



# Probability From Scratch

Bruno Gonçalves

*www.data4sci.com/newsletter*

<https://github.com/DataForScience/Probability>



# Question

<https://github.com/DataForScience/Probability>

• Where are you located?

• Europe

• Asia

• Africa

• US

• Canada

• Latin America

• Oceania

# Question

<https://github.com/DataForScience/Probability>

- What's your job title?

- Data Scientist

- Statistician

- Data Engineer

- Researcher

- Business Analyst

- Software Engineer

- Other

# Question

<https://github.com/DataForScience/Probability>

- How experienced are you in Python?

- Beginner (<1 year)
- Intermediate (1 -5 years)
- Expert (5+ years)

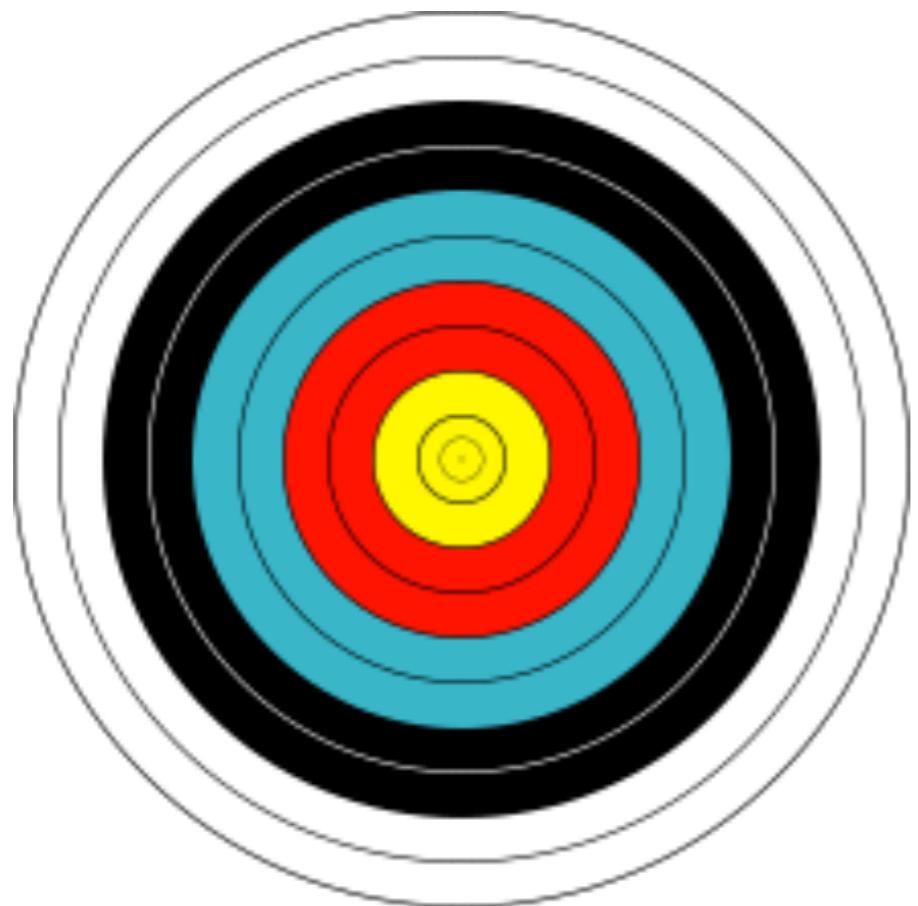


## Lesson I:

# Basic Definitions and Intuition

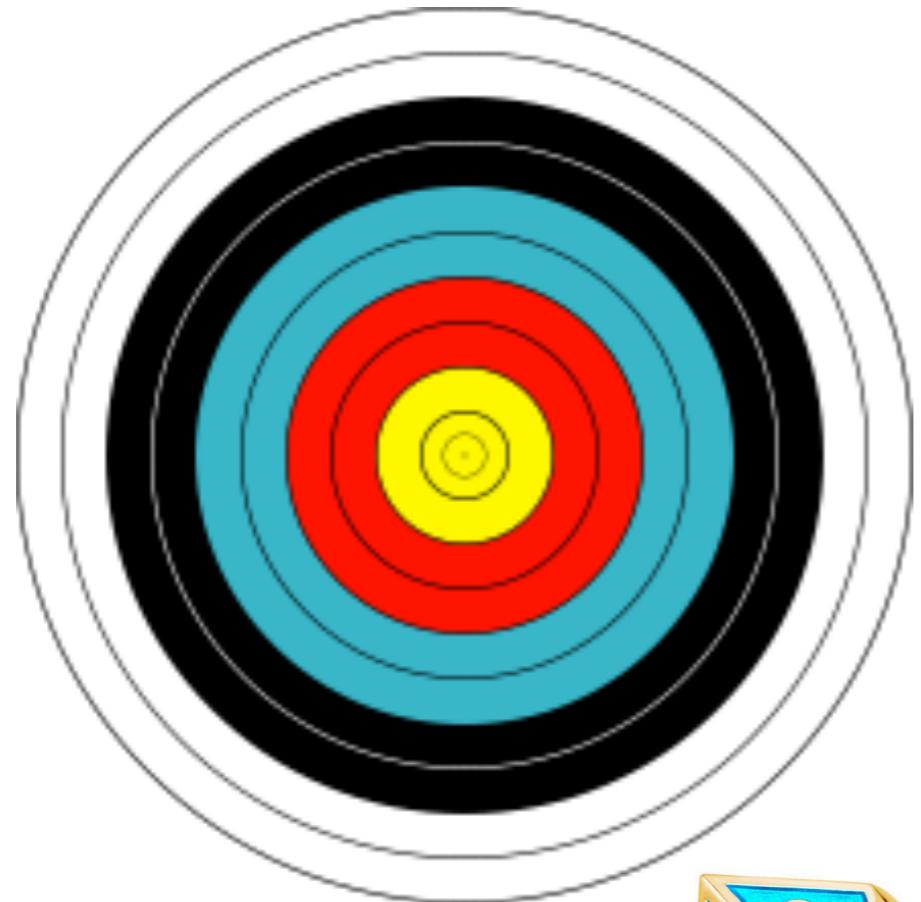
# Randomness

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# Randomness / Uncertainty

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# Kolmogorov's Probability Axioms

[https://en.wikipedia.org/wiki/Probability\\_axioms](https://en.wikipedia.org/wiki/Probability_axioms)

- **Axiom 1:** Probability is a real number **greater or equal to 0**.
- **Axiom 2: Total** probability is equal to **1**.
- **Axiom 3:** Probability of **mutually exclusive** events is the **sum of the probabilities**.

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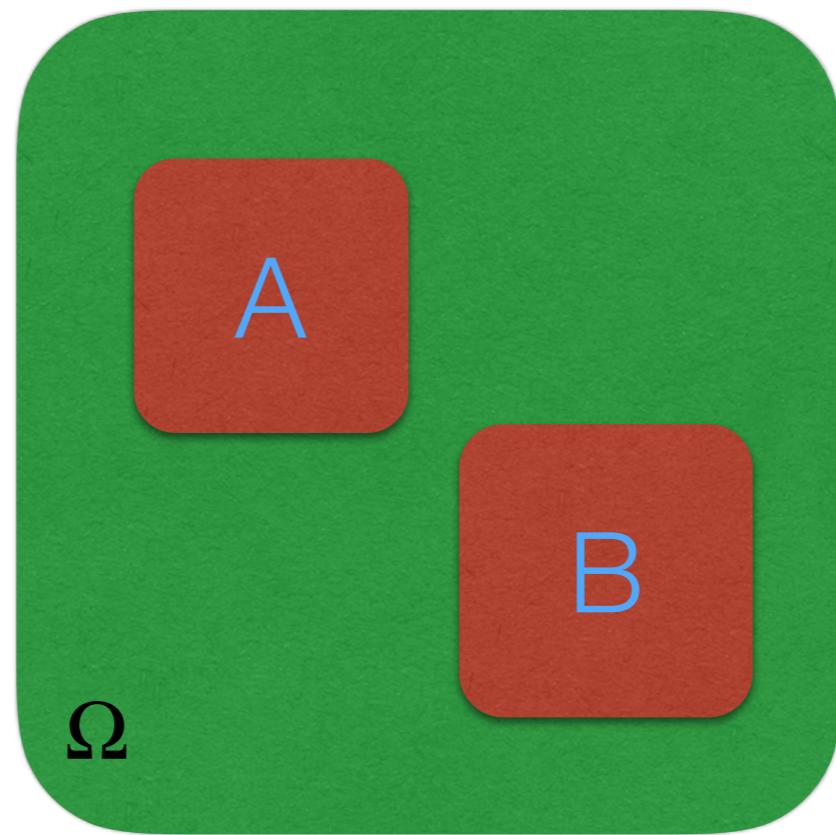
Probability = Area

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Probability = Area



$$0 \leq P(A) \leq 1$$

$$P(\Omega) \equiv 1$$

$$P(A, B) = P(A) + P(B)$$

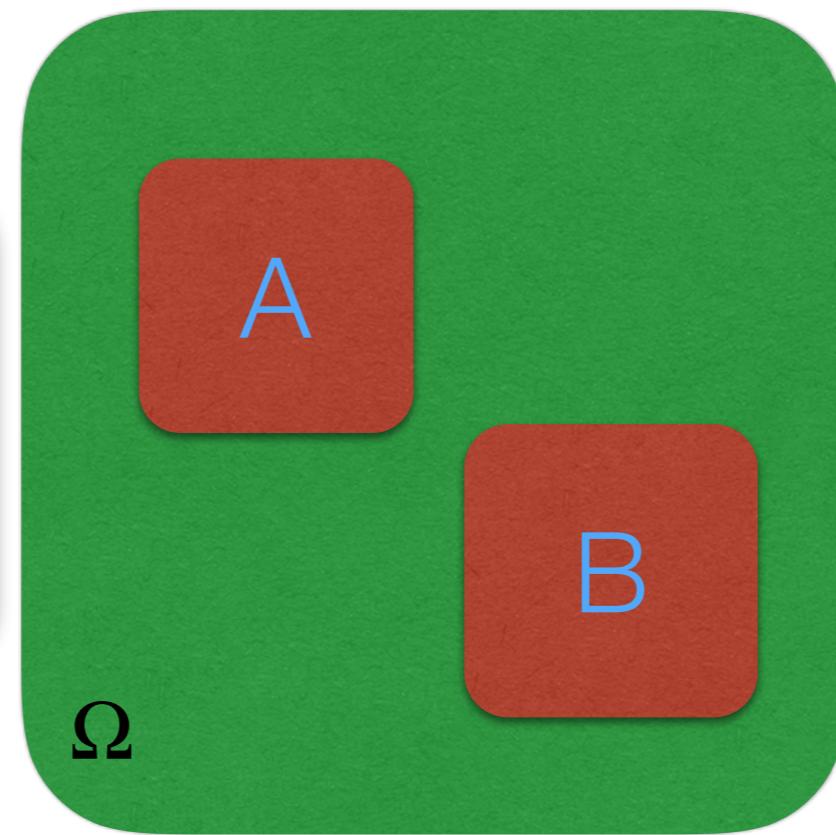
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Probability = Area

Prob(A) = Area A  
Total Area = 1  
Prob(A, B) = Area A + Area B



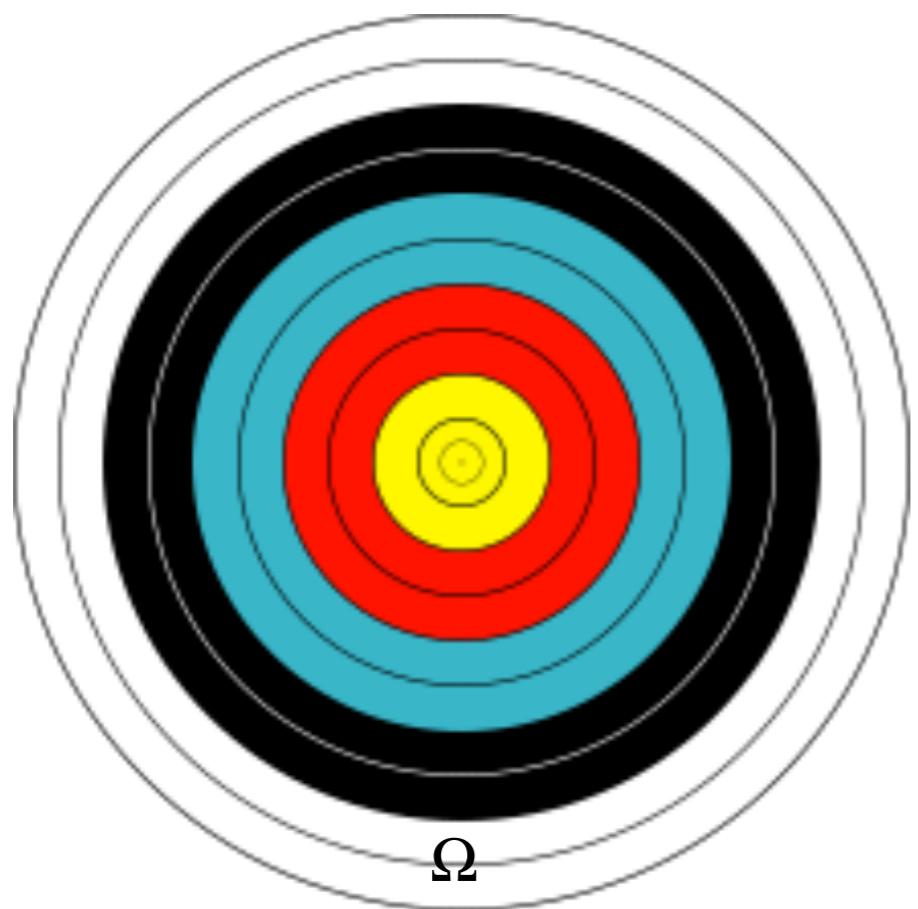
$$0 \leq P(A) \leq 1$$

$$P(\Omega) \equiv 1$$

$$P(A, B) = P(A) + P(B)$$

Probability = Frequency

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# Rolling Dice

- 6 sided die
- 6 possibilities



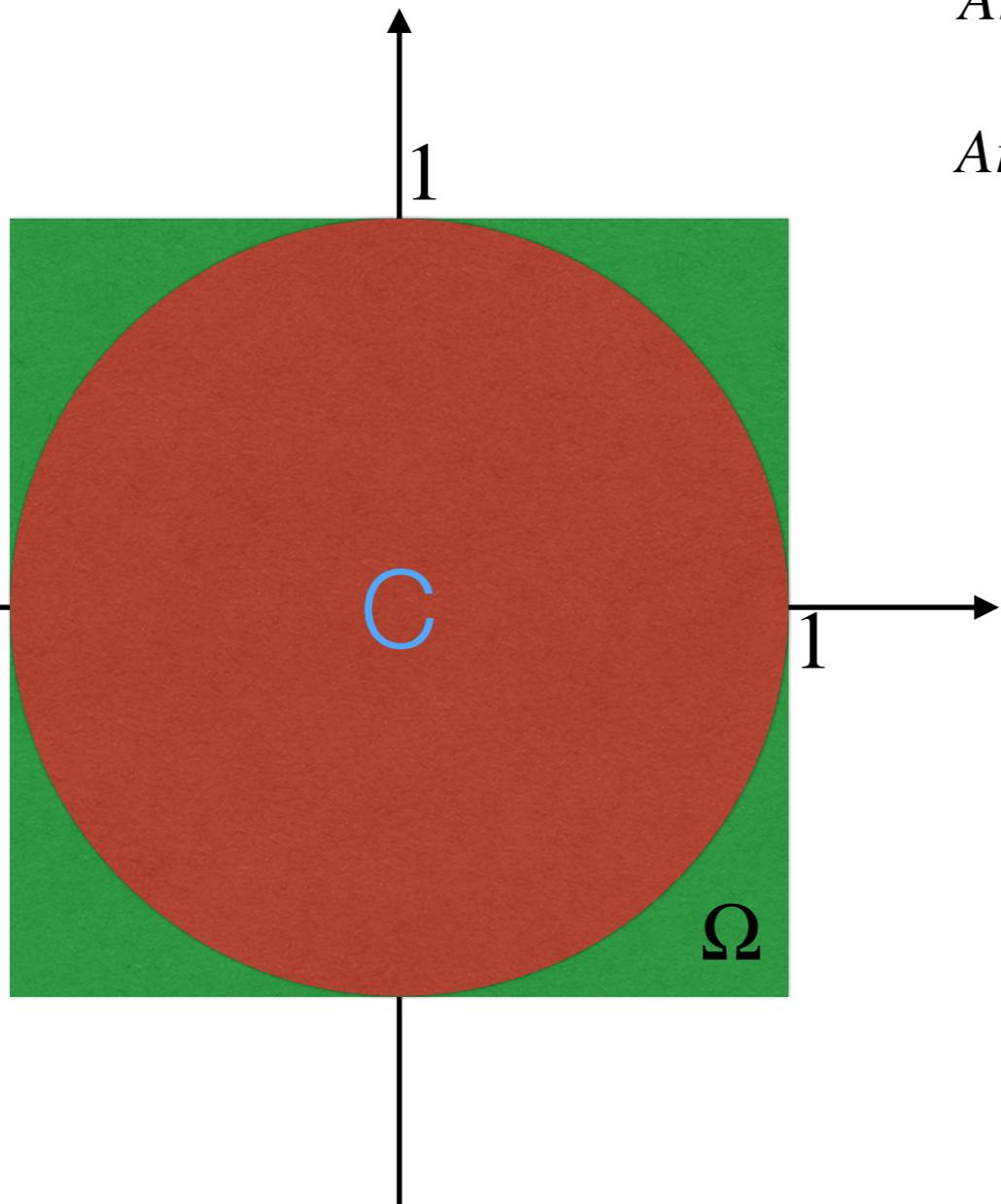
- Each side equally likely:  $P(x) = \frac{1}{6}, x \in [1,6]$

1	2	3
4	5	6
$\Omega$		



# Continuous Version

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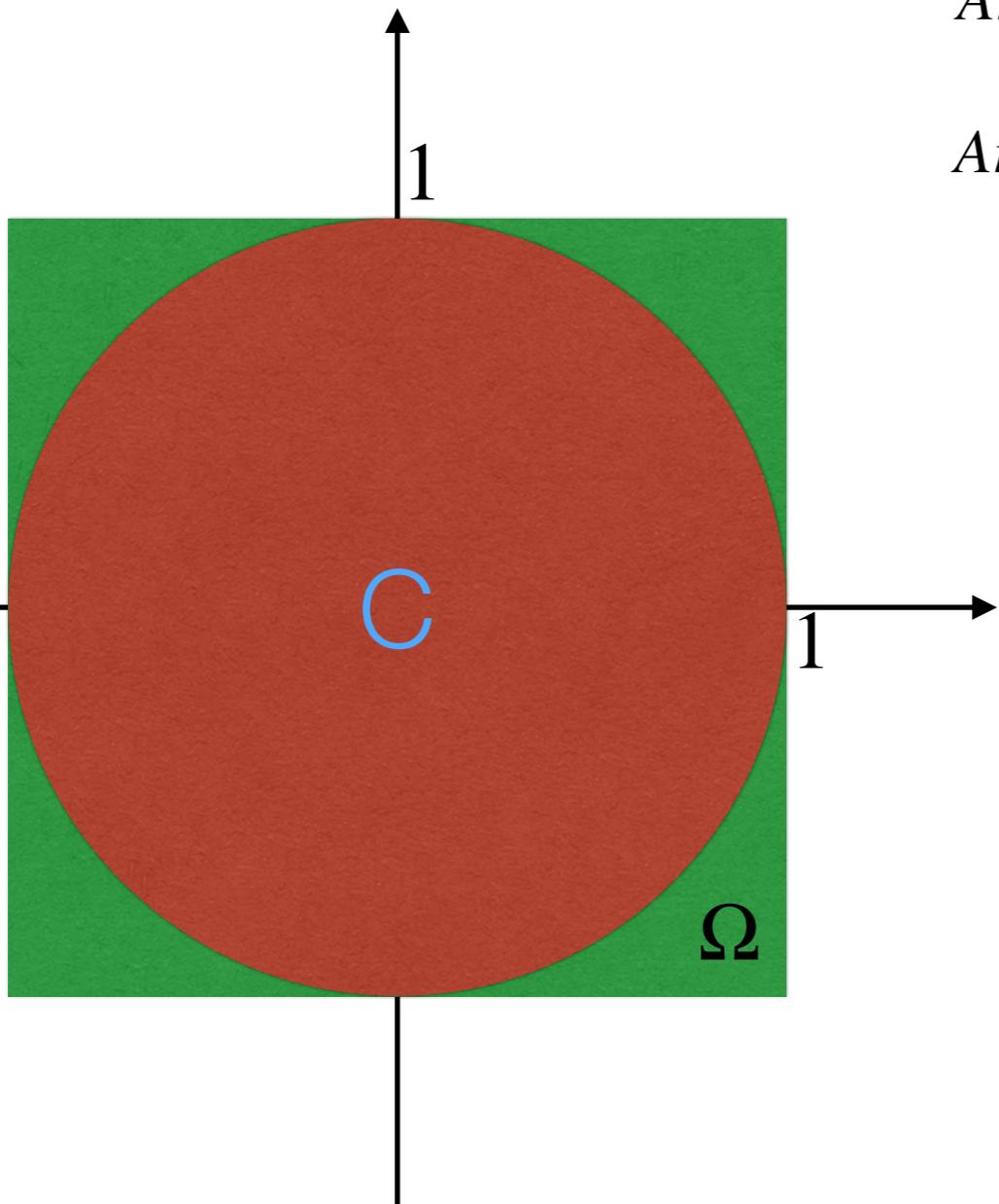


$$Area(C) = \pi$$

$$Area(\Omega) = 4$$

# Continuous Version

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$$Area(\Omega) = 4$$

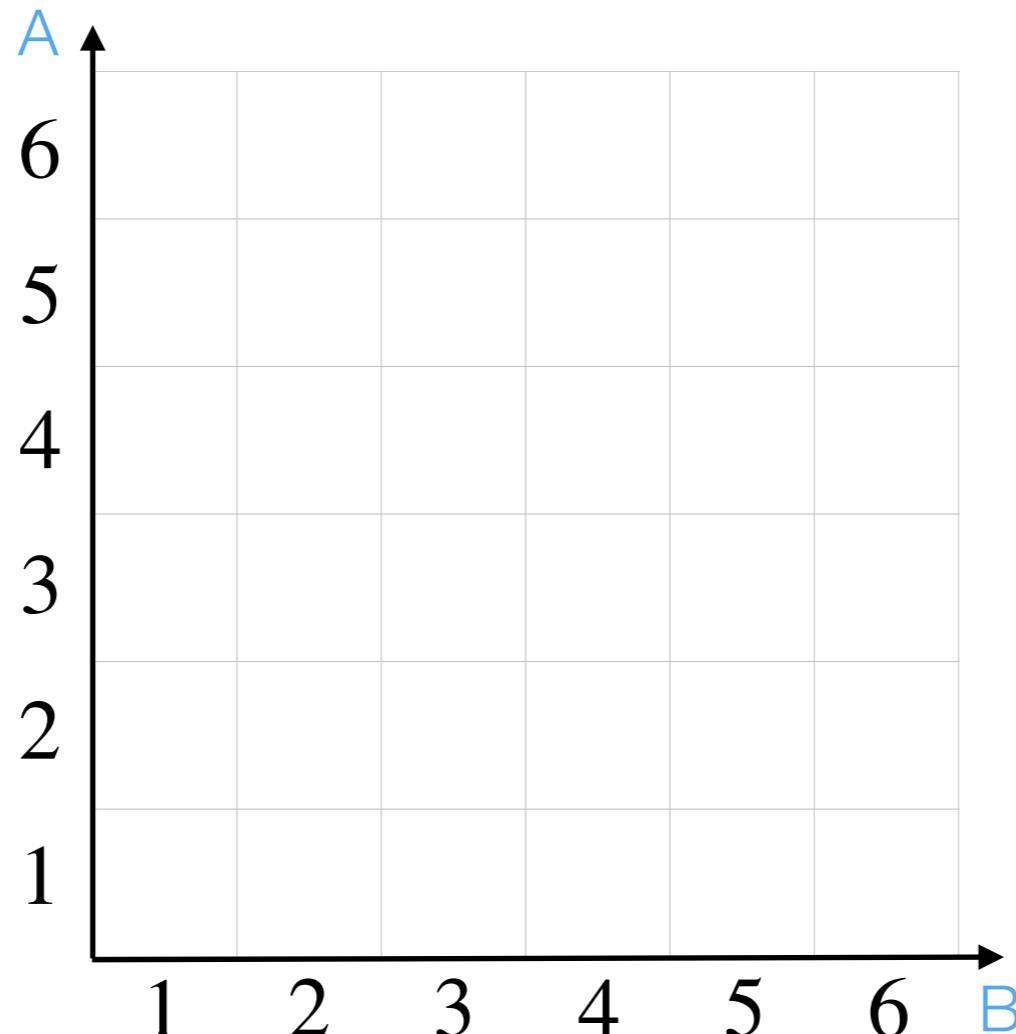
$$P(C) = \frac{Area(C)}{Area(\Omega)} = \frac{\pi}{4}$$



# Sequences of Events

---

- A happens, then B happens
  - I roll a 2 and a 6
  - I roll an odd number followed by an even number

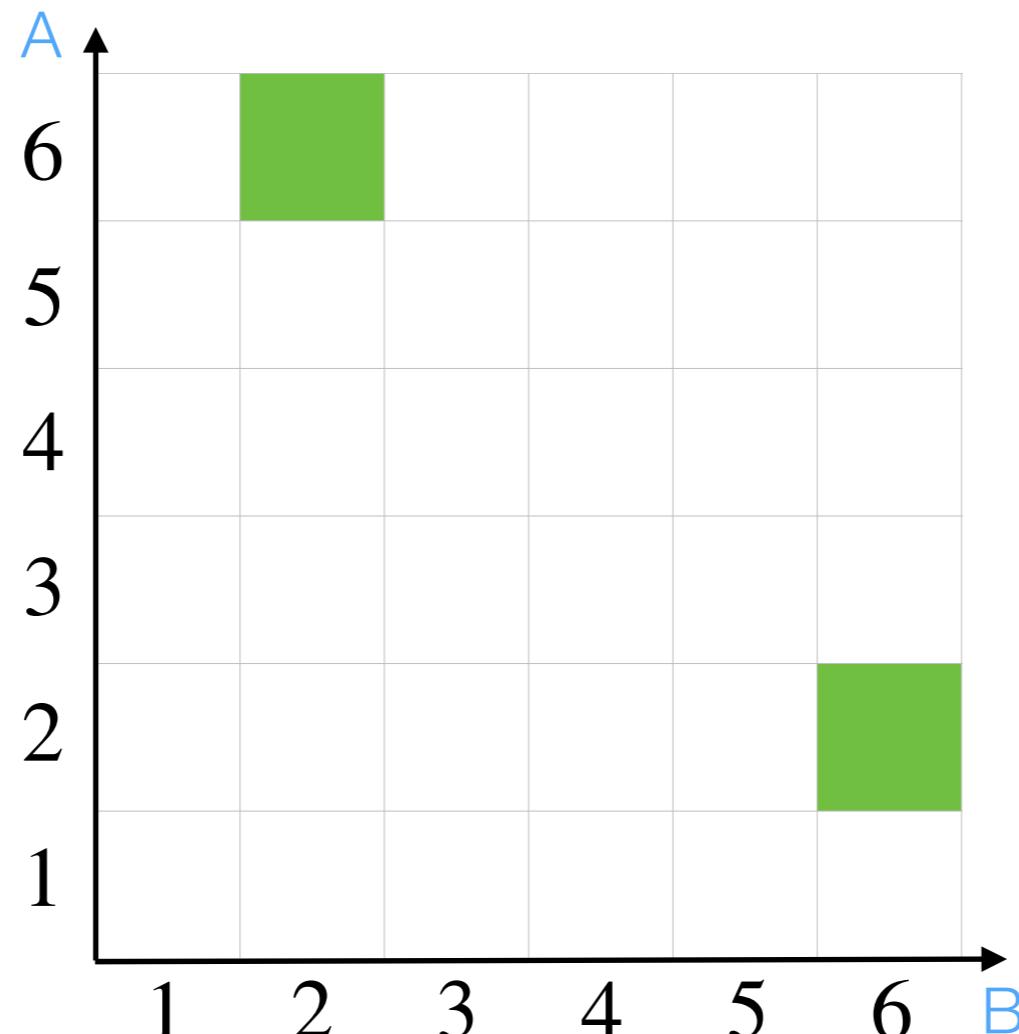


Multiply  
Probabilities

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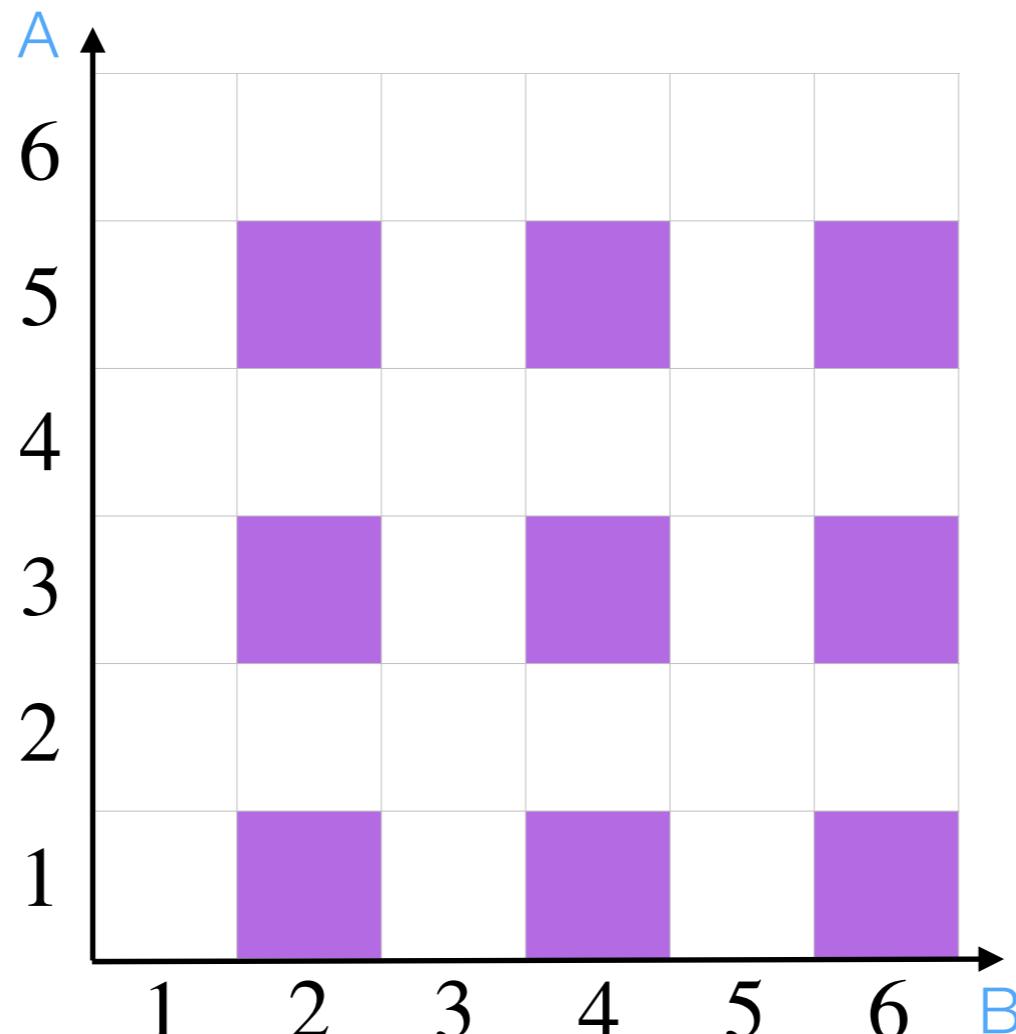
$$P(2,6) = \frac{2}{36}$$

Multiply  
Probabilities

# Sequences of Events

- A happens, then B happens

- I roll a 2 and a 6
- I roll an odd number followed by an even number



$$P(\text{odd, even}) = \frac{9}{36}$$

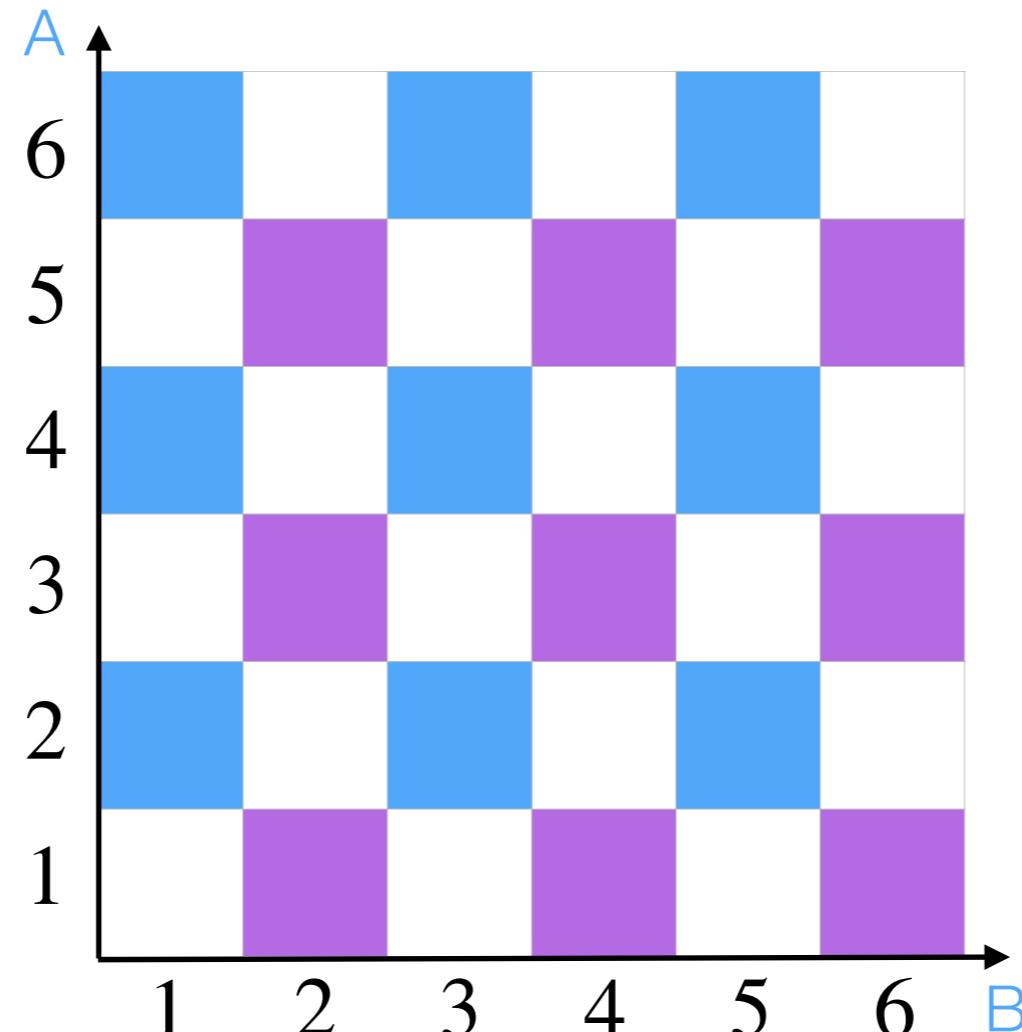
Multiply  
Probabilities

# Sequences of Events

- A happens, then B happens

- I roll a 2 and a 6

- I roll an odd number and an even number (two possibilities)



$$P(\text{even, odd}) = \frac{9}{36}$$

$$P(\text{odd, even}) = \frac{9}{36}$$

Multiply  
Probabilities

# General Procedure

---

- Enumerate all possible outcomes
- Calculate the “Area” of each outcome
- The Probability of a given outcome is the fraction of the total Area it occupies
- Histogram: Observed Frequency of each outcome  $N(X)$
- Probability Distribution: Probability associated with every possible outcome  $P(X)$
- Cumulative Probability Distribution: Probability associated with outcomes smaller or equal to each outcome  $P(X \leq x)$

# Combinatorics

---

- To enumerate all possible outcomes we often have to make use of ideas from Combinatorics:
- **Permutations:** The total number of possible sequences of  $n$  different elements:  
$$n! = n \cdot (n - 1) \cdot (n - 2) \cdots 1$$
- **Combinations:** The total number of ways of grouping  $N$  elements into two groups of size  $k$  and  $N - k$ :  
$$C_k^N = \frac{N!}{k!(N - k)!}$$
- With just these two ideas we can easily calculate the number of possible outcomes in many situations:

- Number of possible playing card shuffles:

$$52! = 80658175170943878571660636856403766975289505440883277824000000000000$$

- Number of ways of getting **3** heads and **2** tails when flipping **5** coins:

$$C_3^5 = \frac{5!}{3!2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = \frac{5 \cdot 4}{2 \cdot 1} = 10$$

# Combinatorics - Coin flips

---

- Can we calculate the Probability distribution of the outcome of flipping **5** coins that come out heads with probability  $p$ ?

- The probability of getting  $N_h$  heads and  $N - N_h$  tails is:

$$p(N, N_h) = p_h^N (1 - p)^{N - N_h}$$

- As we need to get  $N$  heads with probability  $p$  each time and  $N - N_h$  tails with probability  $1 - p$  each time

- From combinatorics we also know that the number of ways to get  $N_h$  heads out of total  $N$  flips is:

$$C_{N_h}^N = \frac{N!}{N_h!(N - N_h)!}$$

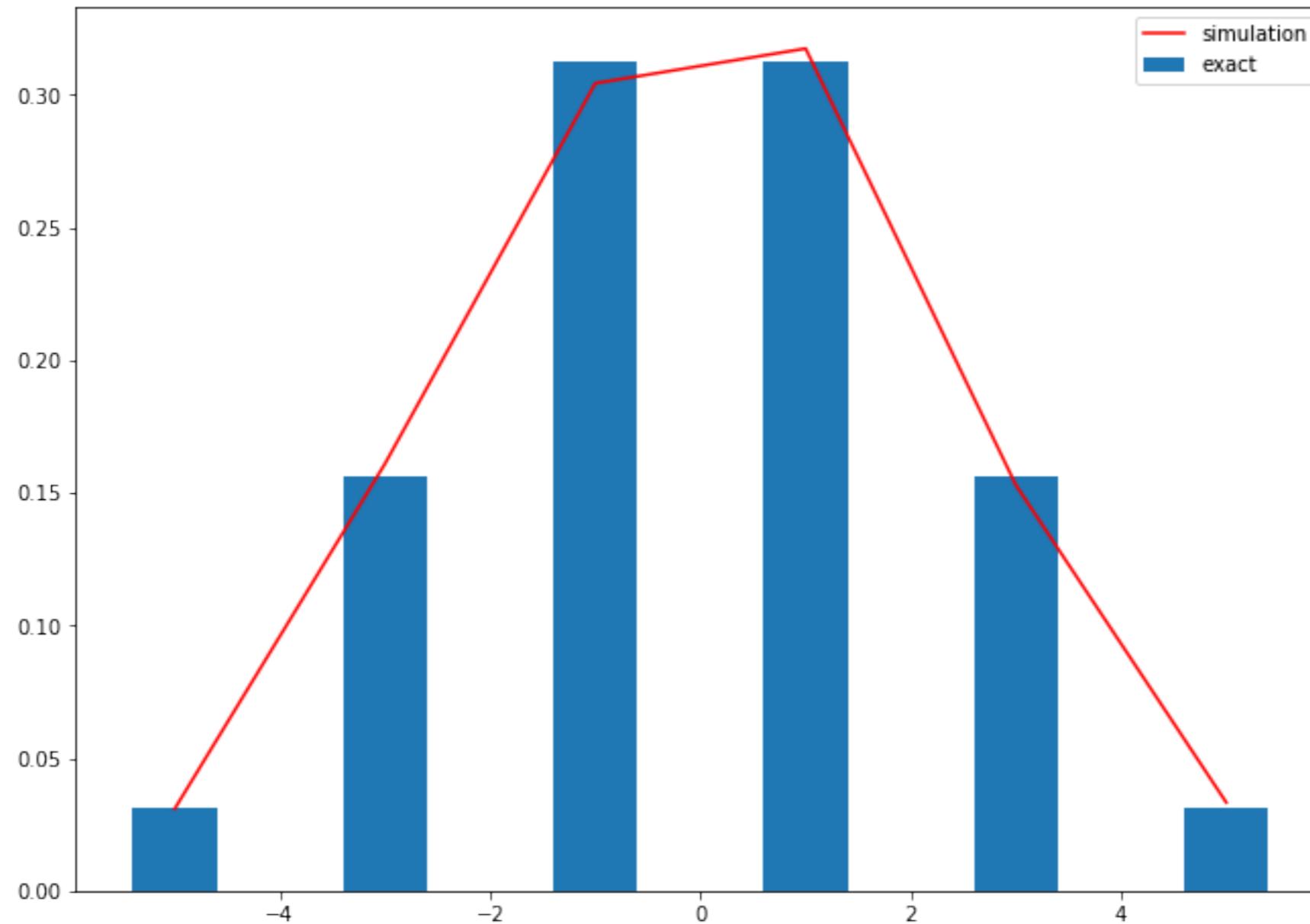
- Therefore

$$P(N, N_h) = \frac{N!}{N_h!(N - N_h)!} p_h^N (1 - p)^{N - N_h}$$

# Combinatorics - Coin Flips

---

- For 5 coins we have:



# Binomial Distribution

- The probability of getting  $k$  successes with  $n$  trials of probability  $p$  ( $k$  heads in  $n$  coin flips):

$$P_B(k, n, p) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

- The mean value is:

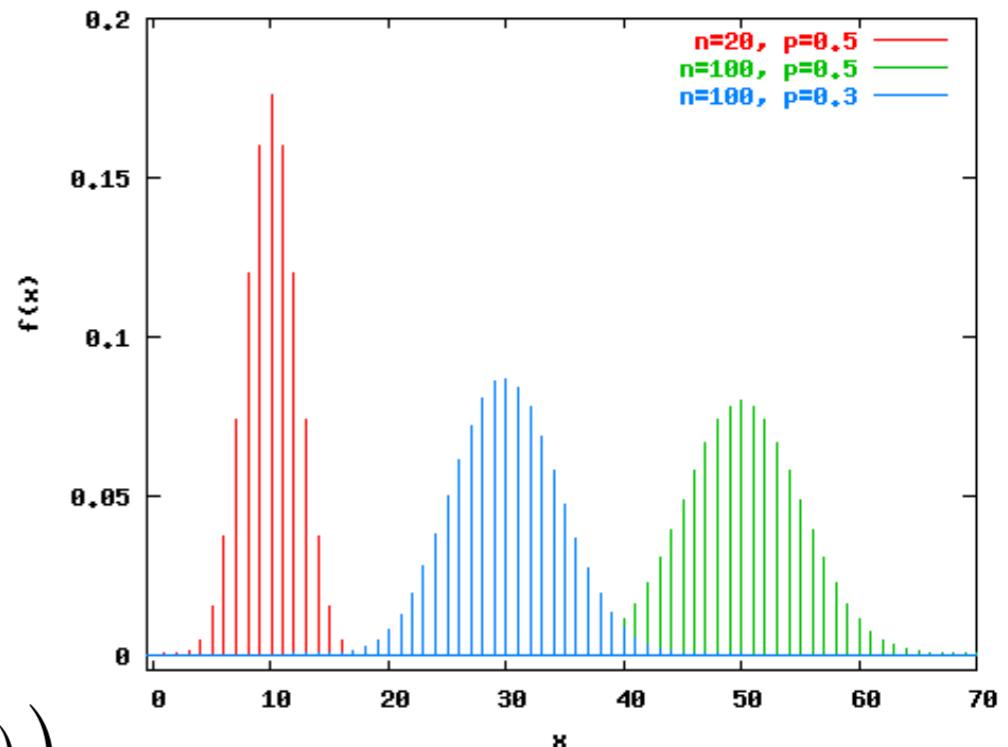
$$\mu = np$$

- and the variance:

$$\sigma^2 = np(1-p)$$

- and for sufficiently large  $n$ :

$$P_B(k, n, p) \sim P_N(np, np(1-p))$$



# Averaging and Expectations

---

- A  $n$ -sided die has a uniform probability  $\frac{1}{n}$  of landing on any of its sides.
- Let's call the value seen after roll  $i$  of the die,  $x_i$
- After 10 rolls of, say, a 6-sided die, we might have:

$$x_i = [4, 6, 4, 3, 5, 1, 1, 5, 2, 4]$$

- The behavior of this variable is **stochastic**, but what about the behavior of functions of this **random variable**?
- For example, the average:

$$\mu_N \equiv \langle \tilde{x} \rangle_N = \frac{1}{N} \sum_{i=1}^N x_i$$

- In this specific example, the average is 3.5 as expected, but if, say, rolls 6 and 7 had been 6s instead of 1s, the average value would be 4.5.

# Averaging and Expectations

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- If we use **10000** rolls of the dice, we find that the average is:

$$\langle \tilde{x} \rangle_{10000} = 3.4888$$

- But if we repeat the same “**experiment**” 10 times, we will find 10 different values:

$\langle \tilde{x} \rangle_{10000}$
3.4888
3.5111
3.4686
3.4893
3.5233
3.4941
3.4975
3.5276
3.4948
3.4775

- So what is the **correct** value?

- For any given set of dice rolls, we are only **estimating** the true value.

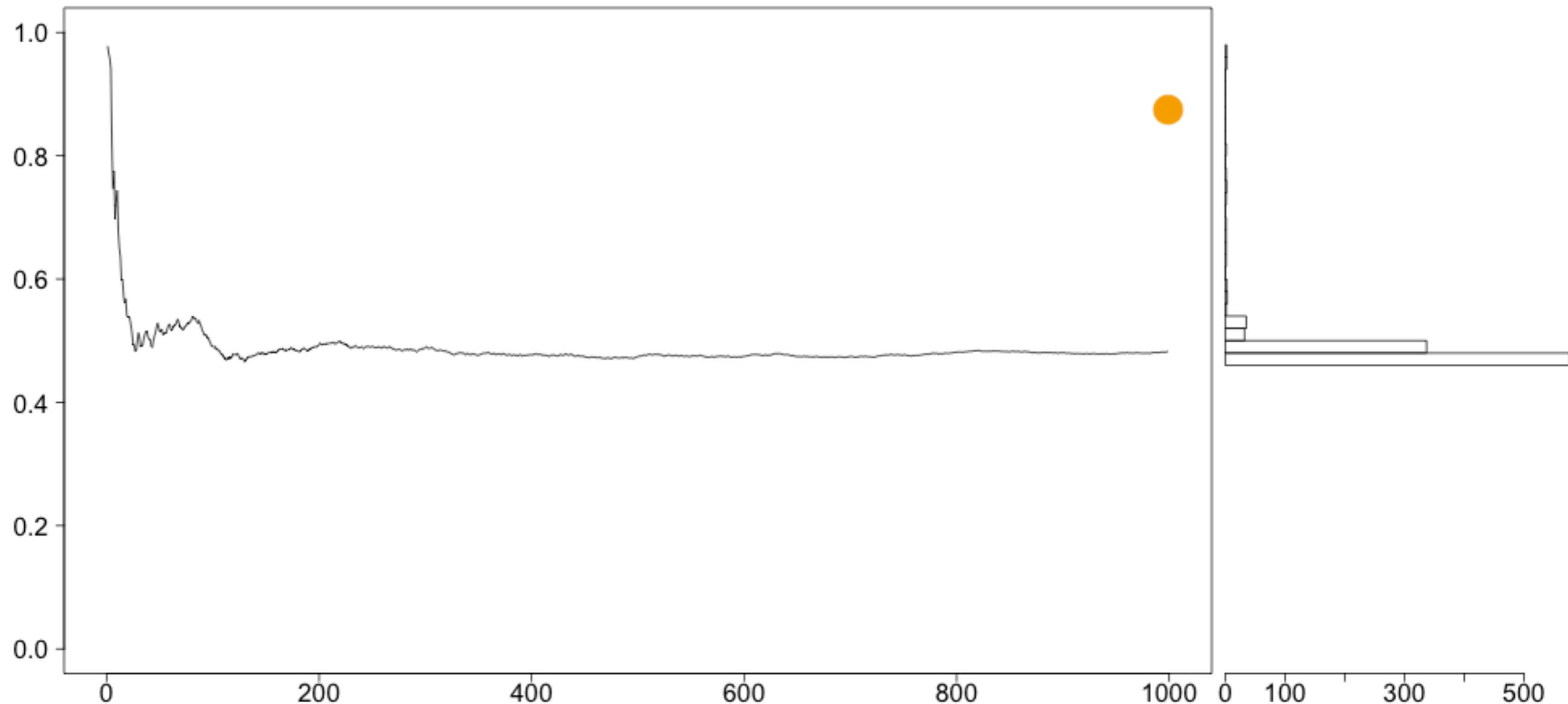
- Estimated values are usually denoted with a  $\sim$  or  $\wedge$  over the variable

- In general, the **higher the number** of rolls we consider in a given realization, the **better our estimate** of the average.

- The true, or expected value, is the one obtained after an infinite number of realizations. This number can be estimated with just a bit of algebra

# Law of Large Numbers

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# Averaging and Expectations

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- As we saw:

$$\mu_N = \frac{1}{N} \sum_{i=1}^N x_i$$

- This can be rewritten as:

$$\mu_N = \sum_{\alpha=1}^n \frac{N_\alpha}{N} x_\alpha$$

- Where  $\alpha$  denotes all **n possible** values of the variable  $x_i$  or, in other words, the values on the sides of the die.

- If we notice that  $\frac{N_\alpha}{N}$  is just our estimate of the value of the probability of observing  $\alpha$  we can write:

$$\mu = \sum_{\alpha=1}^n p_\alpha x_\alpha$$

- where  $p_\alpha$  is the **true probability** and  $\mu \equiv \langle x \rangle$  is the true average value of the average, or the **expected value** of  $x$

# Central Limit Theorem

---

- As  $N \rightarrow \infty$  the random variables:

$$\sqrt{N} (\mu_N - \mu)$$

- with:

$$\mu_N = \frac{1}{N} \sum_i x_i$$

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- after some manipulations, we find:

$$\mu_N \sim \mu + \frac{\mathcal{N}(0, \sigma^2)}{\sqrt{N}}$$

The estimation of the mean converges to the true mean with the square root of the number of samples

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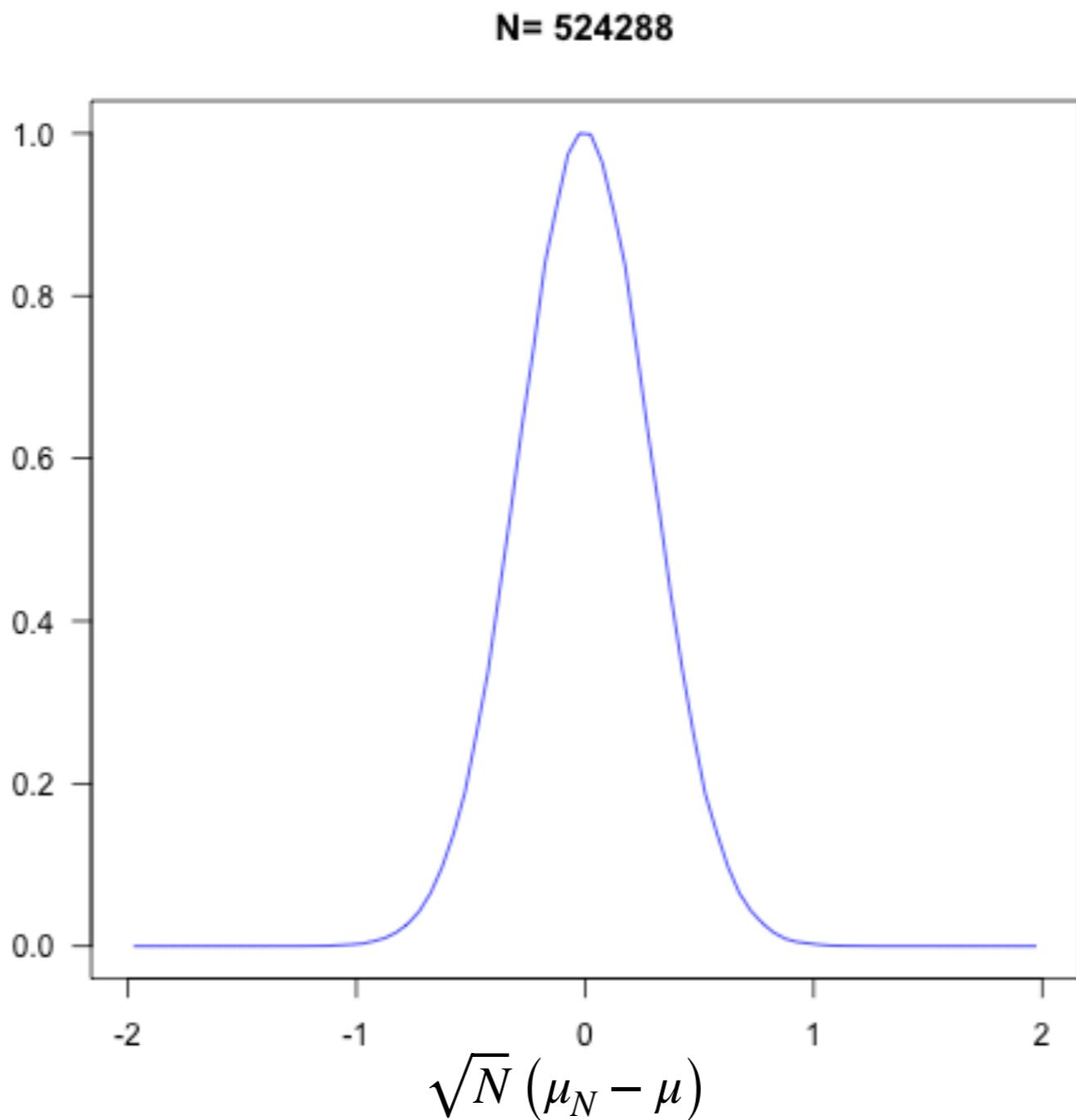
- after some manipulations, we find:

$$\mu_N \sim \mu + \frac{\mathcal{N}(0, \sigma^2)}{\sqrt{N}} \rightarrow SE = \frac{\sigma}{\sqrt{N}} \quad \sigma^2 = \frac{1}{N-1} \sum_{i=1}^N (x_i - \mu)^2$$

The estimation of the mean converges to the true mean with the square root of the number of samples

# Central Limit Theorem

---



# Gaussian Distribution

- The probability of observing value  $x$  from a normal distribution centered at  $\mu$  and with variance  $\sigma^2$ :

$$P_N(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

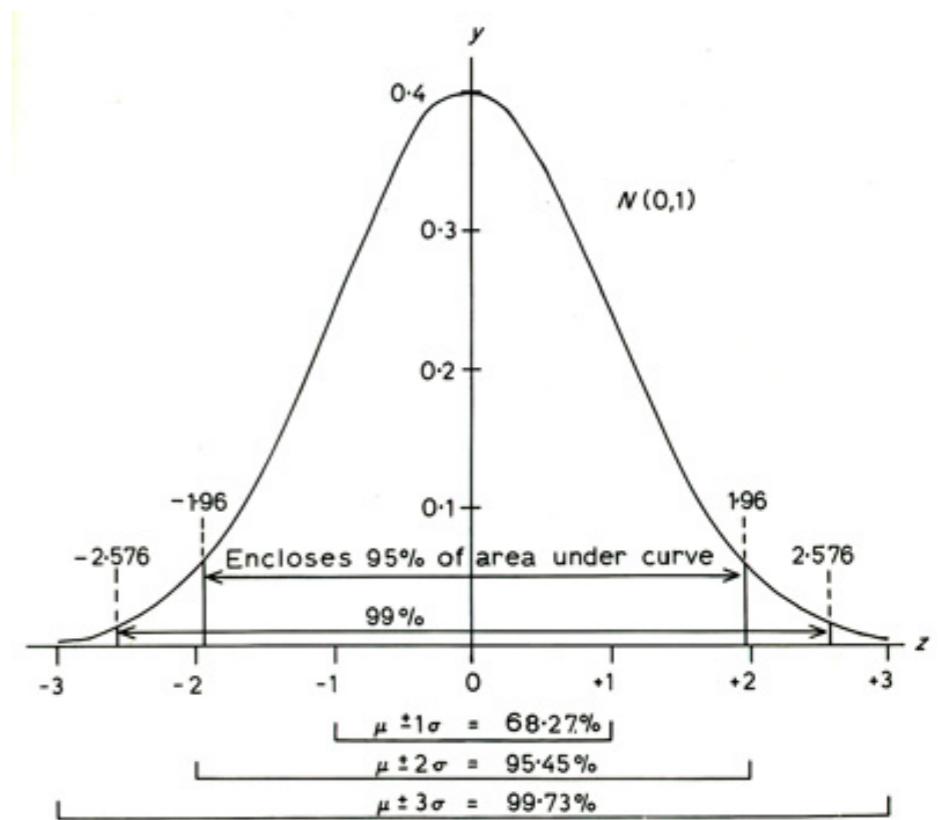
- The mean value is:

$$\mu = \frac{1}{N} \sum_i x_i$$

- and the variance:

$$\sigma^2 = \frac{1}{N-1} \sum_{i=1}^N (x_i - \mu)^2$$

- and for sufficiently large n:



# Experimental Measurements

---

- Experimental errors commonly assumed gaussian distributed
- Many experimental measurements are actually averages:
  - Instruments have a finite response time and the quantity of interest varies quickly over time
- Stochastic Environmental factors
- Etc

# MLE - Fitting a theoretical function to experimental data

---

- In an experimental measurement, we **expect** (CLT) the experimental values to be normally distributed around the theoretical value with a certain variance. Mathematically, this means:

$$P(y - f(x)) \approx \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{(y - f(x))^2}{2\sigma^2} \right]$$

- where  $\mathbf{y}$  are the experimental values and  $f(\mathbf{x})$  the theoretical ones. The likelihood is then:

$$\mathcal{L} = -\frac{N}{2} \log [2\pi\sigma^2] - \sum_i \left[ \frac{(y - f(x_i))^2}{2\sigma^2} \right]$$

- Where we see that to **maximize** the likelihood we must **minimize** the sum of squares

Least Squares Fitting

# MLE - Linear Regression

---

- Let's say we want to fit a straight line to a set of points:

$$y = w \cdot x + b$$

- The Likelihood function then becomes:

$$\mathcal{L} = -\frac{N}{2} \log [2\pi\sigma^2] - \sum_i \left[ \frac{(y - w \cdot x_i - b)^2}{2\sigma^2} \right]$$

- With partial derivatives:

$$\frac{\partial \mathcal{L}}{\partial w} = \sum_i [2x_i(y_i - w \cdot x_i - b)]$$

$$\frac{\partial \mathcal{L}}{\partial b} = \sum_i [(y_i - w \cdot x_i - b)]$$

- Setting to zero and solving for  $\hat{w}$  and  $\hat{b}$ :

$$\hat{w} = \frac{\sum_i (x_i - \langle x \rangle)(y_i - \langle y \rangle)}{\sum_i (x_i - \langle x \rangle)^2}$$

$$\hat{b} = \langle y \rangle - \hat{w}\langle x \rangle$$

# MLE - Coin Flips

---

- Biased coin with unknown probability of heads ( $p$ )
- In a sequence of  $N$  flips, the likelihood of  $N_h$  heads and  $N_t = N - N_h$  tails is proportional to:

- or simply:

$$\mathcal{L} = \log \left[ \frac{N!}{N_h! N_t!} \right] + \log \left[ p^{N_h} (1-p)^{N-N_h} \right]$$

$$\mathcal{L} \propto N_h \log [p] + (N - N_h) \log [1 - p] \quad \text{Ignoring the combinatorial factor!}$$

- Taking the derivative:

$$\frac{\partial \mathcal{L}}{\partial p} = \frac{N_h}{p} - \frac{N - N_h}{1 - p}$$

- Setting to zero and solving for  $p$ :

$$p = \frac{N_h}{N}$$

- which is how we estimated the probability above

# Probability distributions

---

- We already saw three **probability distributions**:
  - **Uniform** - Dice
  - **Binomial** - Dice
  - **Gaussian** - Errors
- Common programming languages (**Python**, **R**, **C++**, **Matlab**, etc) have a wide variety of random number generators either built in or as add on packages.
- But how can we generate numbers following a specific (possibly empirical) distribution?
- First we must define the cumulative distribution:

$$P(X \leq x)$$

- representing the probability of observing a value smaller than some threshold.

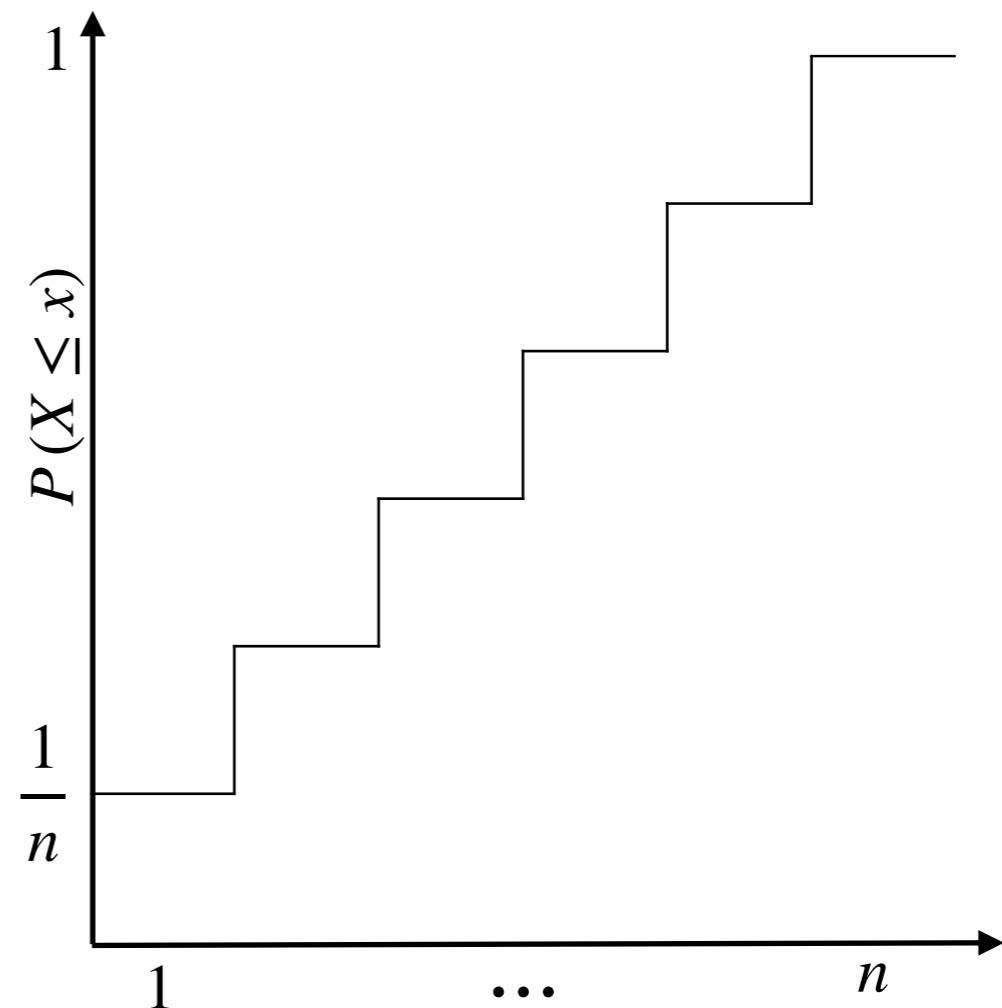
# Uniform Distribution

Probability  
Distribution

In a standard uniform distribution (a fair die) all outcomes are equally likely



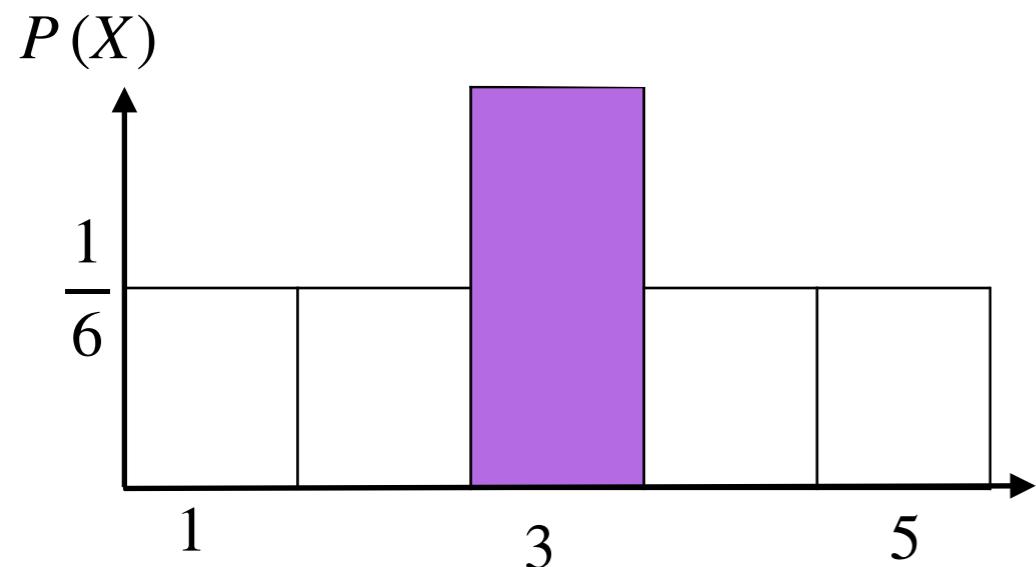
Cumulative  
Distribution



# Non-Uniform Distribution

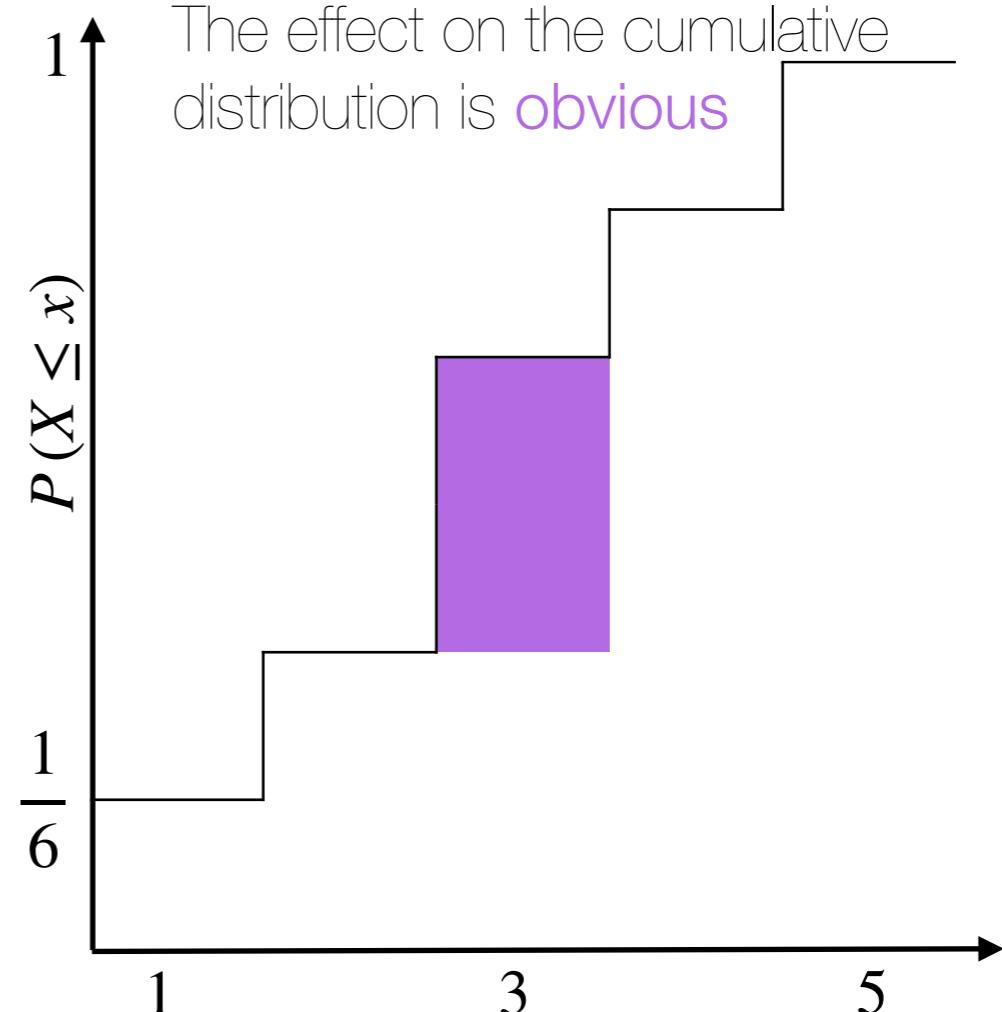
## Probability Distribution

Now we have a funny die  
with two sides labeled as 3



## Cumulative Distribution

The effect on the cumulative distribution is **obvious**



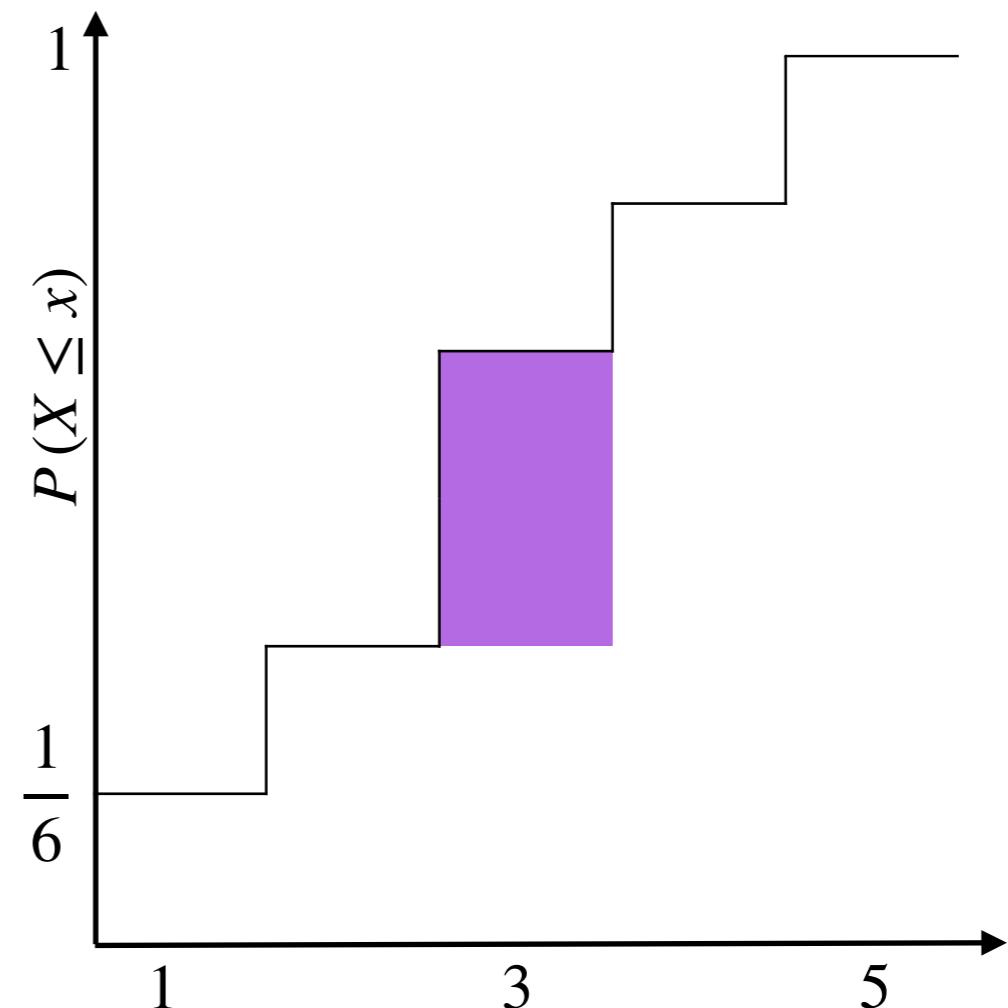
# Non-Uniform Distribution

- The step larger step we see in cumulative distribution givens some clues as to how we might generate random numbers following this distribution
- What if our blindfolded monkey is throwing darts along the  $y$  axis, where will they hit on the  $x$  axis?



- Naturally, the bin with the largest step will receive the largest number of darts!

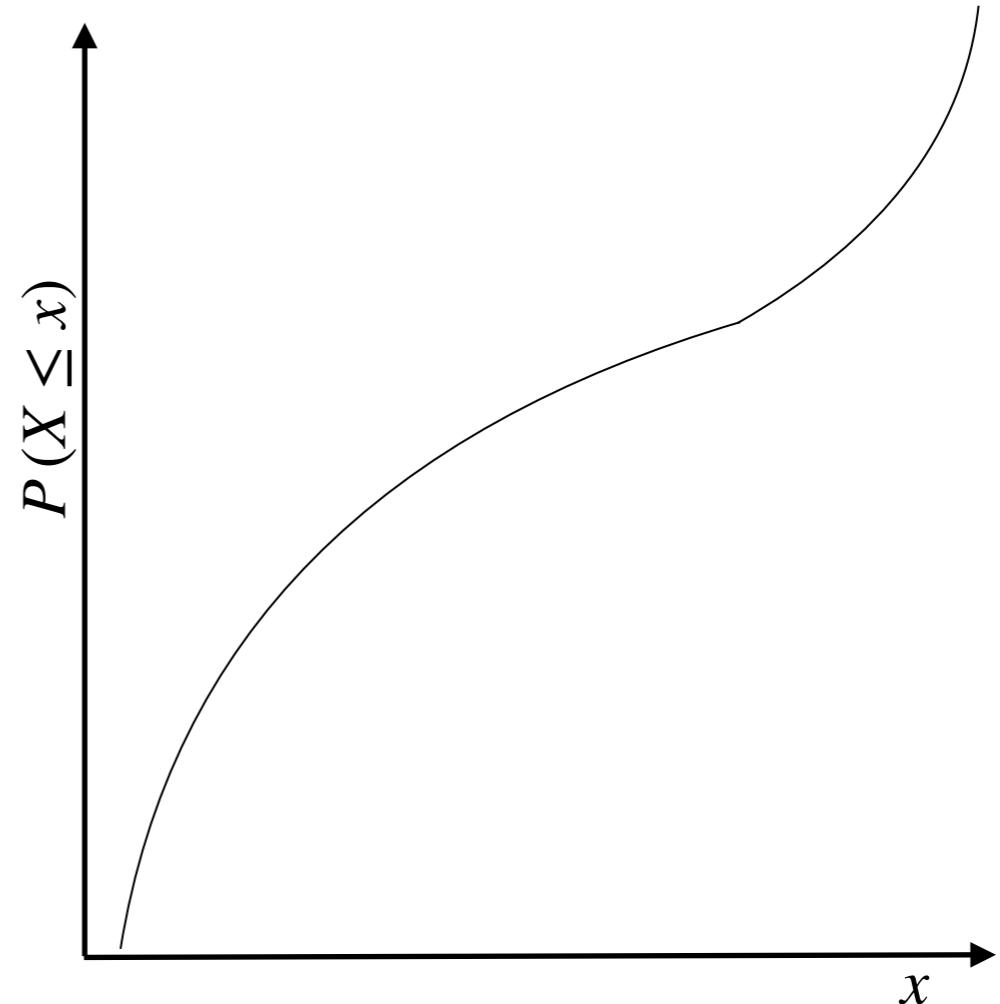
Cumulative  
Distribution



# Generating arbitrary distributions

- Mathematically, this procedure is known as function inversion
- We "invert" the function  $y = f(x)$  to find what would be the value of  $x$  that would produce a specific value of  $y$
- If the analytical expression of the cumulative distribution is known we can invert it analytically, otherwise, we can do it numerically

Cumulative  
Distribution



# Population Sizes

[https://en.wikipedia.org/wiki/Lincoln\\_index](https://en.wikipedia.org/wiki/Lincoln_index)

- So far we've known everything about the range of values we might encounter, the number of sides on the die, etc.
- But what if our goal is precisely to estimate the number of possible elements?
- “**mark and recapture**” is one strategy used in ecology to estimate population sizes:

- visit a site and mark a number  $K$  of individuals (say, rabbits) that are released

- After the first visit a fraction  $\frac{K}{N}$  of the total population has been marked

- On a second visit capture a number  $n$  of individuals if  $k$  of those captured are marked we can assume

$$\frac{k}{n} \approx \frac{K}{N}$$

- And obtain:

$$\hat{N} = \frac{Kn}{k}$$

- This is known as the Lincoln Index estimator and it assumes that the probability of any animal being chosen is  $\frac{1}{N}$  in both visits



## Lesson II: Bayesian Statistics

# Kolmogorov's Probability Axioms

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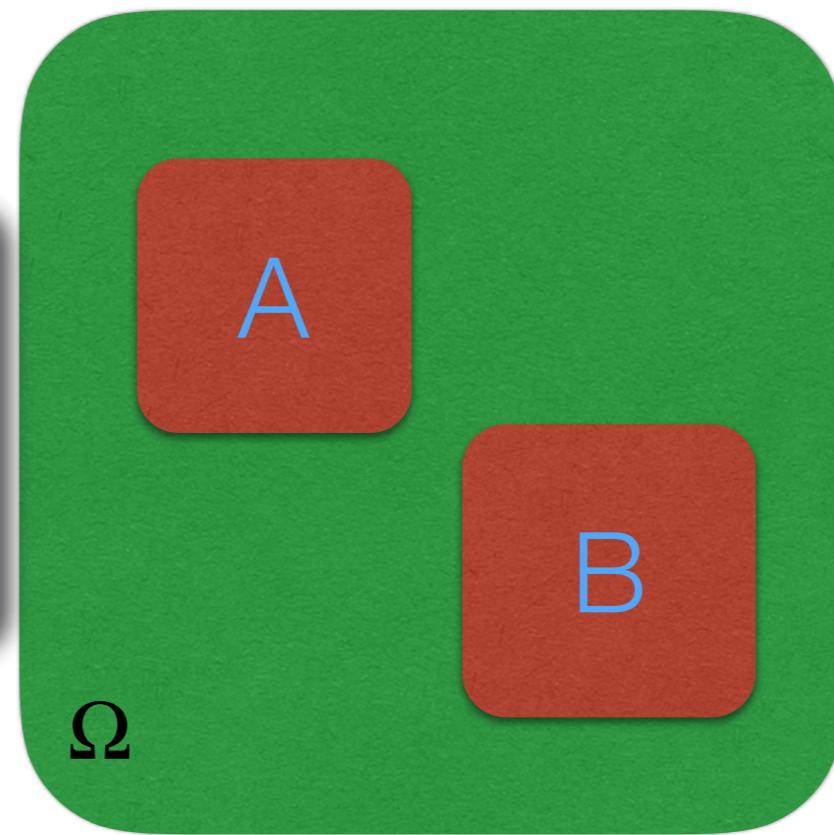
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Probability = Area

Prob(A) = Area A

Total Area = 1

Prob(A or B) = Area A + Area B



$$0 \leq P(A) \leq 1$$

$$P(\Omega) \equiv 1$$

$$P(A \text{ or } B) = P(A) + P(B)$$

# Kolmogorov's Probability Axioms

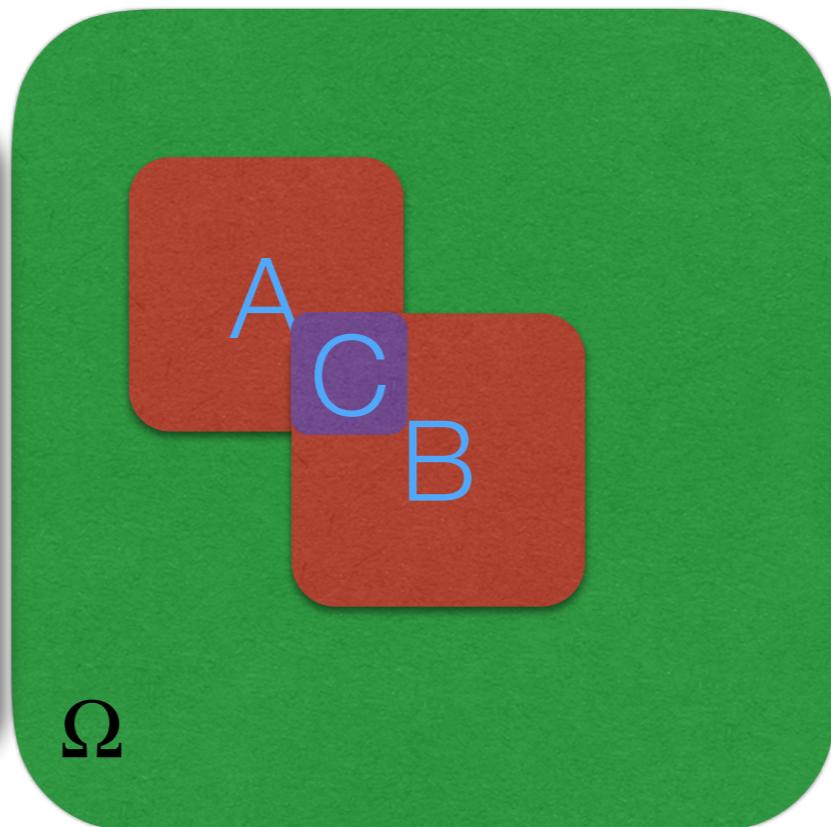
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Probability = Area

Prob(A) = Area A

Total Area = 1

Prob(A or B) = Area A + Area B  
- Area C



$0 \leq P(A) \leq 1$

$P(\Omega) \equiv 1$

$P(A \text{ or } B) = P(A) + P(B) - P(C)$

$P(C) = P(A \text{ and } B) = \text{overlap of A and B}$

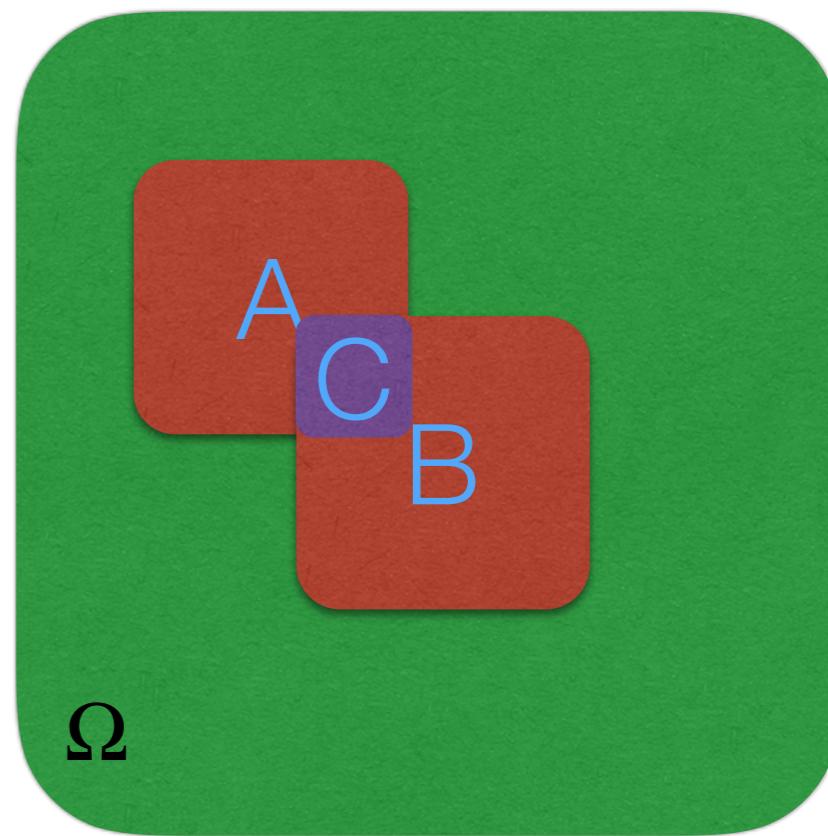
What's the probability that  
I'm in B given that I'm in A?

# Conditional Probability

- What is the **Probability of A given B?**
- What is the probability that I'm in **A** if I **know** that I'm in **B**?
- What is the fraction of **B** that is overlapped by **A**?
- **Normalize** the area of the **overlap (C)** by the area you're **conditioning** on (**B**).
- The **conditional probability** of **A given B** is defined as:

$$P(A|B) = \frac{P(C)}{P(B)}$$

- We just have to figure out how to calculate **P(C)**



# Conditional Probability

- Let's play a game by flipping two 2 coins.
- I **Win** if I get at least one Head and **Lose** otherwise
- There's 4 possibilities.
- What's the probability of a **Win**?

$$P(W) = \frac{3}{4}$$

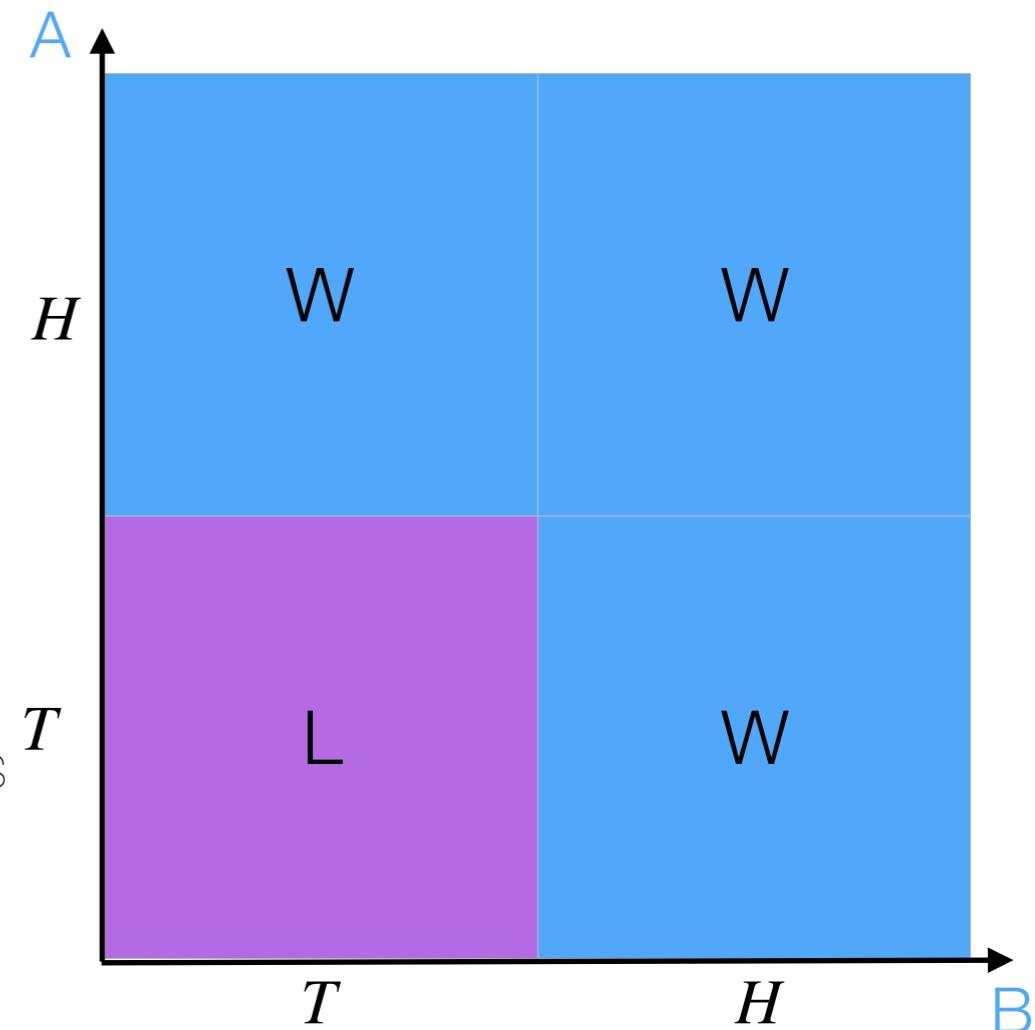
- What's the probability of a **Win** given that the second roll is **Heads**:

$$P(W|H) = \frac{2}{2} = 1$$

- What is the probability that the second rol is **Heads** given a win:

$$P(H|W) = \frac{2}{3}$$

- Symmetric conditional Probabilities can be different. The process of conditioning reflects extra information that we have available



# Bayes Theorem

- We already saw that:

$$P(A|B) = \frac{P(C)}{P(B)}$$

- Conversely:

$$P(B|A) = \frac{P(C)}{P(A)}$$

- From which we can write:

$$P(C) = P(A|B) P(B)$$

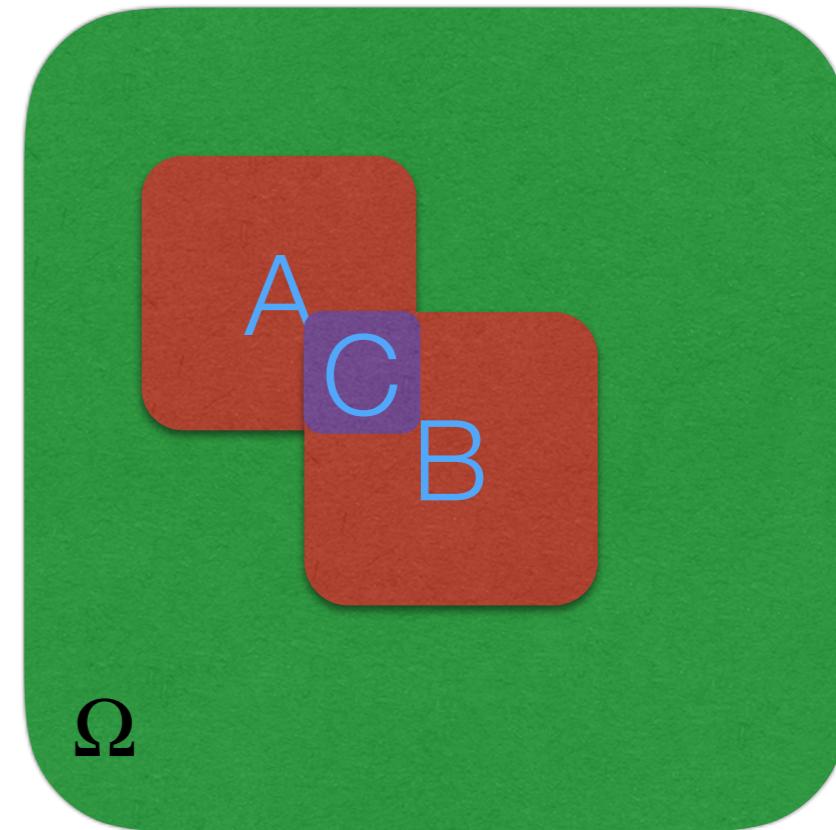
$$P(C) = P(B|A) P(A)$$

- Or, in other words:

$$P(A|B) P(B) = P(B|A) P(A)$$

- And finally:

$$P(A|B) = \frac{P(B|A) P(A)}{P(B)}$$



Bayes Theorem

# Bayes Theorem

- Now we can understand the previous example a bit better: A

$$P(H|W) = \frac{P(W|H)P(H)}{P(W)}$$

- We already know that:

$$P(W|H) = 1$$

- since a single heads is sufficient to give us a win.

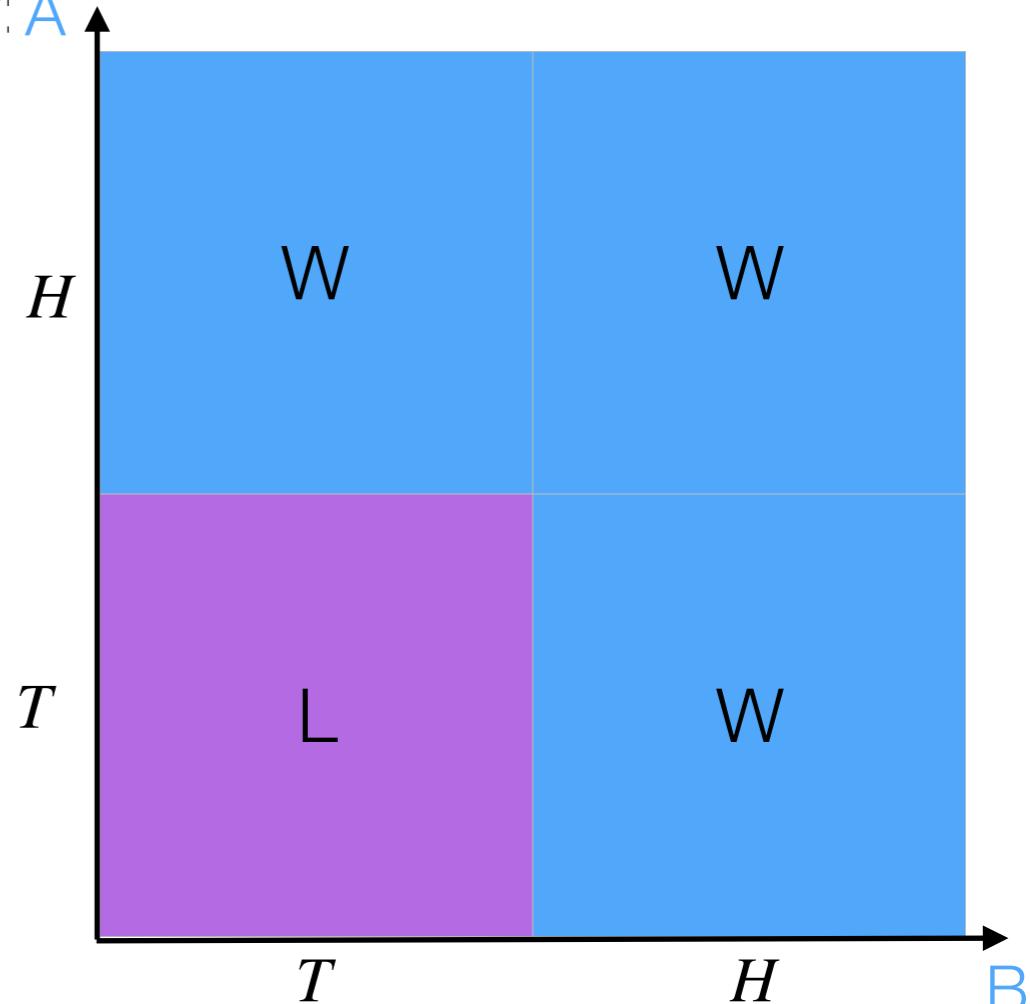
- We also know that

$$P(W) = \frac{3}{4} \quad P(H) = \frac{1}{2}$$

- Therefore:

$$P(H|W) = \frac{P(W|H)P(H)}{P(W)} = \frac{1 \cdot \frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$$

- As we had already seen



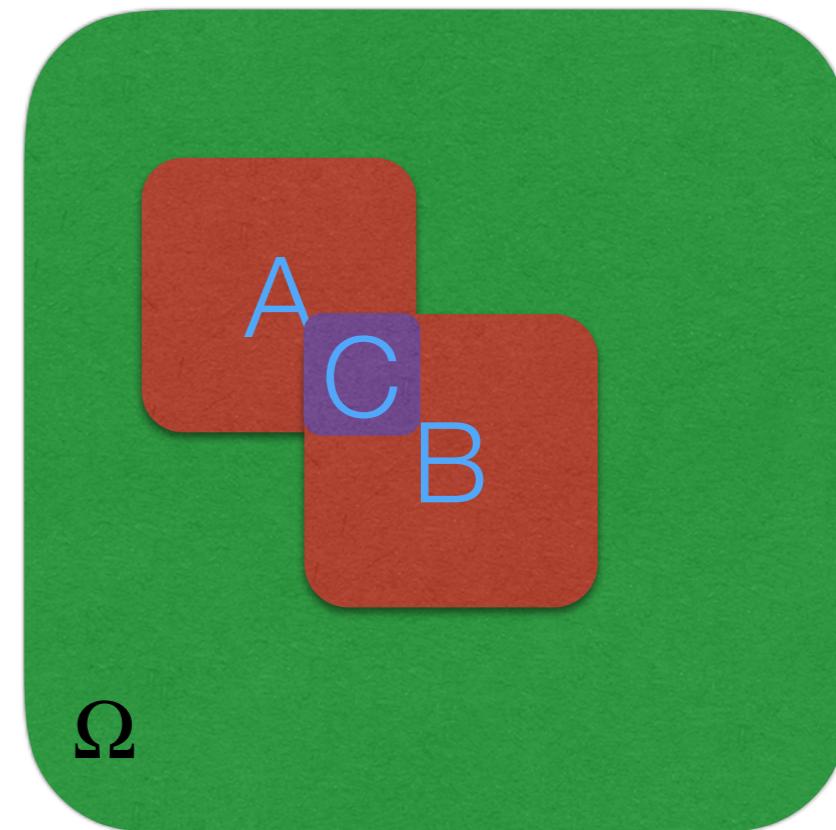
# Bayes Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- A simple way of remembering this formula is to remember that:

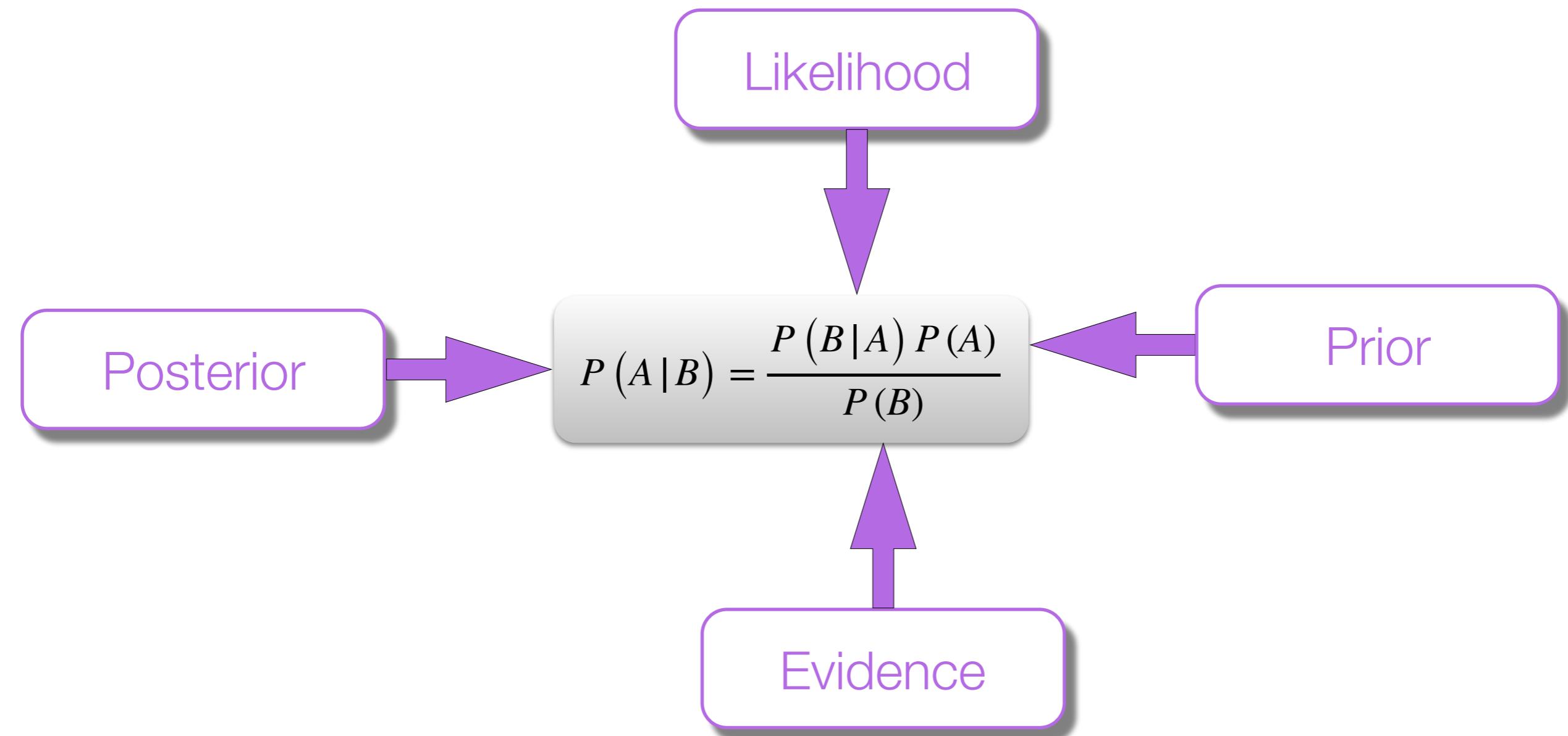
$$P(C) \equiv P(C)$$

- And simply work it out from the two ways of defining  $P(C)$
- Despite its simplicity, **Bayes' Theorem** is extremely powerful and resulted in the flourishing of a whole new branch of statistics, **Bayesian Statistics**
- The process of conditioning reflects the inclusion of added information. You can Bayes' Theorem as a way of **updating your belief** about a given situation in the presence of **new information**

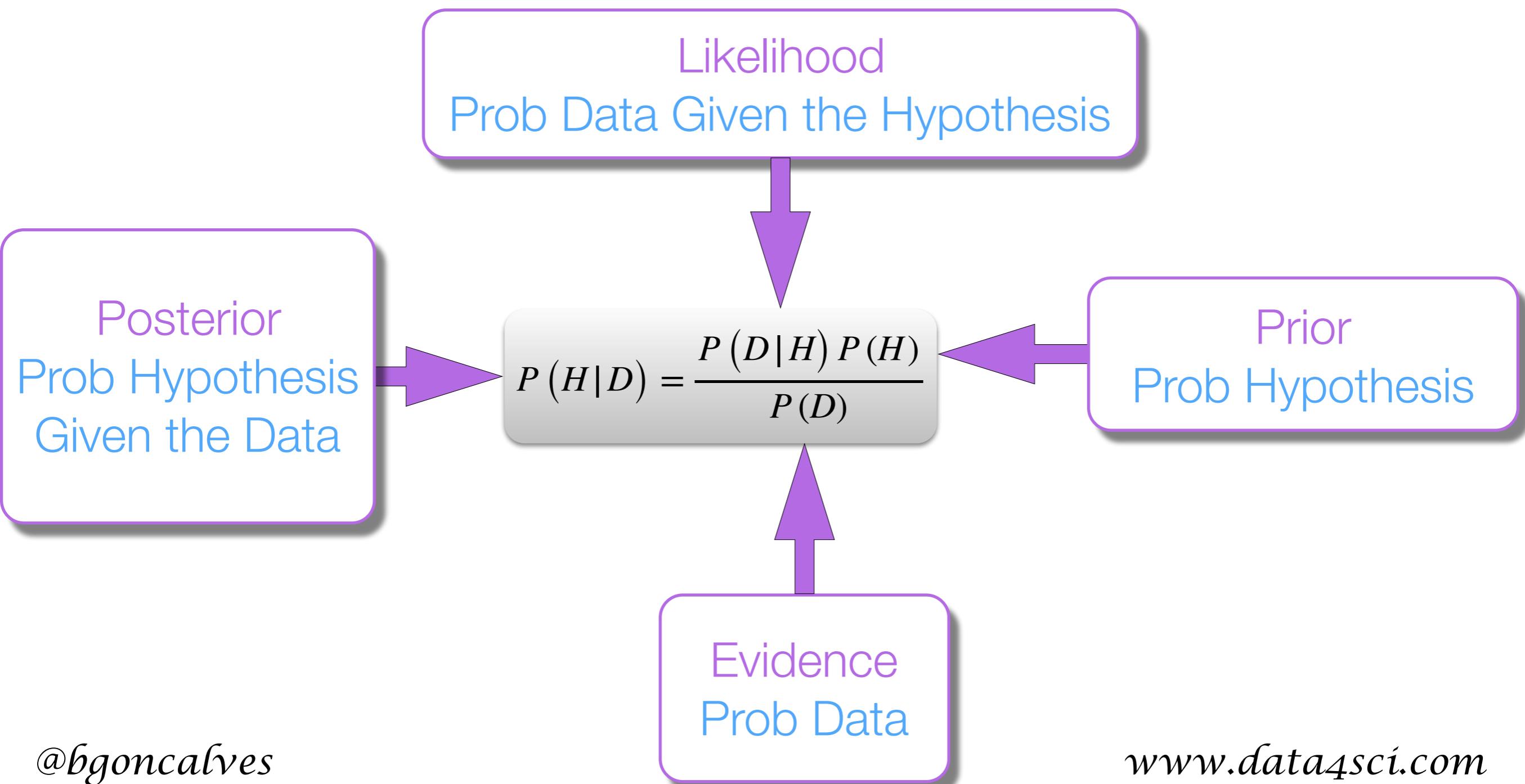


# Bayes Theorem - Terminology

---



# Bayes Theorem - Terminology



# Medical Tests

---

Your doctor thinks you might have a rare disease that affects **1** person in **10,000**. A test that is **99%** accurate comes out **positive**. What's the probability of you having the disease?

Bayes Theorem:

$$P(\text{disease}|\text{positive test}) = \frac{P(\text{positive test}|\text{disease}) P(\text{disease})}{P(\text{positive test})}$$

Total Probability:

$$\begin{aligned} P(\text{positive test}) &= P(\text{positive test}|\text{disease}) P(\text{disease}) \\ &\quad + P(\text{positive test}|\text{no disease}) P(\text{no disease}) \end{aligned}$$

Finally:

$$P(\text{disease}|\text{positive test}) = 0.0098$$

# Medical Tests

---

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Finally:

$$P(\text{disease}|\text{positive test}) = 0.0098$$

Base Rate Fallacy

Low Base Rate Value  
+  
Non-zero False Positive Rate

# Medical Tests

---

Consider a population of 1,000,000 individuals. The numbers we should expect are:

	disease	no disease
positive	99	9,999
negative	1	989,901

$$P(\text{disease}|\text{positive test}) = \frac{TP}{TP + FP} = 0.0098$$

$$P(\text{no disease}|\text{negative test}) = \frac{TN}{TN + FN} = 0.99999$$

# Medical Tests

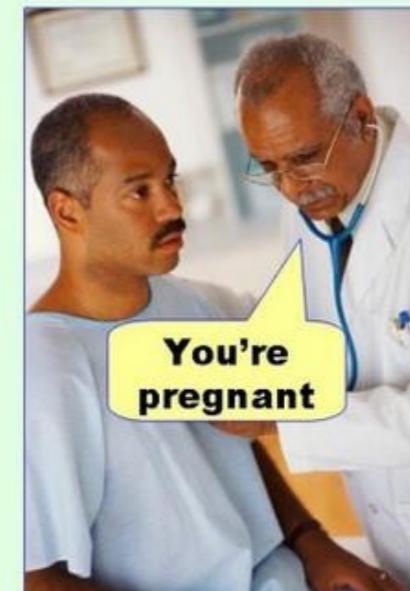
Consider a population of 1,000,000 individuals. The numbers we should expect are:

		disease	no disease	Marginals
		positive	negative	
positive	disease	99	9,999	10,098
	no disease	1	989,901	989,902
Marginals		100	999,900	

$$P(\text{disease}|\text{positive test}) = \frac{TP}{TP + FP} = 0.0098$$

$$P(\text{no disease}|\text{negative test}) = \frac{TN}{TN + FN} = 0.99999$$

@bgoncalves



Type I error  
(false positive)



Type II error  
(false negative)

# A second Test

---

Bayes Theorem still looks the same:

$$P(\text{disease} | \text{positive test}) = \frac{P(\text{positive test} | \text{disease}) P(\text{disease})}{P(\text{positive test})}$$

but now the probability that we have the disease has been **updated**:

$$P^\dagger(\text{disease}) = 0.0098$$

So this time we find:

$$P(\text{disease} | \text{positive test}) = 0.4949$$

Each test is providing new **evidence**, and Bayes theorem is simply telling us how to use it to **update our beliefs**.

# Bayesian Coin Flips

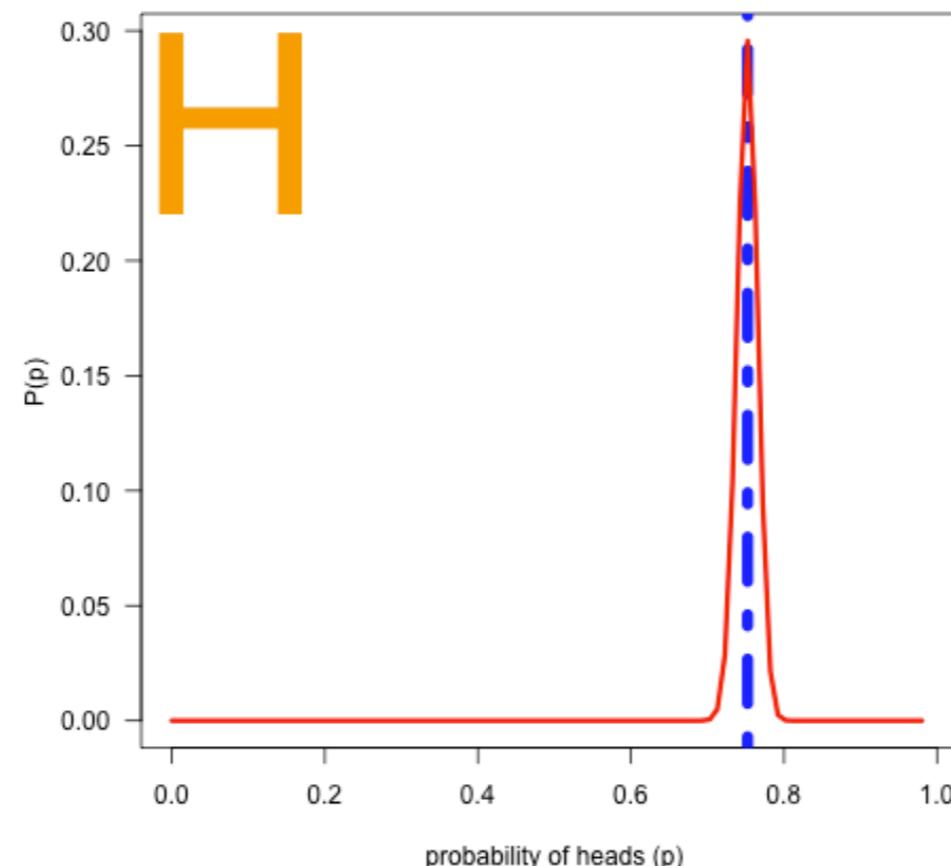
<http://youtu.be/GTx0D8VY0CY>

- Biased coin with unknown probability of heads ( $p$ )
- Perform  $N$  flips and update our belief after each flip using Bayes Theorem

$$P(p \mid \text{heads}) = \frac{P(\text{heads} \mid p) P(p)}{P(\text{heads})}$$

$$P(p \mid \text{tails}) = \frac{P(\text{tails} \mid p) P(p)}{P(\text{tails})}$$

update 1000



# Beta Distribution

$$P_B(k, n, p) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

- Related to Binomial and has a very similar form:

$$P_\beta(x, \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$$

- with  $x \in [0,1]$  and  $\alpha, \beta > 0$
- Can be thought of as modeling the probability of  $p$  for a given number of  $\alpha$  heads and  $\beta$  tails

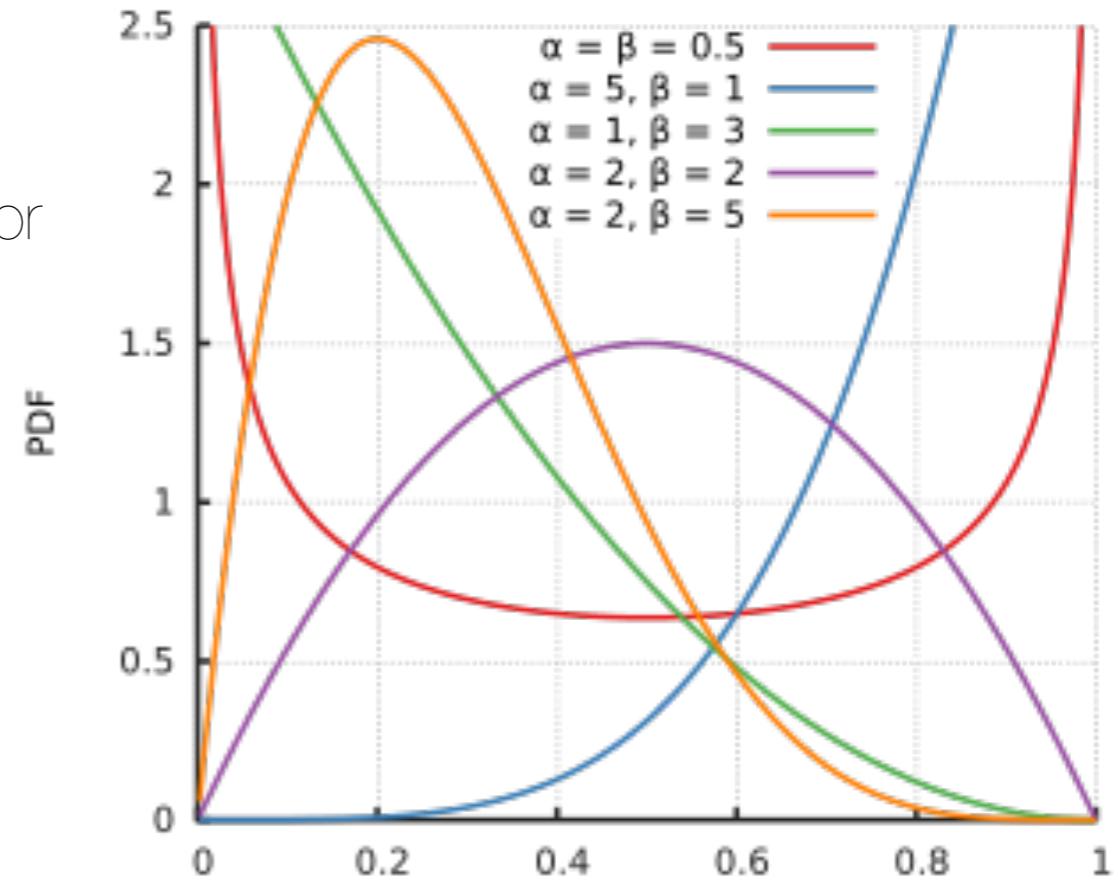
- $\Gamma(\alpha)$  is the continuous version of  $\alpha!$

- The mean is:

$$\mu = \frac{\alpha}{\alpha + \beta}$$

- And the variance:

$$\sigma^2 = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

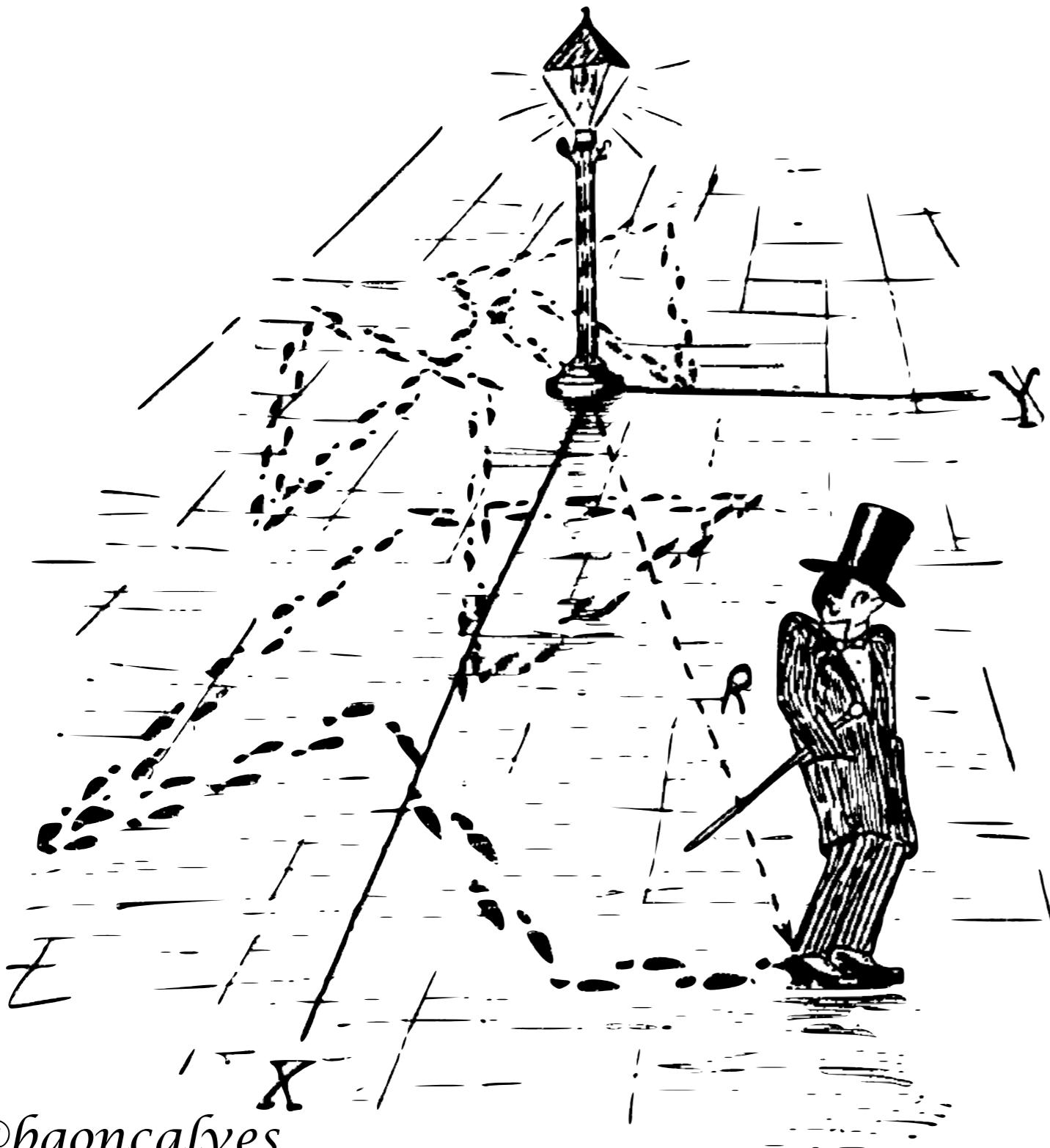




## Lesson III: Random Walks and Markov Chains

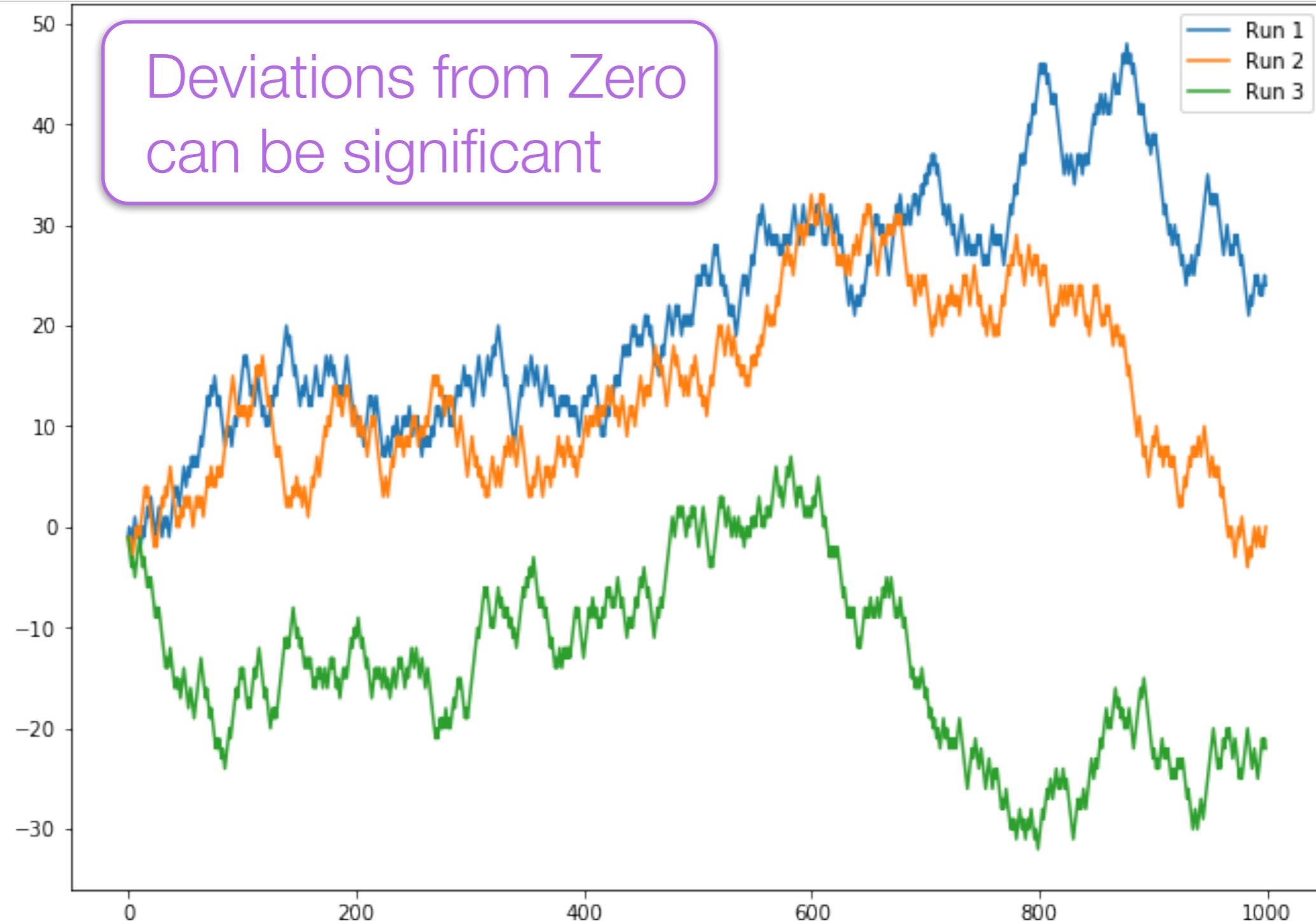
# Random Walks

Illustration by George Gamow

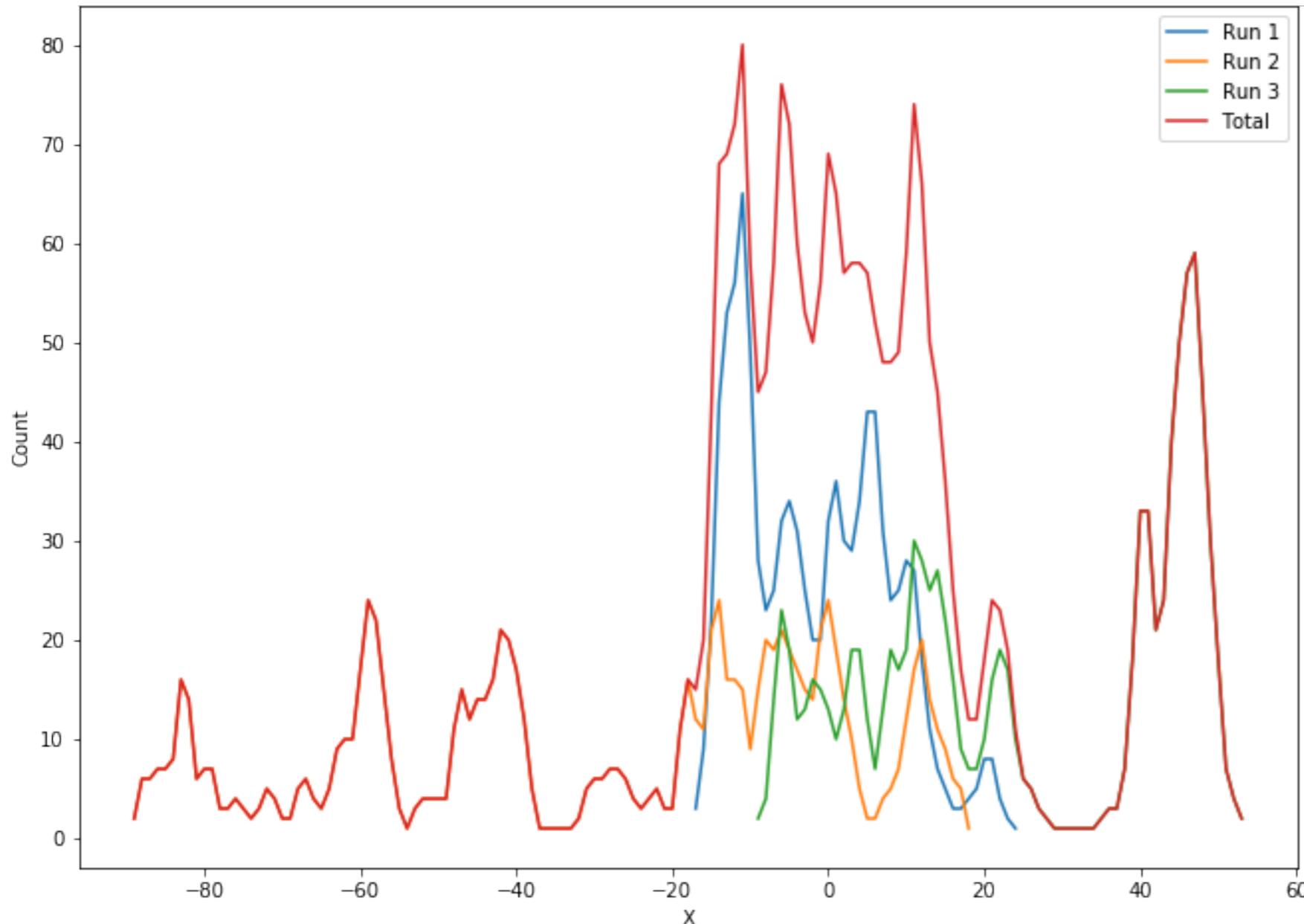


- At each step flip a coin
  - Heads: Move right
  - Tails: Move left
- If you start at position **0**, do you ever reach position **L**?
- On average, we expect the position to be always close to **0**.
- What if the coin is biased as in the previous example?

# Random Walks

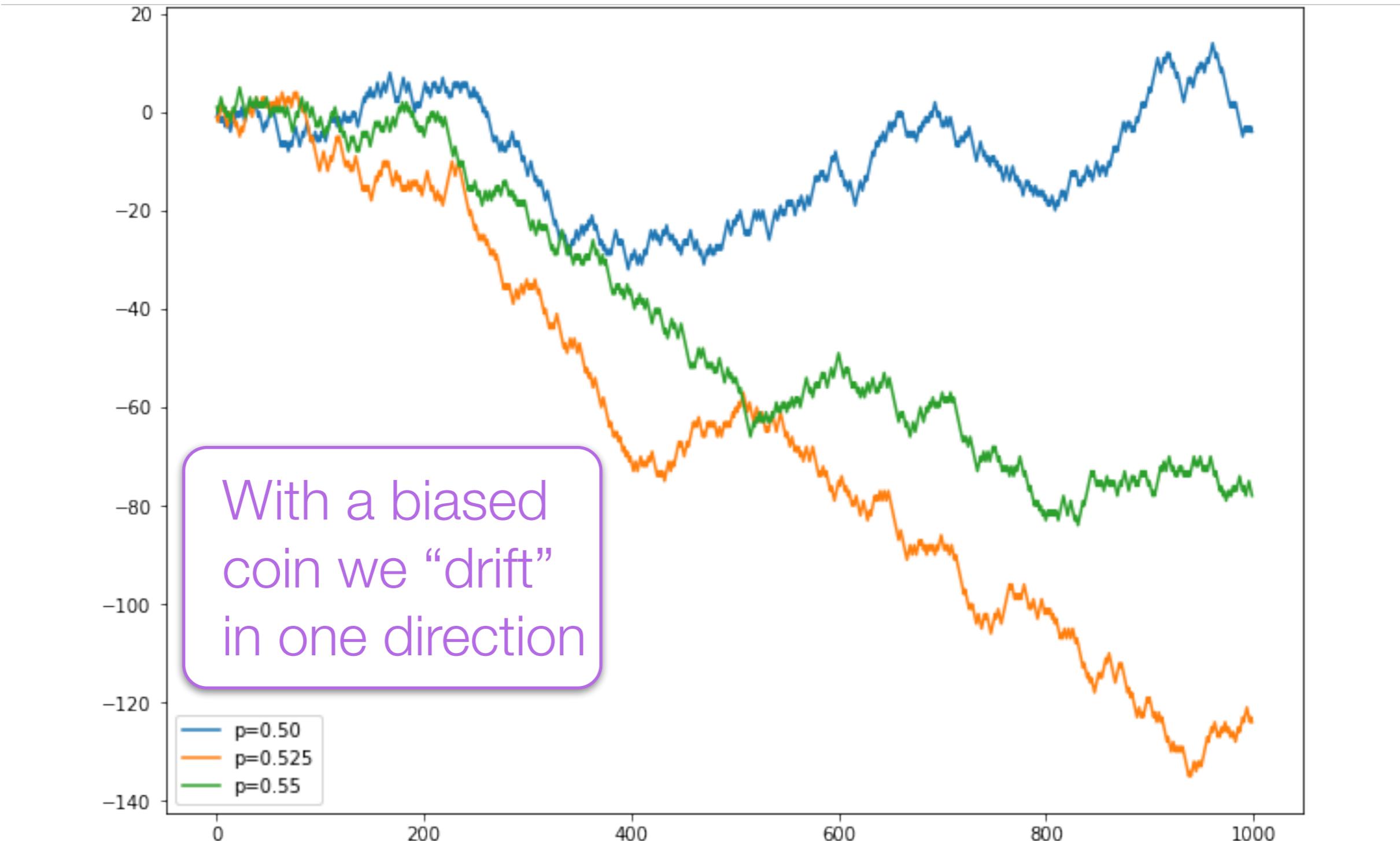


# Visited Positions

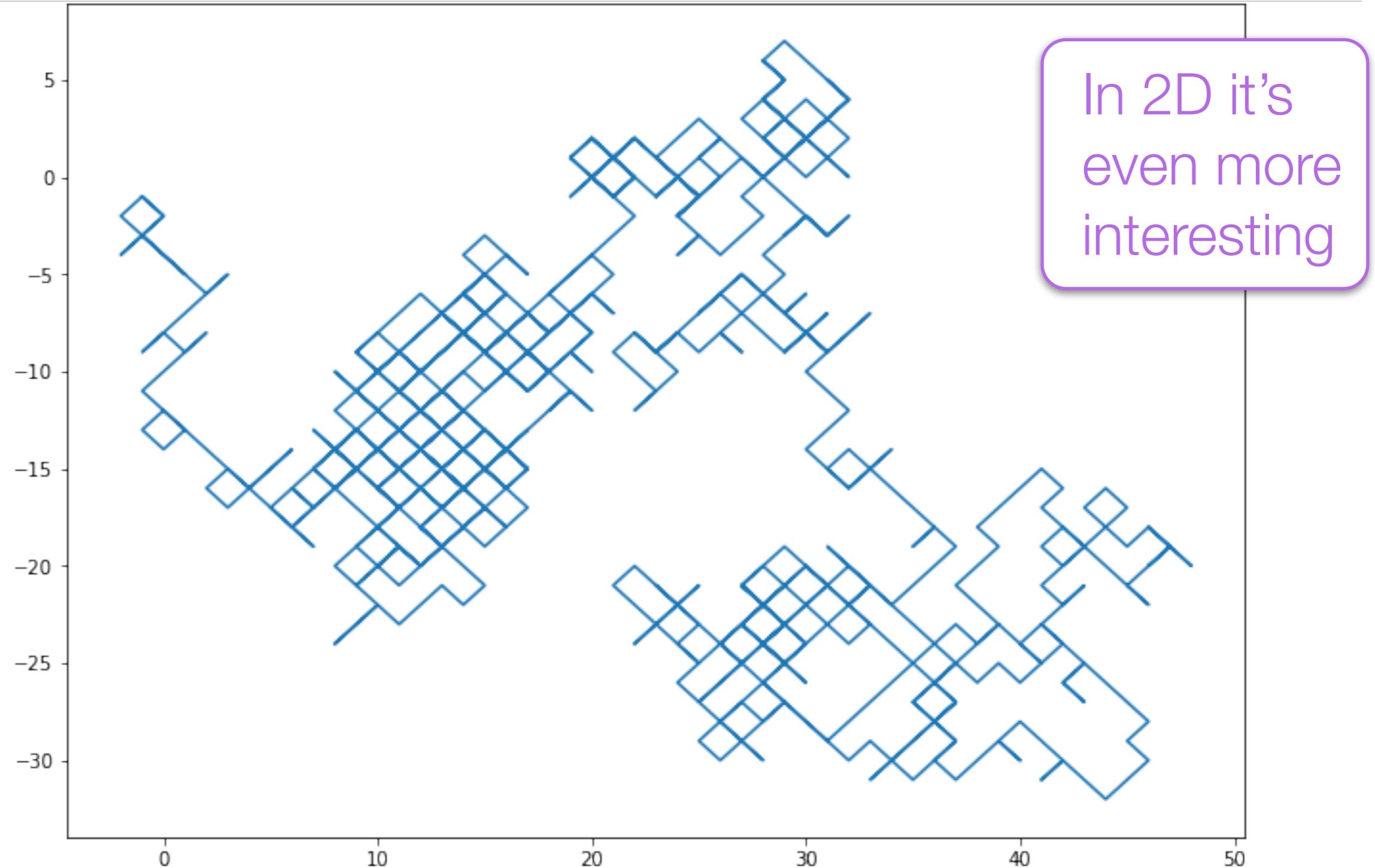


- The most visited positions, across multiple realizations, are the ones closest to the origin, **0**
- This reflects the fact that the number of ways of reach **0** is much larger than the number of ways of reaching the extremes due to the combinatorial factor  $C_k^N$
- **0** is better connected
- With a biased coin, the better connected location would be somewhere else due to drift

# Random Walk with a drift



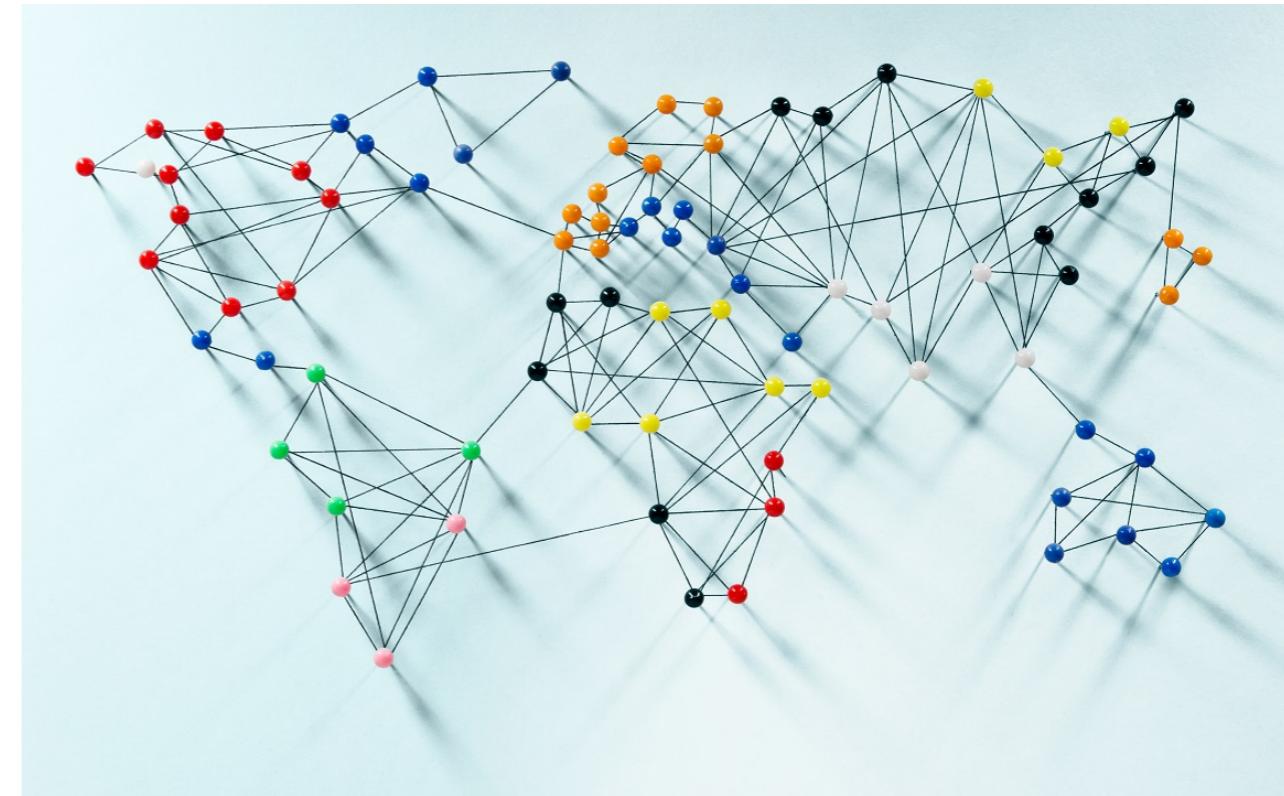
## 2 Dimensions



# What if we're walking on a network?

<https://www.analyticsvidhya.com/blog/2018/04/introduction-to-graph-theory-network-analysis-python-codes/>

- Each possible location is a **node (dot)**
- Each path is an **edge (line)** connecting two nodes
- At each step  $i$  **randomly choose** among the  $k_i$  available edges
- Just a generalization of the previous 1D and 2D examples with fixed size steps
- Nodes can be cities, **websites**, street intersections, etc and edges can be, respectively, airline connections, **links**, roads, etc
- Following a similar idea, we also expect that the **most visited locations** are also the **better connected** ones



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**Manhattan** [+ ↗](#)

From Wikipedia, the free encyclopedia Coordinates:  40°47'N 73°58'W

*This article is about the New York City borough. For other uses, see [Manhattan \(disambiguation\)](#).*

**Manhattan** (/mæn'hætən, mən-/), often referred to locally as **the City**,<sup>[1]</sup> is the most densely populated of the five boroughs of **New York City** and its economic and administrative center, **cultural** identifier,<sup>[5]</sup> and historical birthplace.<sup>[6]</sup> The borough is coextensive with **New York County**, one of the **original counties** of the **U.S. state of New York**. The borough consists mostly of Manhattan Island, bounded by the **Hudson**, **East**, and **Harlem** rivers; **several small adjacent islands**; and **Marble Hill**, a small neighborhood now on the **U.S. mainland**, physically connected to **the Bronx** and separated from the rest of Manhattan by the **Harlem River**. Manhattan Island is divided into three informally bounded components, each aligned with the borough's long axis: **Lower**, **Midtown**, and **Upper Manhattan**.

Manhattan has been described as the cultural, financial, **media**, and **entertainment** capital of the world,<sup>[7][8][9][10]</sup> and the borough hosts the **United Nations Headquarters**.<sup>[11]</sup> Anchored by **Wall Street** in the **Financial District** of **Lower Manhattan**, New York City has been called both the most economically powerful city and the leading financial center of the world,<sup>[12][13][14][15][16][17]</sup> and Manhattan is home to the world's two largest stock exchanges by total **market capitalization**: the **New York Stock Exchange** and **NASDAQ**.<sup>[18][19]</sup> Many **multinational media conglomerates** are based in Manhattan, and the borough has been the **setting** for numerous books, **films**, and television shows. **Manhattan real estate** has since become among the most expensive in the world, with the value of Manhattan Island, including real estate, estimated to exceed US\$3 trillion in 2013;<sup>[6][20]</sup> median residential property sale prices in Manhattan approximated US\$1,600 per square foot (\$17,000/m<sup>2</sup>) as of 2018,<sup>[21]</sup> with **Fifth Avenue** in **Midtown Manhattan** commanding the highest **retail** rents in the world, at US\$3,000 per square foot (\$32,000/m<sup>2</sup>) in 2017.<sup>[22]</sup>

Manhattan traces its origins to a **trading post** founded by **colonists** from the **Dutch Republic** in 1624 on Lower Manhattan; the post was named **New Amsterdam** in 1626. Manhattan is historically documented to have been purchased by **Dutch colonists** from **Native Americans** in 1626 for 60 **guilders**, which equals roughly \$1038 in current terms.<sup>[23][24][25]</sup> The territory and its surroundings came under English control in 1664<sup>[25]</sup> and were renamed **New York** after King **Charles II of England** granted the lands to his brother, the **Duke of York**.<sup>[26]</sup> New York, based in present-day Manhattan, served as the **capital of the United States** from 1785 until 1790.<sup>[27]</sup> The **Statue of Liberty** greeted millions of **immigrants** as they came to the Americas by ship in the late 19th and early 20th centuries<sup>[28]</sup> and is a world symbol of the United States and its ideals of liberty and peace.<sup>[29]</sup> Manhattan became a borough during the **consolidation of New York City** in 1898.

New York County is the **United States' second-smallest county by land area** (larger only than **Kalawao County, Hawaii**), and is also the

**Manhattan**  
New York County, New York

**Borough and county**



Midtown Manhattan facing south toward Lower Manhattan

 Flag

Etymology: **Lenape**: **Manaháhtaan** (the place where we get bows)  
Nickname(s): **The City**<sup>[1]</sup>



# Markov Chain

---

- Random walks are simple examples of the more general class of Markov Chain processes
- **Memoryless** - "Future depends only on present state and not on past history"

$$\vec{\pi}(t+1) = \mathcal{G} \vec{\pi}(t)$$

- Has a stationary state  $\vec{\pi} = \mathcal{G} \vec{\pi}$  if the transition matrix  $\mathcal{G}$  is:

- **Stochastic** - Column vectors normalized to 1.
- **Irreducible** - All nodes are accessible (graph is connected)
- **Aperiodic** - Return to node i does not occur periodically

# PageRank

- PageRank is just a random walk on the web graph
- The underlying intuitions that "A page is important if it is pointed to by other important pages"

$$\pi_i = \sum_j a_{ji} \frac{\pi_j}{k_j}$$

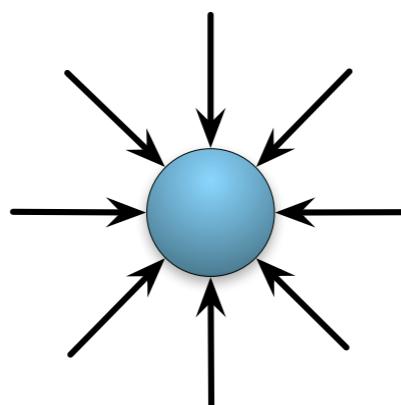
- The PageRank score of a page is the stationary state of a discrete time random walk on the network of pages and links

$$\vec{\pi}(t+1) = \mathcal{T} \vec{\pi}(t)$$

- Where  $\mathcal{T} = K_0^{-1} A^T$  is the operator that performs one step of the random walk

Problems:

"Dangling nodes"



Slow convergence

Solution:

Connect dangling nodes to every other node

Add damping factor (reset button)

# Power Method

---

- The google transition operator (also called the google matrix) is:

$$\mathcal{G} = (1 - \alpha) \mathcal{S} + \alpha \mathcal{E}$$


$\mathcal{E}$  - Fully connected matrix  
 $\alpha$  - damping factor

- The google matrix has a stationary state (converges) if:

$$\vec{\pi} = \mathcal{G} \vec{\pi}$$

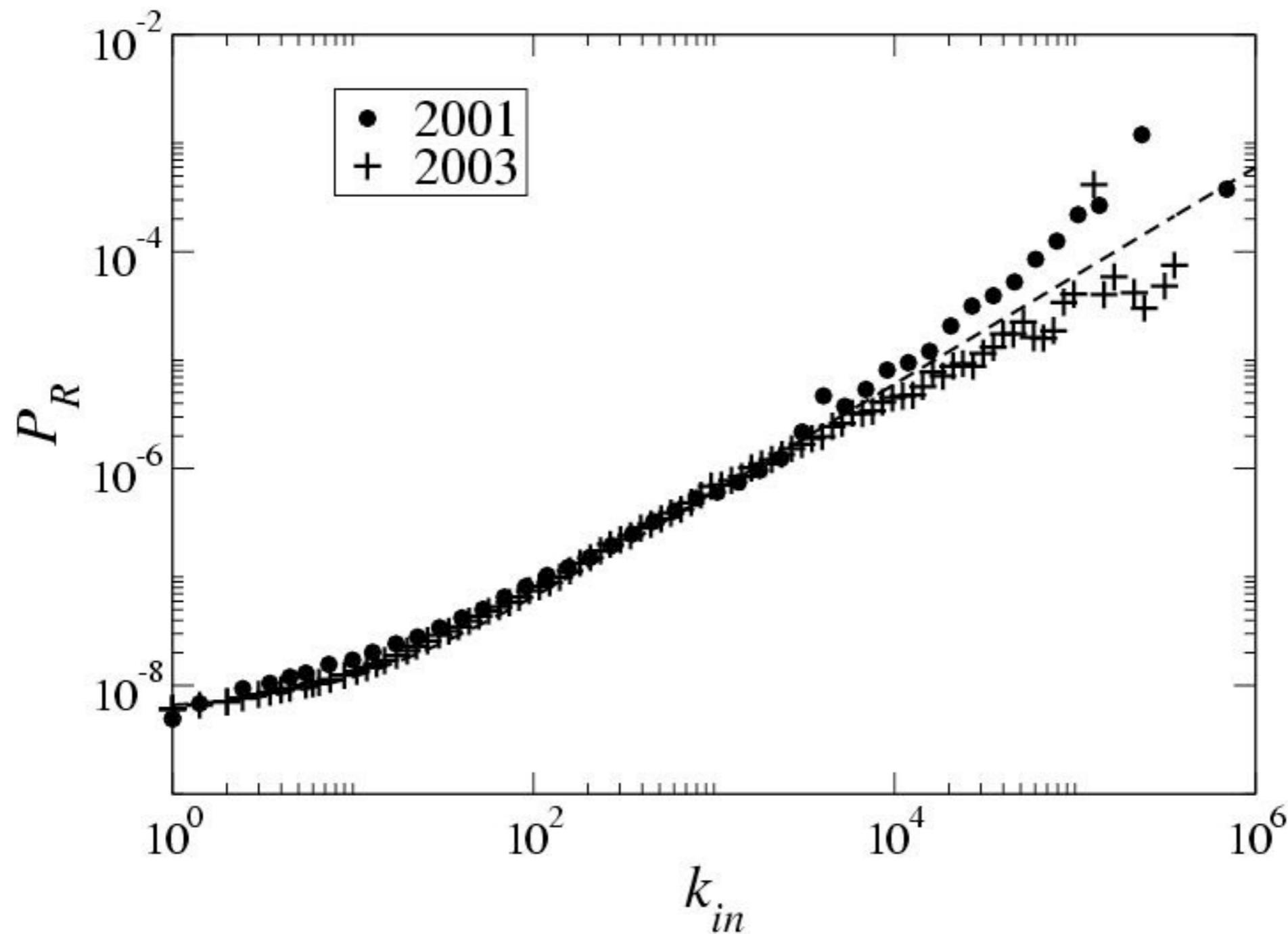
- The Power-Method consists of iterating this process until:

$$\vec{\pi}(t+1) \approx \vec{\pi}(t)$$

- This is equivalent to computing successive powers of

# Web Graph

[https://link.springer.com/chapter/10.1007%2F978-3-540-78808-9\\_6](https://link.springer.com/chapter/10.1007%2F978-3-540-78808-9_6)





## Lesson M: A / B Testing

# Hypothesis Testing

---

- The classical question we are trying to answer is: Is my **intervention** having an actual **effect**?
- **Our hypothesis** is that our intervention is effective
- The **null-hypothesis** is that there is no effect
- The main goal of Hypothesis Testing is to determine under what circumstances we can **reject the null-hypothesis** with a certain **degree of certainty**?
- In other words: How sure are we that we're not observing this difference **just by chance** (due to fluctuations as per the **CLT**)
- Select an appropriate test statistic to compare the two approaches

# Hypothesis Testing

---

- In the case of binary outcomes, conversions follow a binomial distribution and the test statistic is the **Z** score:

$$Z = \frac{p_A - p_B}{\sqrt{SE_A^2 + SE_B^2}}$$

- where:

$$SE = \sqrt{\frac{p(1-p)}{N}}$$

- is the standard error for each instance.
- Under common assumptions, Z follows a Gaussian (normal) distribution centered at **zero** and with width **one**.

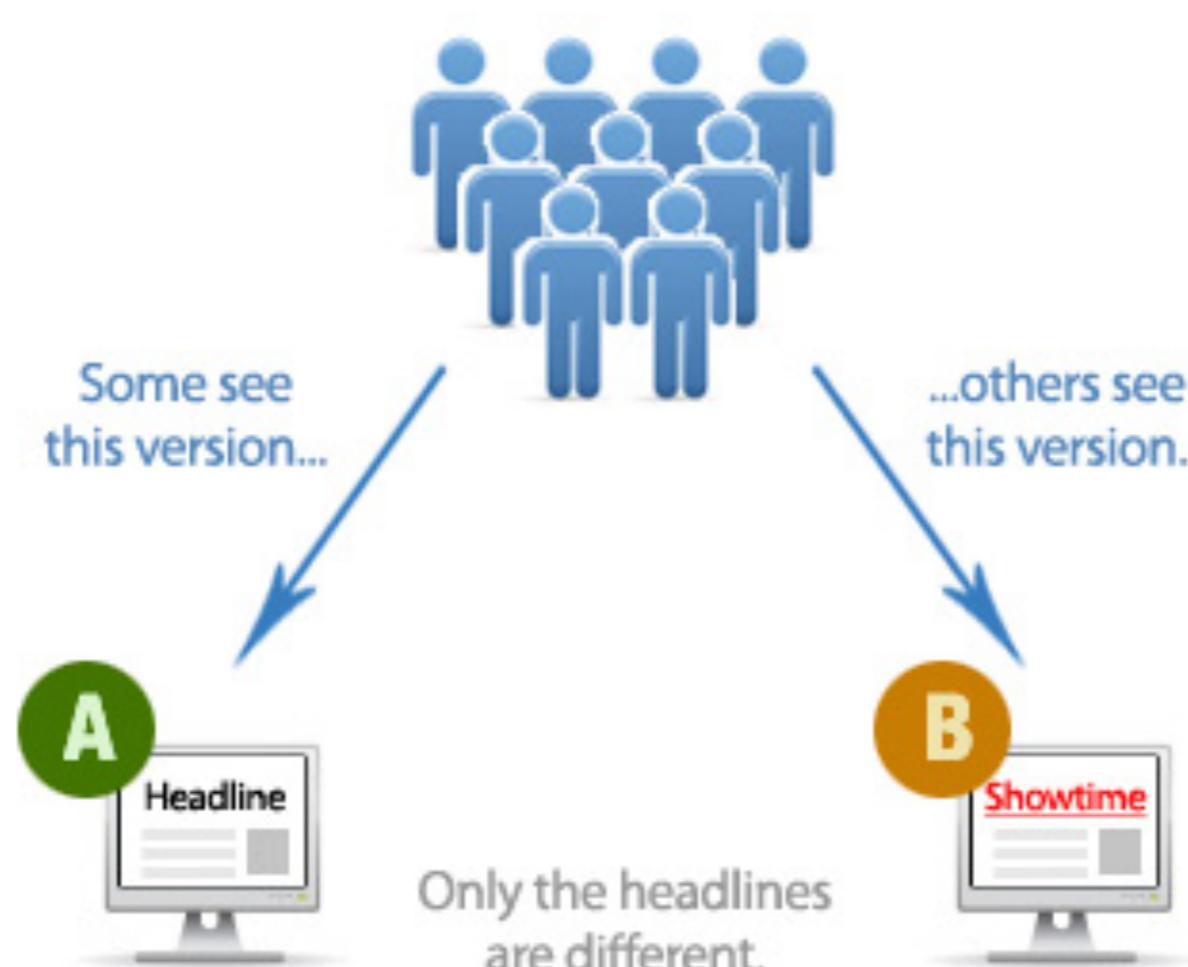
$$\mathcal{N}(0,1)$$

- Let's consider a practical example to clarify things

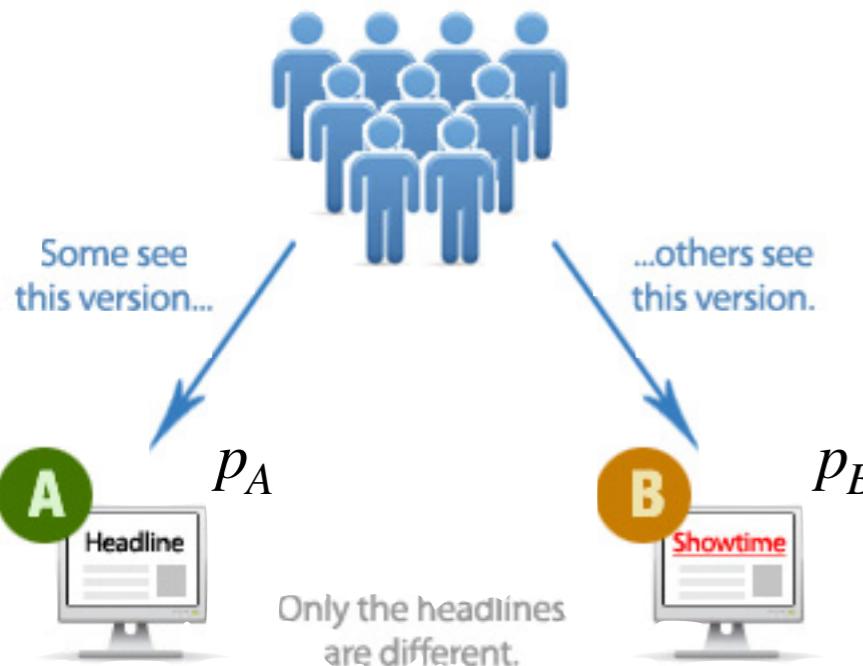
# A/B Testing

---

- Which version of a headline results in more clicks?
- Divide users into two groups **A** and **B** and show each of them just one version
- Measure the click probability in each group,  $p_A$  and  $p_B$
- The null hypothesis is that  $p_A = p_B$ . Can we reject it?

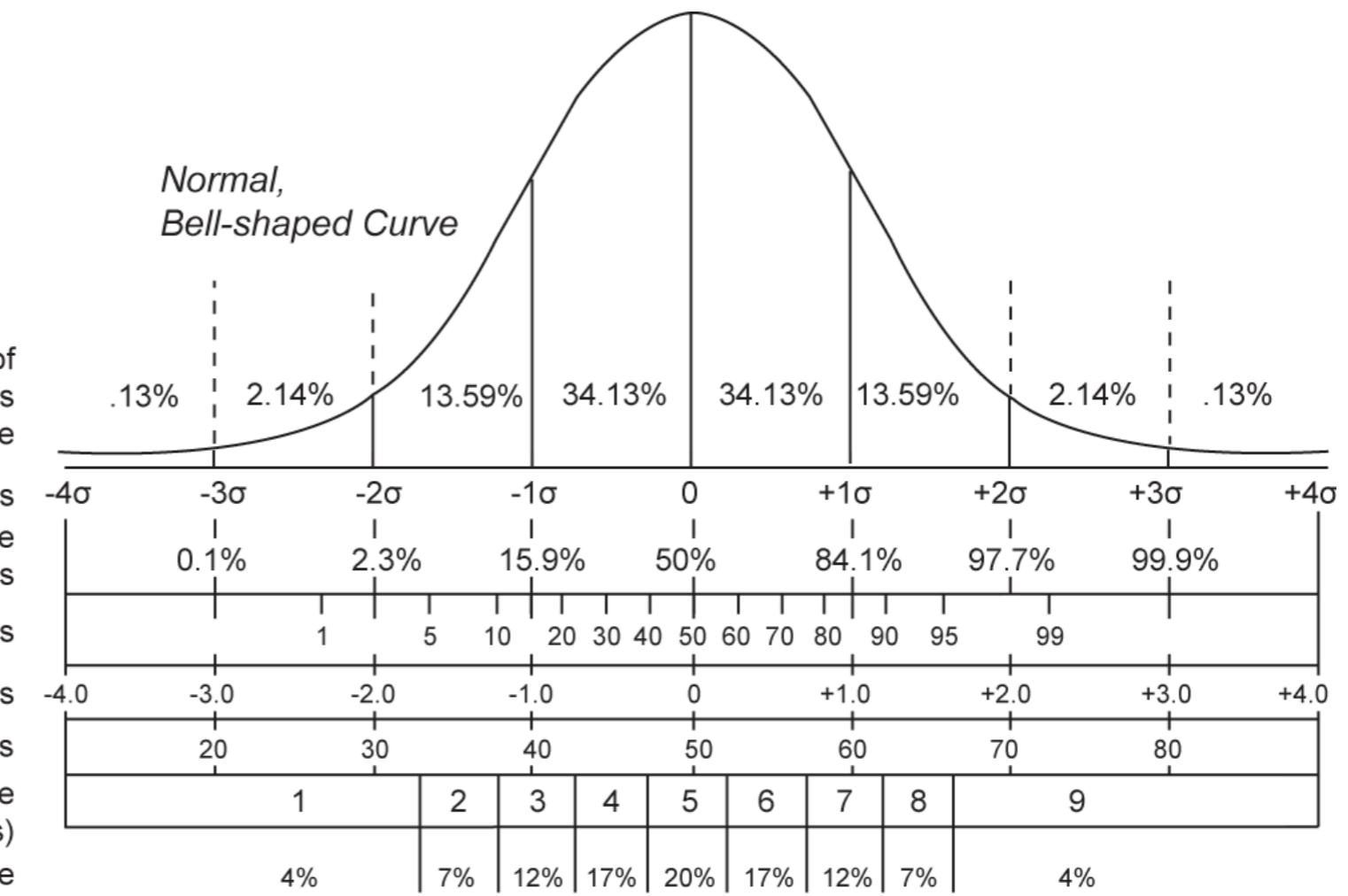


# A/B Testing



$$Z = \frac{p_A - p_B}{\sqrt{SE_A^2 + SE_B^2}}$$

Percentage of cases in 8 portions of the curve  
Standard Deviations  
Cumulative Percentages  
Percentiles  
Z scores  
T scores  
Standard Nine (Stanines)  
Percentage in Stanine

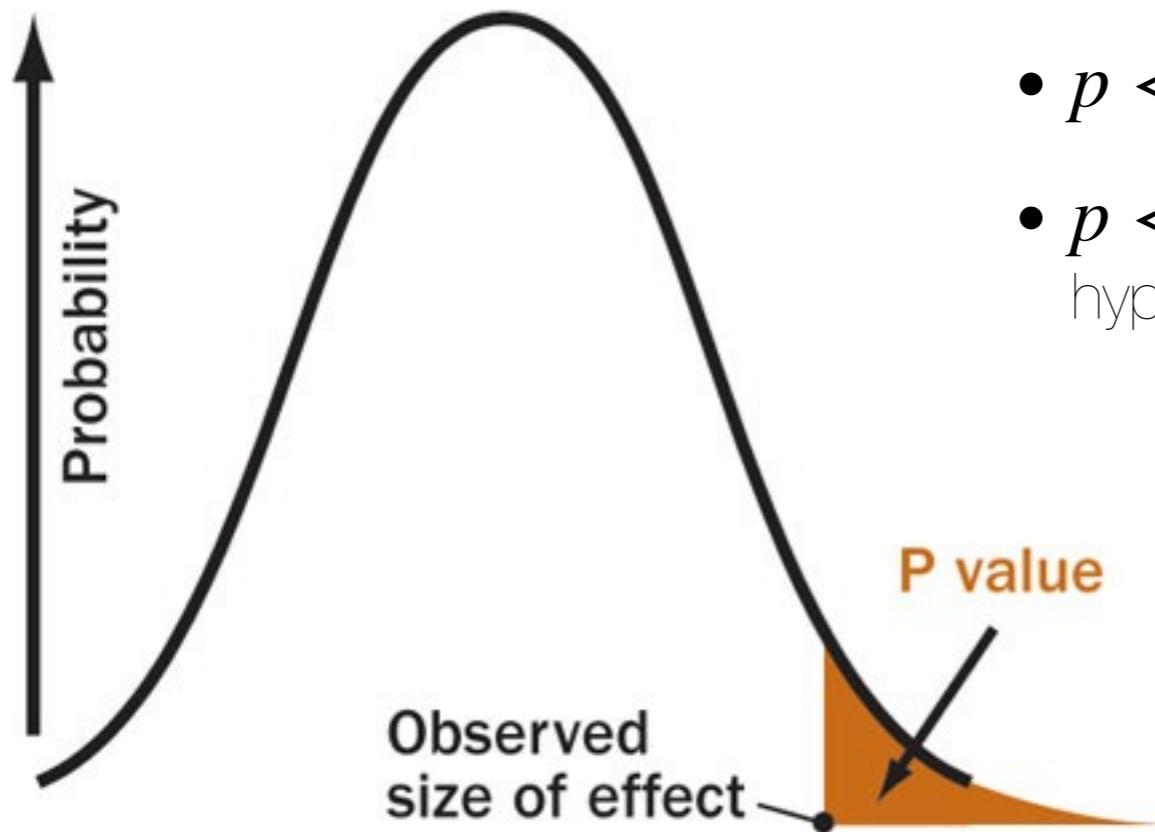


- The value of  $P(Z)$  effectively tells us how likely we are to observe this difference between  $p_A$  and  $p_B$  just due to sampling effects
- $P(Z)$  is often called the  $p$ -value

# p-value

---

- Calculate the probability of an event more extreme than the observation under the “null hypothesis”
  - $p < 0.05$  **Moderate** evidence against null-hypothesis
  - $p < 0.01$  **Strong** evidence against null-hypothesis
  - $p < 0.001$  **Very strong** evidence against the null-hypothesis



- The smaller the p-value the better.

# Berkeley Discrimination Case Part I

---

	Candidates	Acceptance Rate
Men	8442	0.44
Women	4321	0.35

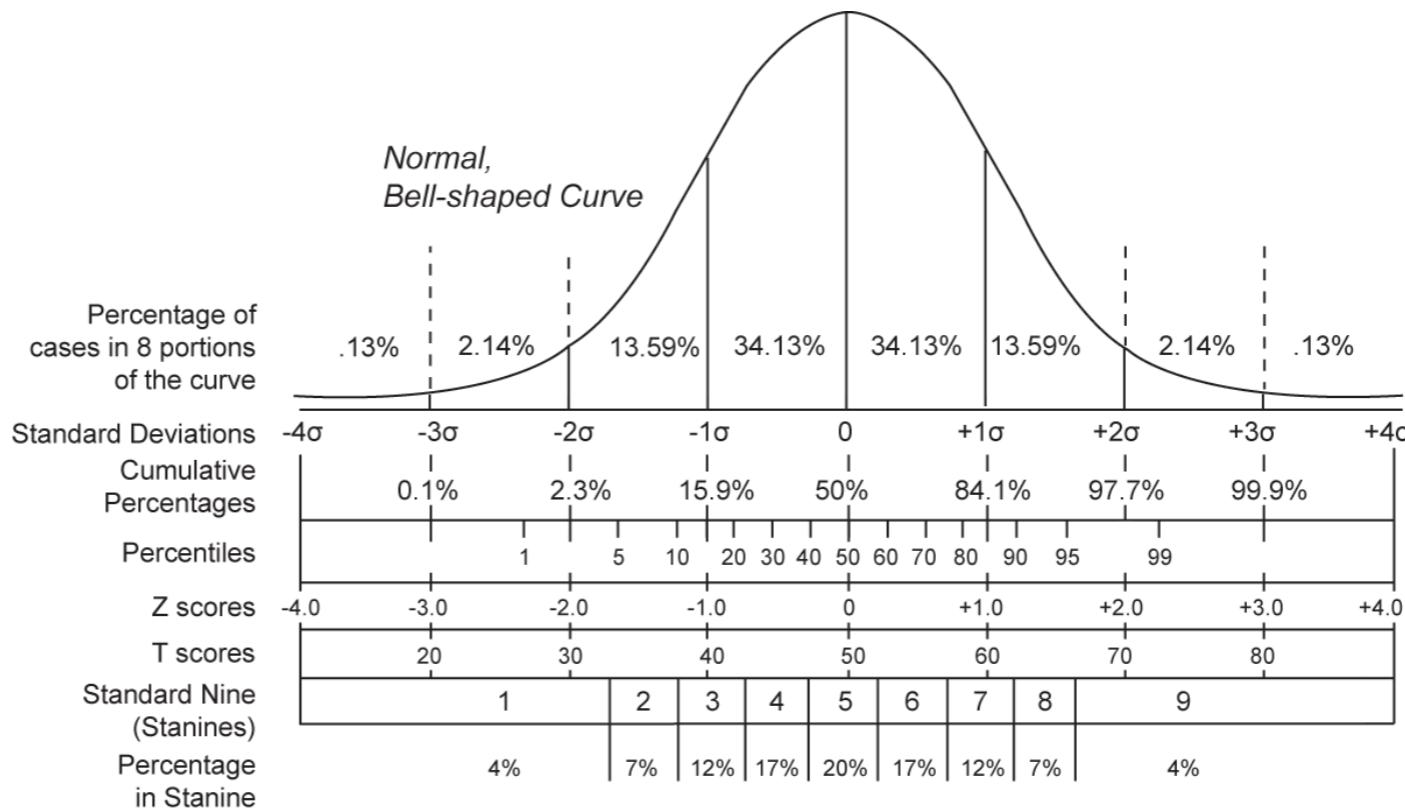
Were women being discriminated against when they applied to Berkley?

# Berkeley Discrimination Case Part I

	Candidates	Acceptance Rate	SE
Men	8442	0.44	$5.4 \times 10^{-3}$
Women	4321	0.35	$7.2 \times 10^{-3}$

Were women being discriminated against when they applied to Berkley?

$$Z = \frac{p_A - p_B}{\sqrt{SE_A^2 + SE_B^2}}$$



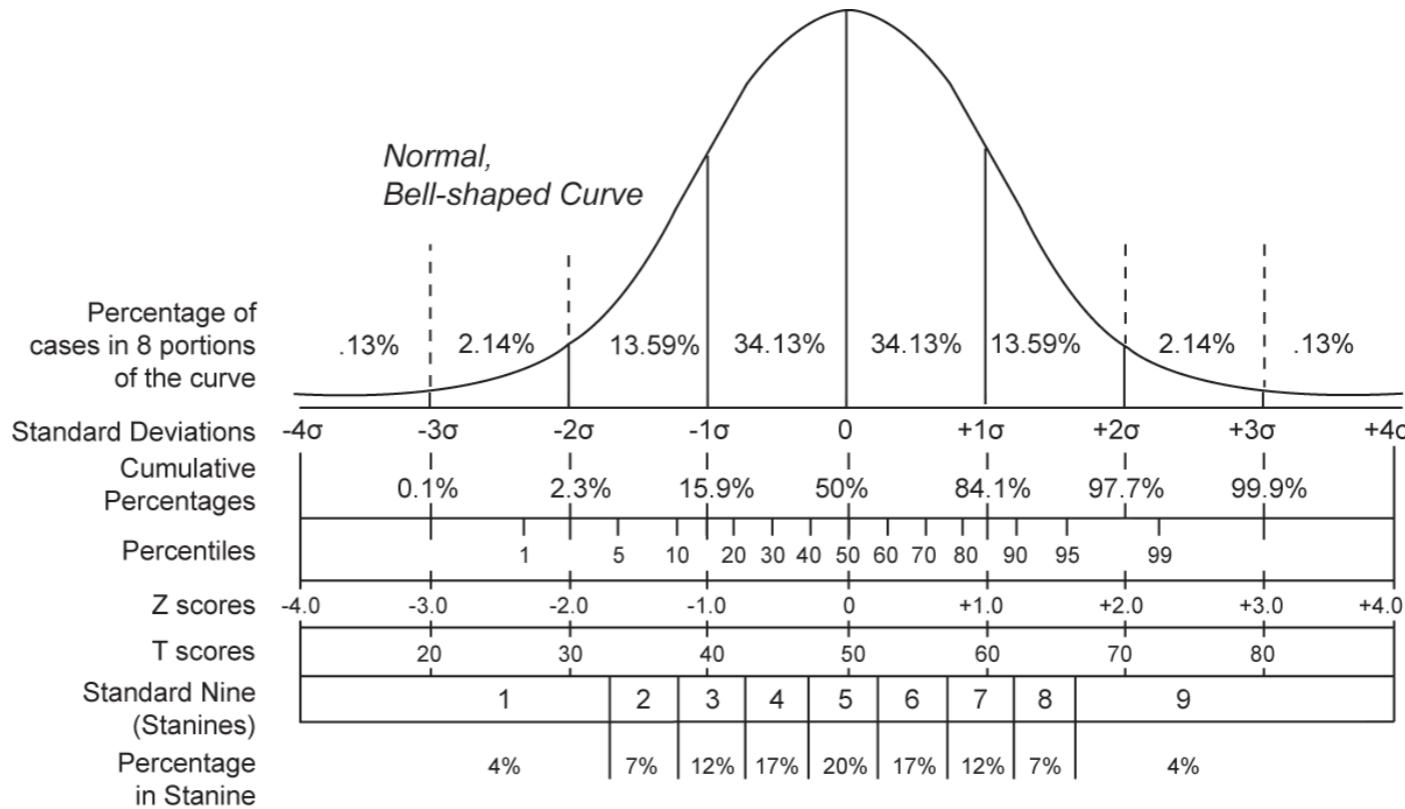
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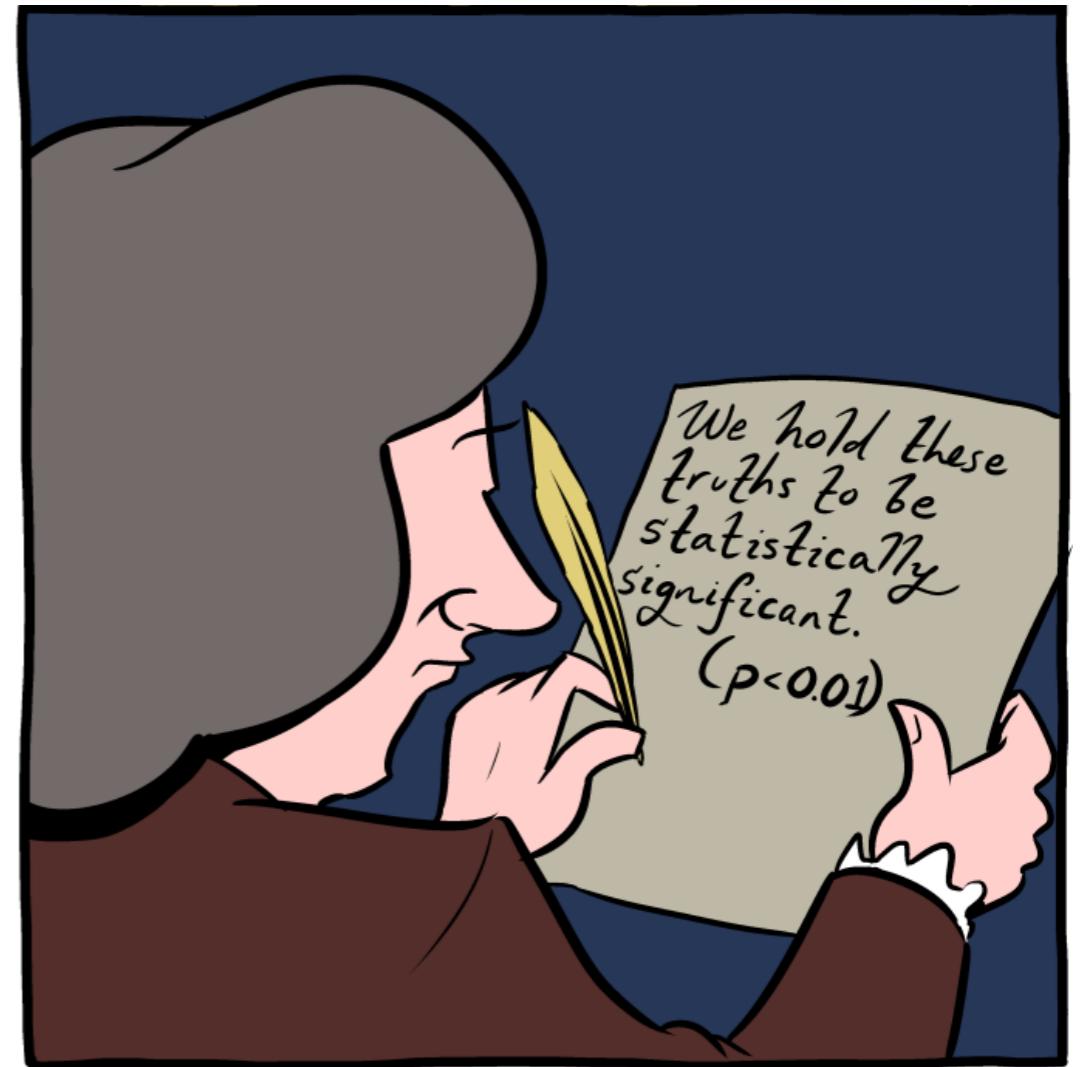
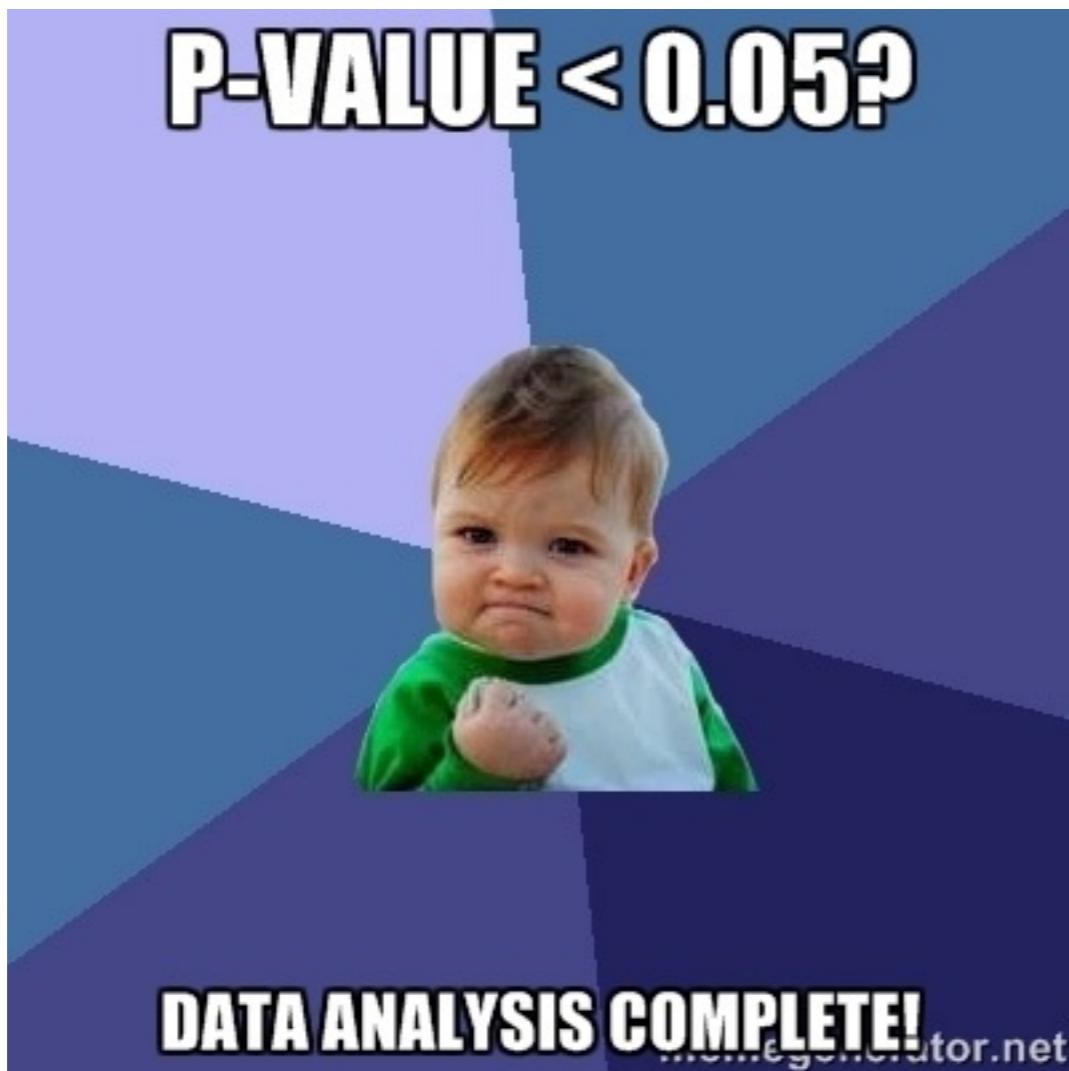
$$Z = \frac{p_A - p_B}{\sqrt{SE_A^2 + SE_B^2}}$$

$$p \approx 10^{-23}$$



# p-value

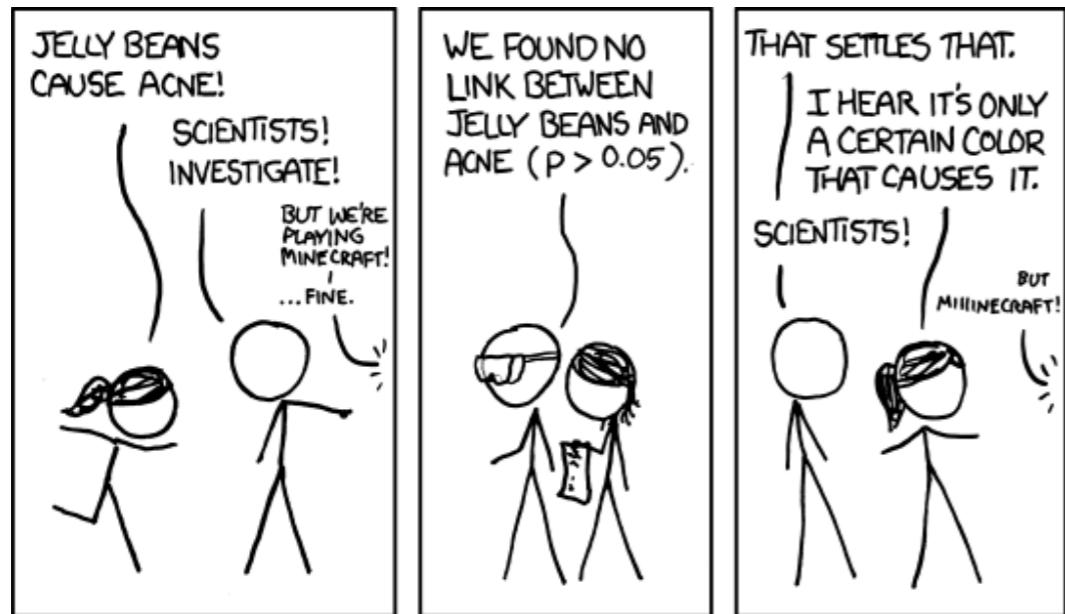
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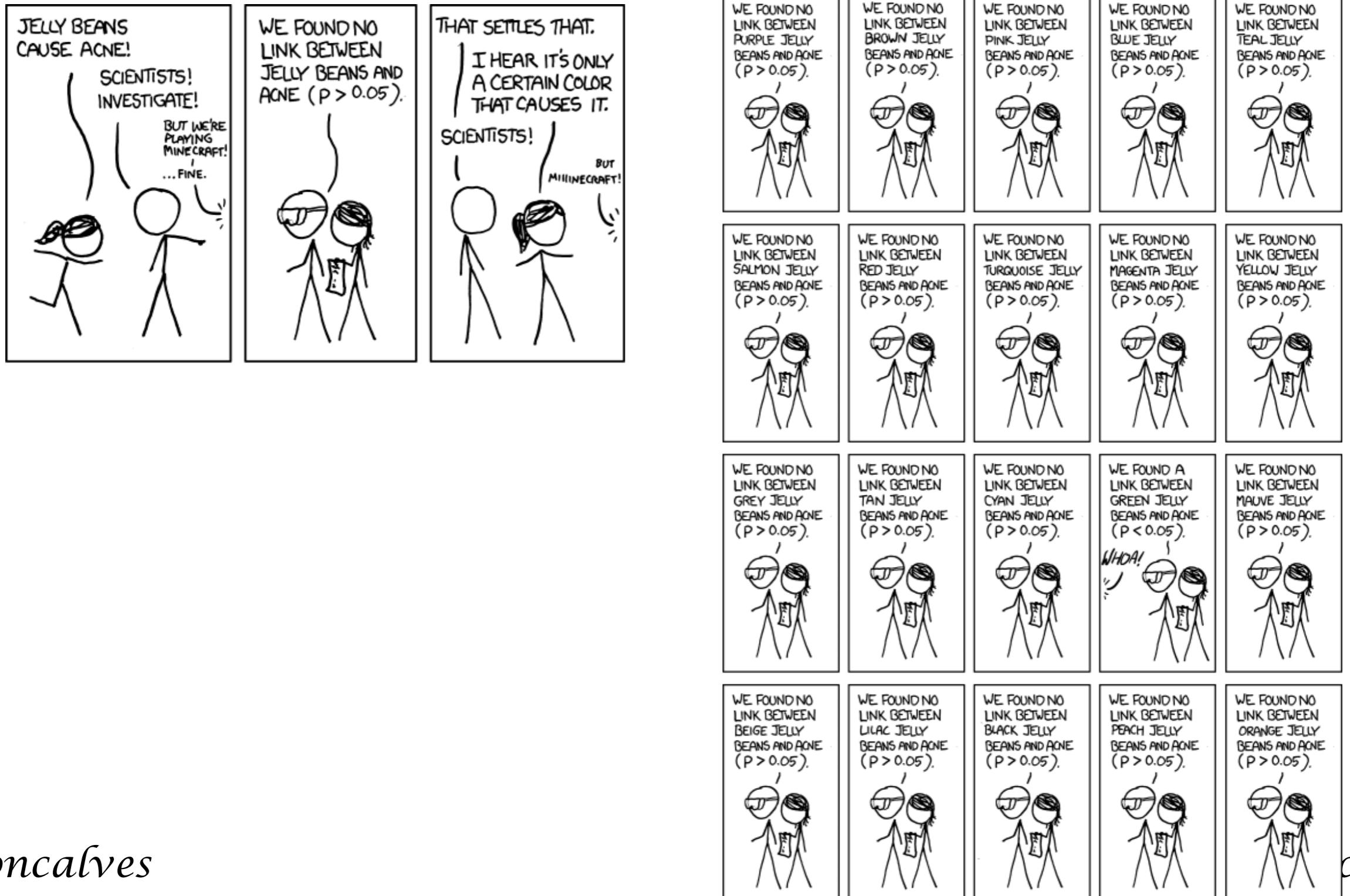
“Statistical significance does not imply scientific significance”

# Bonferroni Correction

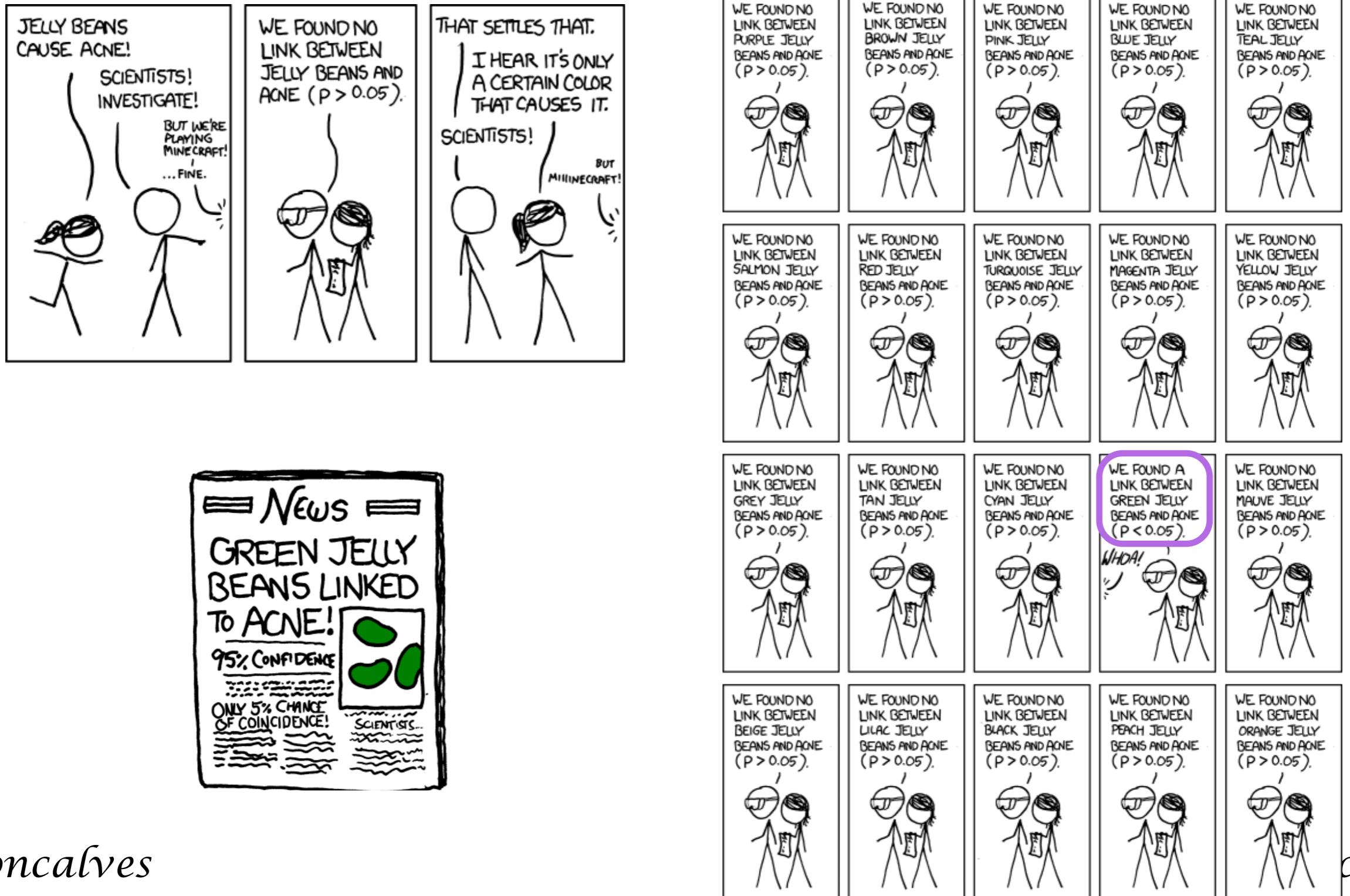
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# Bonferroni Correction



# Bonferroni Correction



# Bonferroni Correction

---

- You can think of the  $p$ -value as the probability of observing a result as extreme by chance. With  $n$  comparisons, this probability becomes:

$$p_n = 1 - (1 - p)^n$$

which quickly goes to 1 as  $n$  increases.

- However, if we replacing  $p$  by  $\frac{p}{n}$  for each individual comparison, we obtain:

$$p_n = 1 - \left(1 - \frac{p}{n}\right)^n$$

- and for sufficiently large  $n$ :

$$p_n \approx 1 - e^{-p} \approx p$$

- allowing us to keep the **probability of false results arbitrarily low** even with arbitrarily **large numbers of comparisons**.

# Simpsons Paradox

Science 187, 398 (1975)

	Candidates	Acceptance Rate
Men	8442	0.44
Women	4321	0.35

Berkeley Discrimination Case Part II:  
The statisticians strike back.

Dept	Men		Women	
	Candidates	Acceptance	Candidates	Acceptance
A	825	0.62	108	0.82
B	560	0.63	25	0.68
C	325	0.37	594	0.34
D	417	0.33	375	0.35
E	191	0.28	393	0.24
F	272	0.06	341	0.07
	<b>2590</b>	<b>0.46</b>	<b>1835</b>	<b>0.30</b>

# Simpsons Paradox

Science 187, 398 (1975)

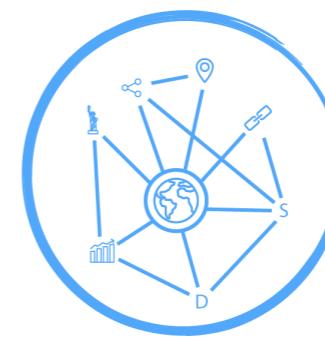
	Candidates	Acceptance Rate
Men	8442	0.44
Women	4321	0.35

“aggregated data can appear to reverse important trends in the numbers being combined”

WSJ, Dec 2, 2009

Dept	Men		Women	
	Candidates	Acceptance	Candidates	Acceptance
A	825	0.62	108	0.82
B	560	0.63	25	0.68
C	325	0.37	594	0.34
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F	272	0.06	341	0.07
	2590	0.46	1835	0.30





### Natural Language Processing (NLP) from Scratch

- Aug 5, 2019 5am-9am (PST)

### Data Visualization with matplotlib and seaborn

- Aug 9, 2019 5am-9am (PST)

### Deep Learning From Scratch

- Sept 24, 2019



### Data Visualization with matplotlib and seaborn

- Sept 30, 2019 5am-9am (PST)



### Natural Language Processing (NLP) from Scratch

<http://bit.ly/LiveLessonNLP> - On Demand