4. Stochastic Models

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Backshift Operator

The **backshift operator** ($\mathbf{B}$) is defined as:

$$\mathbf{B}^n x_t = x_{t-n}$$

For example:

$$\begin{aligned}\mathbf{B}x_t &= x_{t-1},\\ \mathbf{B}^2 x_t &= x_{t-2},\\ \mathbf{B}x_{t-1} &= x_{t-2}\end{aligned}$$

and

$$\mathbf{B}(\mathbf{B}x_t) = x_{t-2}$$

Difference Operator

The **difference operator** (∇) is defined as:

$$\nabla^n x_t = x_t - x_{t-n}$$

For example:

$$\begin{aligned}\nabla x_t &= x_t - x_{t-1}\\ \nabla^2 x_t &= x_t - x_{t-2}\\ \nabla x_{t-1} &= x_{t-1} - x_{t-2}\end{aligned}$$

and

$$\nabla(\nabla x_t) = (x_t - x_{t-1}) - (x_{t-1} - x_{t-2}) = x_t - 2x_{t-1} - x_{t-2}$$

White Noise

(**Gaussian**) White Noise >A time series, $\{w_t\}$, is white noise if $w_1, w_2, ..., w_n$ are independent and identically distirbuted (IID) with $\mu_w = 0$. A time series is **Gaussian White Noise** if it is white noise that is also normally distributed.

Notes: - IID implies that $\sigma^2_{w_1} = \sigma^2_{w_2} = \cdots = \sigma^2_{w_n} = \sigma^2_w$ - $\mathbf{C}(w_i, w_j) = 0 \ \forall i \neq j$ - Past values of w tell us nothing about future values of w.

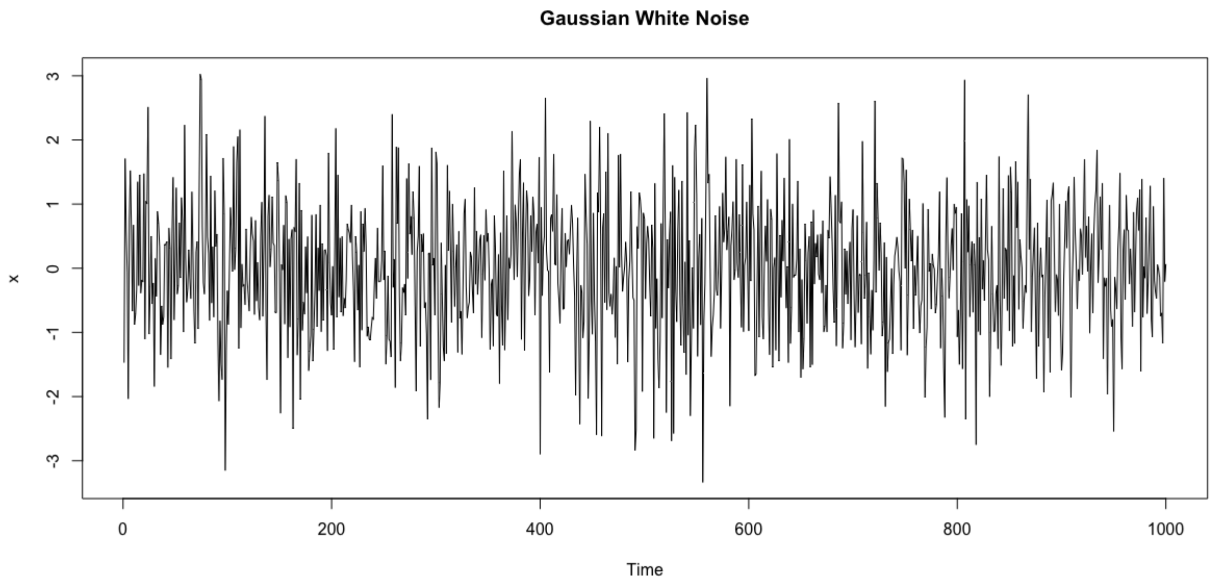

Figure 1:

White Noise

White Noise

Random Walk

A time series, $\{x_t\}$, is a **random walk** if

$$x_t = x_{t-1} + w_t$$

where $w_t \sim WN$

Notice that $x_{t-1} = x_{t-2} + w_{t-1}$. Repeated substitution then yields:

$$x_t = w_t + w_{t-1} + \cdots + w_1$$

where w_1 is the first realization of $\{x_t\}$.

Random Walk using B

$$\begin{aligned} x_t &= \mathbf{B}x_t + w_t \\ (1-\mathbf{B})x_t &= w_t \\ x_t &= (1-\mathbf{B})^{-1}w_t \end{aligned}$$

Recall that $(1-\mathbf{B})^{-1} = (1 + \mathbf{B} + \mathbf{B}^2 + \cdots)$, and thus

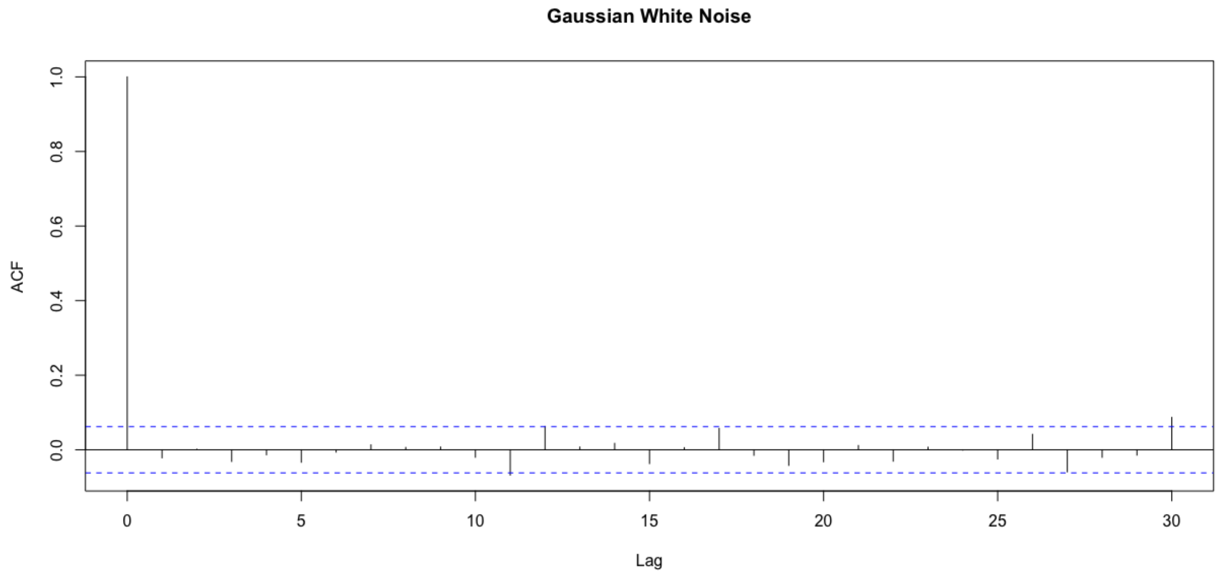

Figure 2:

$$
\begin{aligned}
x_t &= (1-\mathbf{B})^{-1} w_t \\
&= (1+\mathbf{B}+\mathbf{B}^2+\cdots)w_t \\
&= w_t + w_{t-1} + w_{t-2}
\end{aligned}
$$

Properties of the Random Walk

Expected Value

$$\mathbf{E}(x_t) = x_0 = 0 = \mu_x$$

Variance

$$
\begin{aligned}
\mathbf{V}(x_t) &= \mathbf{E}\left[(x_t-\mu_x)^2\right] \\
&= \mathbf{E}\left[\left(\sum_{t=1}^{n} w_t\right)^2\right] \\
&= \sum_{t=1}^{n} \sigma_w^2 \\
&= t\sigma_w^2
\end{aligned}
$$

$t\sigma_w^2$ is clearly not a constant $\implies$ x_t is nonstationary.

ACF for the Random Walk

Assume that $k > 0$.

$$
\begin{aligned}
r_k(x_t, x_{t+k}) &= \frac{\mathbf{C}(x_t, x_{t+k})}{\sqrt{(\mathbf{V}(x_t)\mathbf{V}(x_{t+k}))}} \\
&= \frac{t\sigma_w^2}{\sqrt{t\sigma_w^2(t+k)\sigma_w^2}} \\
&= \frac{1}{\sqrt{1+\frac{k}{t}}}
\end{aligned}
$$

What if t is large and k is relatively small?

Plot of a Random Walk (n=100)

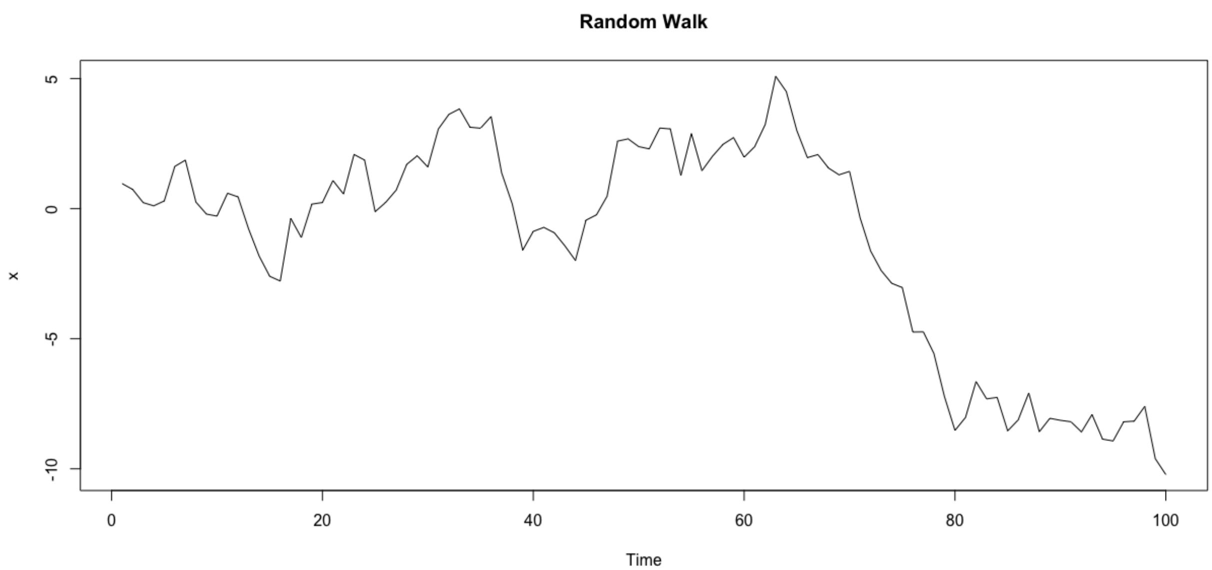

Figure 3:

ACF of a Random Walk (n=100)

Plot of a Random Walk (n=1000)

ACF of a Random Walk (n=1000)

Identifying a Random Walk

We know…

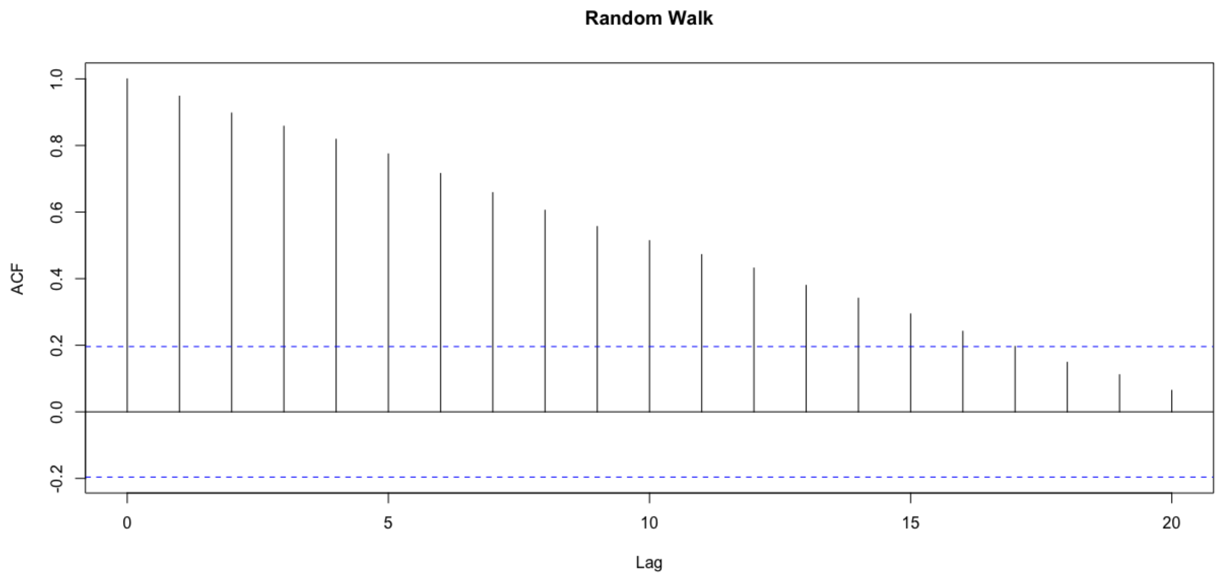

Figure 4:

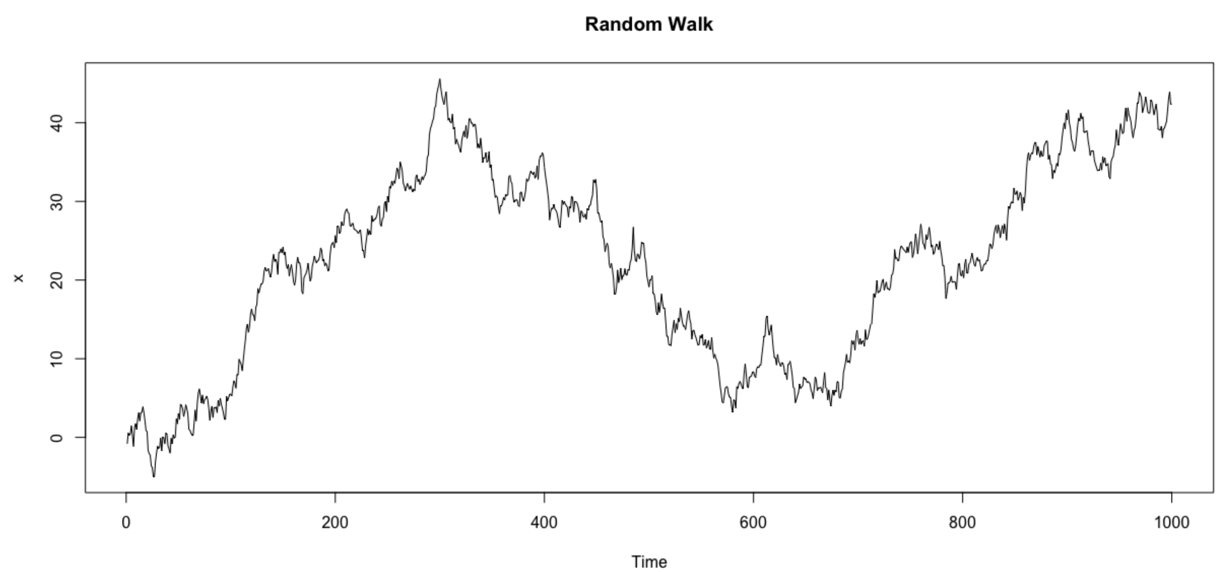

Figure 5:

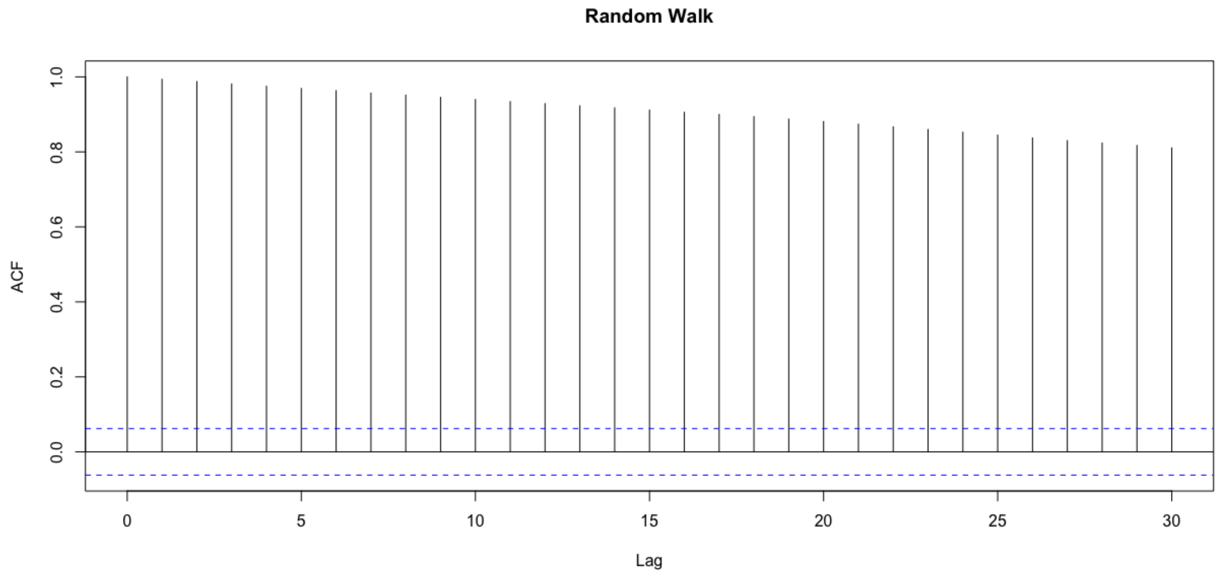

Figure 6:

$$
\begin{aligned}
x_t &= x_{t-1} + w_t \\
x_t - x_{t-1} &= w_t \\
\nabla x_t &= w_t
\end{aligned}
$$

We've seen the ACF of x_t, but just to be sure, let's take a look at the plot and ACF of ∇x_t.

Difference of a Random Walk

ACF of the Difference of a Random Walk

Forecasting a Random Walk

So if $x_{t+1} = x_t + w_{t+1}$ and $w_t \sim WN$...

What's our best predictor for $x_{t+1}, \hat{x}_{t+1}$?

- An Econometric Model?

Forecasting a Random Walk

So if $x_{t+1} = x_t + w_{t+1}$ and $w_t \sim WN$...

What's our best predictor for $x_{t+1}, \hat{x}_{t+1}$?

- An Econometric Model?
- Exponential Smoothing?

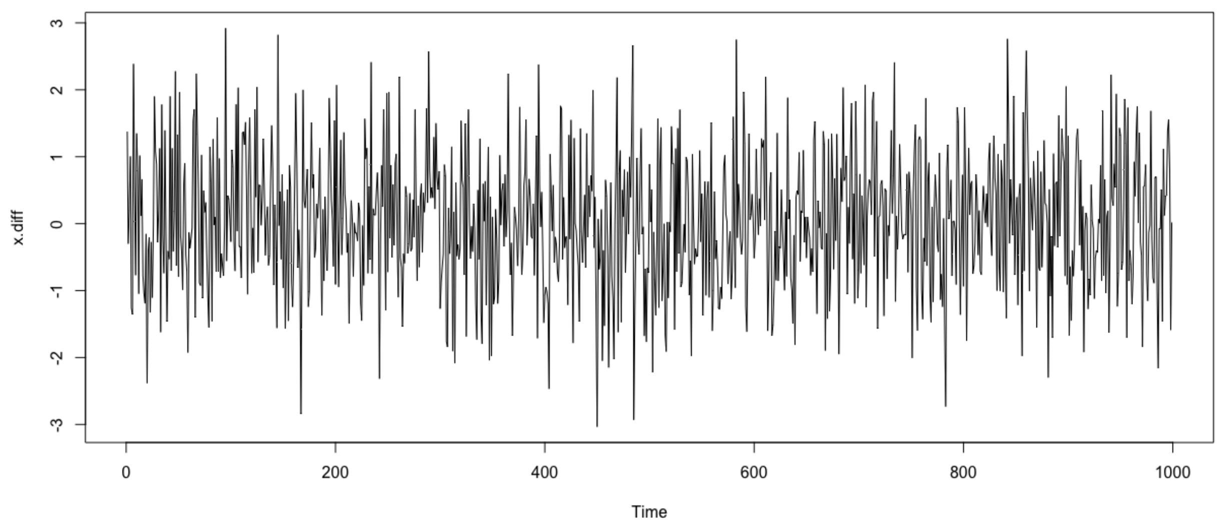

Figure 7:

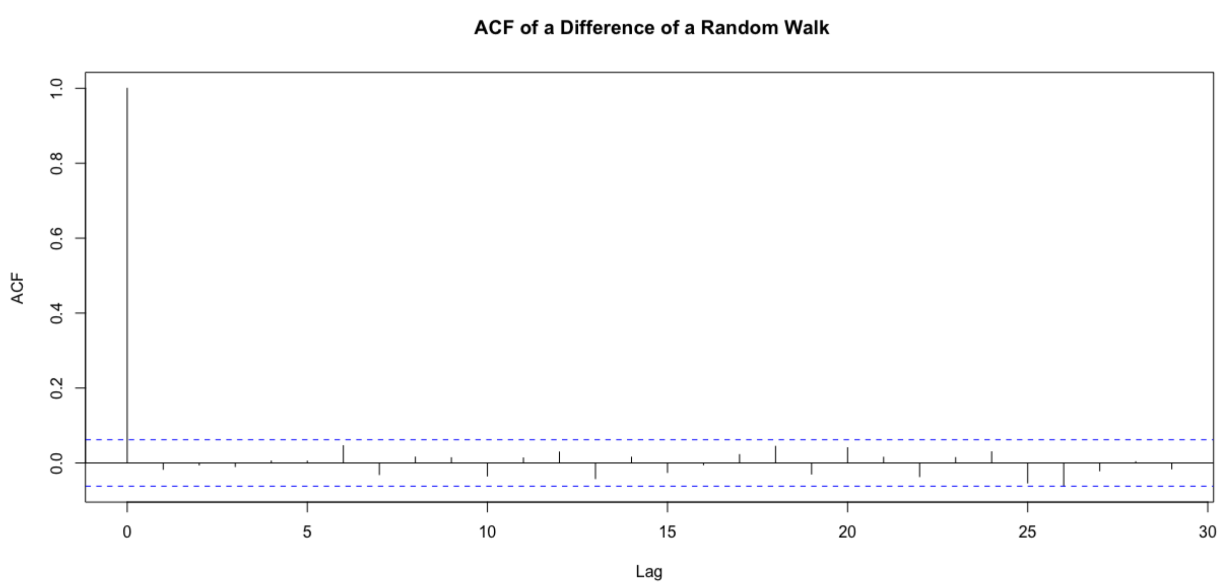

Figure 8:

Forecasting a Random Walk

So if $x_{t+1} = x_t + w_{t+1}$ and $w_t \sim WN$...

What's our best predictor for $x_{t+1}, \hat{x}_{t+1}$?

- An Econometric Model?
- Exponential Smoothing?
- Averaging?

Forecasting a Random Walk

So if $x_{t+1} = x_t + w_{t+1}$ and $w_t \sim WN$...

What's our best predictor for $x_{t+1}, \hat{x}_{t+1}$?

- An Econometric Model?
- Exponential Smoothing?
- Averaging?
- Naive Model?

Random Walk with Drift

A time series, $\{x_t\}$, is a **random walk with drift** if

$$x_t = \delta + x_{t-1} + w_t \quad \delta \neq 0$$

where $w_t \sim WN$

Notice that $x_t = \delta + x_{t-2} + w_{t-1}$. Repeated substitution yields:

$$x_t = t\delta + w_t + w_{t-1} + \cdots + w_t$$

where w_1 is the first realization of $\{x_t\}$

Properties of a Random Walk with Drift

$$\begin{aligned}
x_t &= \delta + x_0 + w_1 \\
x_2 &= \delta + x_1 + w_2 \\
&= \delta + (\delta + x_0 + w_1) + w_2 \\
&\dots \\
x_t &= t\delta + x_0 + \sum_{t=1}^{n} w_t
\end{aligned}$$

$\mathbf{E}(x_t) = t\delta + x_0$

$\mathbf{V}(x_t) = \mathbf{E}\left[(x_t - \mathbf{E}(x_t))^2\right] = \mathbf{E}\left[(\sum_{t=1}^{n} w_t)^2\right] = \sum_{t=1}^{n} \sigma_w^2 = t\sigma_w^2$

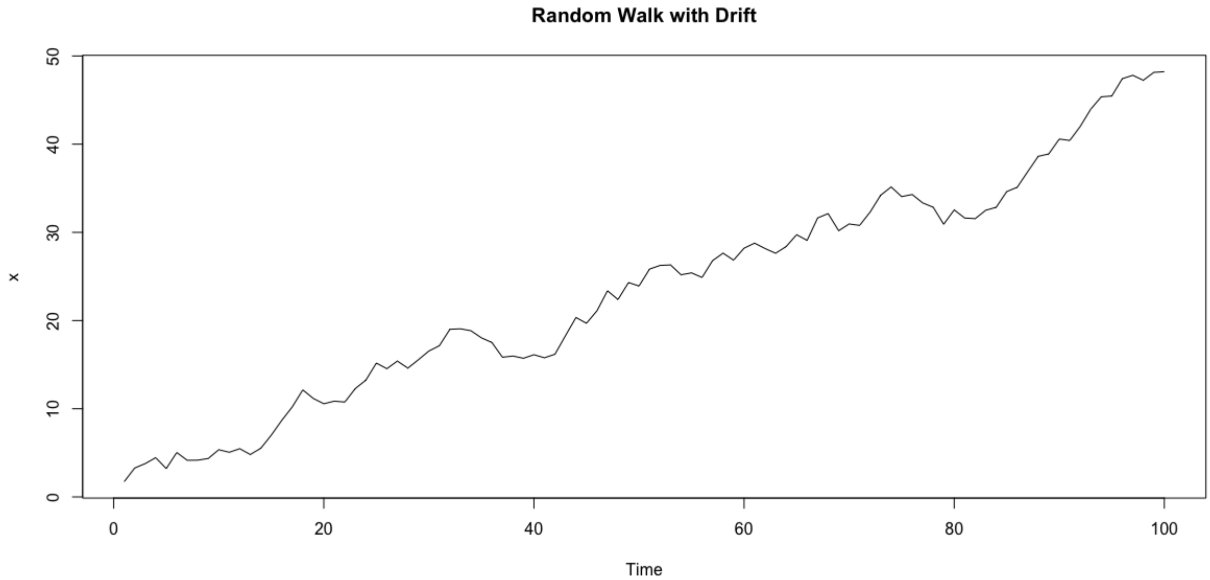

Figure 9:

Plot Random Walk w/ Drift (n=100)

ACF Random Walk w/ Drift (n=100)

Forecasting a Random Walk with Drift

Since the naive model was arguably the best for forecasting a random walk model, perhaps the naive model with trend is the best way to forecast a random walk with drift.

An alternative would be to calculate:

$$\overline{\nabla x_t} = \frac{1}{n-1}\sum_{t=2}^{n} \nabla x_t$$

as an estimator for δ. Then, an estimator could be

$$\hat{x}_{t+1} = \hat{\delta} + x_t$$

Random Walk Wrap-Up

- Visual Inspection of both the time series plots and ACFs.
- Random walks without trend have no long-run pattern
  - ACF decays slowly
    - The longer the time period, the slower the decay
- Random walks with trend do have long-run upward/downward patterns
  - ACF decays slowly

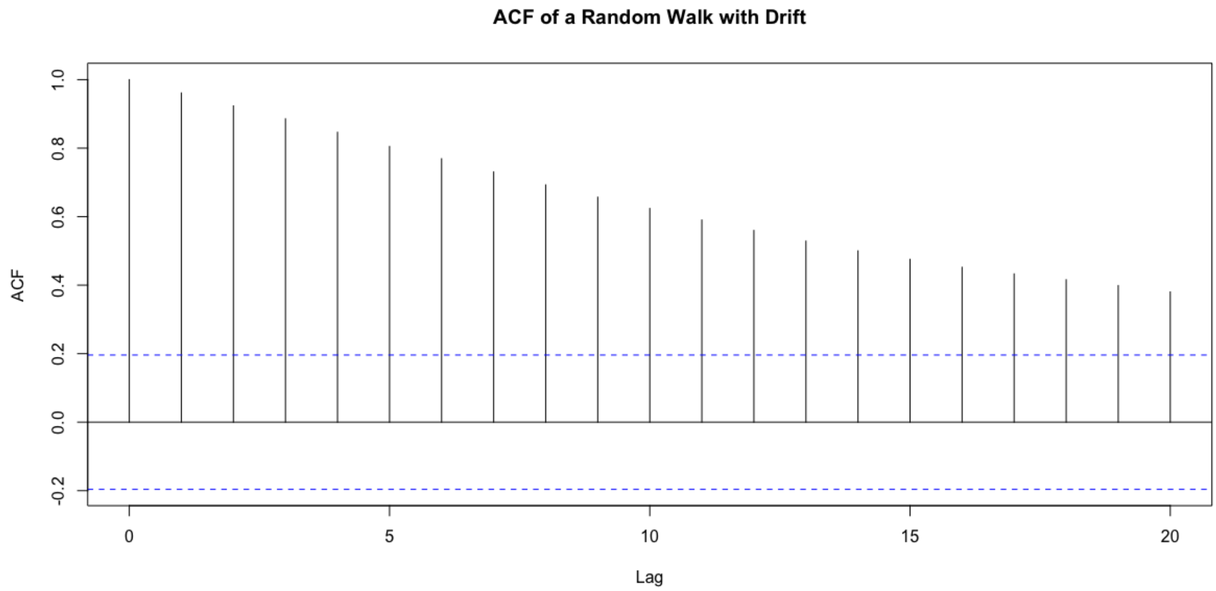

Figure 10:

- Check ACFs for the differenced series - should be stationary