a_0 + a_1 \epsilon_{t-1}^2}$$

- Allowing for higher order ARCH(q)

$$\epsilon_t = w_t \sqrt{a_0 + \sum_{i=1}^{q} a_1 \epsilon_{t-i}^2}$$

Properties of ARCH(1)

Expected Value

$$
\begin{aligned}
\mathbf{E}(\epsilon_t) &= \mathbf{E}\left(w_t\sqrt{a_0 + a_1\epsilon_{t-1}^2}\right) \\
&= \mathbf{E}(w_t)\cdot\mathbf{E}\left(\sqrt{a_0 + a_1\epsilon_{t-1}^2}\right) \\
&= 0\cdot\mathbf{E}\left(\sqrt{a_0 + a_1\epsilon_{t-1}^2}\right)
\end{aligned}
$$

since $\mathbf{E}(w_t) = 0$ and $\mathbf{V}(\epsilon_t) < \infty$

Properties of ARCH(1) Continued...

Variance

$$
\begin{aligned}
\mathbf{V}(\epsilon_t) &= \mathbf{E}\left[(\epsilon_t - \mathbf{E}(\epsilon_t))^2\right] \\
&= \mathbf{E}[\epsilon_t^2] \\
&= \mathbf{E}\left[\left(w_t\sqrt{a_0 + a_1\epsilon_{t-1}^2}\right)^2\right] \\
&= \mathbf{E}[w^2]\cdot\mathbf{E}[a_0 + a_1\epsilon_{t-1}^2]
\end{aligned}
$$

$$
= a_0 + a_1\mathbf{V}(\epsilon_t) \tag{1}
$$

(1) can be modeled as an AR(1) process

- Look at the ACF of squared residuals
- Residuals must be white noise (i.e. an appropriate model will generate this)

GARCH

Generalized ARCH (**GARCH**) is simply an extension of the ARCH model allowing for lagged versions of the variance

Modeling the error:

$$
e_t = w_t\sqrt{h_t}
$$

where

$$
h_t = a_0 + \sum_{i=1}^{q} a_i e_{t-i}^2 + \sum_{i=1}^{p} b_i h_{t-i}^2
$$

== GARCH(0,1) = ARCH(1) - In general, GARCH(0, q) = ARCH(q)

GARCH(**p** , **q**) - '**p**' is the moving average term for the disturbance - '**q**' is the autoregressive term for the disturbance

Typically, higher order ARCH models can be specified as lower order GARCH models

== The key component of GARCH models is that the disturbance term (e) constitutes an ARMA process

To attempt to identify the GARCH model we must look at the squared residuals - If the model of $\{y_t\}$ is adequate, the the residuals will be white noise - Otherwise, the residuals will contain information that we can use

Constructing the Correlogram

Step 1: use the "best" model to estimate the dependent variable, $\{y_t\}$. Obtain the residuals and square them. Also, calculate the sample variance:

$$\widehat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T} \hat{e_t}^2$$

Step 2: Calculate and plot the sample autocorrelations of the squared residuals:

$$\rho_i = \frac{\sum_{t=i+1}^{T}(\hat{e_t}^2 - \widehat{\sigma}^2)(\hat{e_{t-1}}^2 - \widehat{\sigma}^2)}{\sum_{t=1}^{T}(\hat{e_t}^2 - \widehat{\sigma}^2)^2}$$

Constructing the Correlogram

Step 3: Use the Ljung-Box Q statistic to test groups of autocorrelations

$$Q = T(T+2)\sum_{i=1}^{n} \frac{\rho_i^2}{T-i}$$

- Rejecting the null hypothesis is equivalent to testing the null hypothesis of no ARCH/GARCH errors
- In practice, consider values of n up to $T/4$

Constructing the Correlogram

Step 3 (alternative): Use LM test to determine significance of ARCH/GARCH errors

This is done on the residuals from the "best" model (ARMA or OLS)

$$e_t = a_0 + a_1 e_{t-1}^2 + a_2 e_{t-2}^2 + \cdots + a_q e_{t-q}^2$$

- Null hypothesis is that there are no ARCH/GARCH terms
- If OLS, remember to include Xs

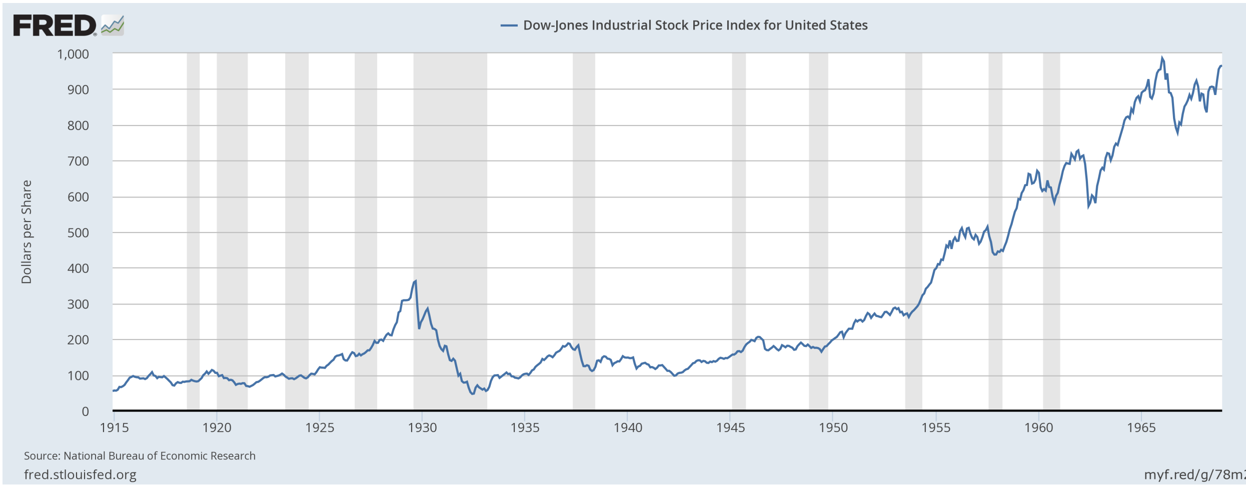

Figure 6:

Dow-Jones Index Example Continued…

ACF of DJI

ACF of the Difference of the DJI

PACF of the Difference of the DJI

== After applying the Box-Jenkins Methodology

```
Call:
arima(x = x, order = c(1, 1, 1), include.mean = F)

Coefficients:
         ar1     ma1
      0.0026  0.2624
s.e.  0.1400  0.1345

sigma^2 estimated as 147.2:  log likelihood = -2536.87,  aic = 5079.75
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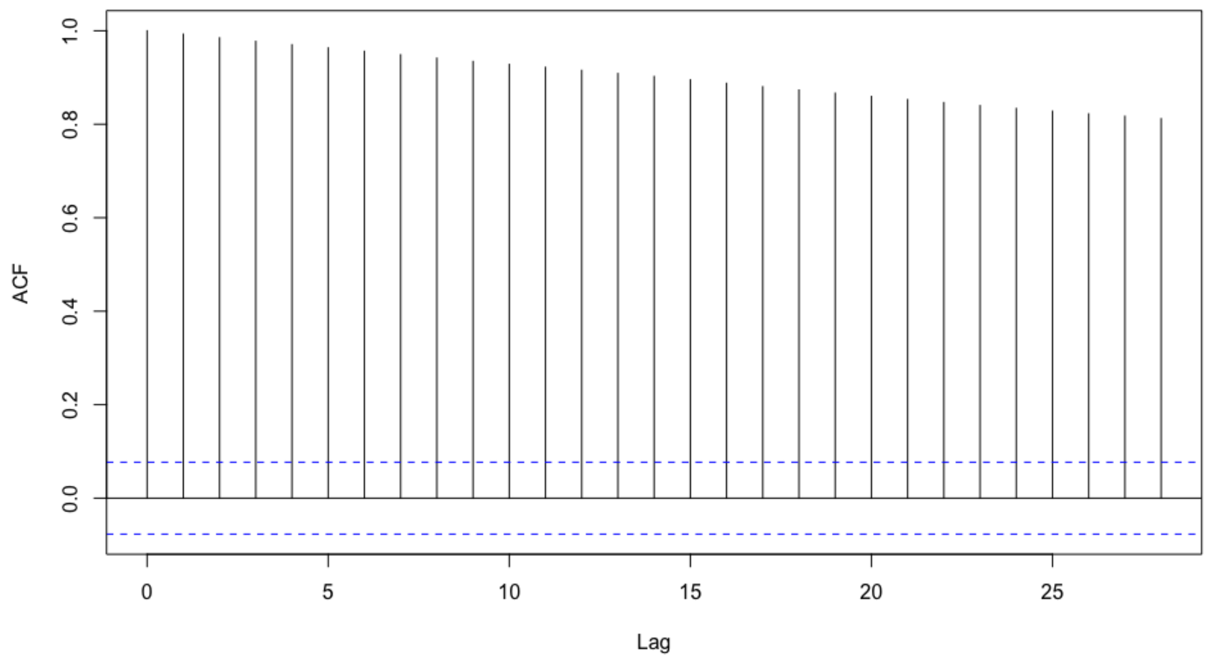

Figure 7:

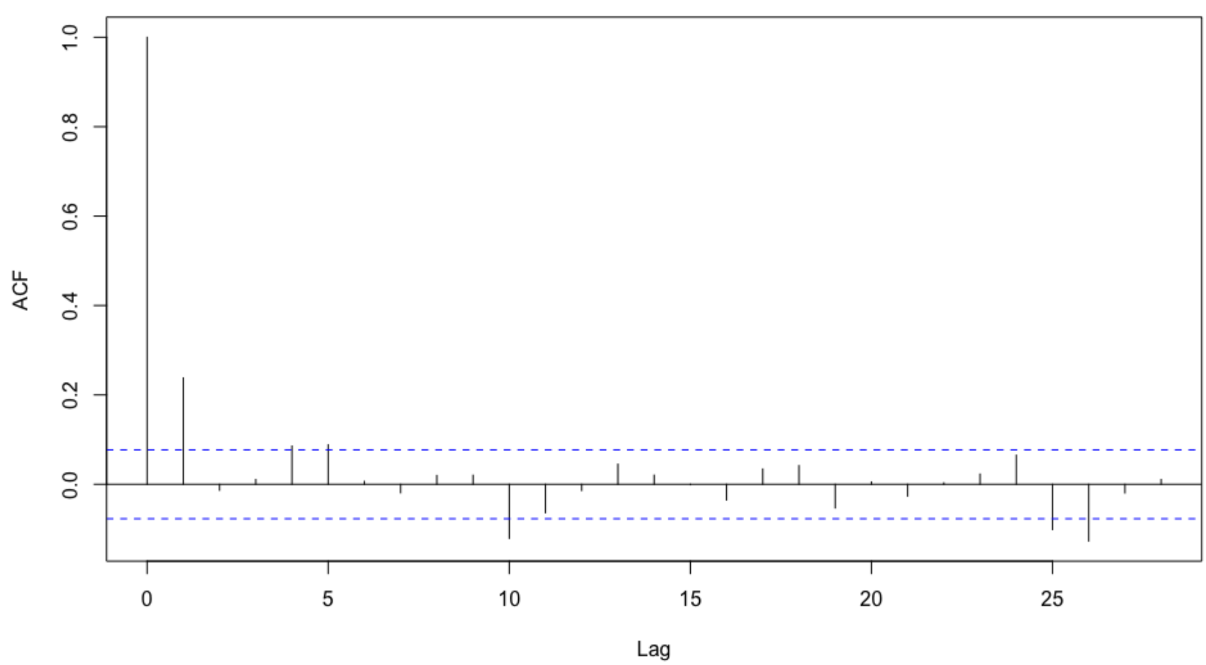

Figure 8:

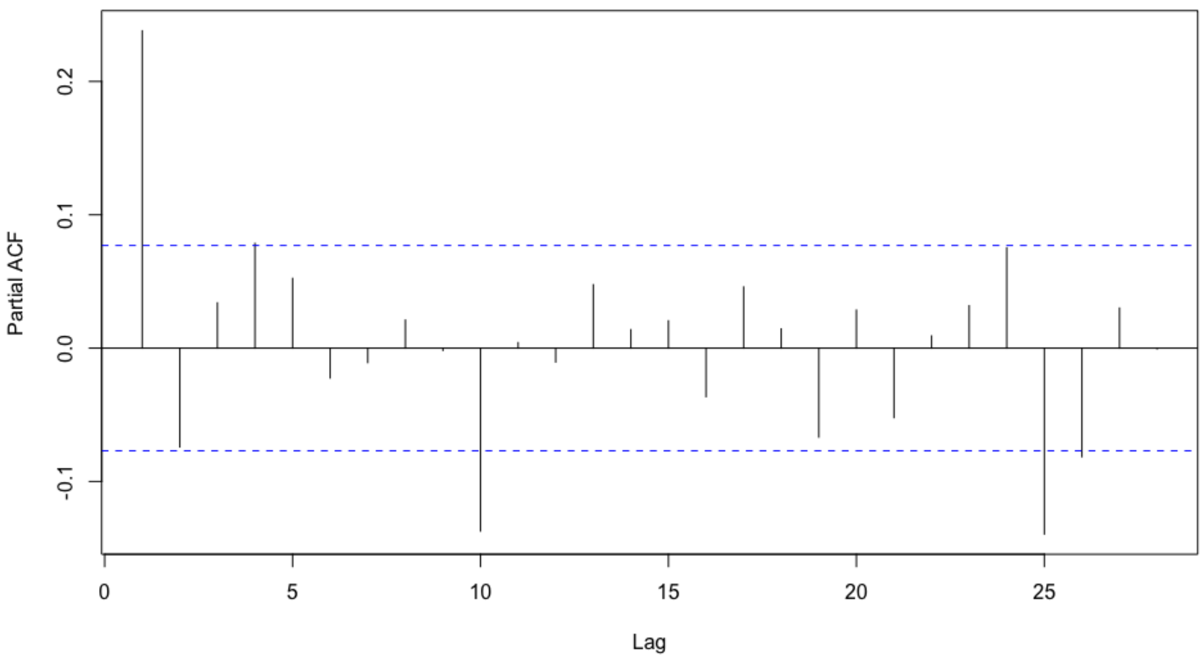

Figure 9:

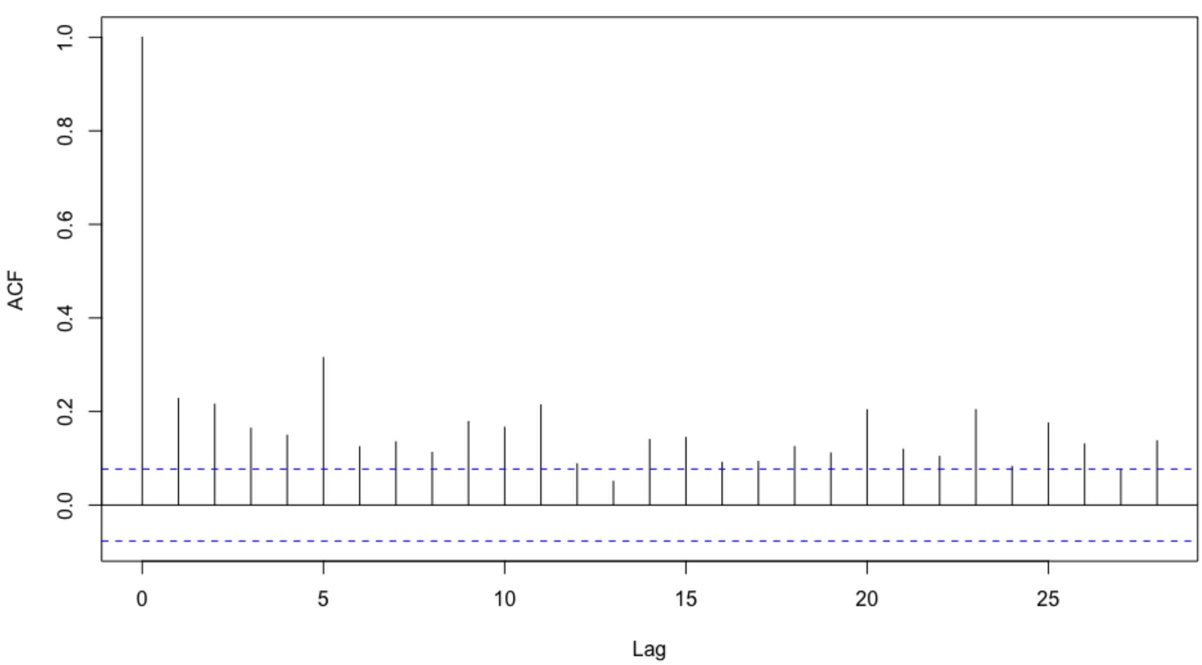

Figure 10:

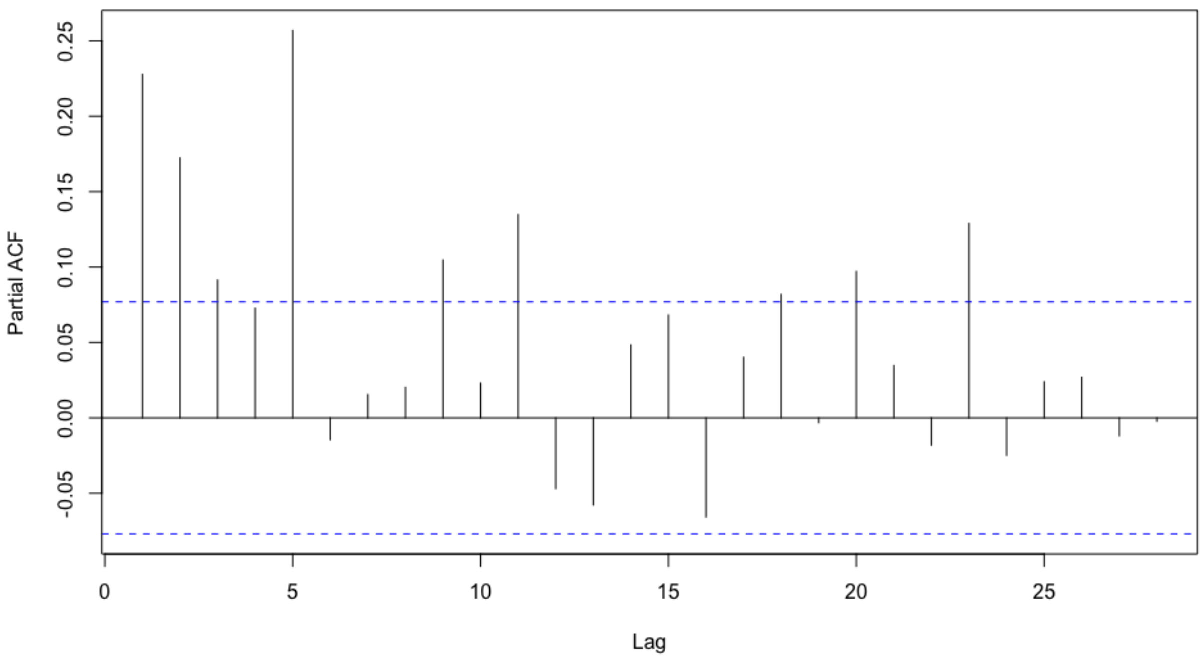

Figure 11:

ACF of Squared Residuals

PACF of Squared Residuals

Starting with GARCH(2,2)

Forecasting with GARCH

Your level/mean forecast from your ARIMA(p,d,q)(P,D,Q)s is still good for your point forecast

Just be aware that periods of high volatility can exist and these should be considered for your confidence intervals

GARCH is mainly used for (financial) time series simulations - $\widehat{\sigma}^2$ is commonly interpreted as a measure of risk

Extensions

ARCH-M - Allows mean of a sequence to depend on its own conditional variance - Useful for study of asset markets

IGARCH - Controls for persistent conditional heteroskedasticity - Useful for stock market analysis

Extensions

TARCH and **EGARCH** - Allows for 'news' to affect model - Bad news is worse (i.e. more volatility) than good news is good

Multivariate GARCH - Allows for exogenous regressors in the GARCH equation - Warning: These can be very difficult to estimate properly

Suggestions

- Make sure that the model estimating the mean is appropriate. Any mean model misspecification spills over to the variance model
- It's easy to overfit models. Make sure to use parsimonious models
- Be wary of structural breaks