Time Series Analysis From the Ground Up

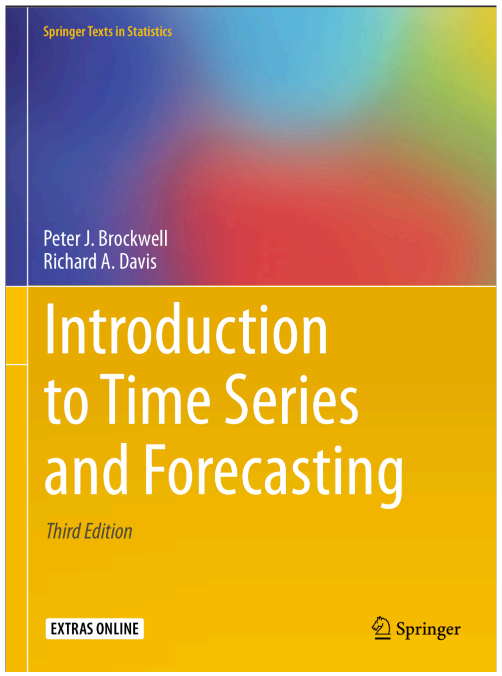
Bruno Gonçalves

www.data4sci.com/newsletter

https://github.com/DataForScience/Timeseries

Lesson I:
Understanding Timeseries

References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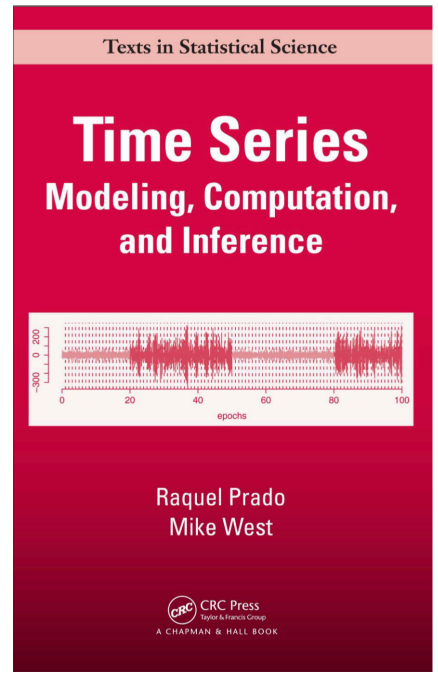
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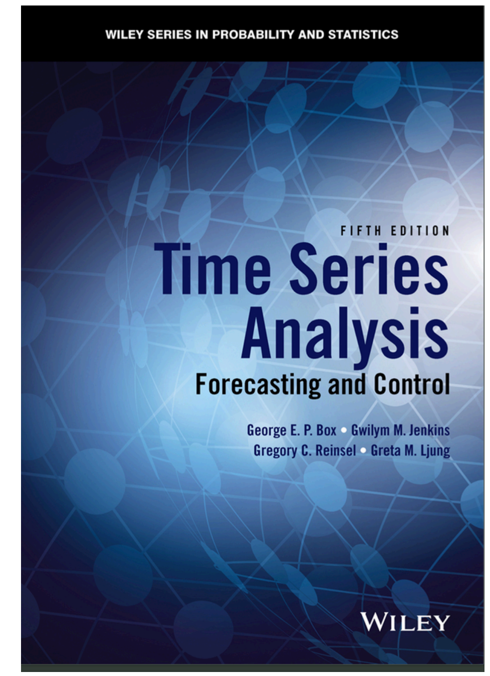
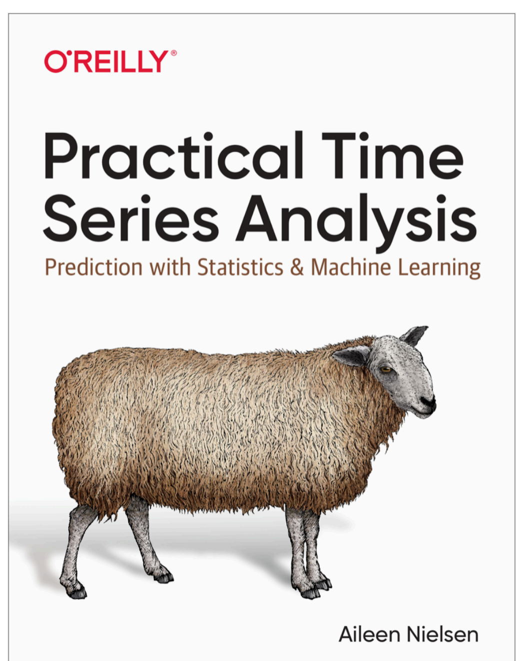
Time Series
Modeling, Computation, and Inference
Raquel Prado
Mike West
CRC Press
A CHAPMAN & HALL BOOK


WILEY SERIES IN PROBABILITY AND STATISTICS
FIFTH EDITION
Time Series Analysis
Forecasting and Control
George E. P. Box • Gwilym M. Jenkins
Gregory C. Reinsel • Greta M. Ljung
WILEY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Practical Time Series Analysis
Prediction with Statistics & Machine Learning
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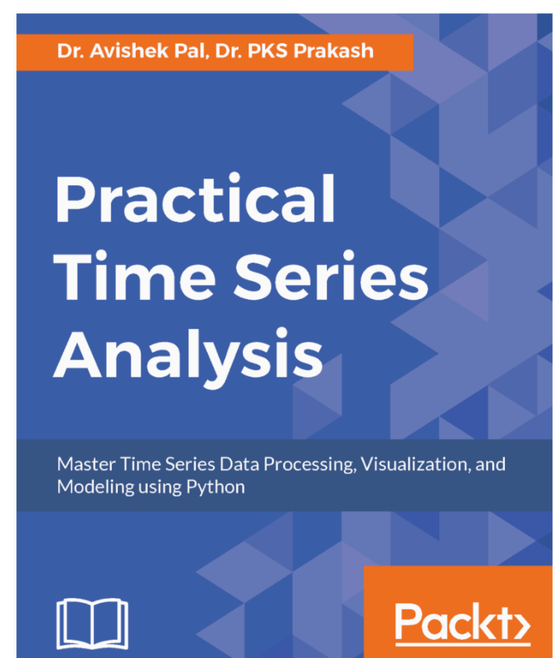
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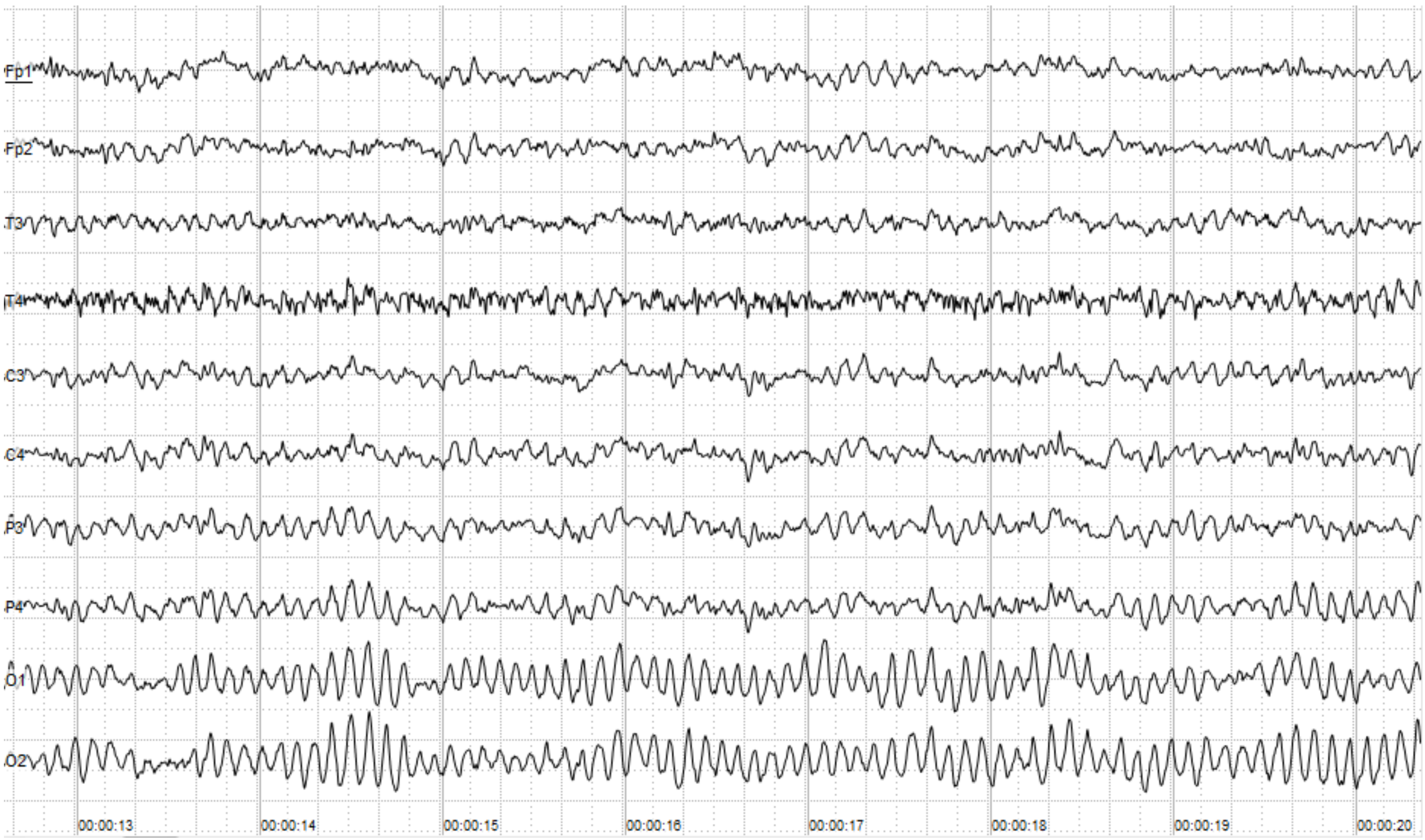

Airline Passengers

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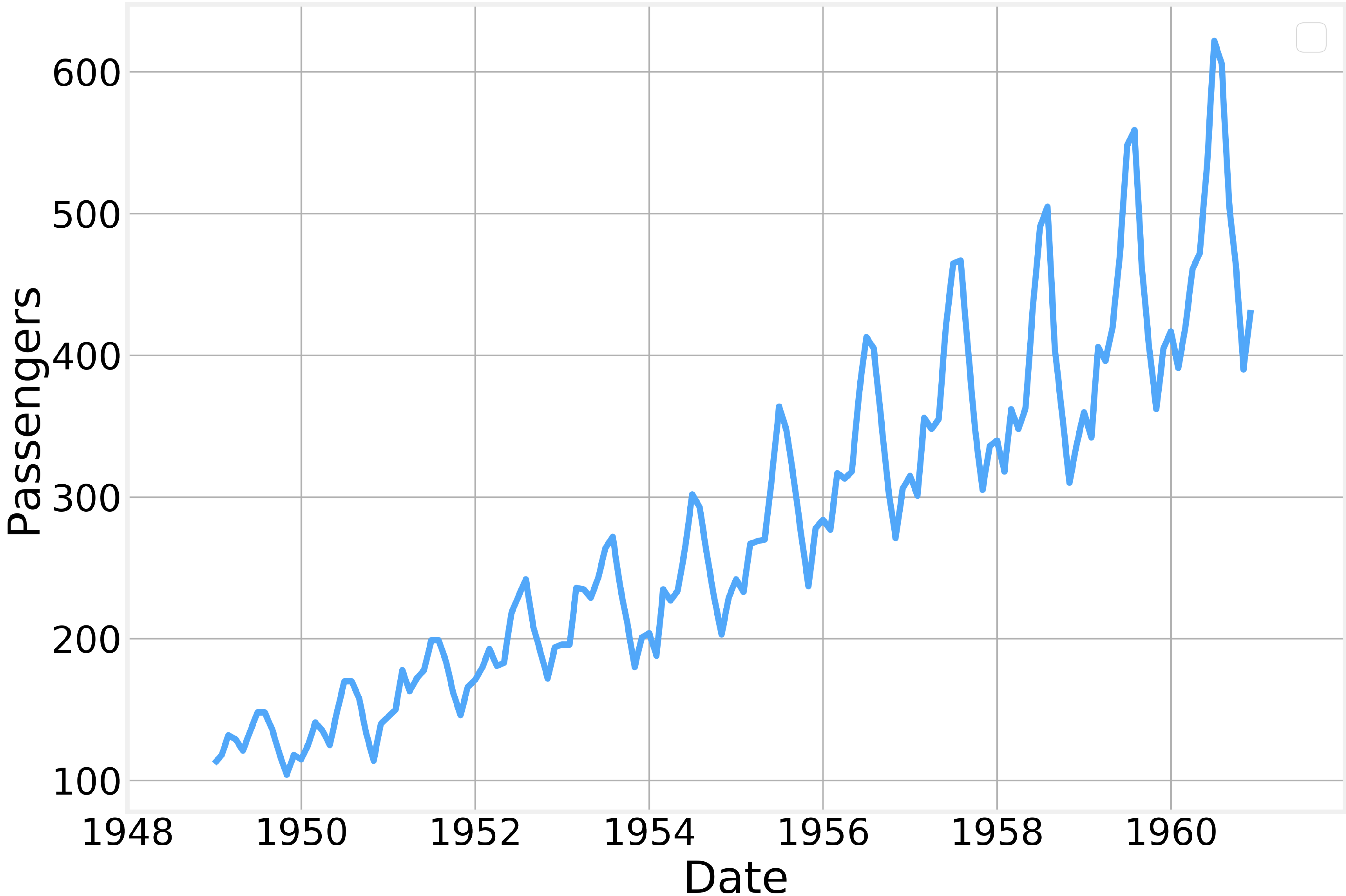

GDP

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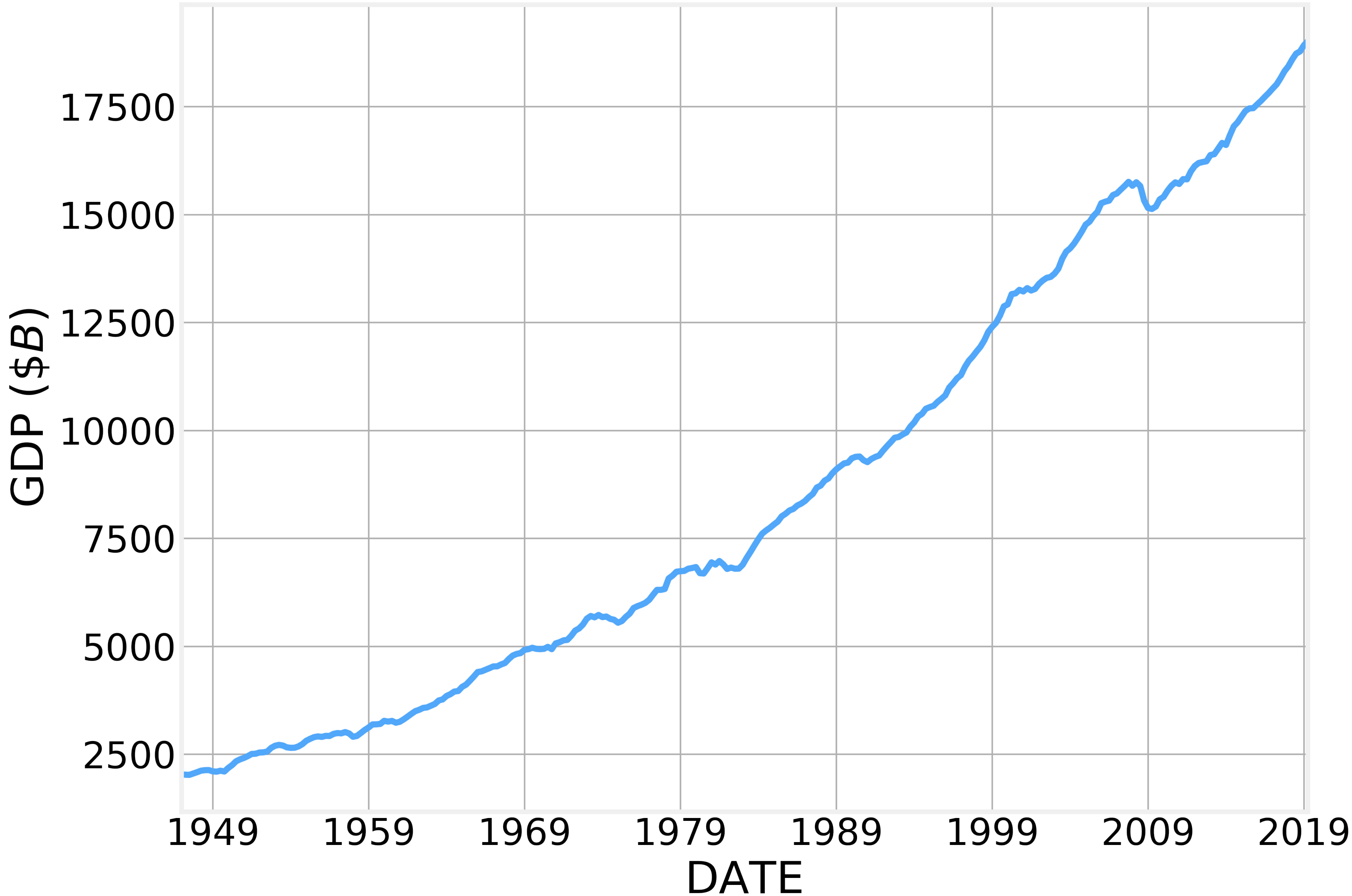
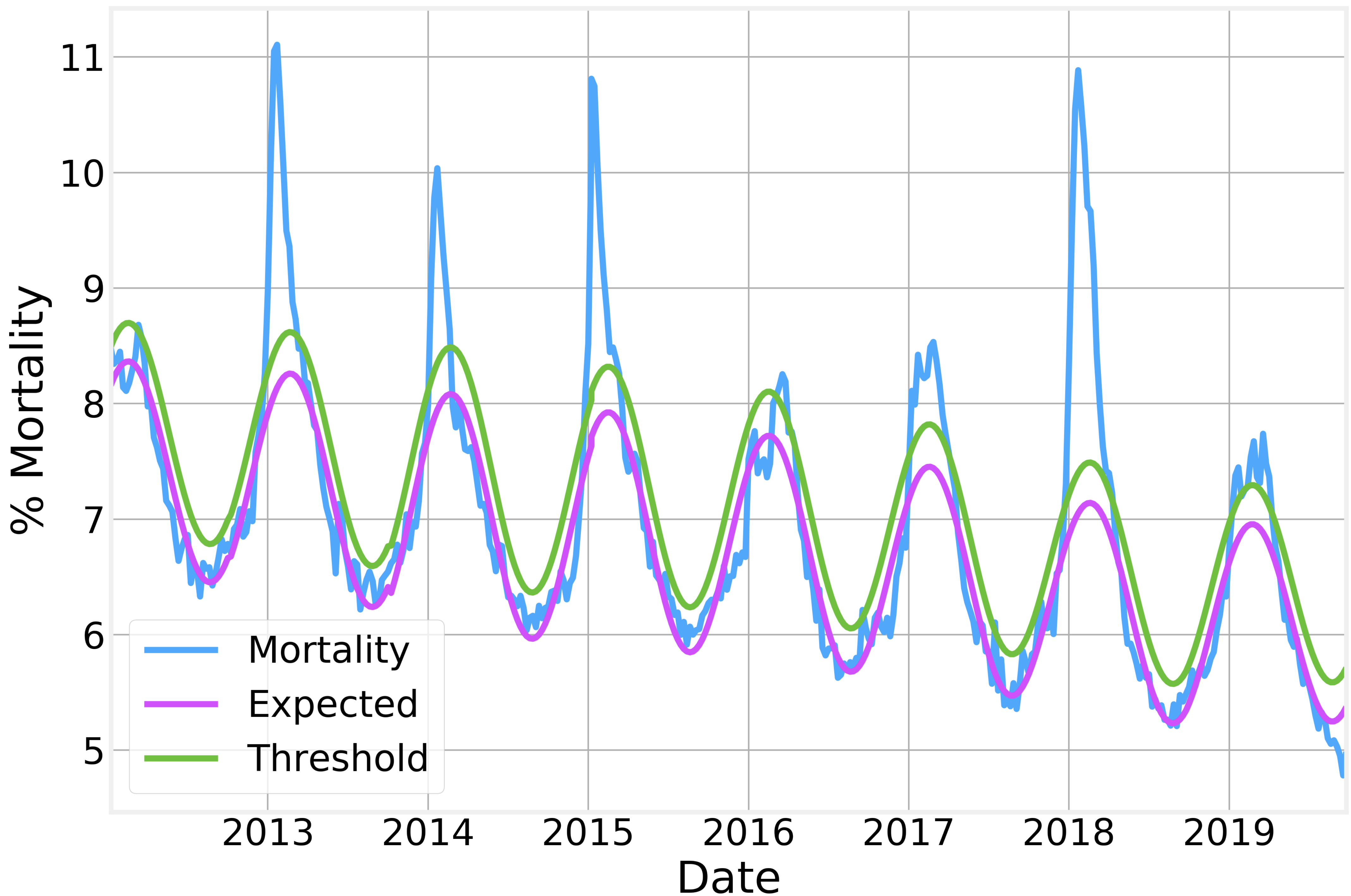


Influenza

www.cdc.gov/flu/weekly/



Sunspot activity

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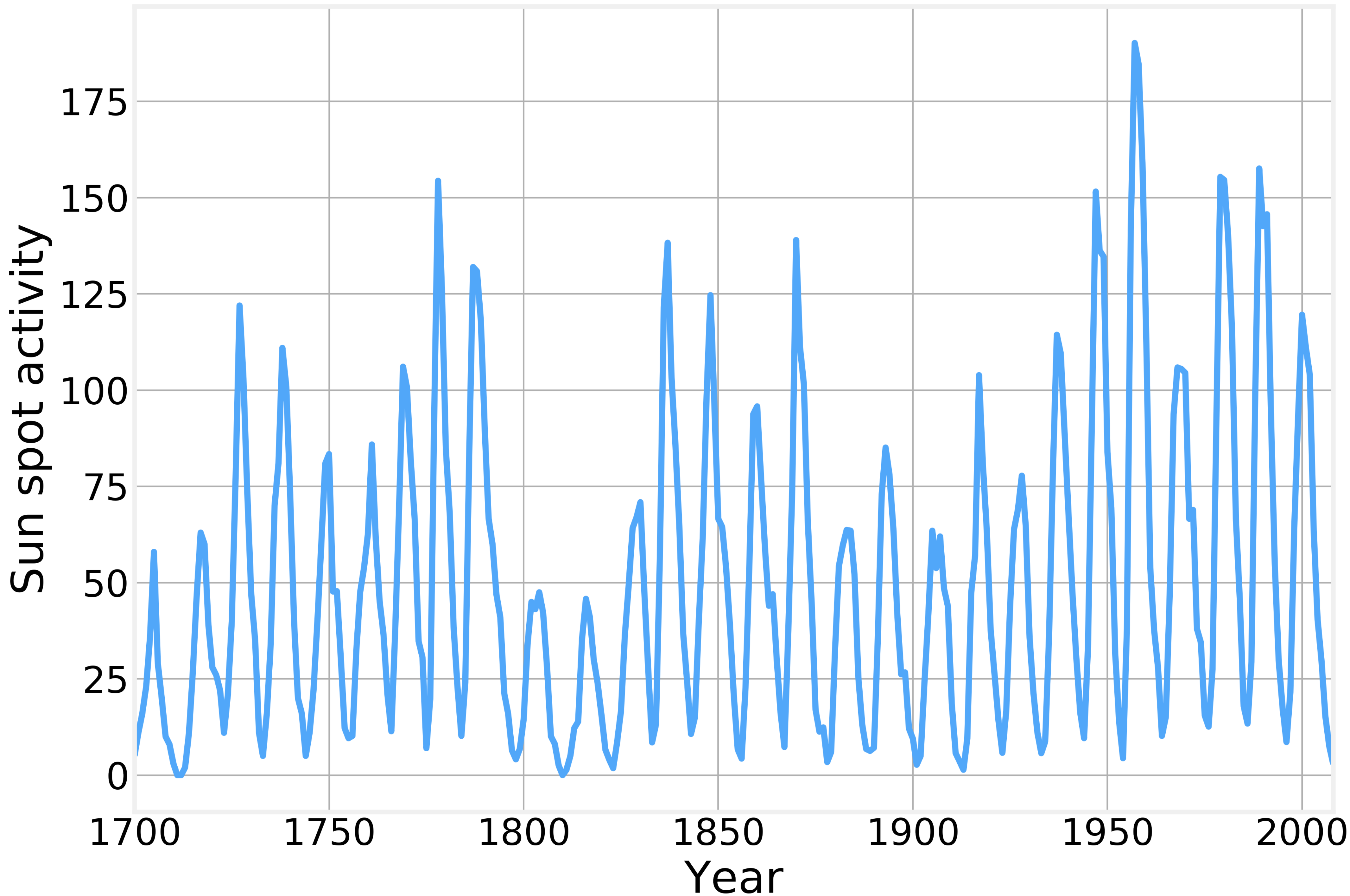
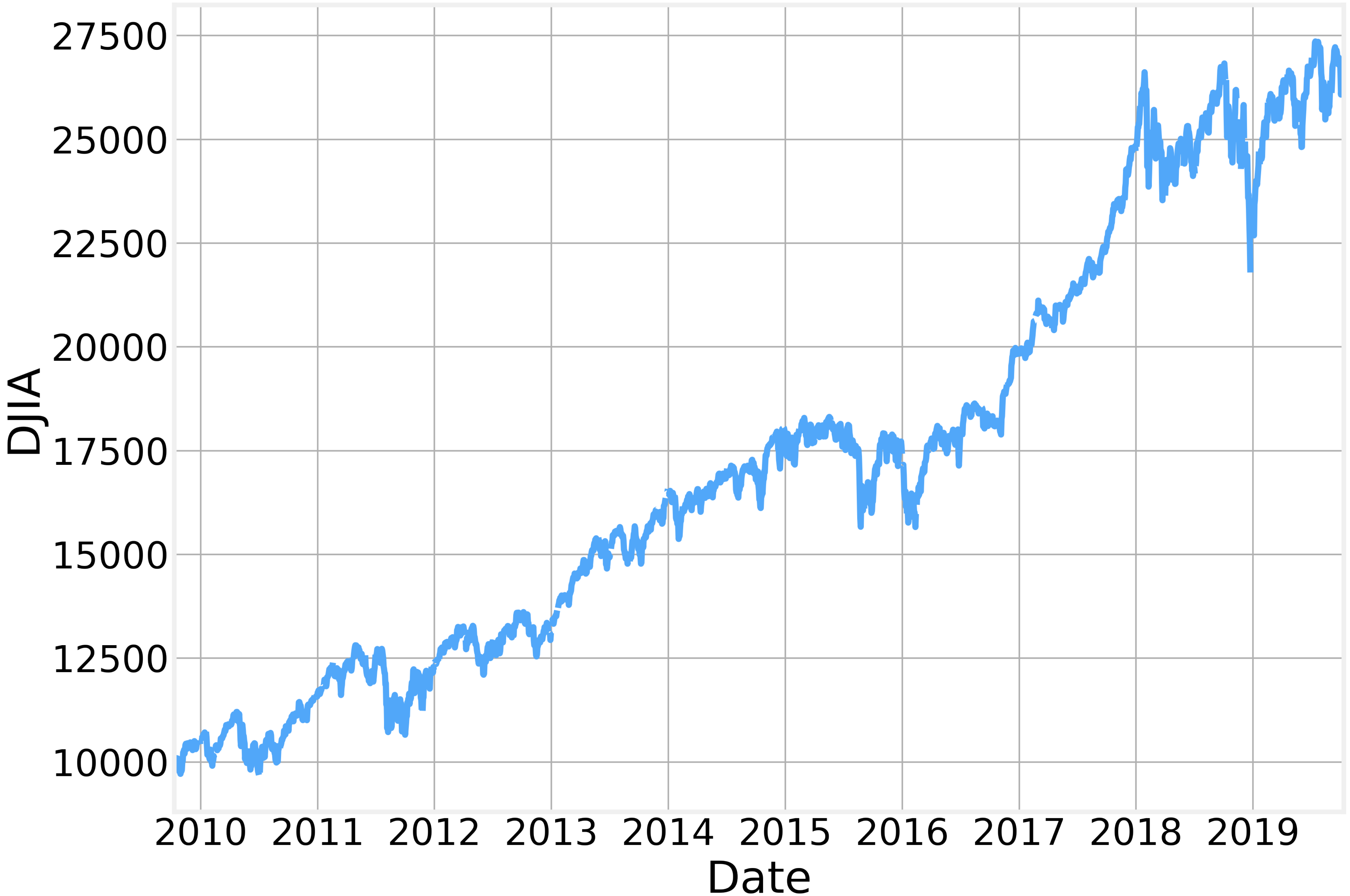


Stock Market - DJIA

https://fred.stlouisfed.org/series/DJIA



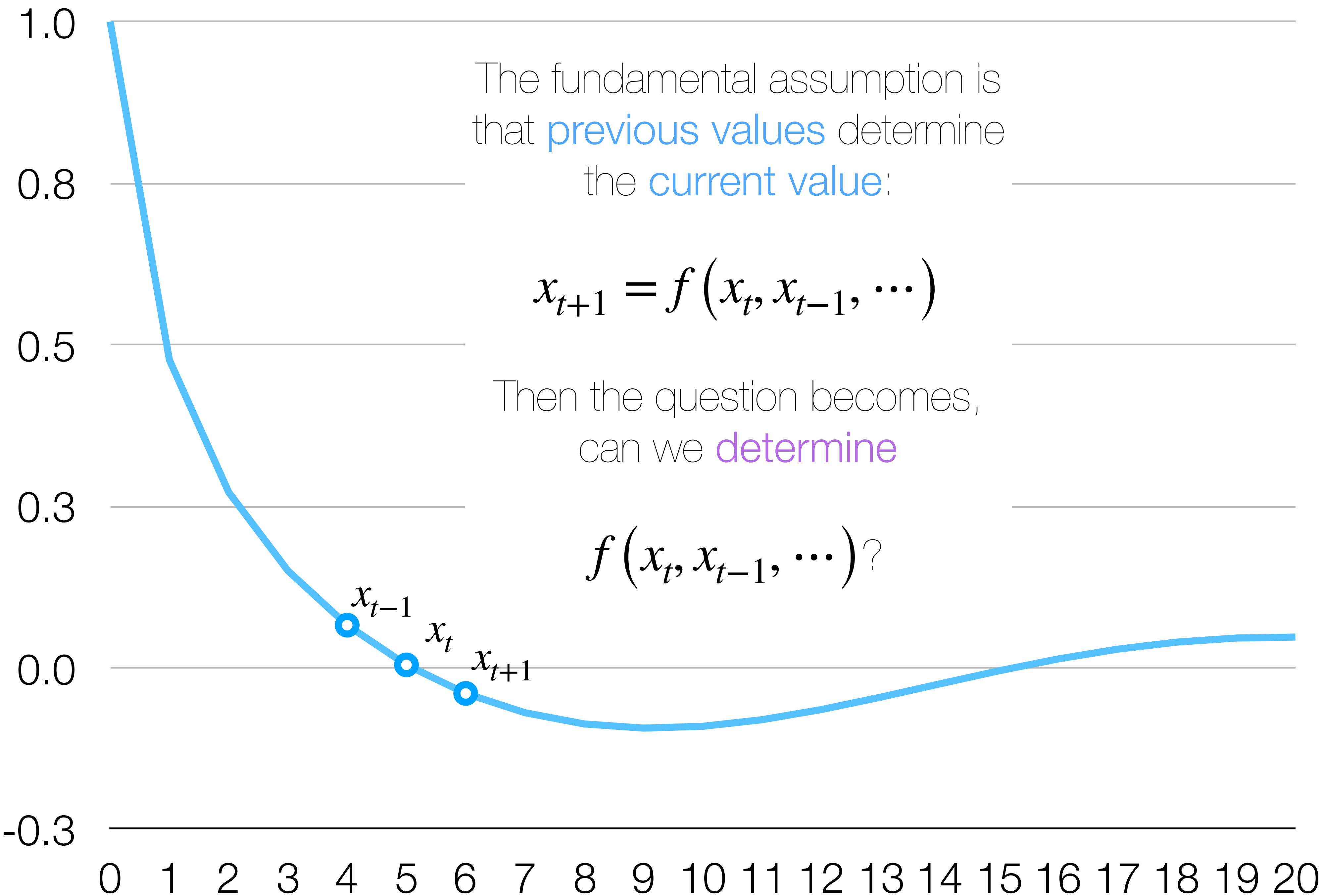
Time series

- A set of values measured sequentially in time
- Values are typically (but not always) measured at equal intervals, x_1, x_2, x_3, x_3, etc…
- Values can be:
    - continuous
    - discrete or symbolic (words).
- Associated with empirical observation of time varying phenomena:
    - Stock market prices (day, hour, minute, tick, etc…)
    - Temperatures (day, minute, second, etc…)
    - Number of patients (week, month, etc…)
    - GDP (quarter, year, etc…)
- Forecasting requires predicting future values based on past behavior

Mathematical Conventions

- The value of the time series at time t is given by x_t
- The values at a given lag l are given by x_{t-l}
- The mean of the overall signal is μ and the corresponding running value is μ_t^w
- The variance of the overall signal is σ and the corresponding running value is σ_t^w
- Running values are calculated over a window of width w

Time series analysis



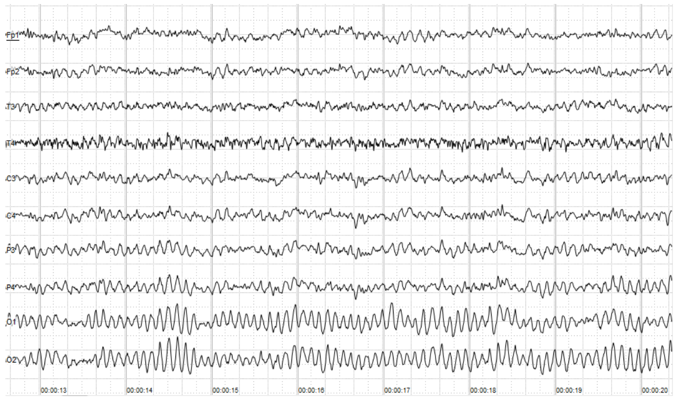
The fundamental assumption is that previous values determine the current value:

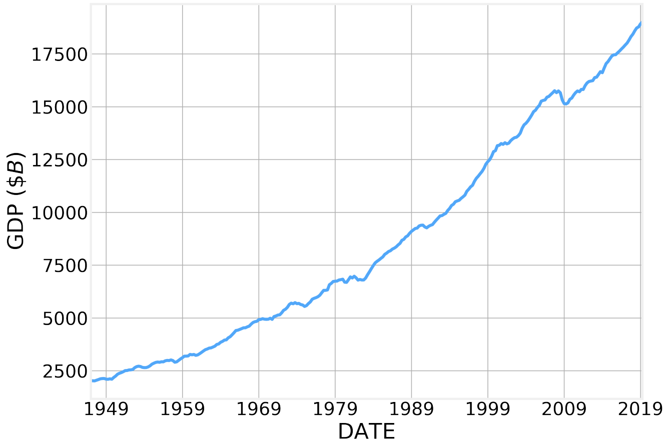
$$x_{t+1} = f\left(x_t, x_{t-1}, \cdots\right)$$

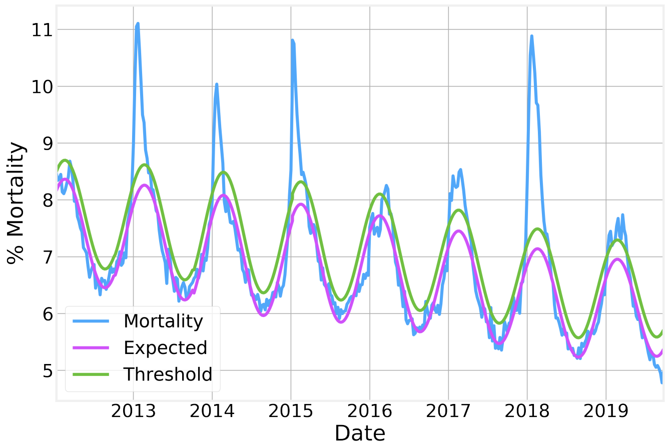
Then the question becomes, can we determine

$$f\left(x_t, x_{t-1}, \cdots\right)?$$

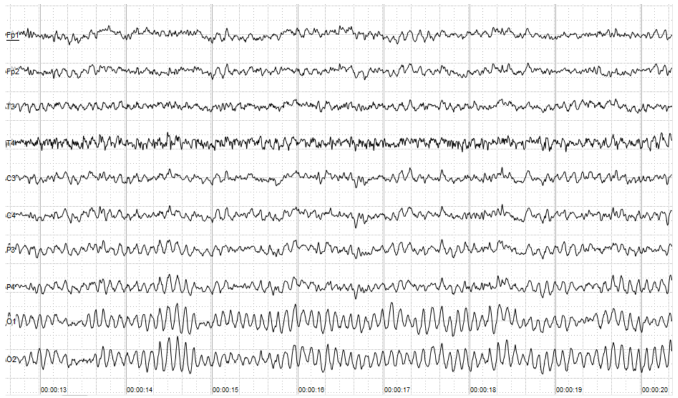
There fundamental behaviors





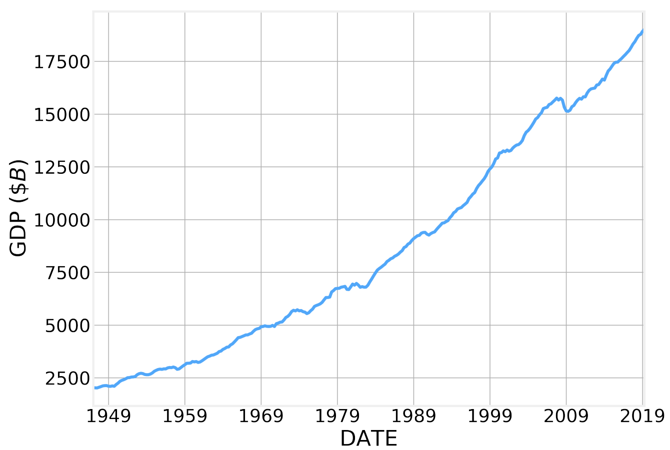


There fundamental behaviors



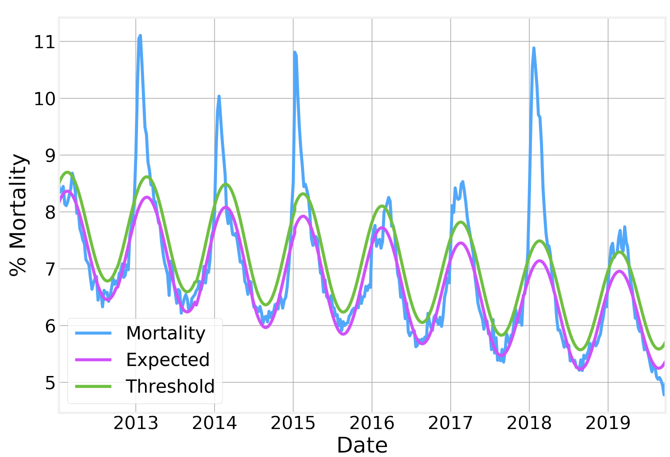
Stationary

$$\langle x_t \rangle \approx \text{constant}$$



Trending

$$\langle x_t \rangle \approx ct$$



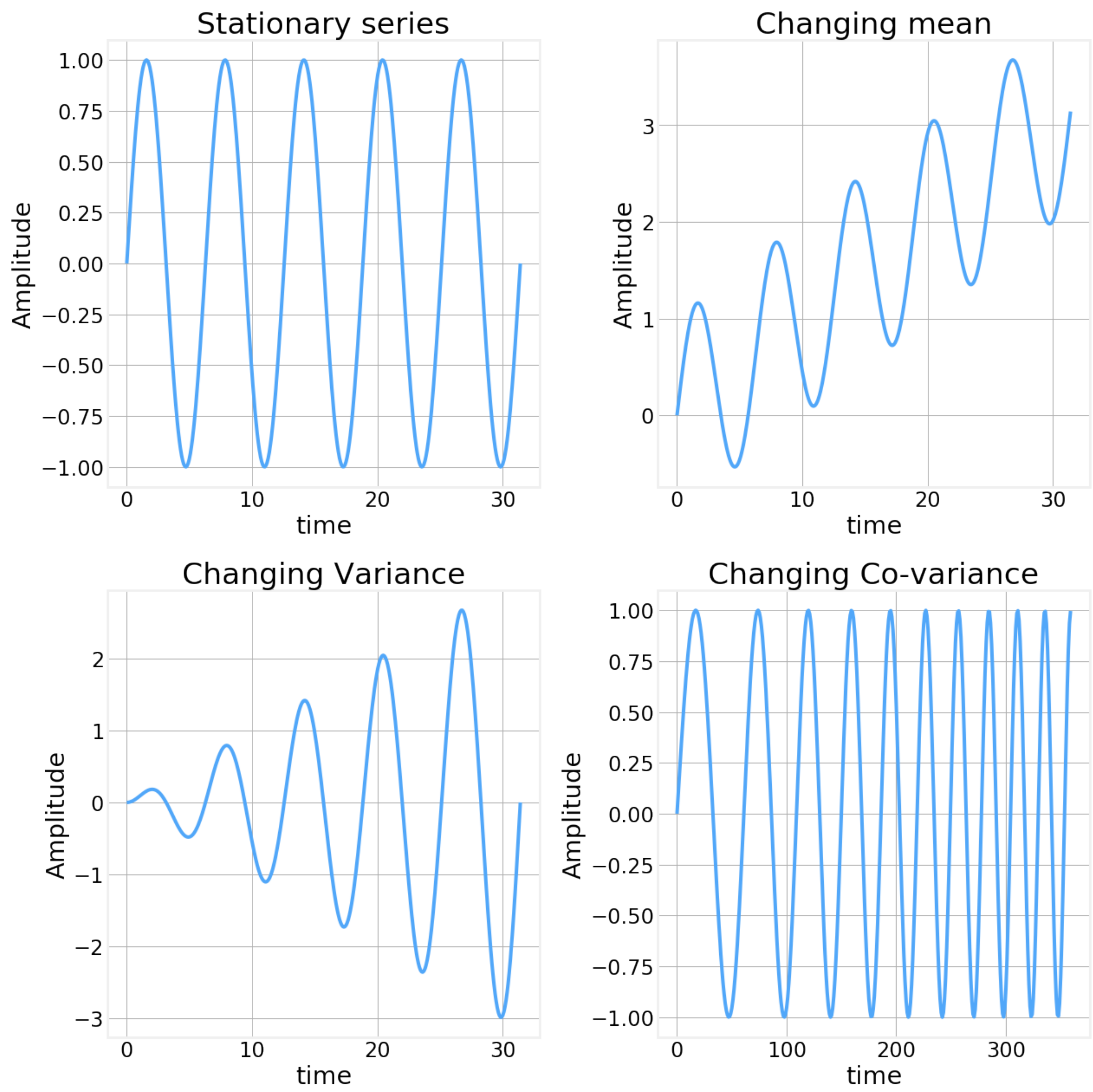
Seasonal

$$x_{t+T} \approx x_t$$

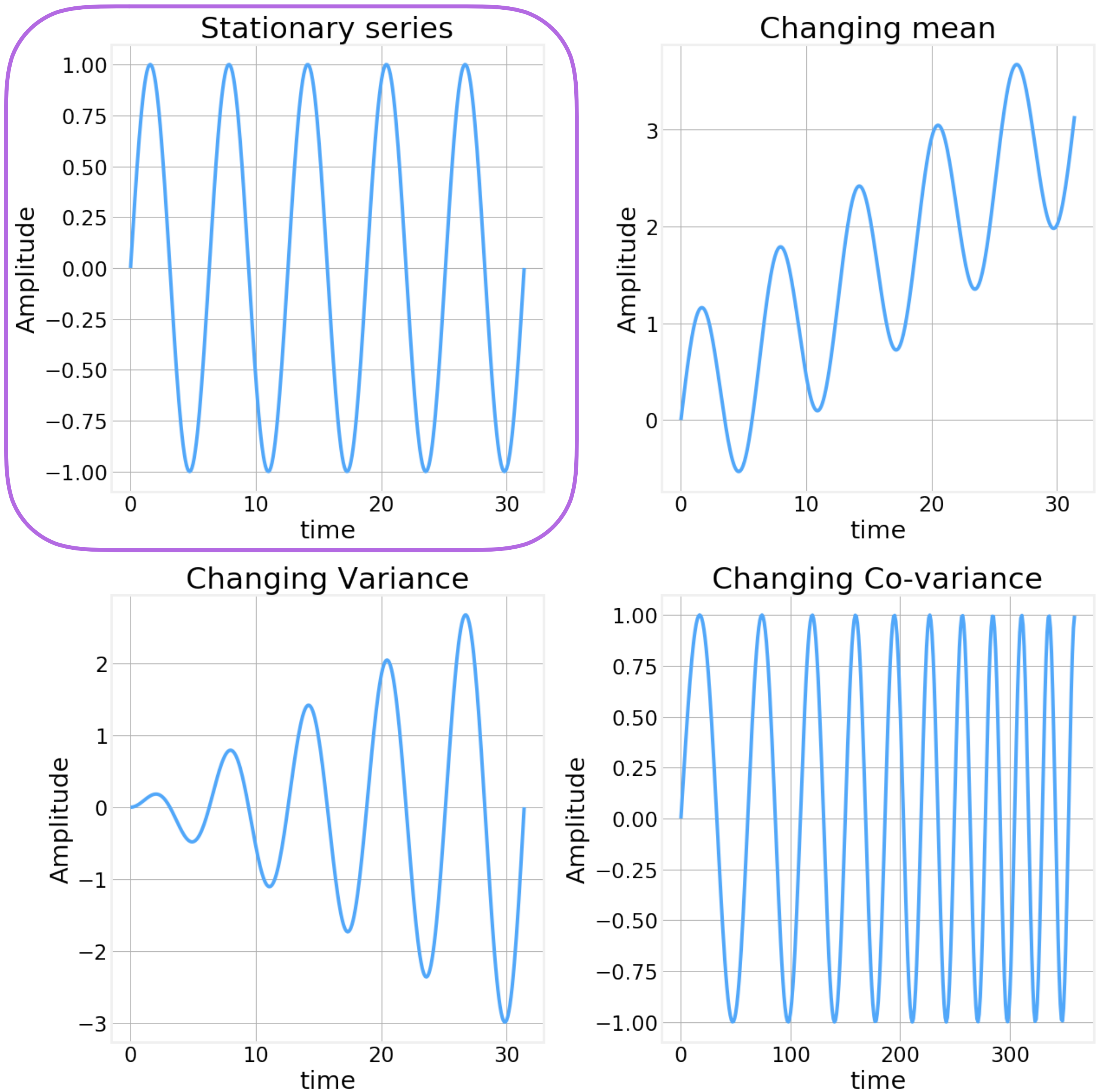
Stationarity

- A time series is said to be stationary if its basic statistical properties are independent of time
- In particular:
    - Mean - Average value stays constant
    - Variance - The width of the curve is bounded
    - Covariance - Correlation between points is independent of time
- Stationary processes are easier to analyze
- Many time series analysis algorithms assume the time series to be stationary
- Several rigorous tests for stationarity have been developed such as the (Augmented) Dickey-Fuller and Hurst Exponent
- Typically, the first step of any analysis is to transform the series to make it stationary

Stationarity



Stationarity



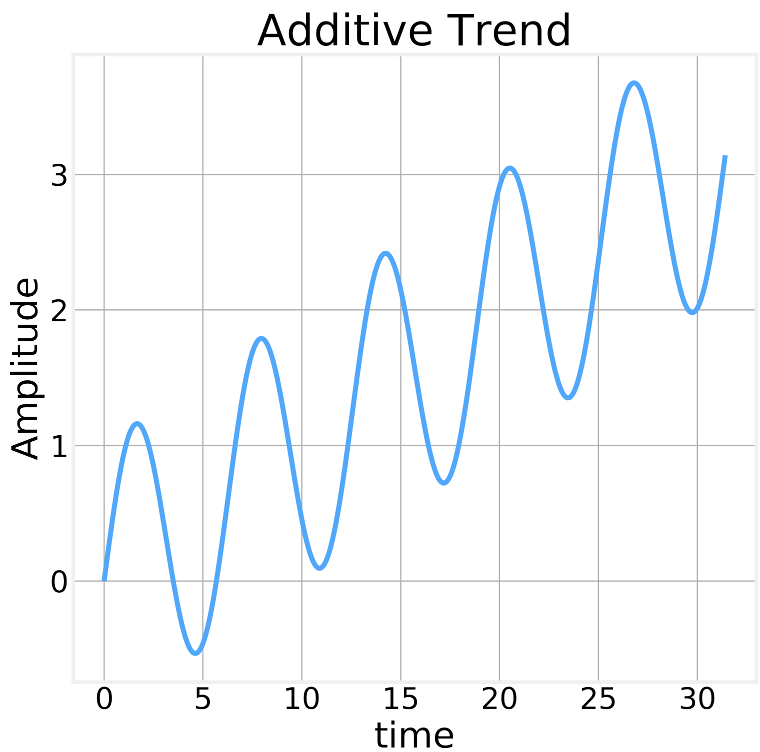
Code - Data Exploration

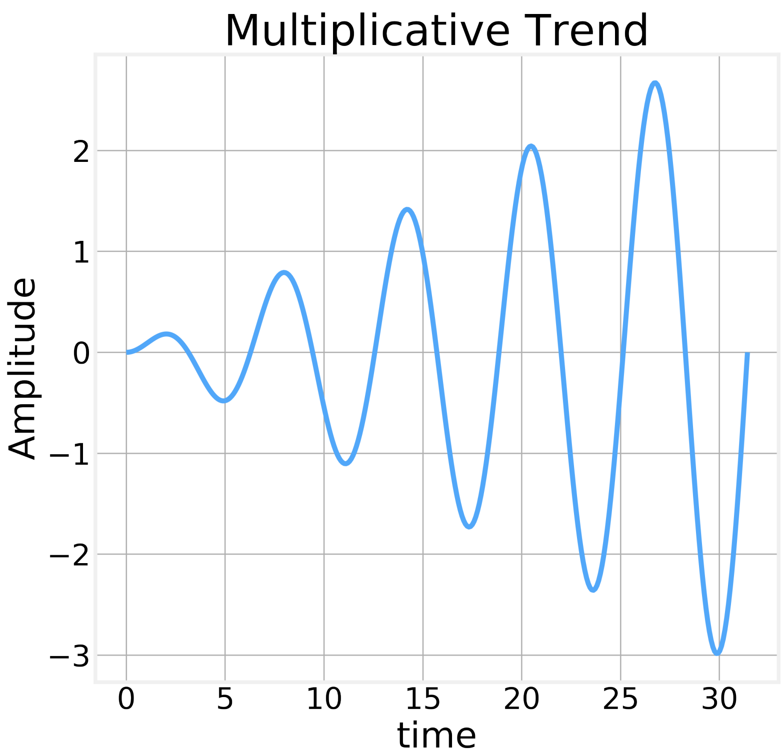
https://github.com/DataForScience/Timeseries

Trend

- Many time series have a clear trend or tendency:
  - Stock market indices tend to go up over time
  - Number of cases of preventable diseases tends to go down over time
  - etc
- Trends can be additive or multiplicative:

$$x = \frac{t}{10} + \sin(t)$$

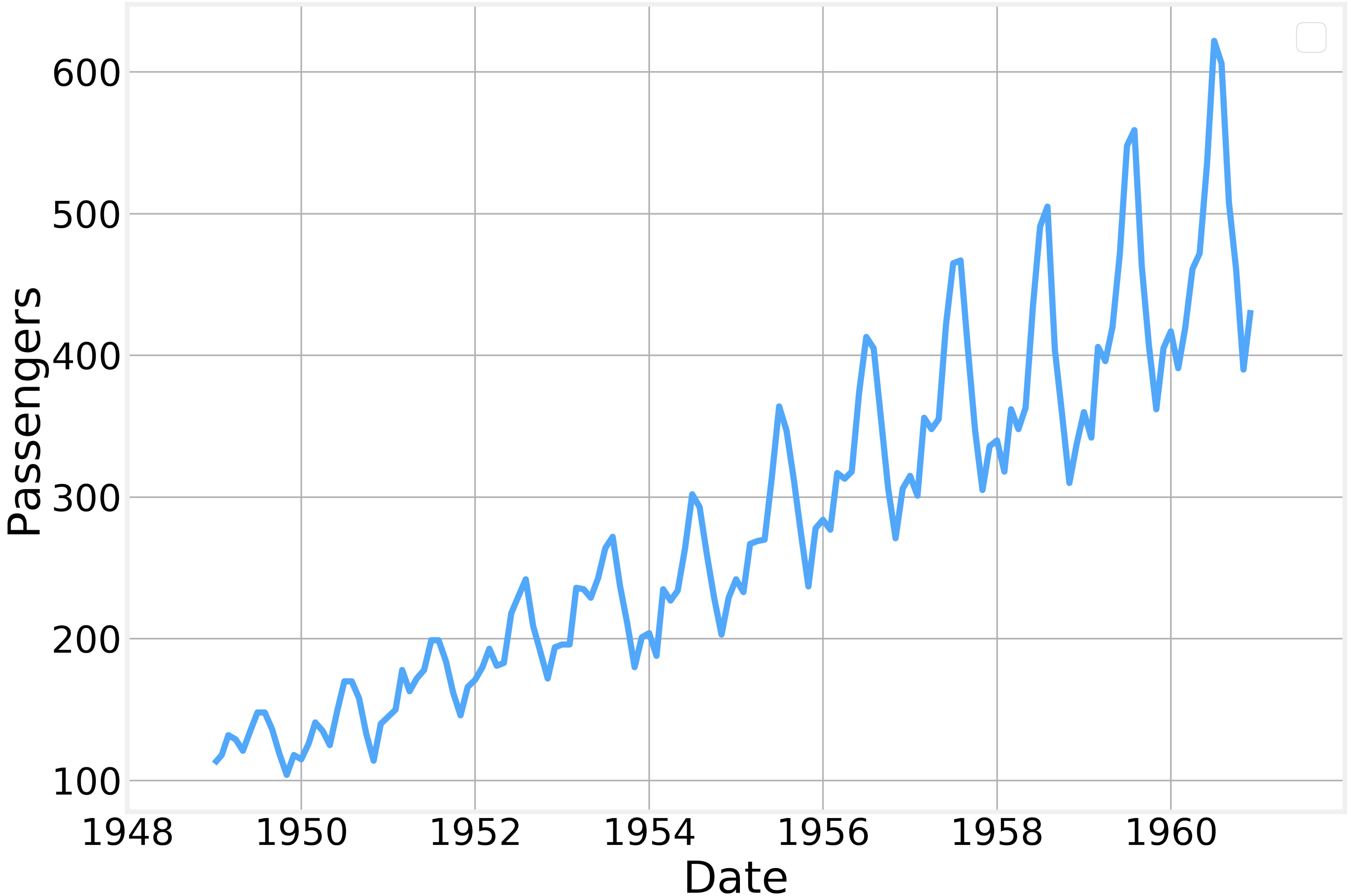




$$x = \frac{t}{10} \sin(t)$$

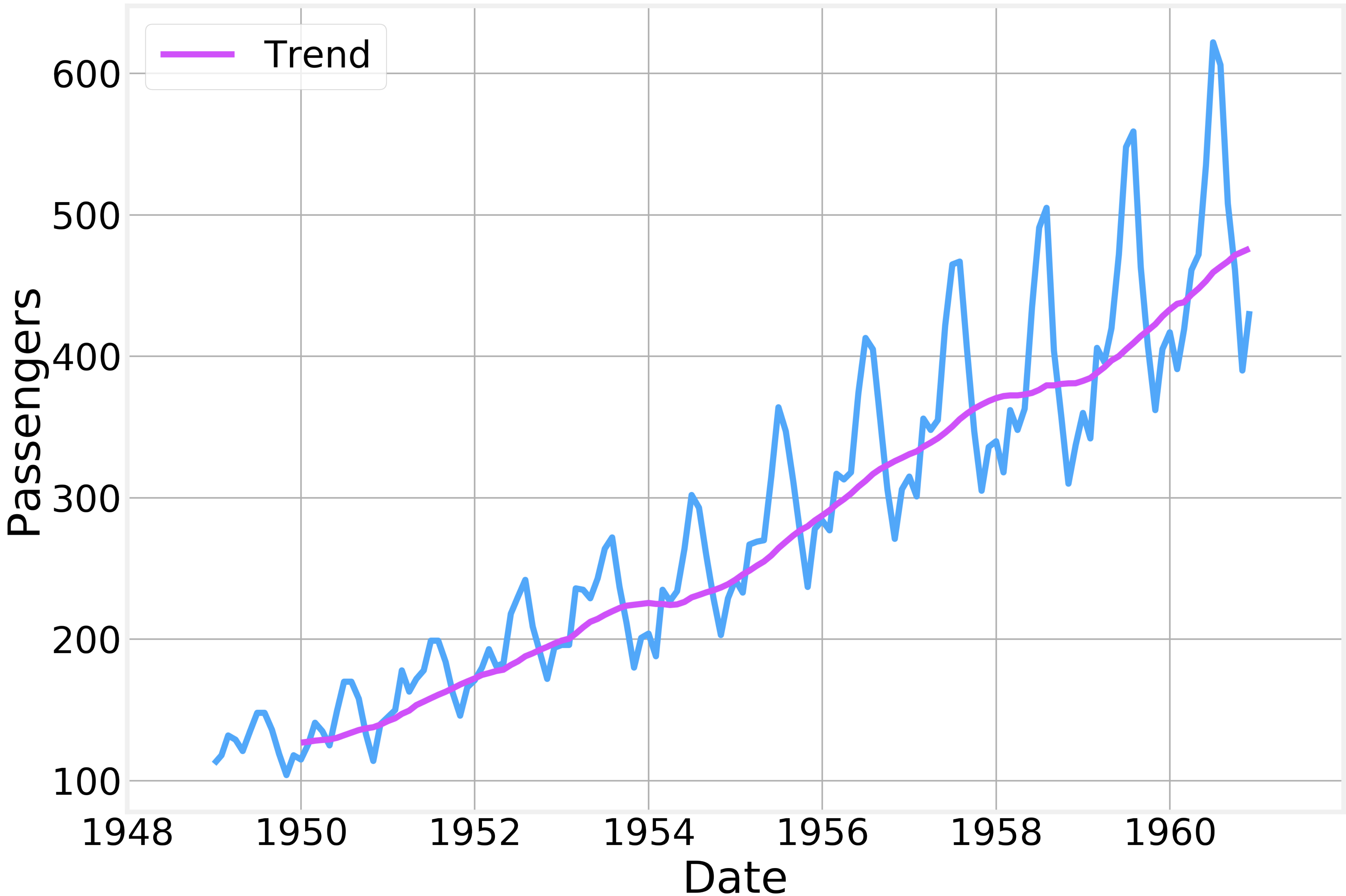
- Trends can be removed by subtraction or division of the correct values
- One simple way to determine the trend is to calculate a running average over the series

Trend



Passengers
600
500
400
300
200
100
1948
1950
1952
1954
1956
1958
1960
Date

Trend



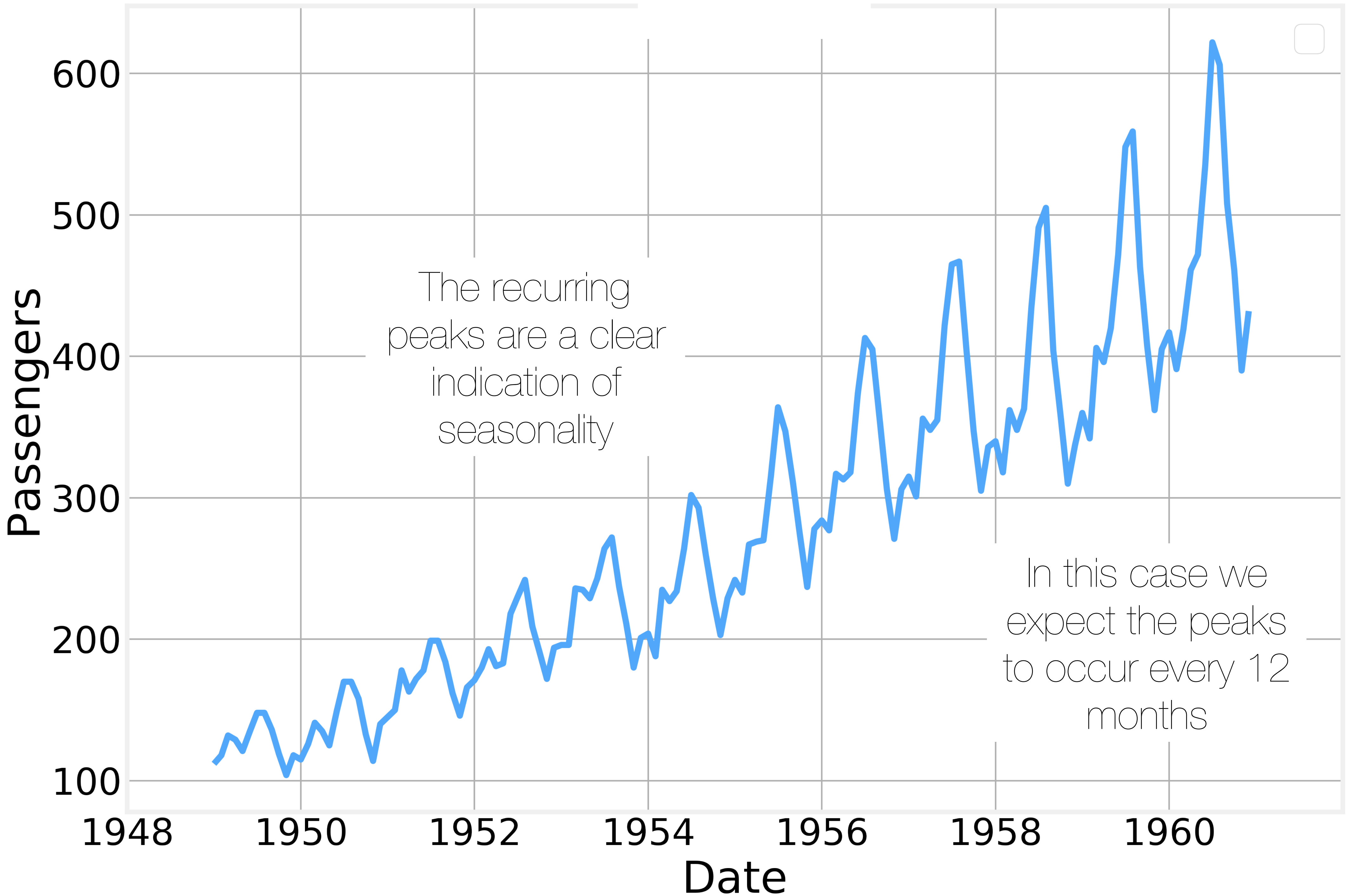
Trend
600
500
400
300
200
100
Passengers
1948
1950
1952
1954
1956
1958
1960
Date

Seasonality

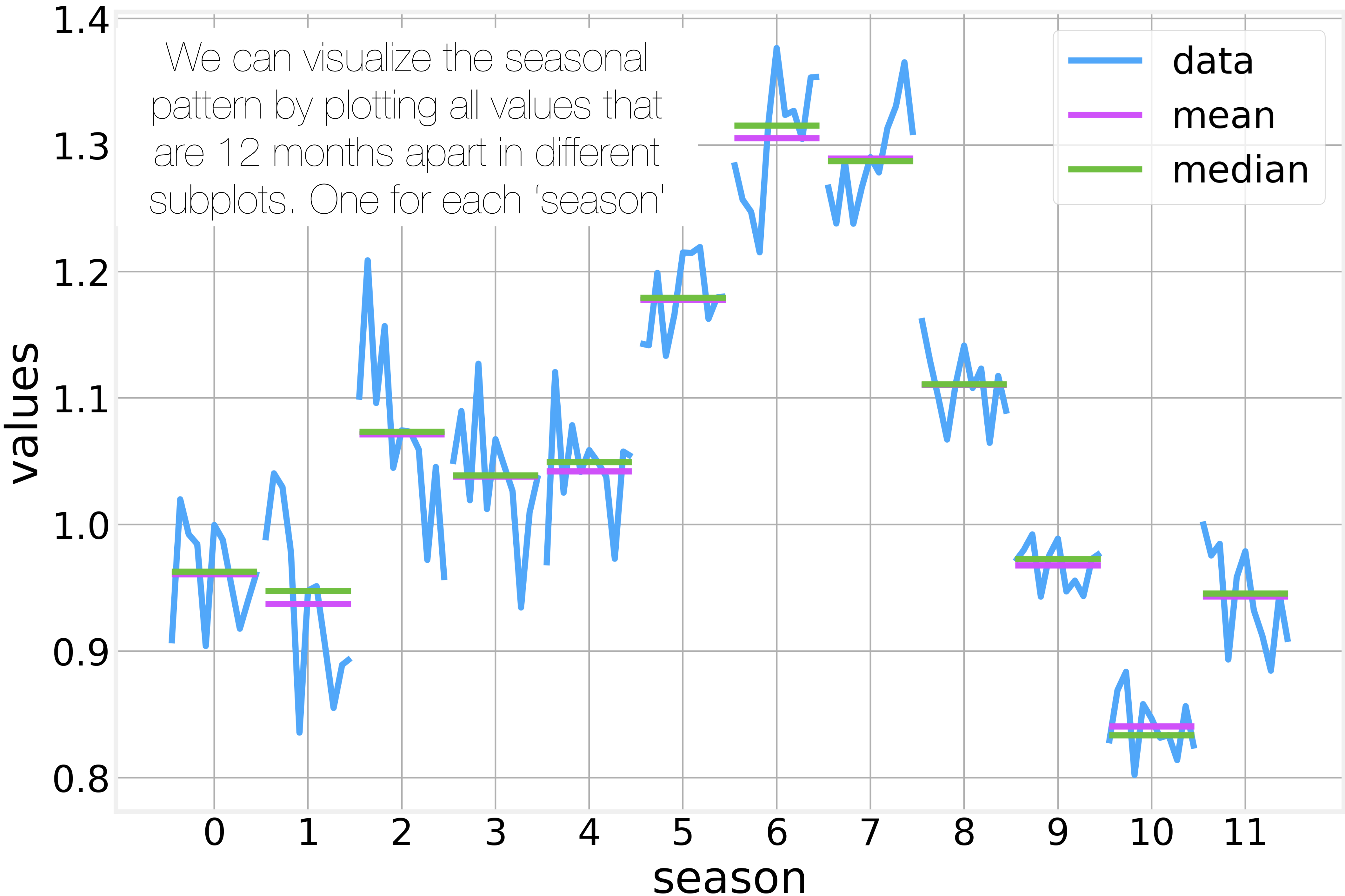
- Many of the phenomena we might be interested in varying in time in a cyclical or seasonal fashion
    - Ice-cream sales peak in the summer and drop in the winter
    - Number of cell phone calls made is larger during the day than at night
    - Many types of crime are more frequent at night than during the day
    - Visits to museums are more frequent in the weekend than in weekdays
    - The stock market grows during bull periods and shrinks during bear periods
    - etc
- Understanding the seasonality of a time series provides important information about its long term behavior and is extremely useful in predicting future values
- If the period is fixed it's called seasonality, while if the period is irregular it's called cyclical

Seasonality

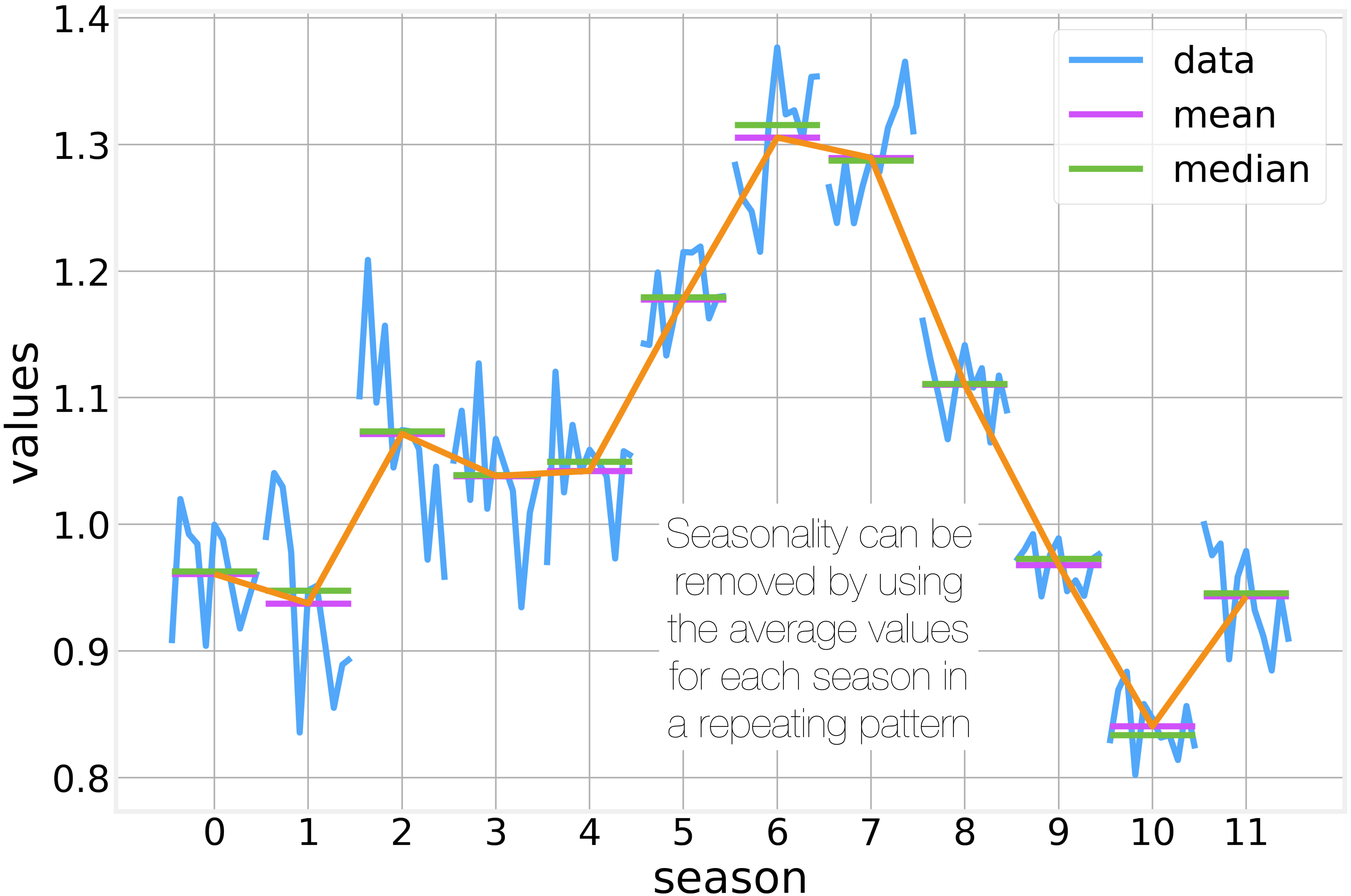
https://www.kaggle.com/chirag19/air-passengers



Seasonality



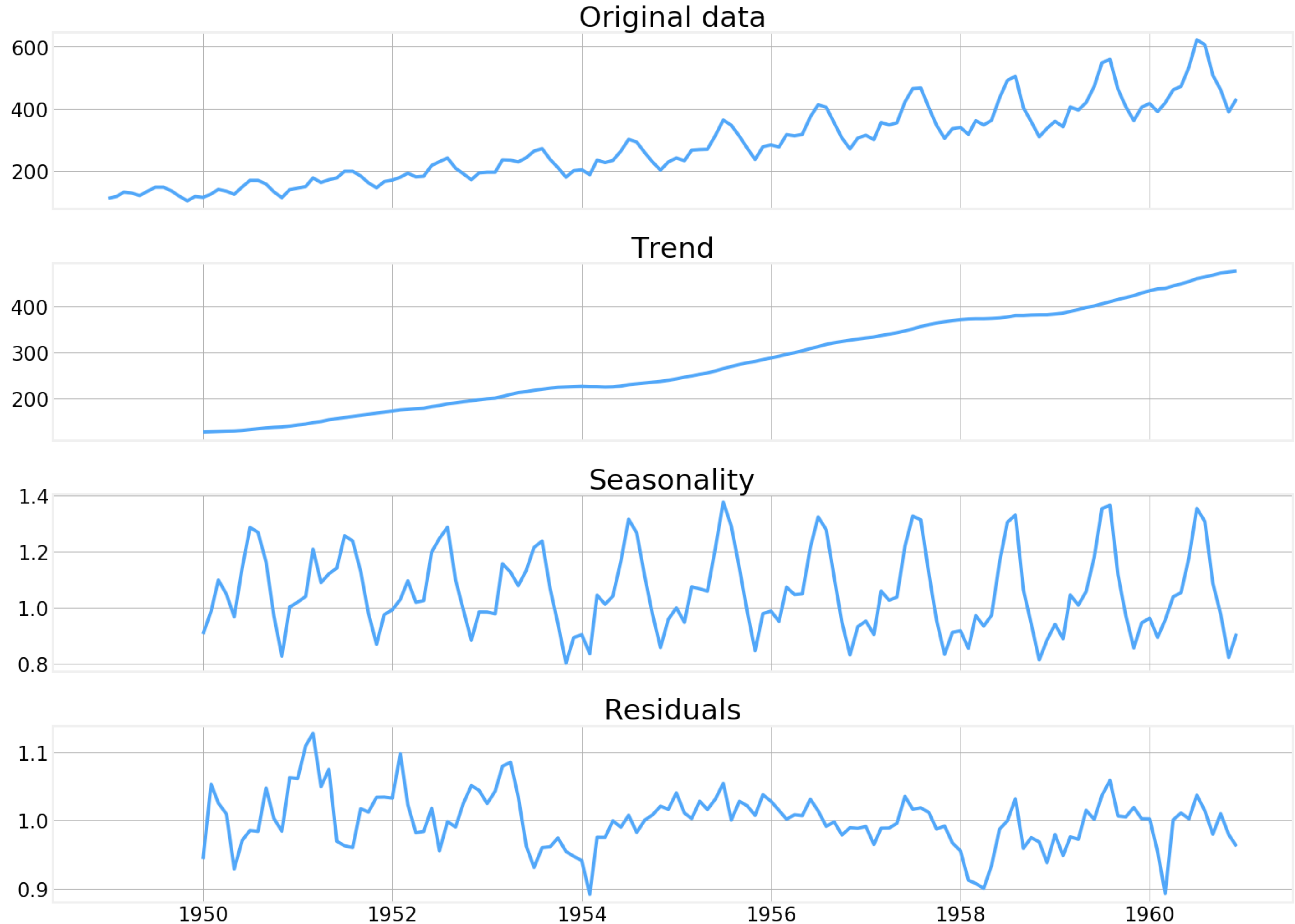
Seasonality



Time series decomposition

- A time series can be decomposed into three components:
    - Trend, T_t
    - Seasonality, S_t
    - Residuals, R_t
- Decompositions can be
    - additive - $x_t = T_t + S_t + R_t$
    - multiplicative - $x_t = T_t \cdot S_t \cdot R_t$
- The residuals are simply what is left of the original signal after we remove the trend and the seasonality
- Residuals are typically stationary

Time series decomposition



Code - Decomposition

https://github.com/DataForScience/Timeseries

Lesson II:
Processing Timeseries data

Lagged values and Differences

- While analyzing time series, we often refer to values that our time series took 1, 2, 3, etc time steps in the past
- These are known as lagged values and denoted:

$$x_{t-l}$$

- where l is the value of the lag we are considering.
- Perhaps the most common case is for the calculation of differences of the form:

$$x_t - x_{t-1}$$

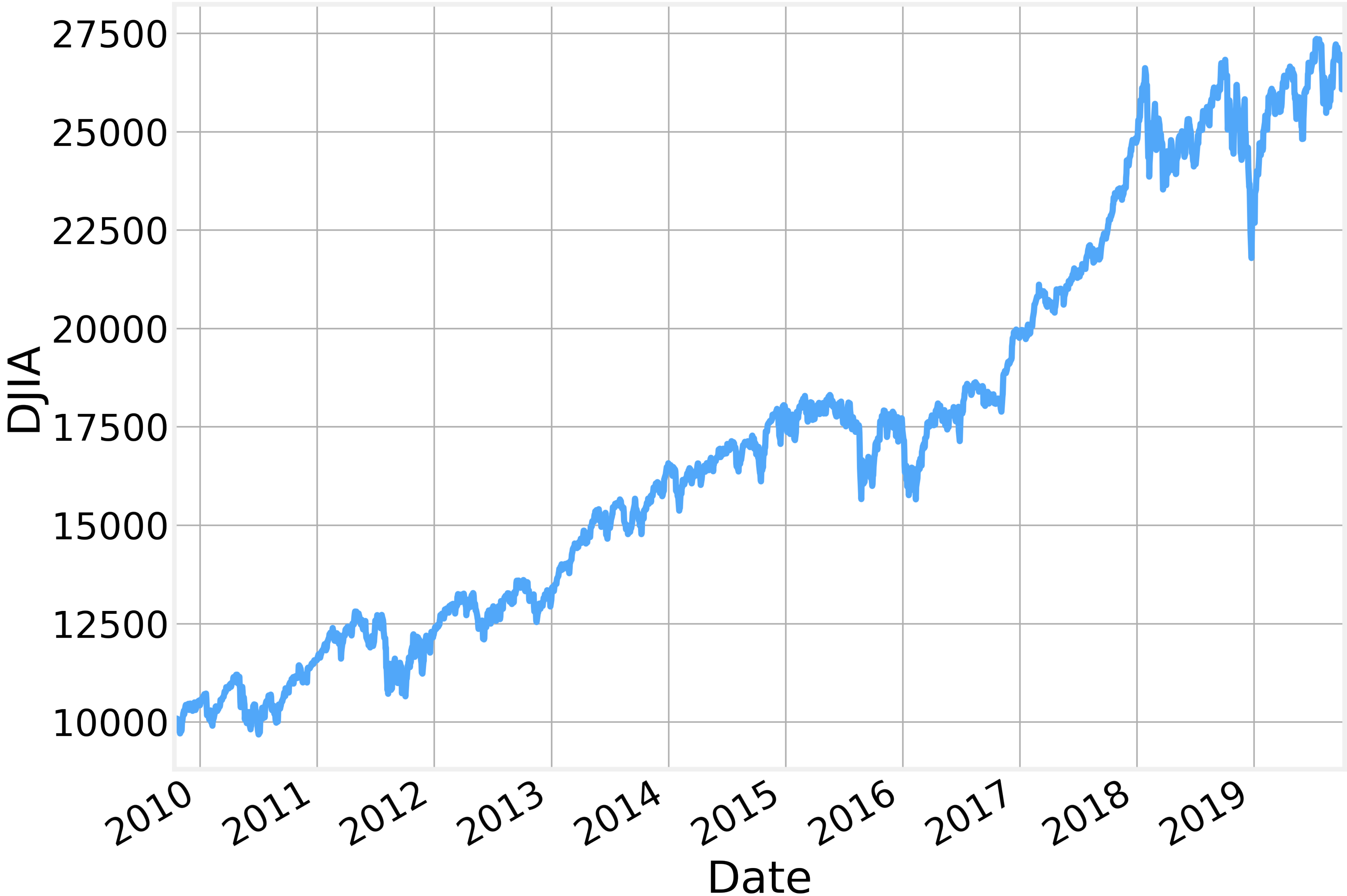
- Naturally, higher order differences can also be used, in which case, the difference of the difference is calculated:

$$y_t = x_t - x_{t-1}$$

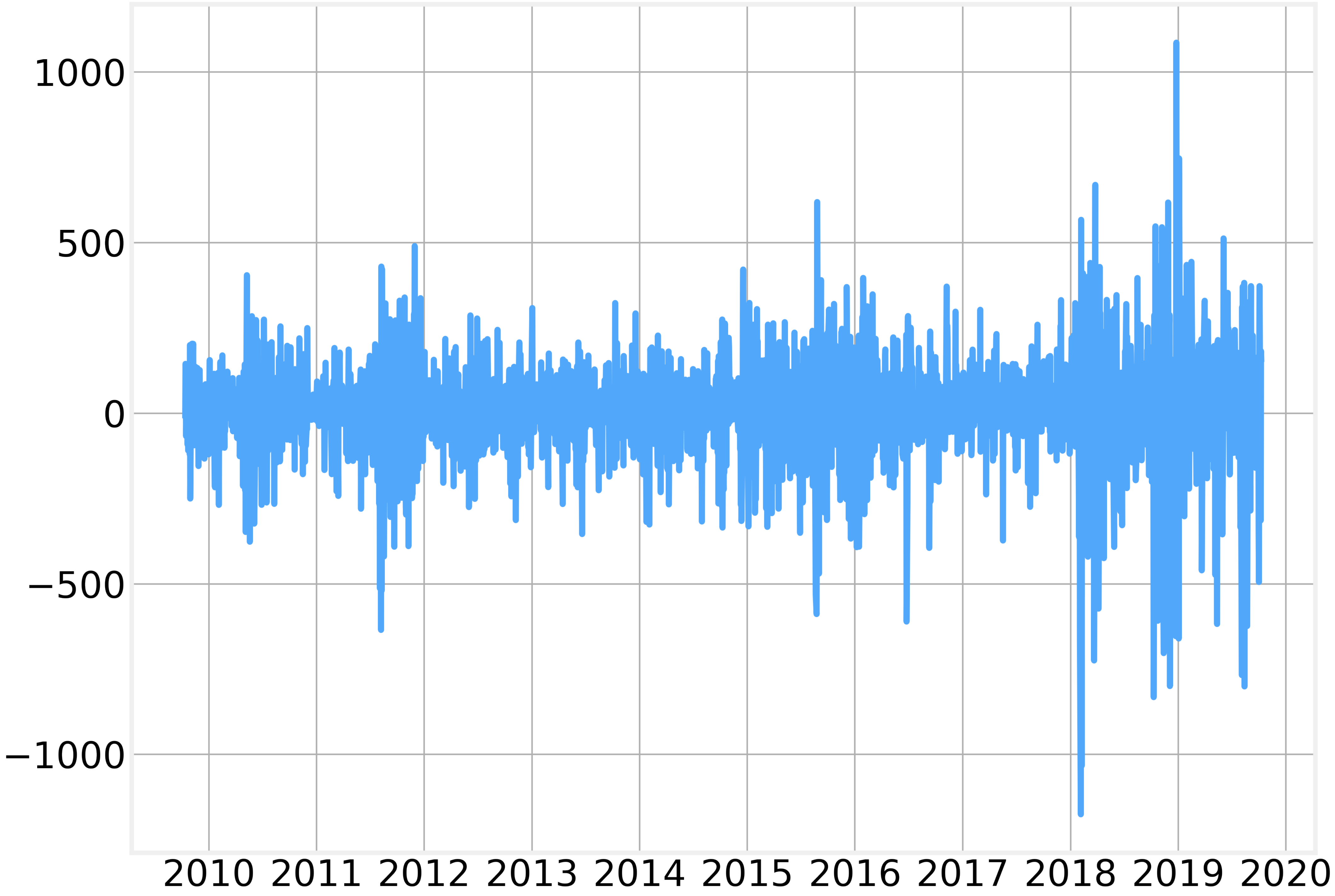
$$z_t = y_t - y_{t-1} \equiv x_t - 2x_{t-1} + x_{t-2}$$

- This can be thought of as a discrete version of the usual derivative of a function.
- Differences are also a particularly simple way to detrend a time series

Differences

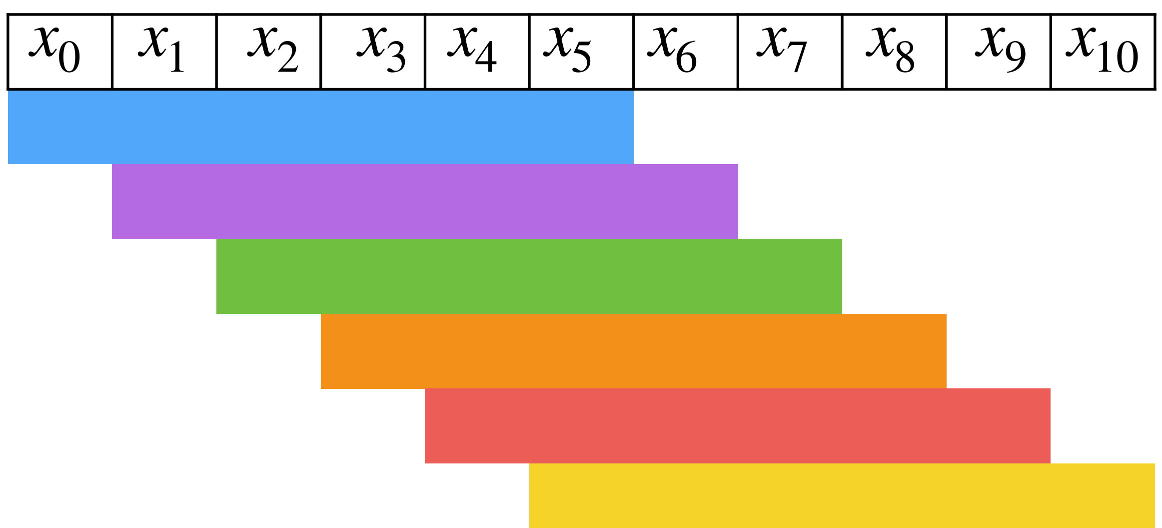


Differences



Windowing

- When analyzing the temporal behavior of of a signal, we often need to evaluate if specific quantities are time varying or not
- A common approach is to use sliding windows of a given length to evaluate the required values
- So a sliding window of width 6 on a series of length 11 would look like:



$$= \begin{matrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{matrix}$$

- and we would calculate the metric of interest within each window.

Running values

- In the first part of the lecture, we already used running averages to detrend a time series
- Other common metrics are:
  - Variance
  - Maximum value
  - Minimum value
  - etc…
- One important detail to note is that while using windowing to calculate running values we "lose" a number of points equal to the width of the window
- Depending on the application we can choose to place the missing values in either or (or even both) extremes of the time interval

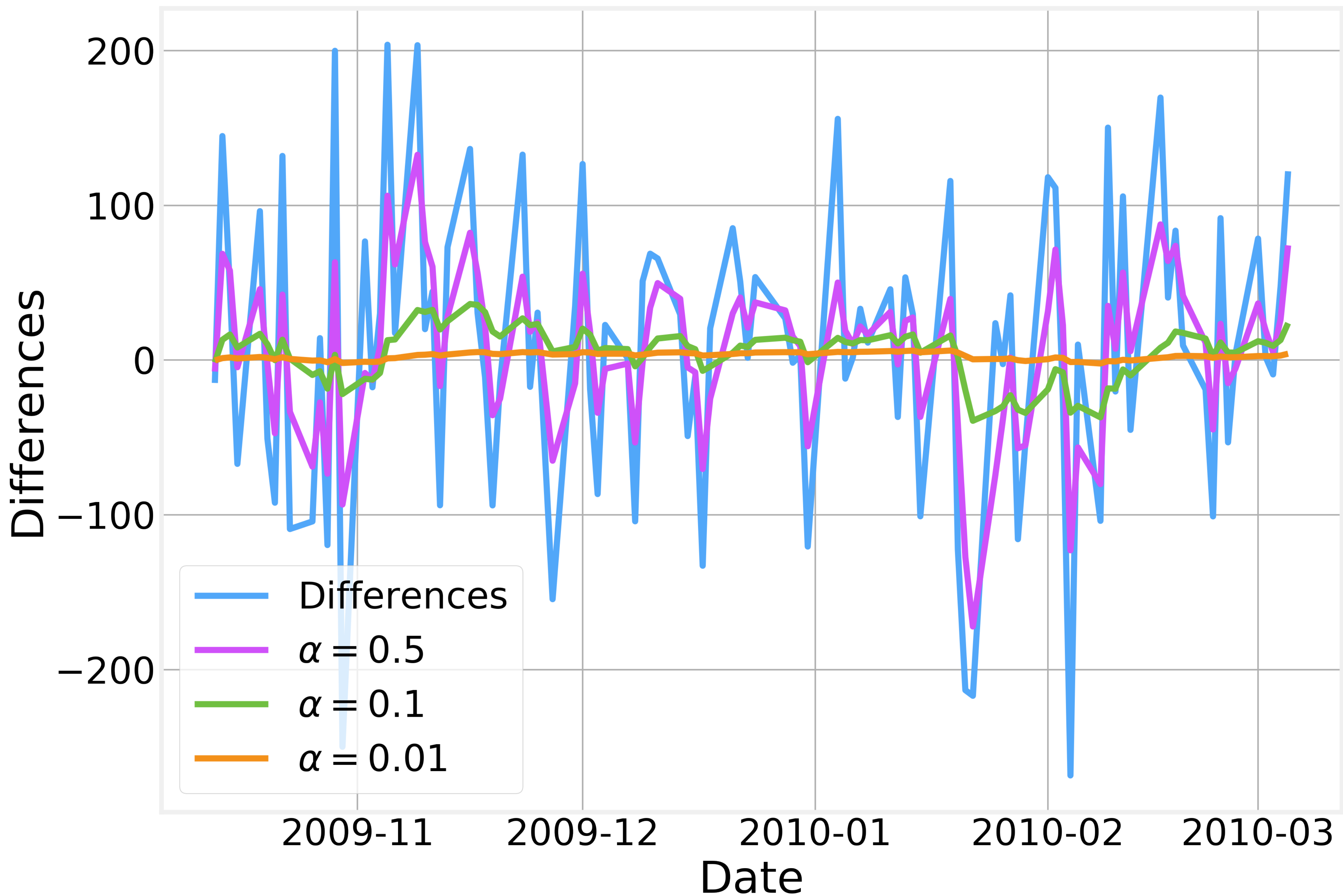
Exponential Smoothing

- One alternative to a simple running average is Exponential Smoothing
- The exponentially"smooth" version of a time series is given by:

$$s_t = \alpha x_t + (1 - \alpha) s_{t-1}$$

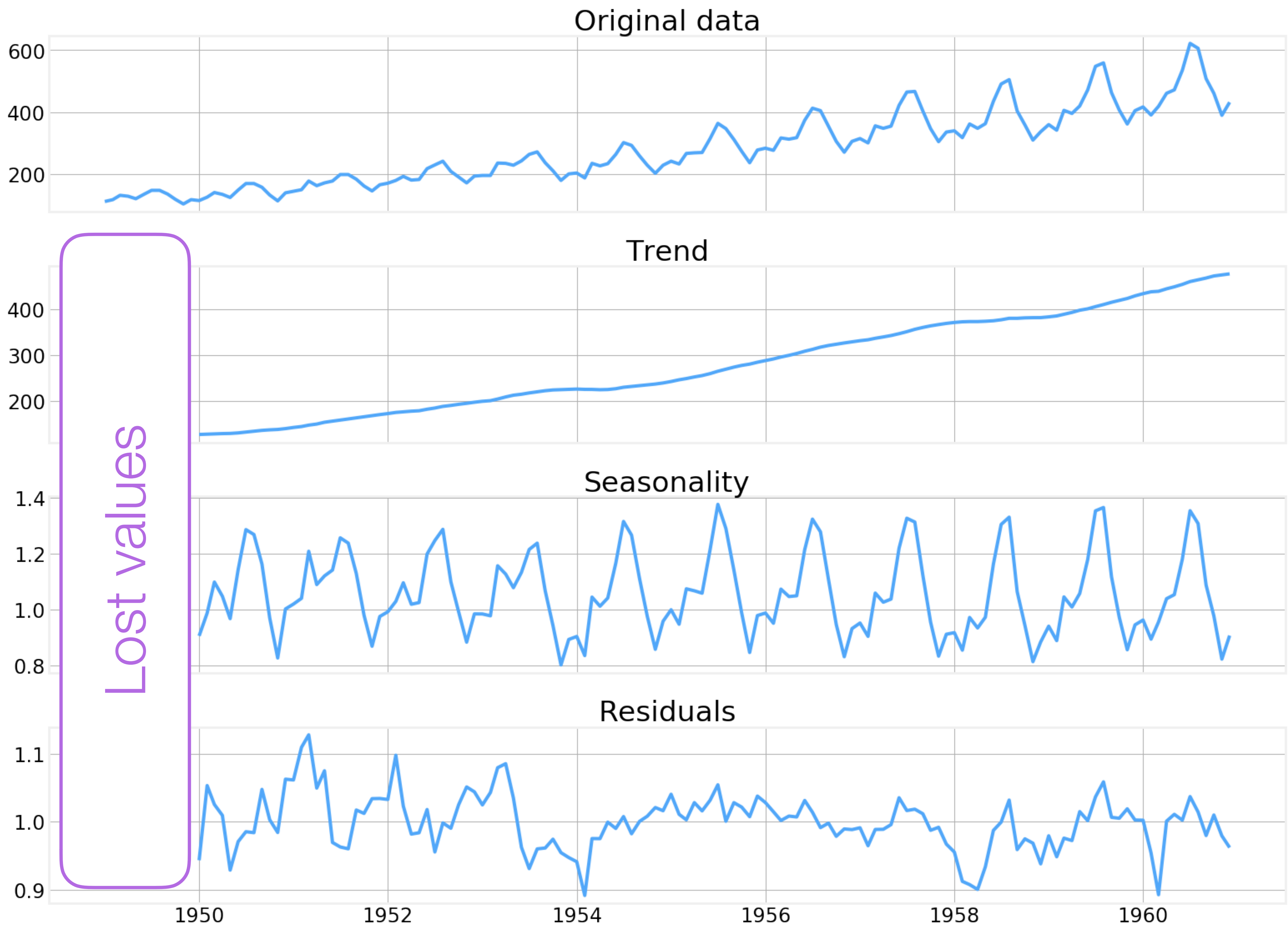
- The smaller the value of the weight α, the less influence each point has on the transformed time series.

Exponential Smoothing



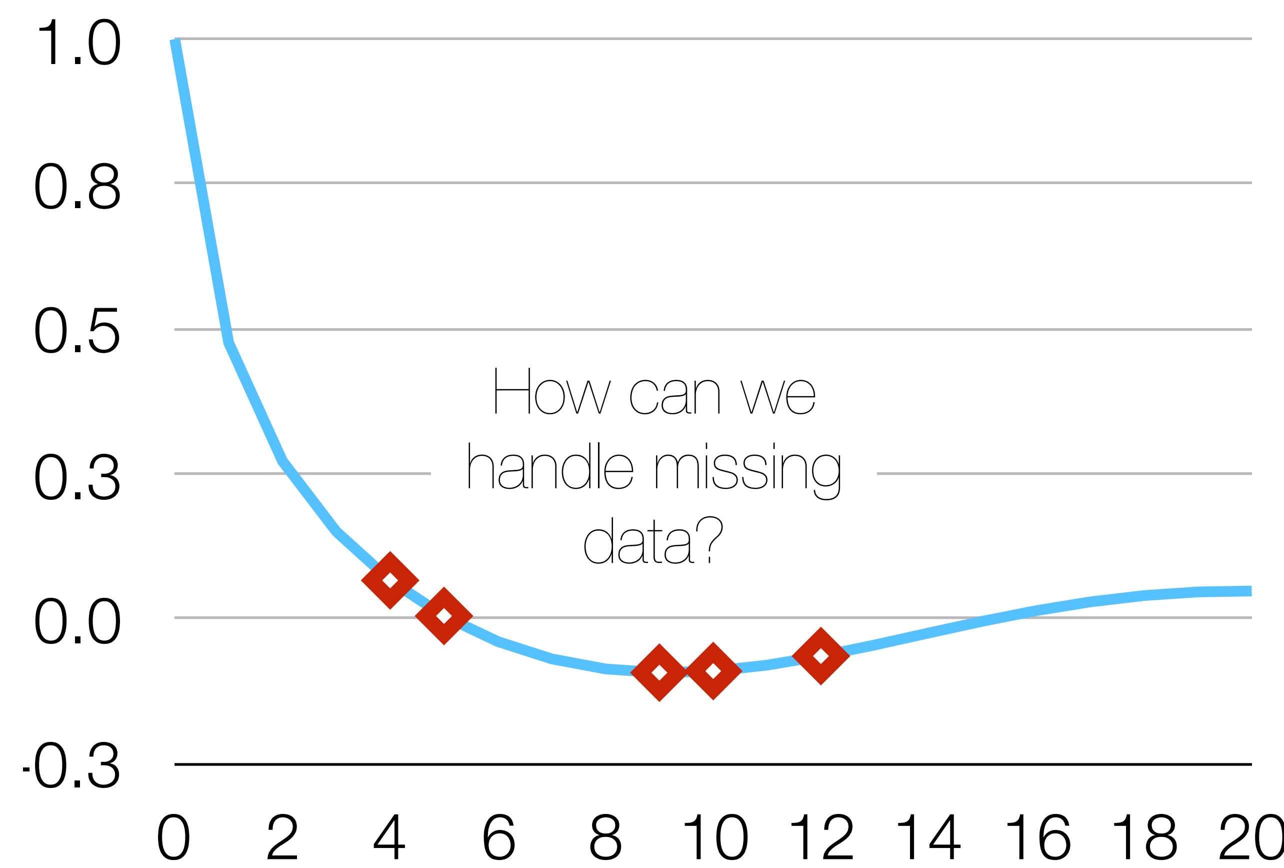
Windowing

One common approach is to place all "lost values" at the beginning as it avoids "future leaking" when splitting the dataset

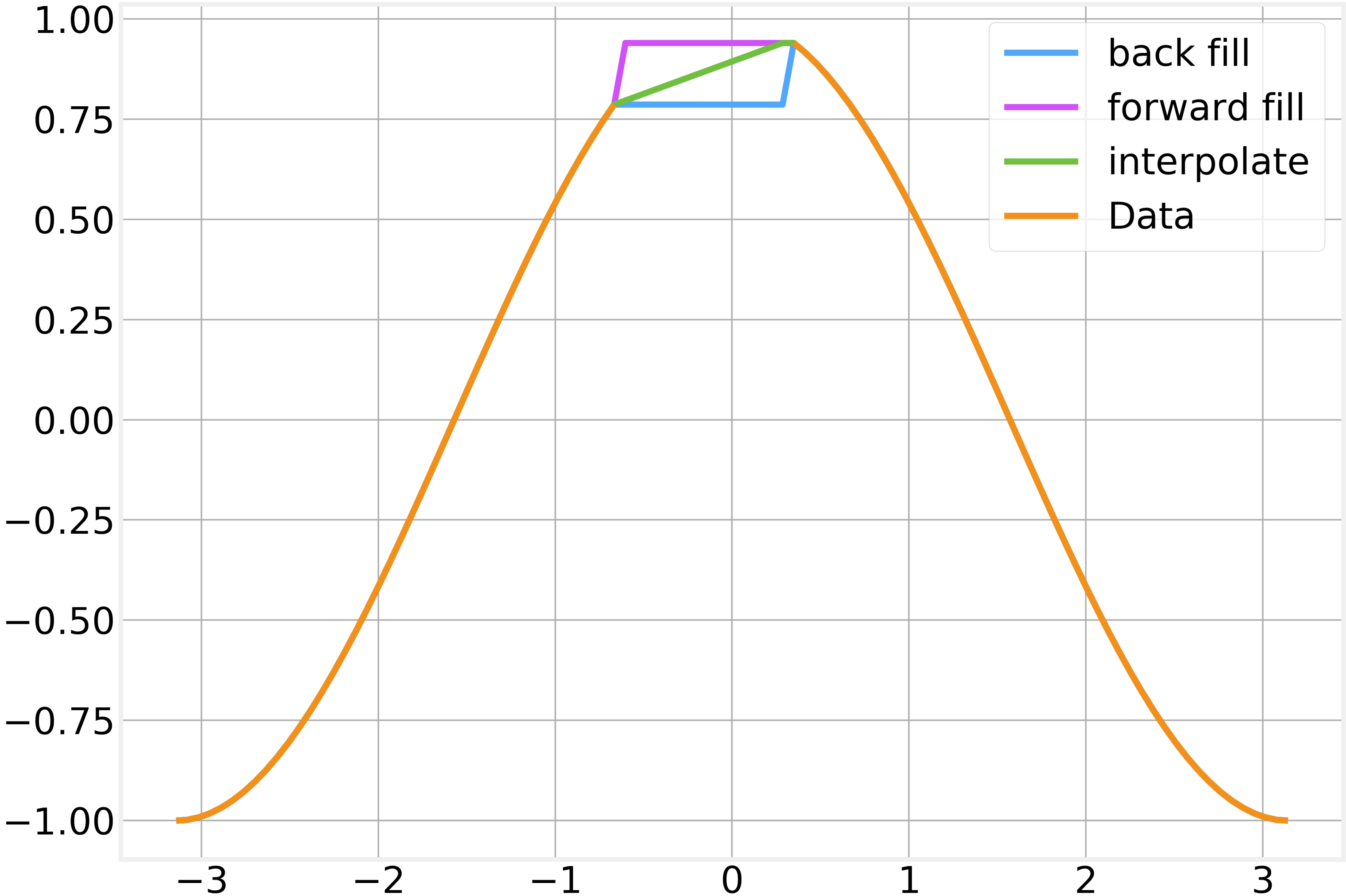


Fill methods

- Sometimes the time series is incomplete.
- Missing data points can be due to data corruption, data collection issues, etc.
- Missing values are represented as nan
- Several techniques have been developed to handle this case:
    - back fill - keep the last previous value
    - forward fill - keep the next value
    - interpolate - add values by interpolating between the previous and the next value
    - imputation - add values based on what we expect the missing values to be



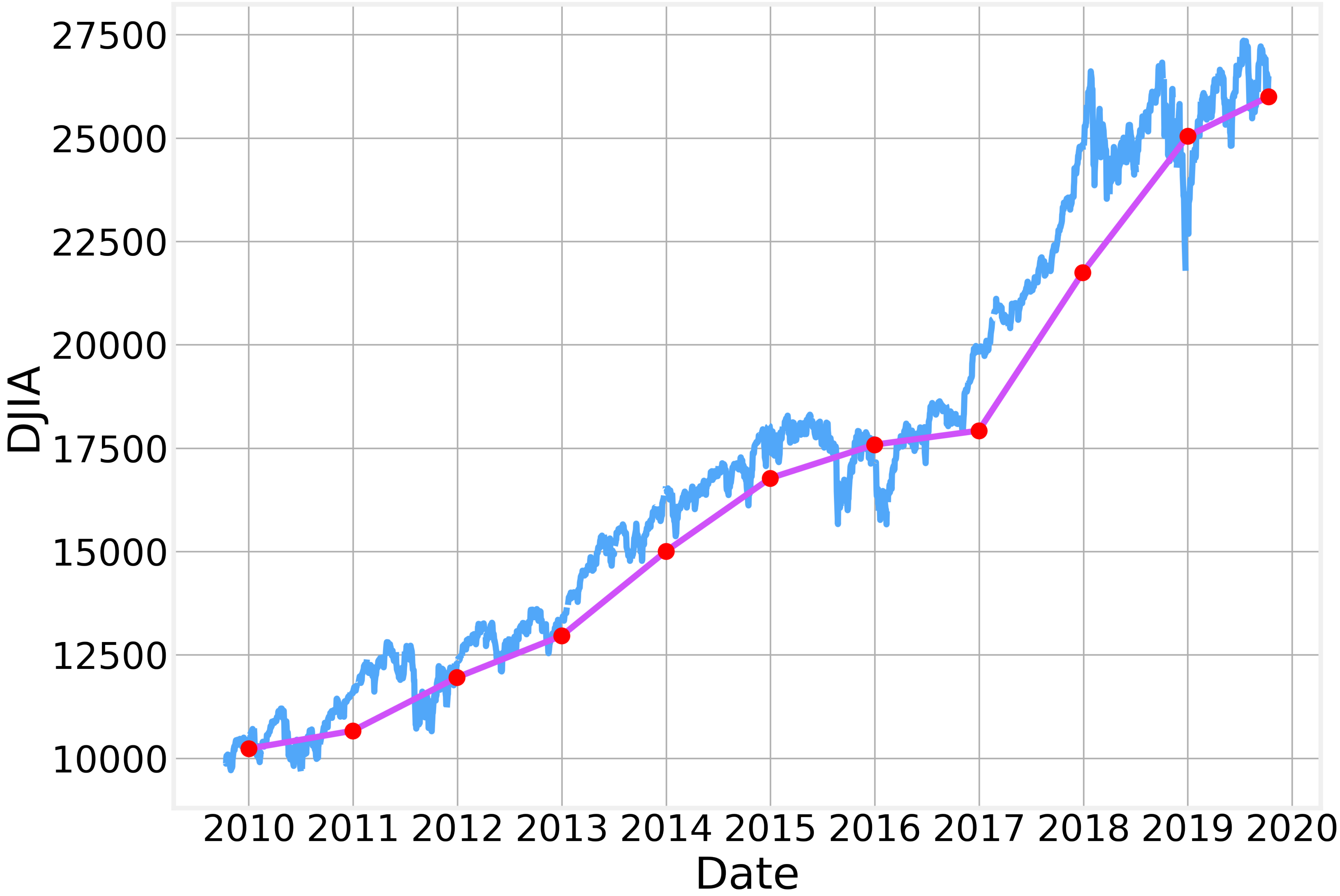
Fill methods



Resampling

- Time series typically have an intrinsic time scale at which the data was collected: ticks, seconds, days, months, etc
- In many cases, our analysis requires that we resample the data to a different time scale
- Resampling to a longer timescale is relatively simple and similar to aggregation:
    - Transforming from daily to weekly frequency requires simply aggregating by week
- Resampling to shorter timescales requires interpolation or imputation to make up for the missing values
    - Going from weekly to daily frequency requires specifying how to allocate the values for each day of the week

Resampling



27500
25000
22500
20000
17500
15000
12500
10000
DJIA
2010 2011 2012 2013 2014 2015 2016 2017 2018 2019 2020
Date

Jackknife Estimation

- Also known as 'leave one out' estimation
- Commonly used for mean and variance estimates
- The Jackknife estimate of a parameter is the average value of the parameter calculated by omitting each of the values one at a time.
- If $\boldsymbol{\mu_i}$ is the mean calculated by omitting the $\boldsymbol{i^{th}}$ value, then the Jackknife estimate of the mean is given by:

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} \mu_i$$

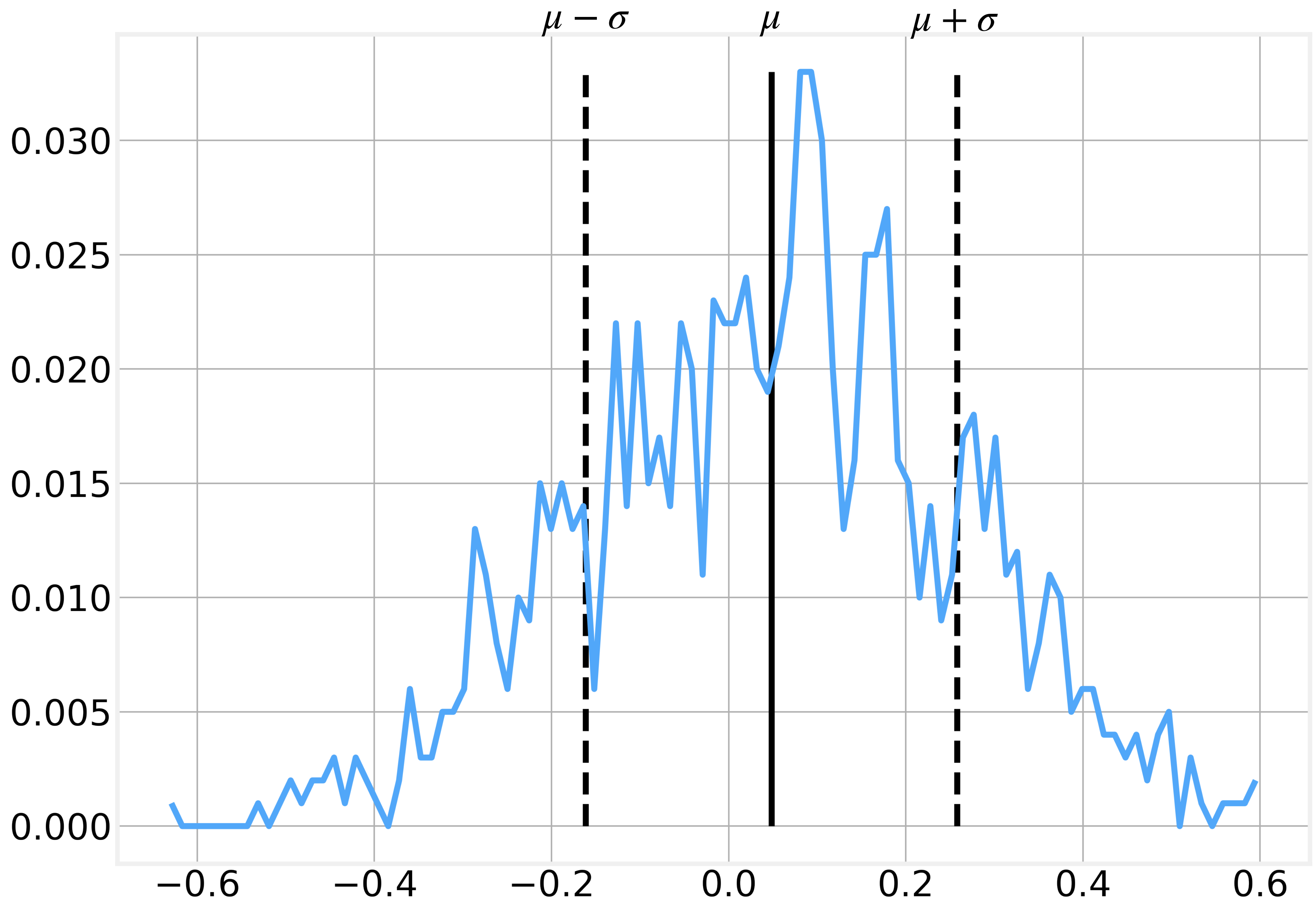
- And the variance of the estimate is:

$$\hat{\sigma}\left(\hat{\mu}\right) = \frac{N-1}{N} \sum_{i=1}^{N} \left(\mu_i - \hat{\mu}\right)^2$$

Bootstrapping

- Bootstrapping is another alternative to estimate statistical properties such as mean and variance
- Bootstrapping measures the desired property in a large number of samples (with replacement), of the observed dataset.
- Each sample has equal size to the observed dataset.
- From the entire population of samples, the empirical bootstrap distribution of the expected values can be obtained to provide information about the distribution in the total population

Bootstrapping



Code - Transformations

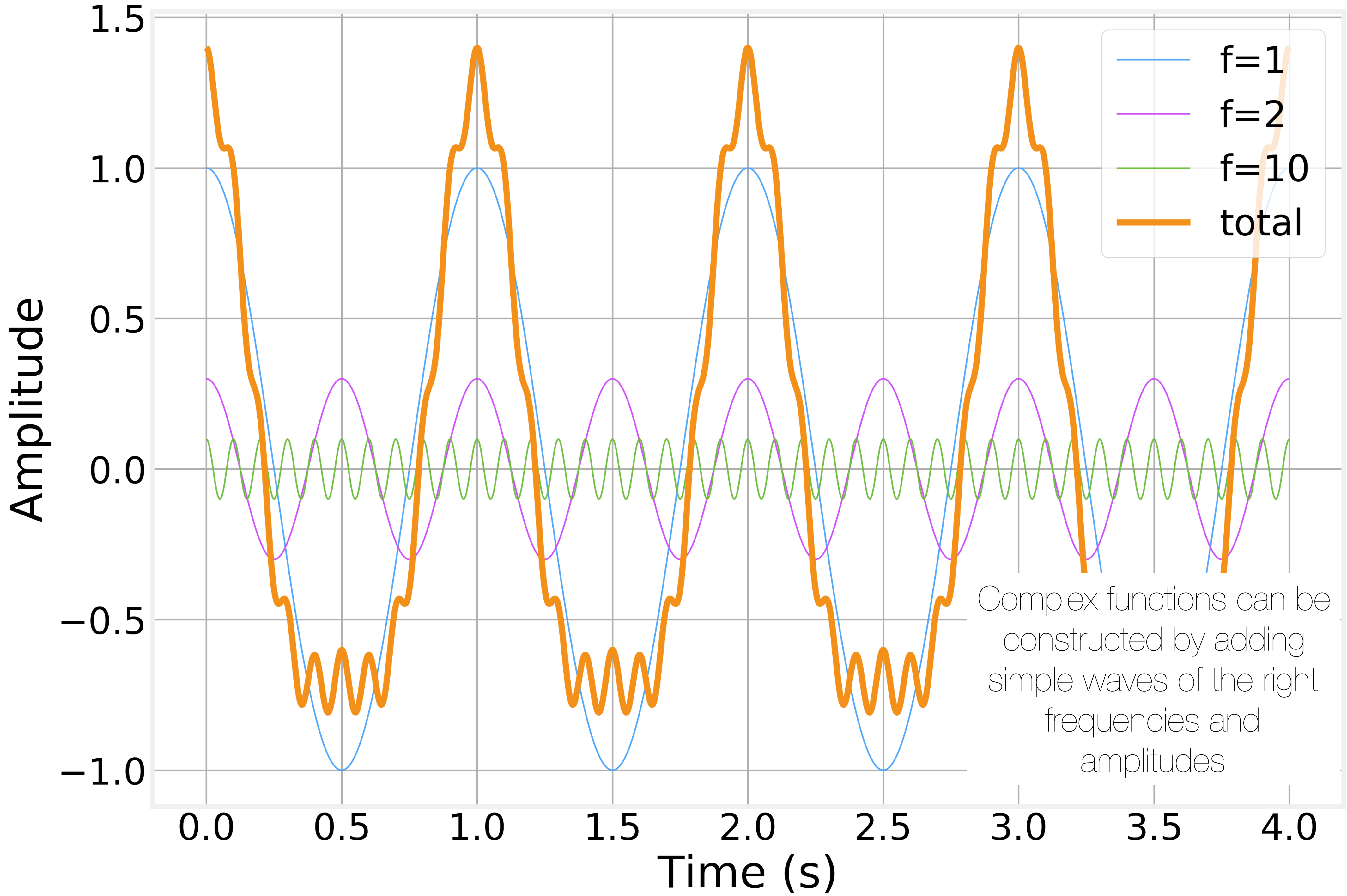
https://github.com/DataForScience/Timeseries

Lesson III:
Fourier Transformations and Filtering

Frequency Domain

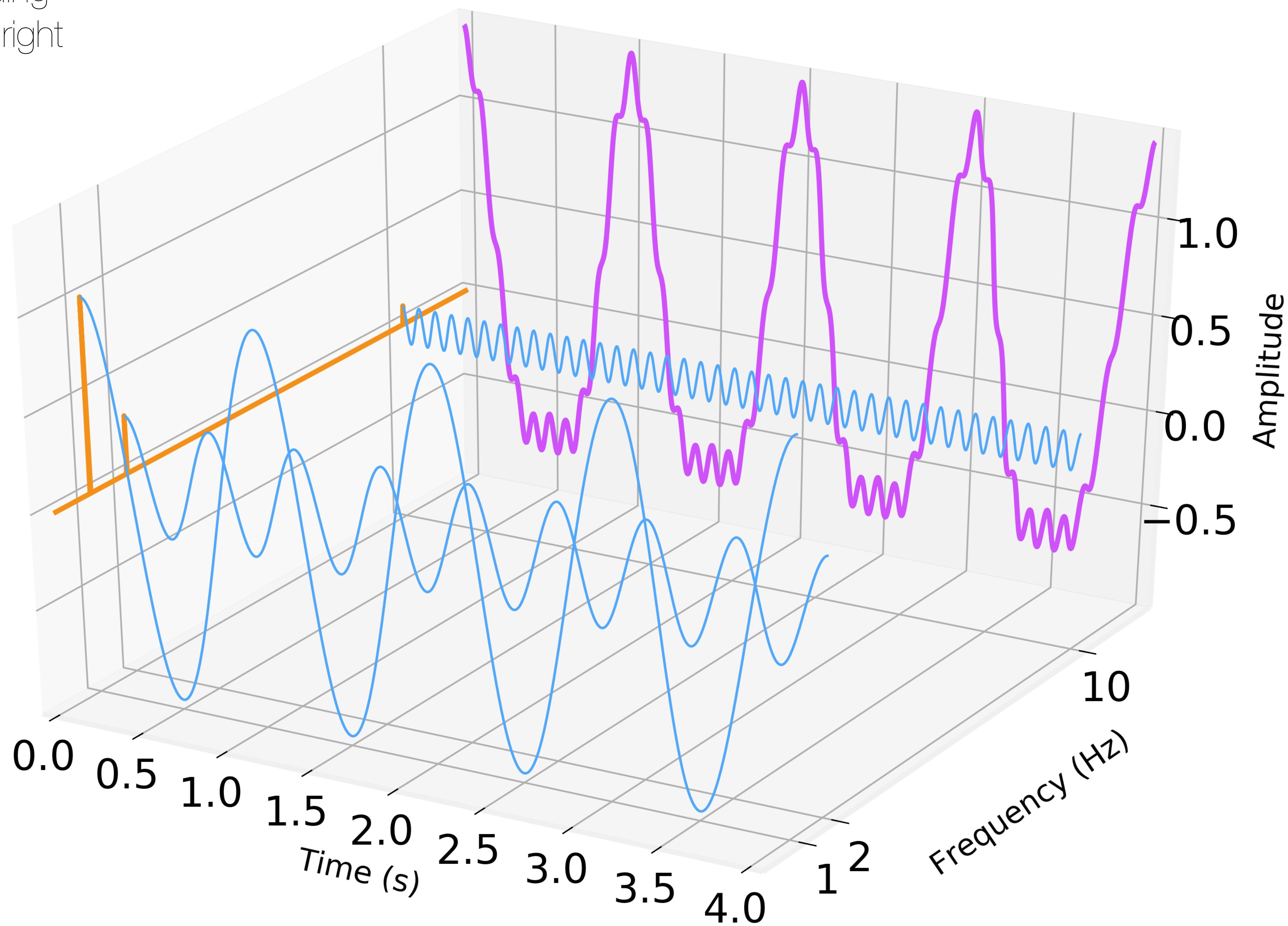
- So far we have focused on the natural (time based) representation of a time series.
- An alternative representation is based on frequencies and was first introduced by Jean Fourier in 1807
- Fourier showed that periodic functions can be decomposed as a sum of trigonometric functions
- Fourier's original result was later extended to all functions
- The Discrete Fourier Transforms provides us with an simple and convenient way to move from the time-domain to the frequency-domain and back.

Adding Frequencies



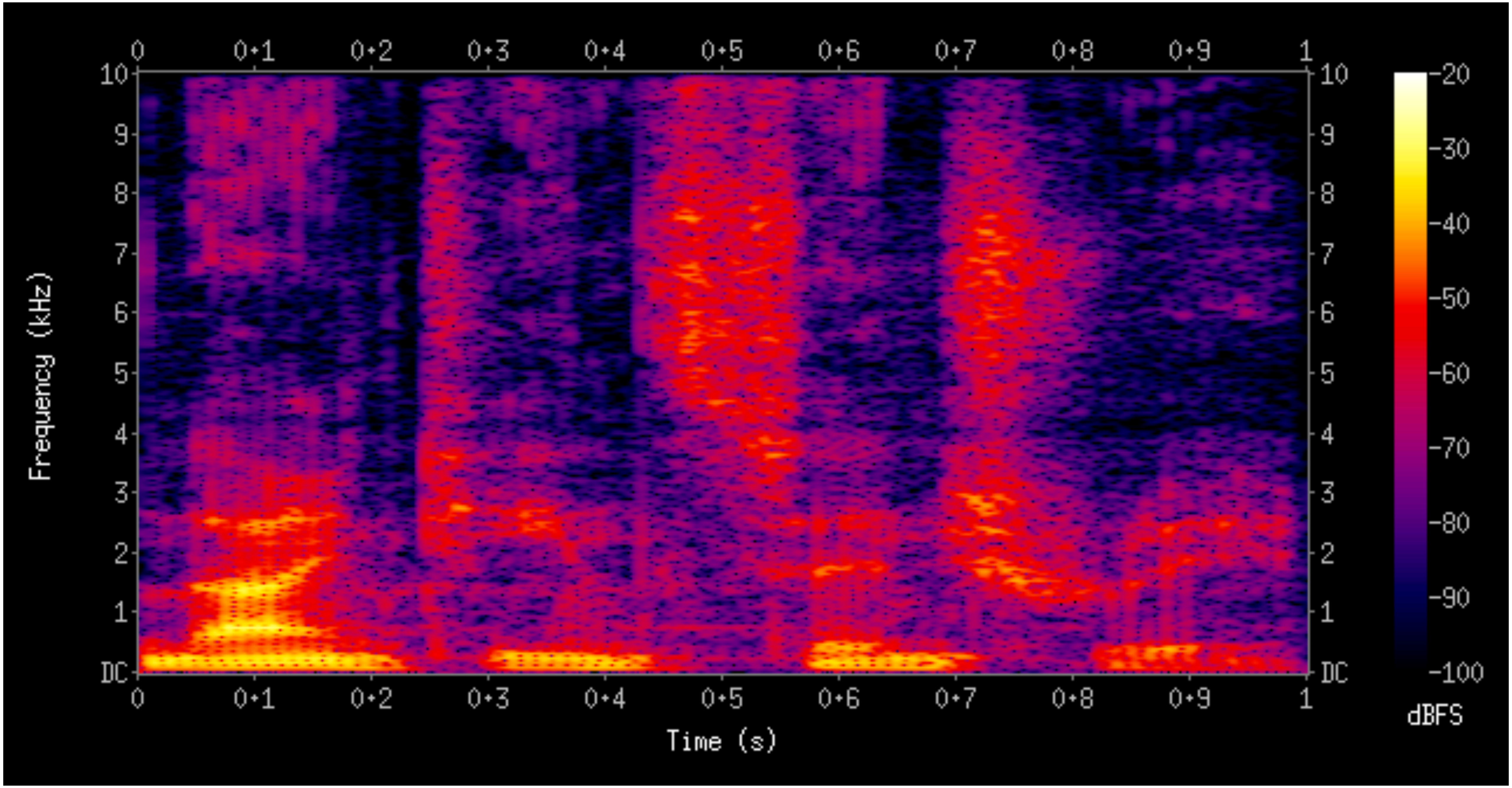
3D Visualization

Complex functions can be constructed by adding simple waves of the right frequencies and amplitudes



Spectrogram

https://en.wikipedia.org/wiki/Spectrogram



(Discrete) Fourier Transform

https://en.wikipedia.org/wiki/Discrete-time_Fourier_transform

- The DFT maps a sequence of N values x_n representing a time series $x(t)$ into N complex numbers X_k defined as:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i\frac{2\pi kn}{N}}$$

- where:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

- To recover the original values we use the Inverse DFT, defined as:

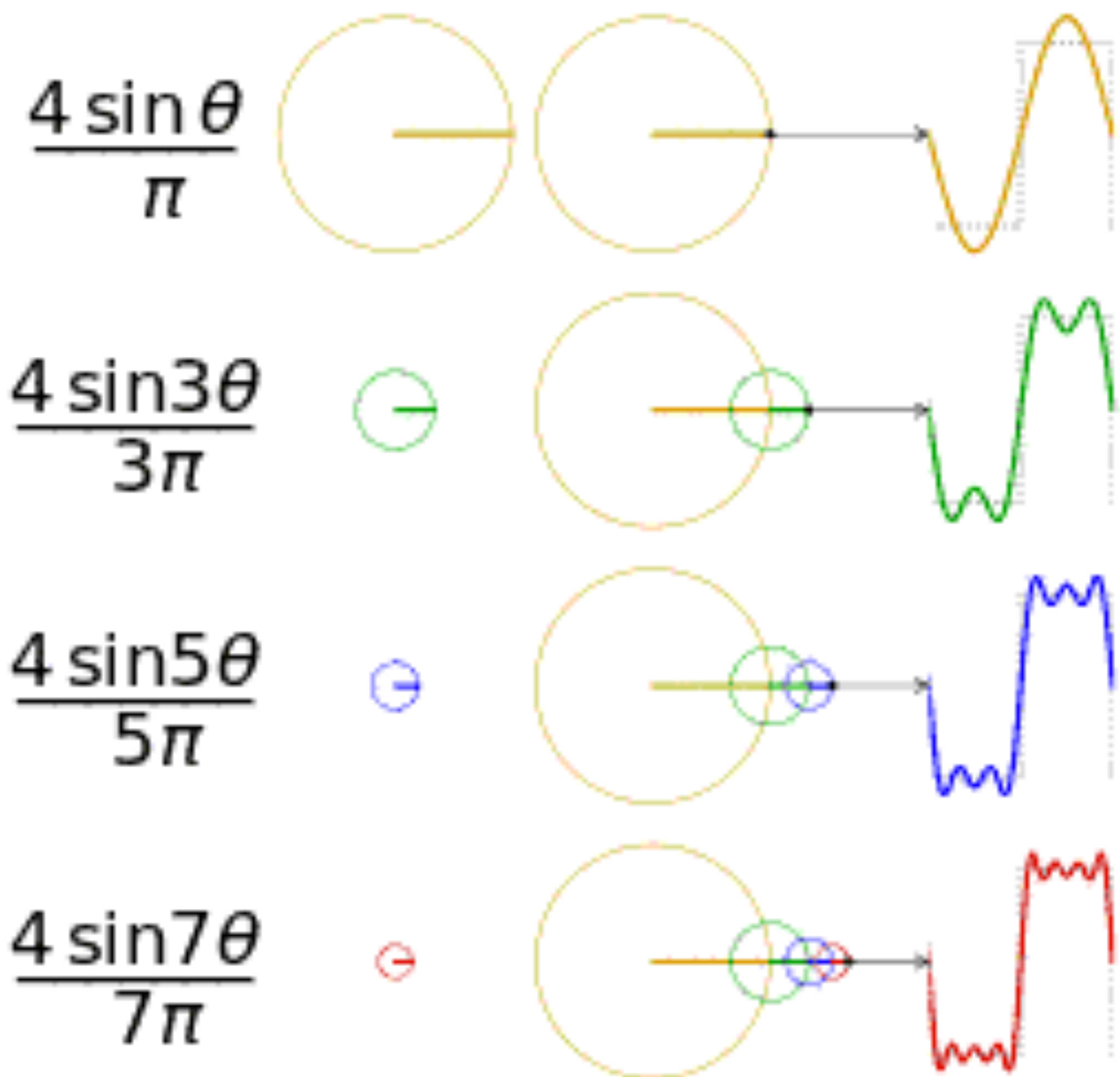
$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} x_n e^{i\frac{2\pi kn}{N}}$$

- The DFT represents the continuous series $x(t)$ as a sum of discrete frequencies:

$$\omega_n = 2\pi\frac{k}{N}$$

(Discrete) Fourier Transform

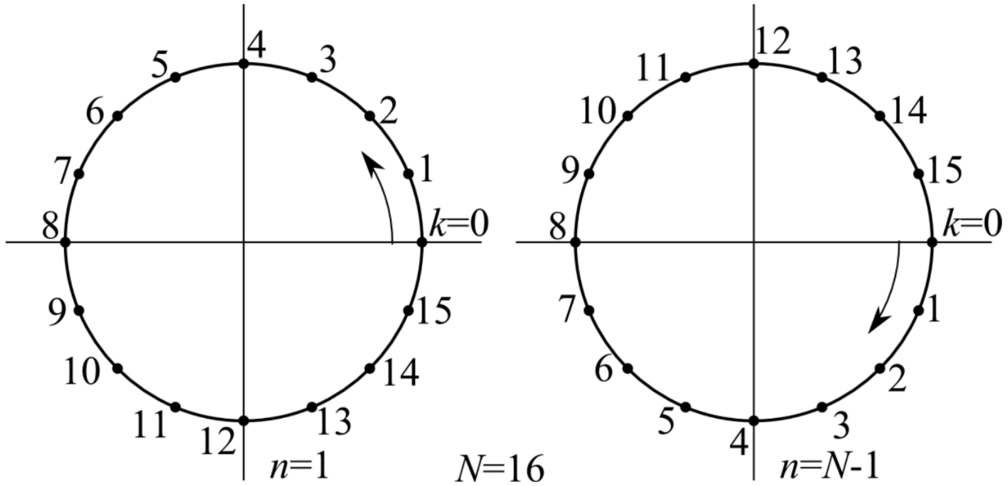
https://en.wikipedia.org/wiki/Discrete-time_Fourier_transform



SciPy

scipy.org

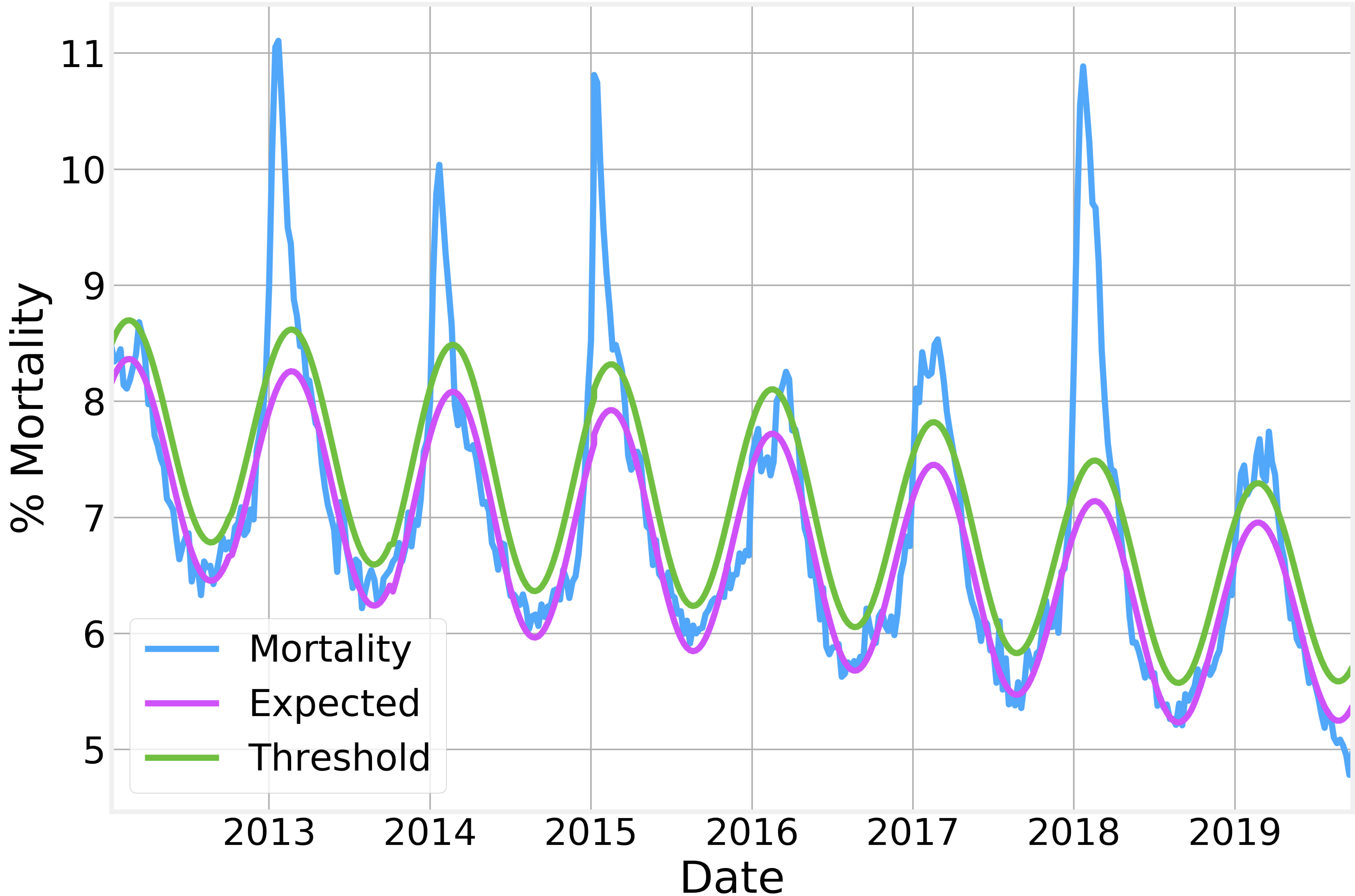
- Python-based ecosystem of open-source software for mathematics, science, and engineering.
- Provides practical implementation of the Fast Fourier Transform an efficient algorithm to compute the DFT and IDFT. See https://en.wikipedia.org/wiki/Fast_Fourier_transform
- scipy.fftpack.fft()/scipy.fftpack.ifft() - DFT and IFT
- scipy.fftpack.fftfreq() - return the list of frequencies
- scipy.fftpack.fftshift()/scipy.fftpack.ifftshift() - Shift the zero-frequency component to the center of the spectrum and back.



Filtering

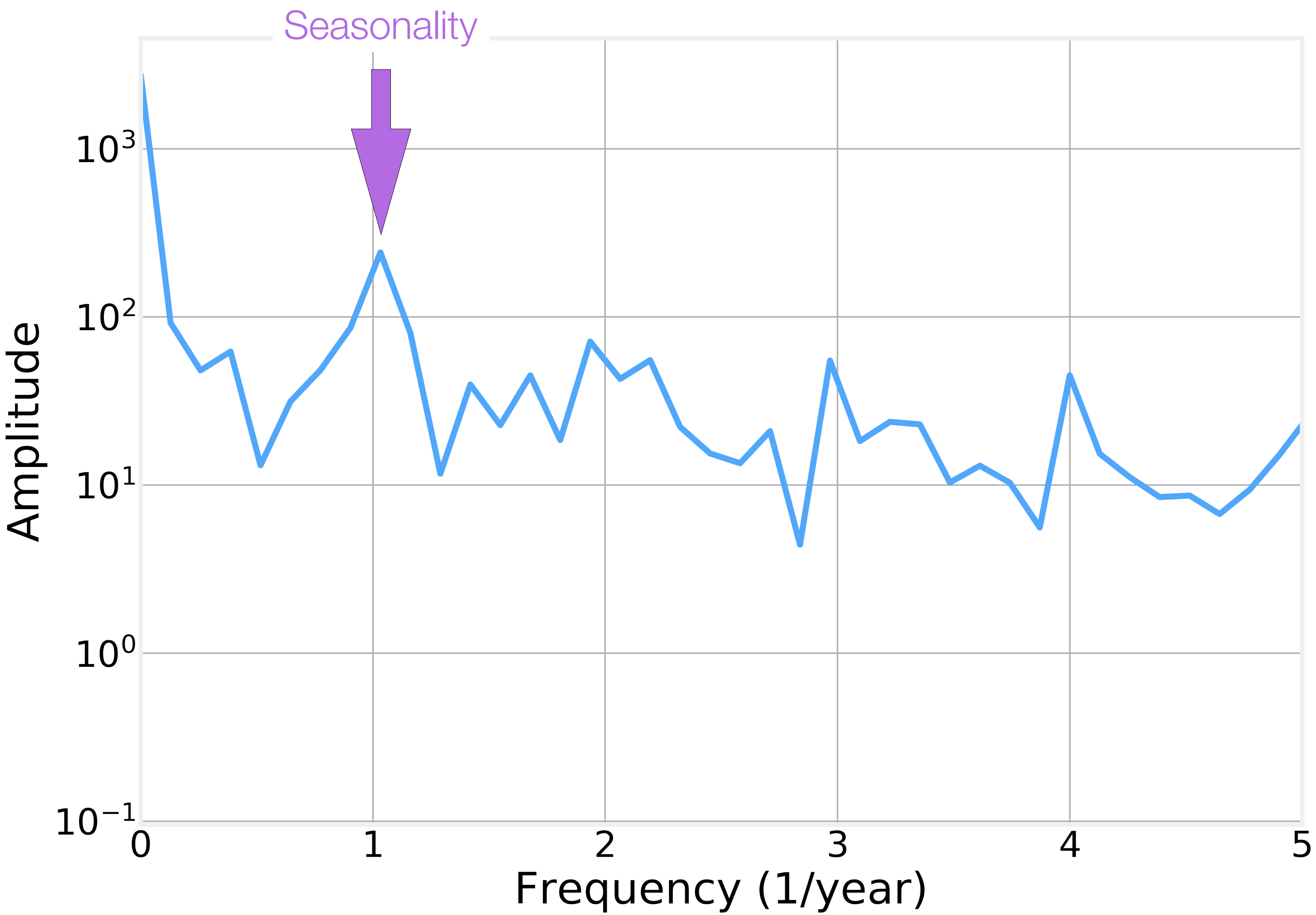
- Common applications of Fourier Analysis are:
    - Seasonality - determine the main frequency underlying a time series
    - Filtering - remove higher order frequencies to eliminate noise
    - Processing - Several signal processing operations are simpler to compute in the frequency-space

Filtering



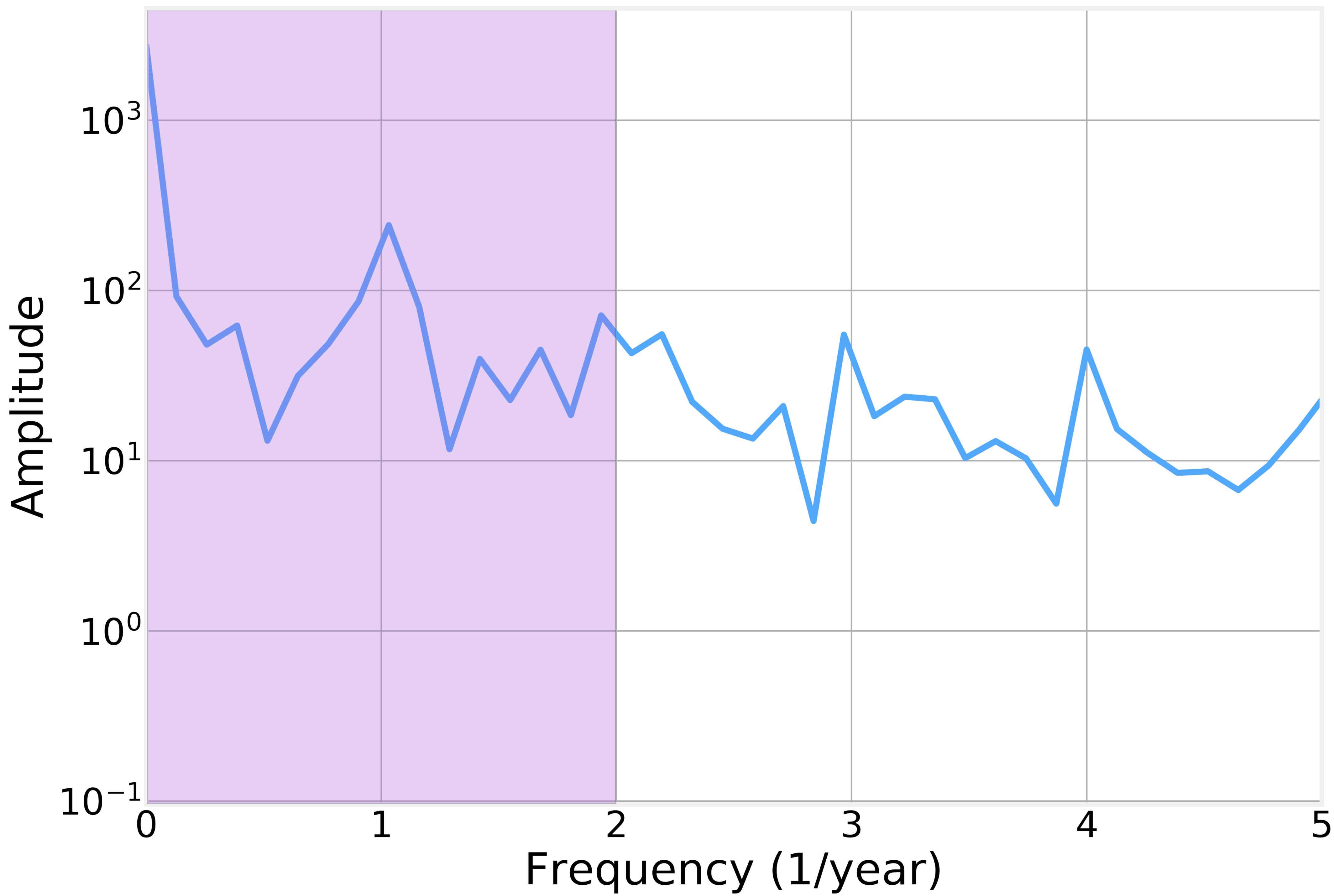
11
10
9
8
7
6
5
% Mortality
2013
2014
2015
2016
2017
2018
2019
Date
Mortality
Expected
Threshold

Filtering

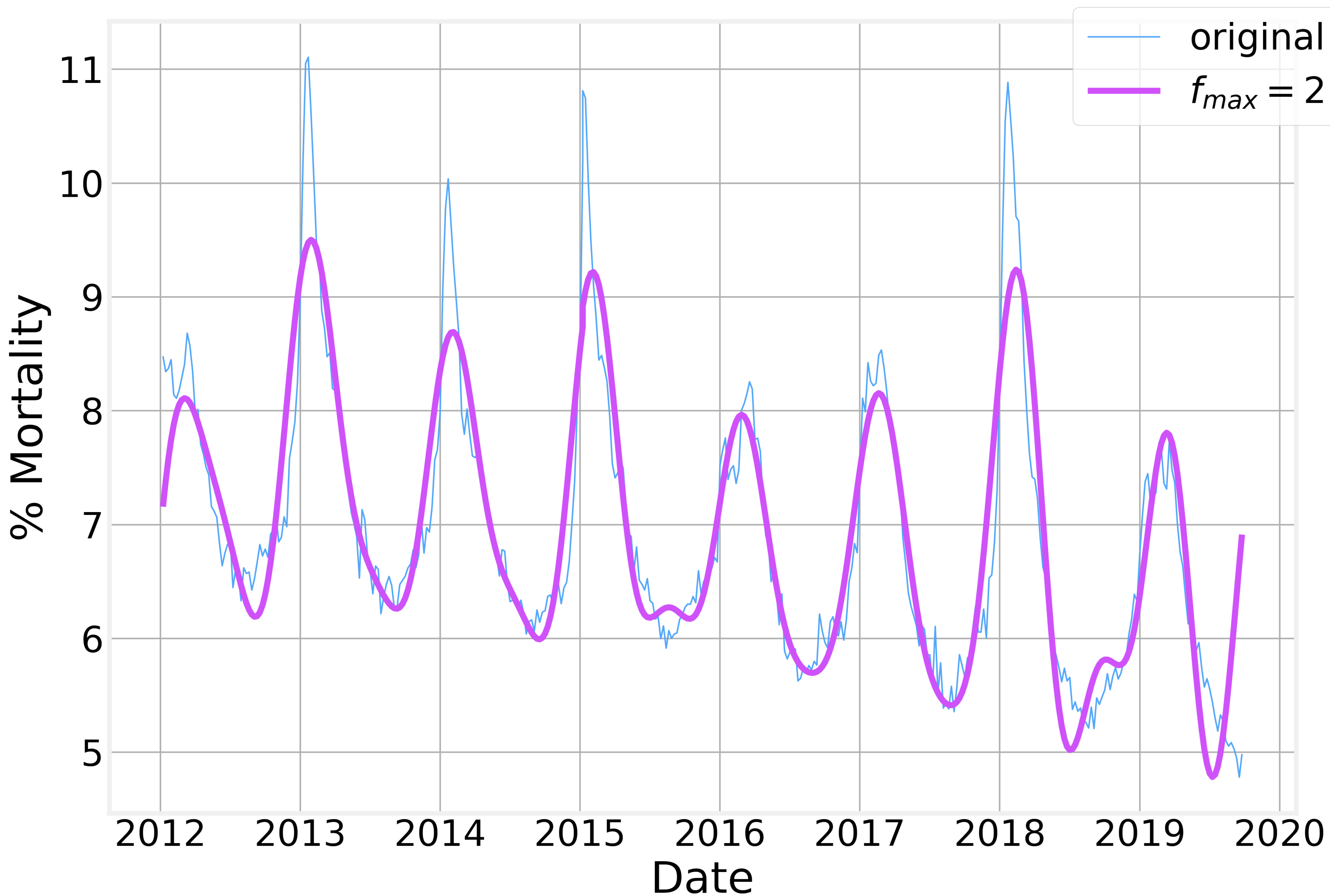


Seasonality
10^3
10^2
10^1
10^0
10^{-1}
Amplitude
0
1
2
3
4
5
Frequency (1/year)

Filtering

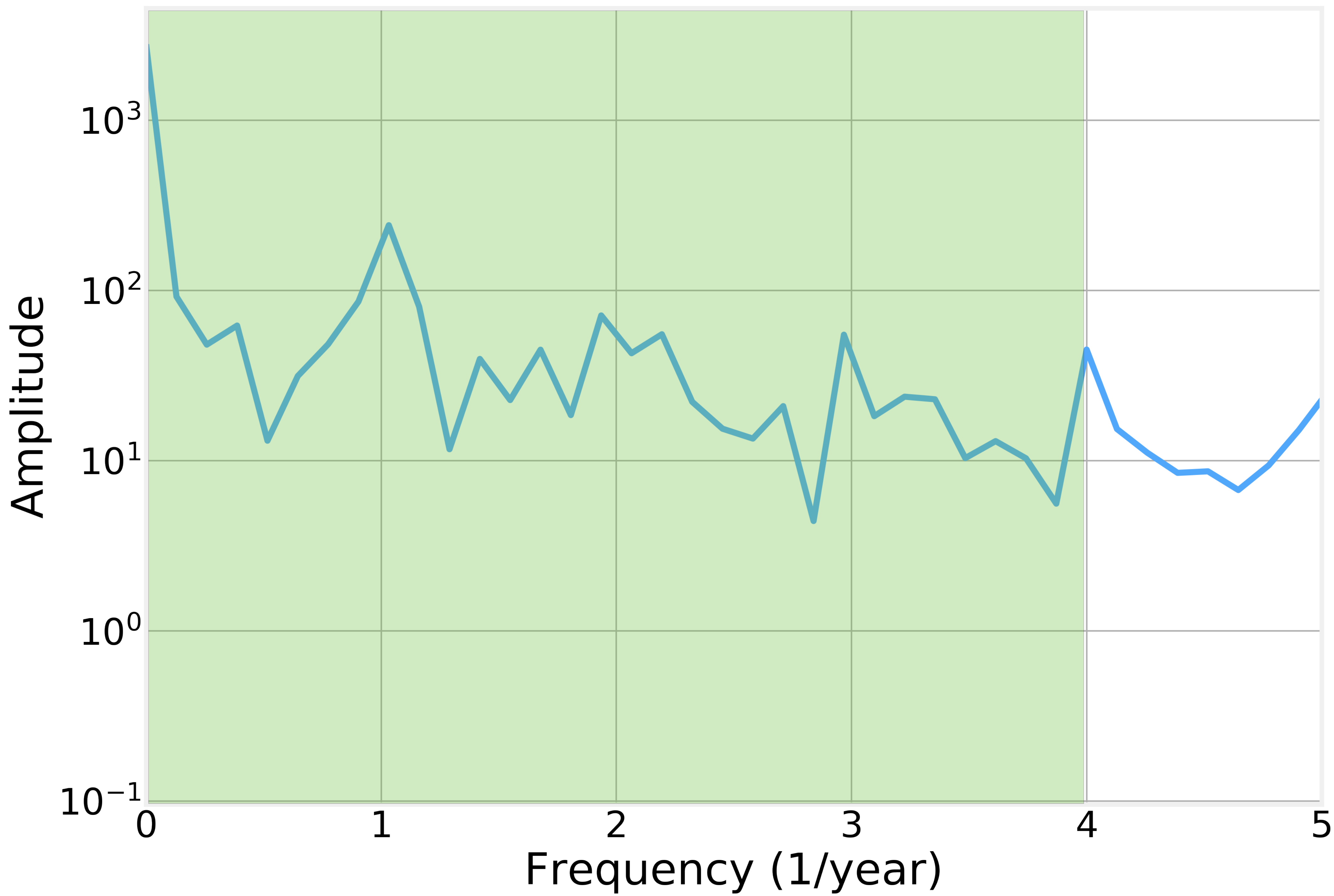


Filtering

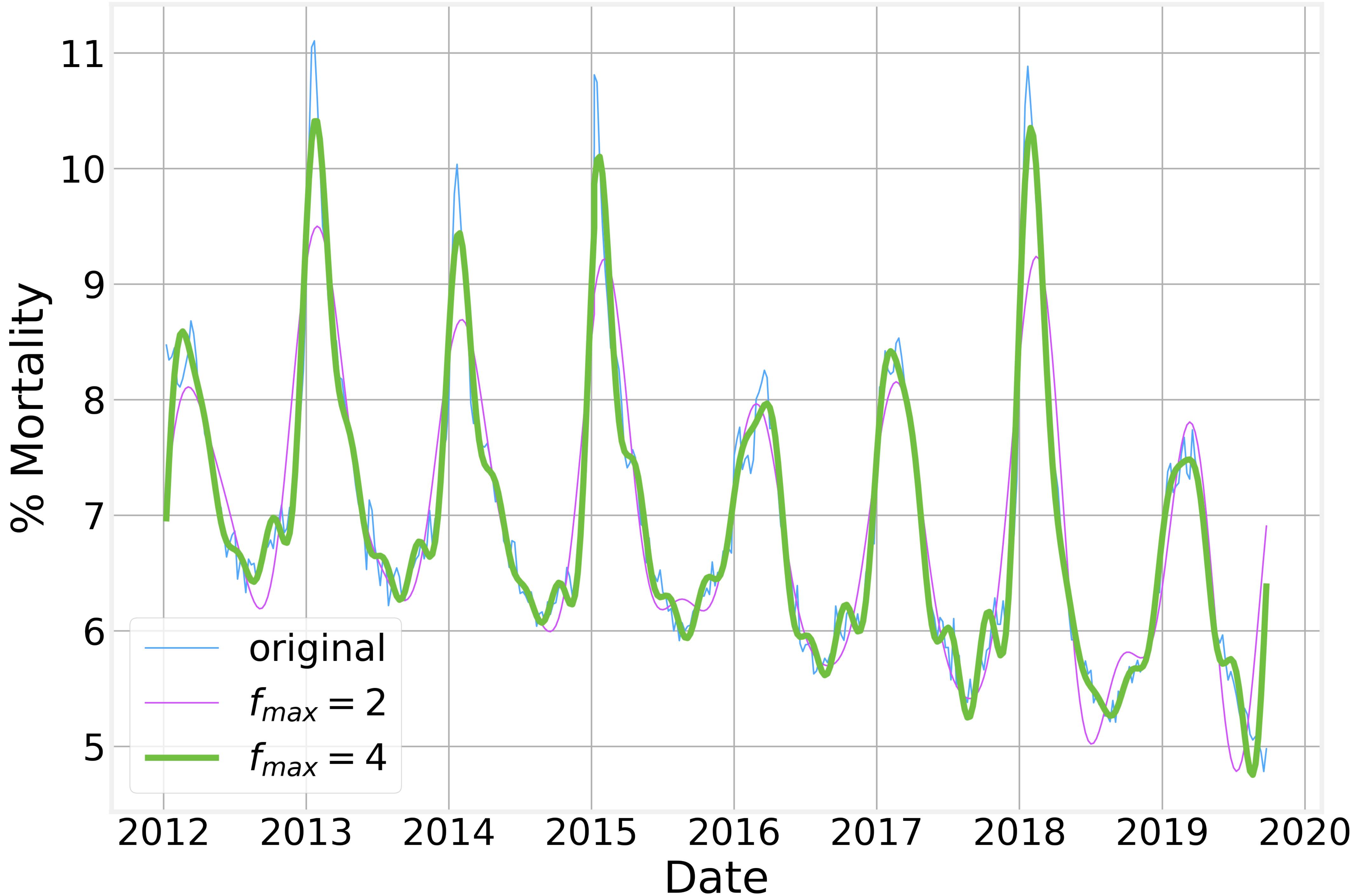


original
$f_{max} = 2$
11
10
9
8
7
6
5
% Mortality
2012
2013
2014
2015
2016
2017
2018
2019
2020
Date

Filtering



Filtering



11
10
9
8
7
6
5
% Mortality
2012
2013
2014
2015
2016
2017
2018
2019
2020
Date
original
$f_{max} = 2$
$f_{max} = 4$

Code - Fourier Analysis

https://github.com/DataForScience/Timeseries

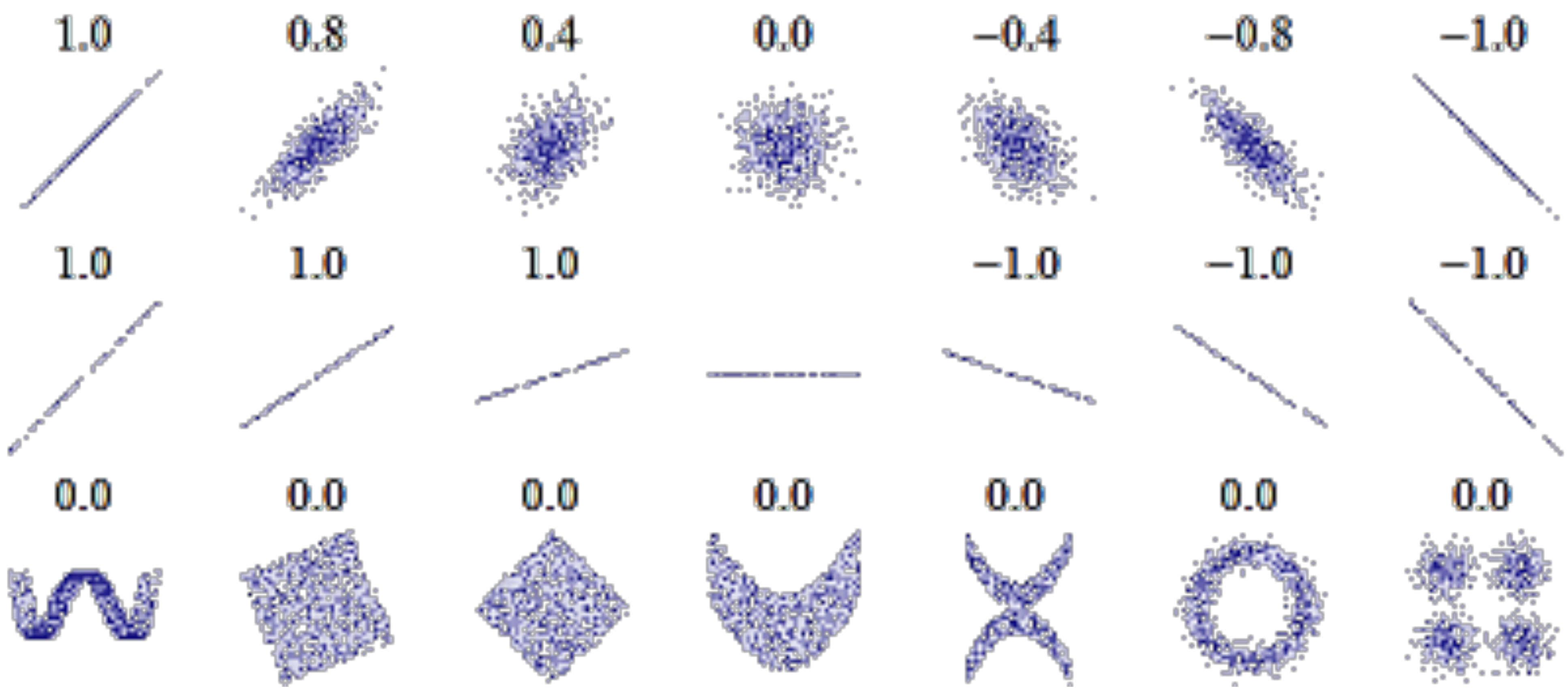
Lesson IV: Correlations

Correlation

- Many correlation measures have been proposed over the years
- The most well known one is the Pearson Correlation

$$\rho\left(x,y\right)=\sum_{i=1}^{N}\frac{\left(x_i-\mu_x\right)\left(y_i-\mu_y\right)}{\sigma_x\sigma_y}$$

- Assumes a linear relationship between x and y.



Correlations of 2 time series

- The correlation of two time series gives you an indication of how similar their behavior is
- Two completely unrelated time series (say, two sequences of random numbers) will have a Pearson correlation coefficient of 0
- However, if we add a trend to both series we immediately observe a significant correlation

The Pearson correlation of two trending series is overwhelmed by the trend

Auto-correlation

https://en.wikipedia.org/wiki/Correlogram

- It follows from the previous slide that a series will have a perfect correlation with itself, but what about lagged versions of itself?

- We define the Auto-correlation function as the Pearson correlation between values of the time series at different lags, as a function of the lag:

$$ACF_x(l) = \rho(x_t, x_{t-l})$$

- By definition, $ACF_x(0) \equiv 1$

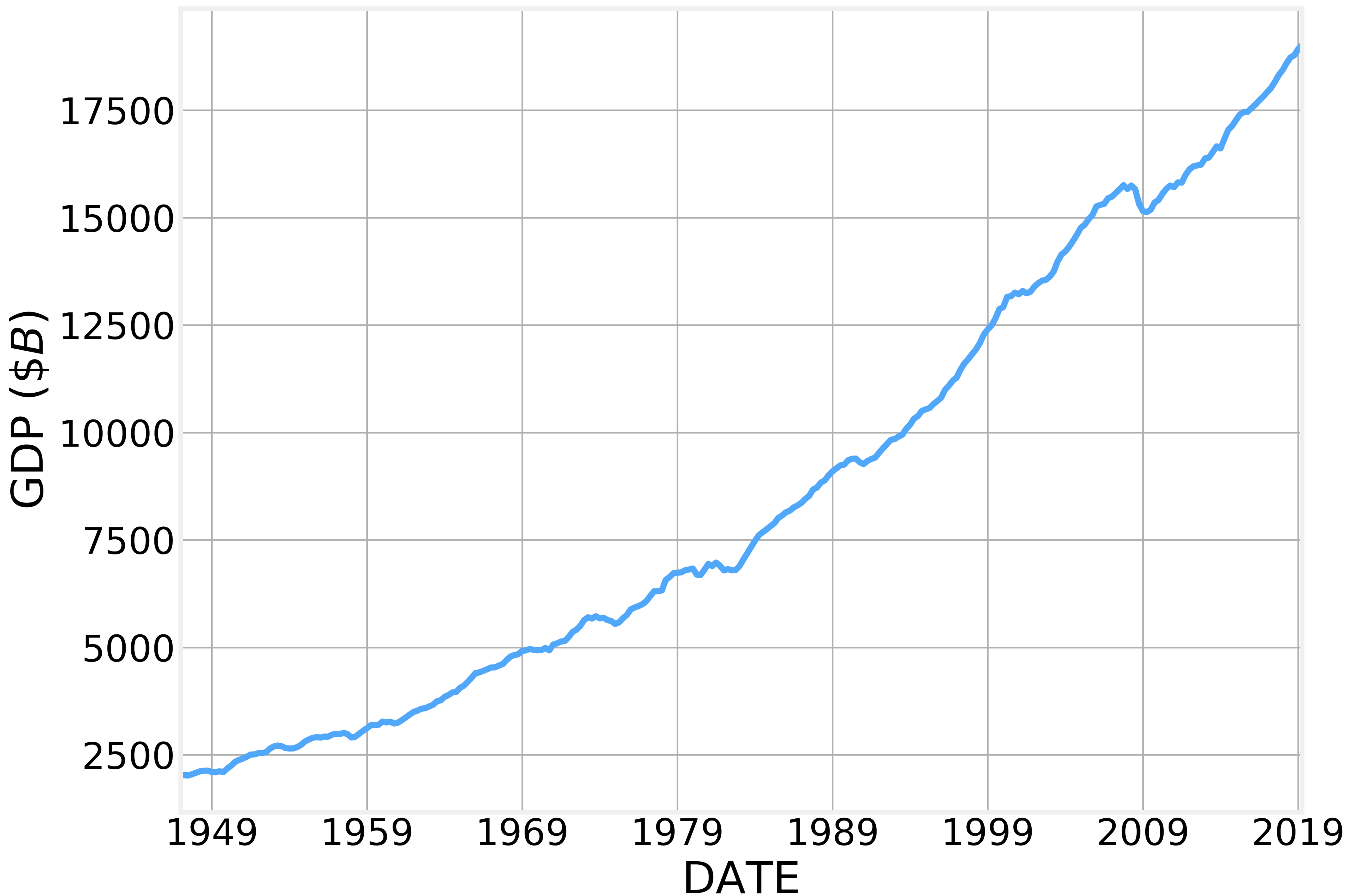
- And as the lag l increases the value of the ACF tends to decrease.

- We can calculate the confidence interval for the ACF using:

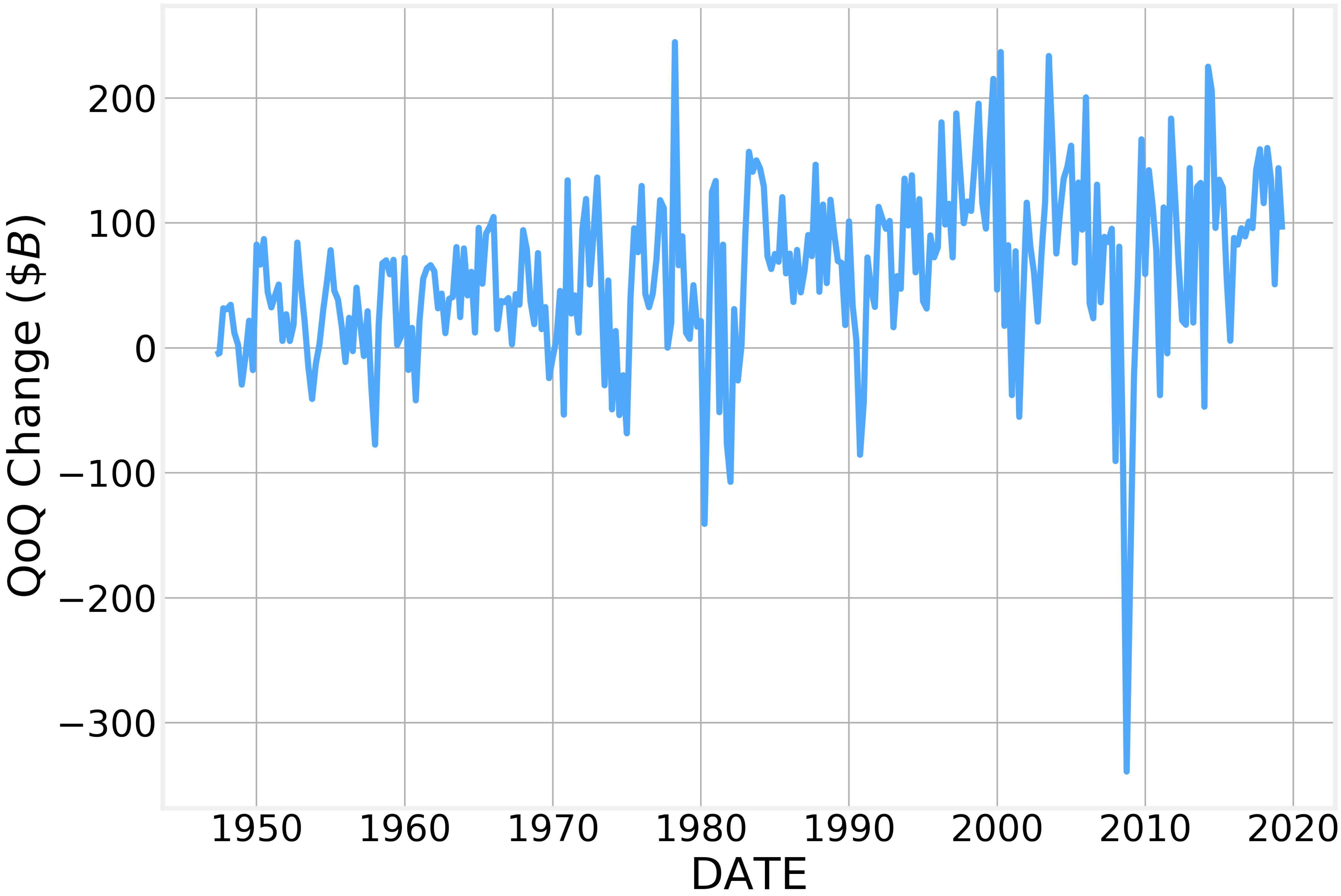
$$CI = \pm z_{1-\alpha/2} \sqrt{\frac{1}{N}\left(1 + 2\sum_{l=1}^{k} r_l^2\right)}$$

- where $z_{1-\alpha/2}$ is the quantile of the normal distribution corresponding to significance level α and r_l are the values of the ACF for a specific lag l

Auto-correlation

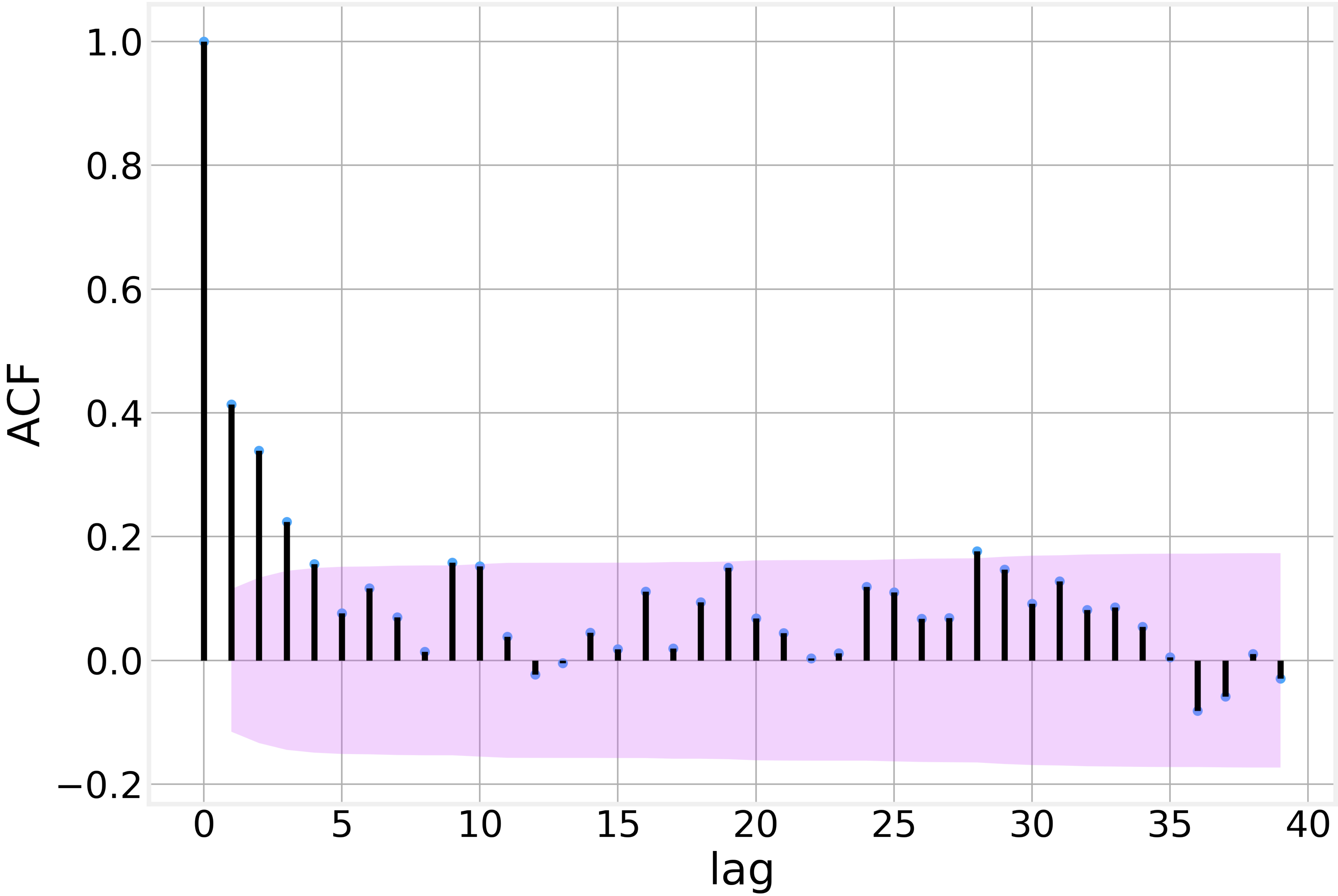


Auto-correlation



Auto-correlation

https://en.wikipedia.org/wiki/Correlogram



Partial Autocorrelation

https://en.wikipedia.org/wiki/Partial_autocorrelation_function

- One of the disadvantages of the Autocorrelation function is that it still considers the intermediate values

- The Partial Autocorrelation function calculate the correlation function between x_t and x_{t-l} after explaining away all the intermediate values $x_{t-1} \cdots x_{t-l+1}$

- Intermediate values are "explained away" by fitting a linear model of the form:

$$\hat{x}_t = f\left(x_{t-1} \cdots x_{t-l+1}\right)$$

$$\hat{x}_{t-l} = f\left(x_{t-1} \cdots x_{t-l+1}\right)$$

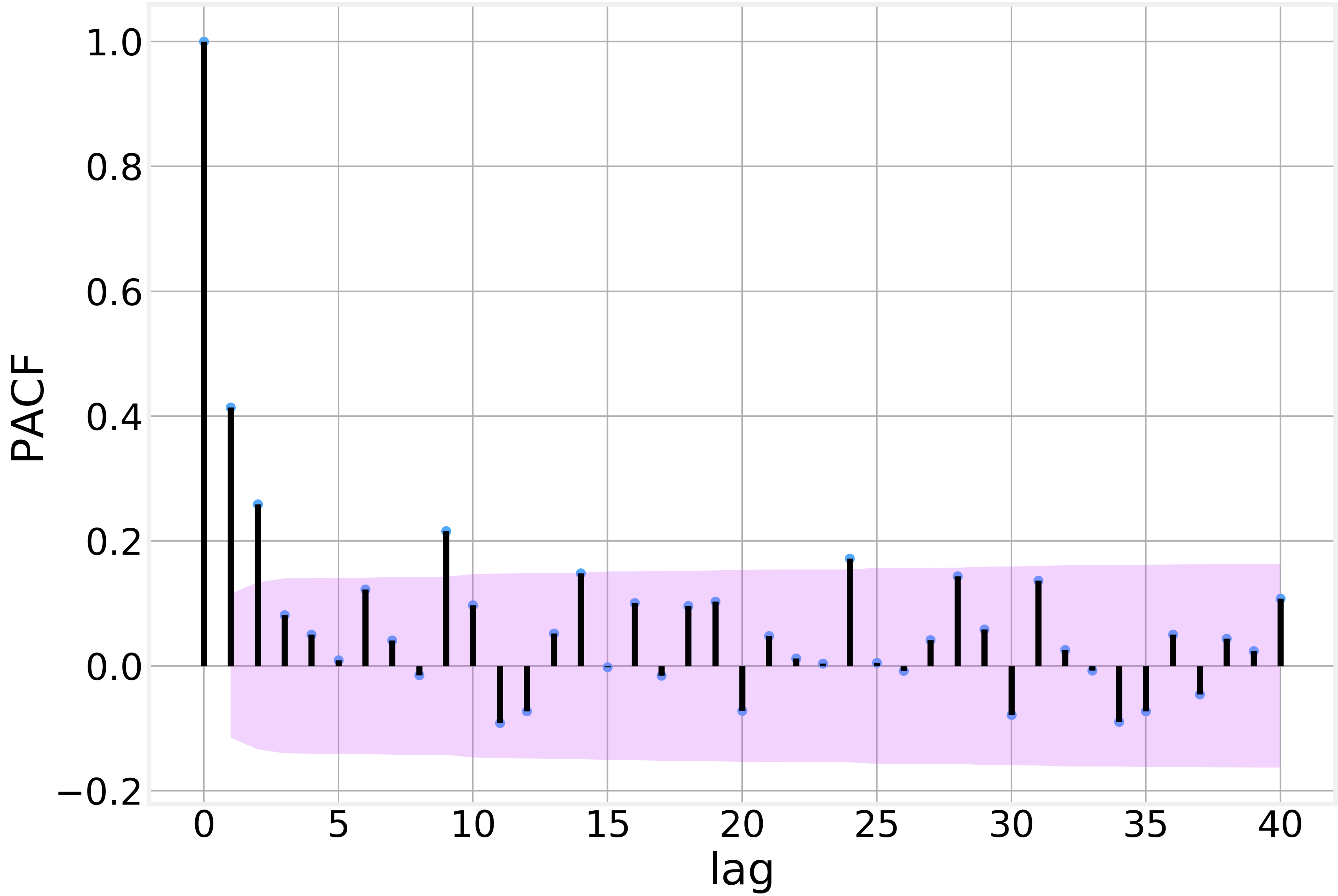
- And then calculating the Pearson correlation function between the values and their residuals:

$$PACF_x\left(l\right) = \rho\left(x_t - \hat{x}_t, x_{t-l} - \hat{x}_{t-l}\right)$$

- Confidence intervals can be computed using the same formula used for the ACF

Partial Autocorrelation

https://en.wikipedia.org/wiki/Partial_autocorrelation_function



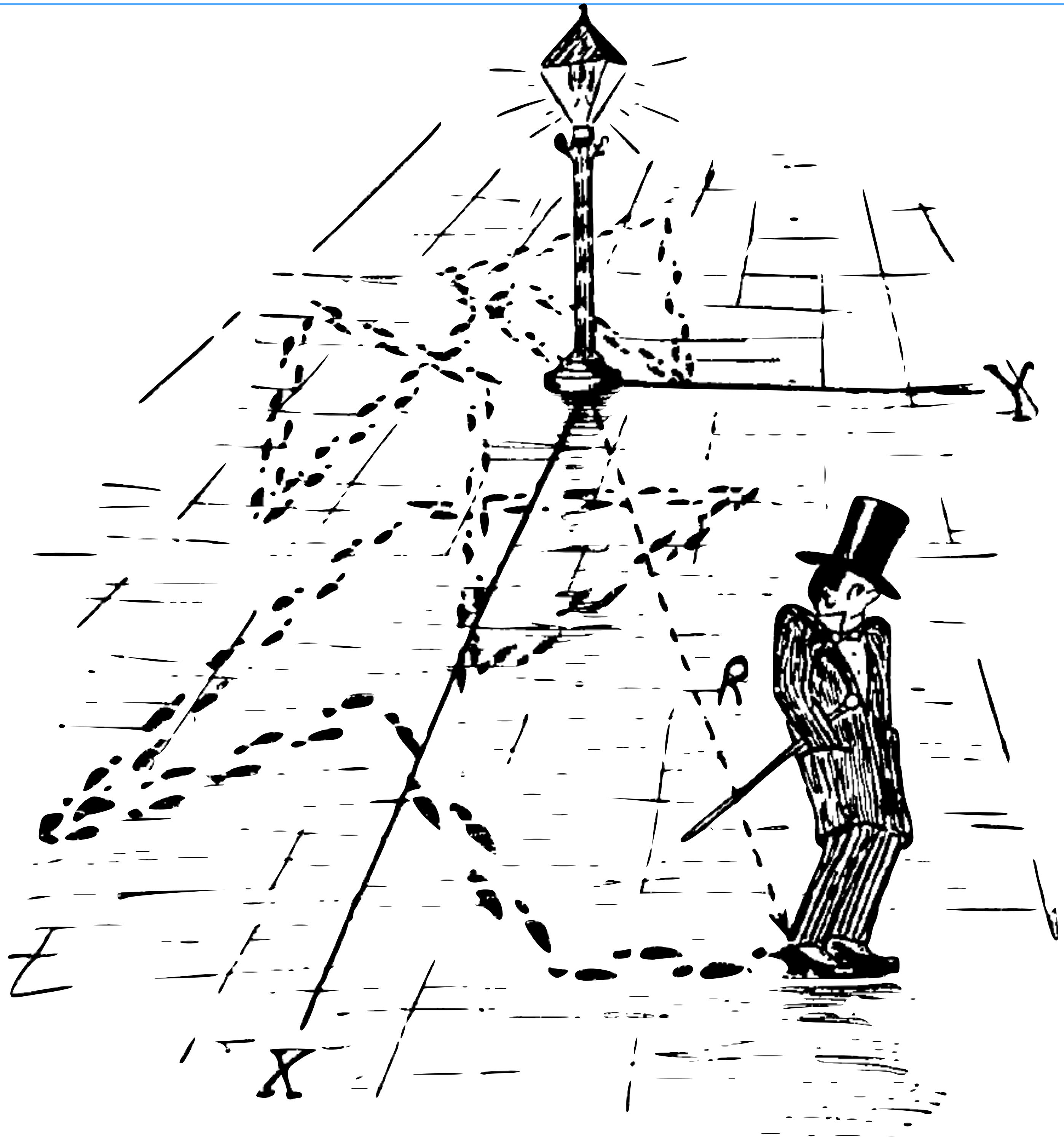
Code - Correlations

https://github.com/DataForScience/Timeseries

Lesson V:
Random Walks

Random Walks

Illustration by George Gamow



- At each step flip a coin
  - Heads: Move right
  - Tails: Move left
- If you start at position 0, do you ever reach position L?
- On average, we expect the position to be always close to 0.
- What if the coin is biased as in the previous example?

Random Walks

- Mathematically, we can describe the position of our random walker at time t as:

$$x_t = x_{t-1} + \epsilon_t$$

- Where ϵ_t is the stochastic value generated by our coin flip ($+1$ or -1)
- We can further write:

$$x_t = x_0 + \sum_i \epsilon_i$$

- which shows that the current position is just the sum across all coin flips in our walk.
- Naturally, we can treat a random walk as a realization of a time series, but is it stationary?
- The mean position is:

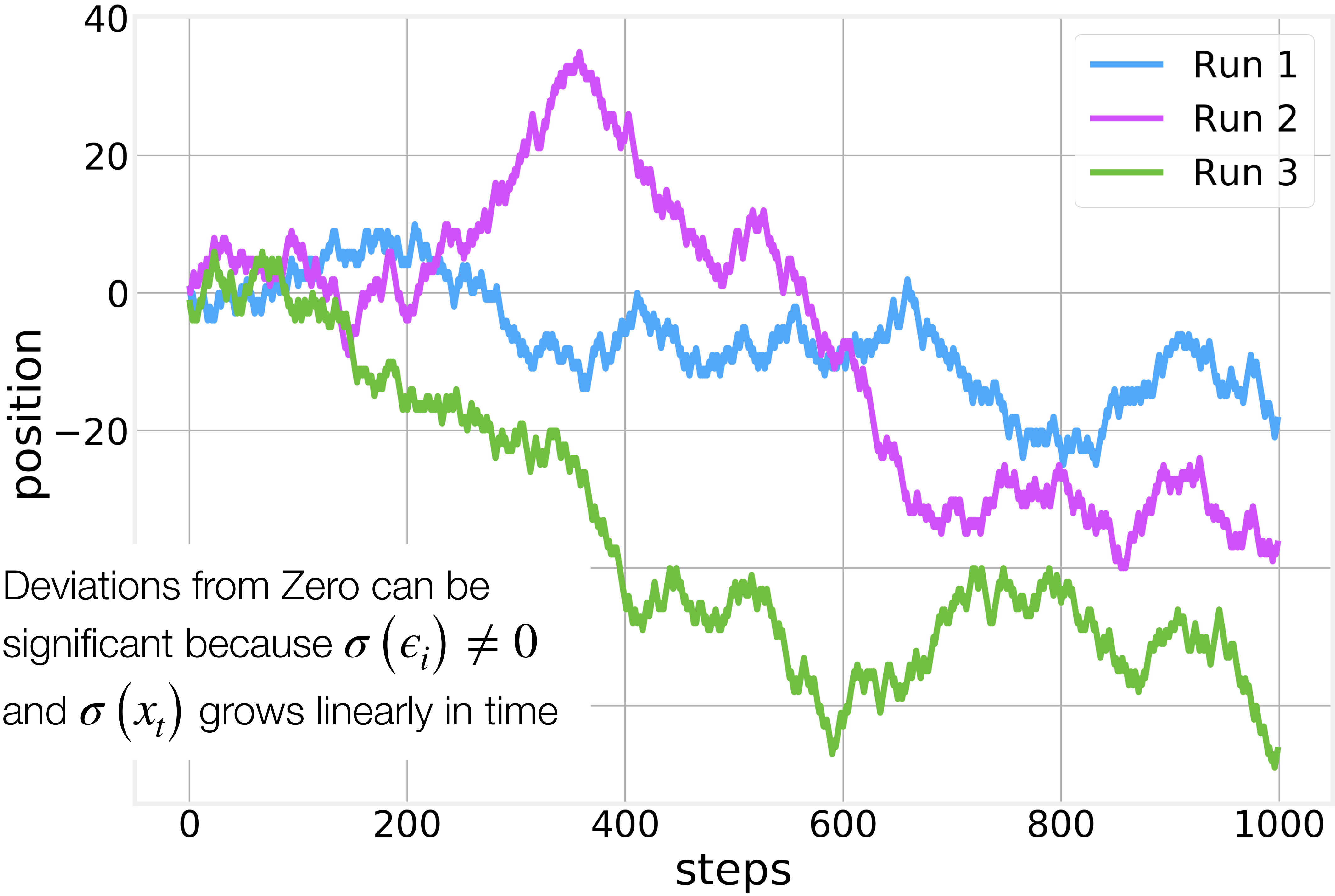
$$\mu = \langle x_t \rangle = \langle x_0 \rangle + \sum_i \langle \epsilon_i \rangle$$

- If the coin is unbiased, $\langle \epsilon_i \rangle = 0$ and the mean is constant. On the other hand, the variance is:

$$\sigma = \sigma\left(x_t\right) = \sigma\left(x_0\right) + \sum_i \sigma\left(\epsilon_i\right) = \sigma\left(x_0\right) + t \cdot \sigma\left(\epsilon\right)$$

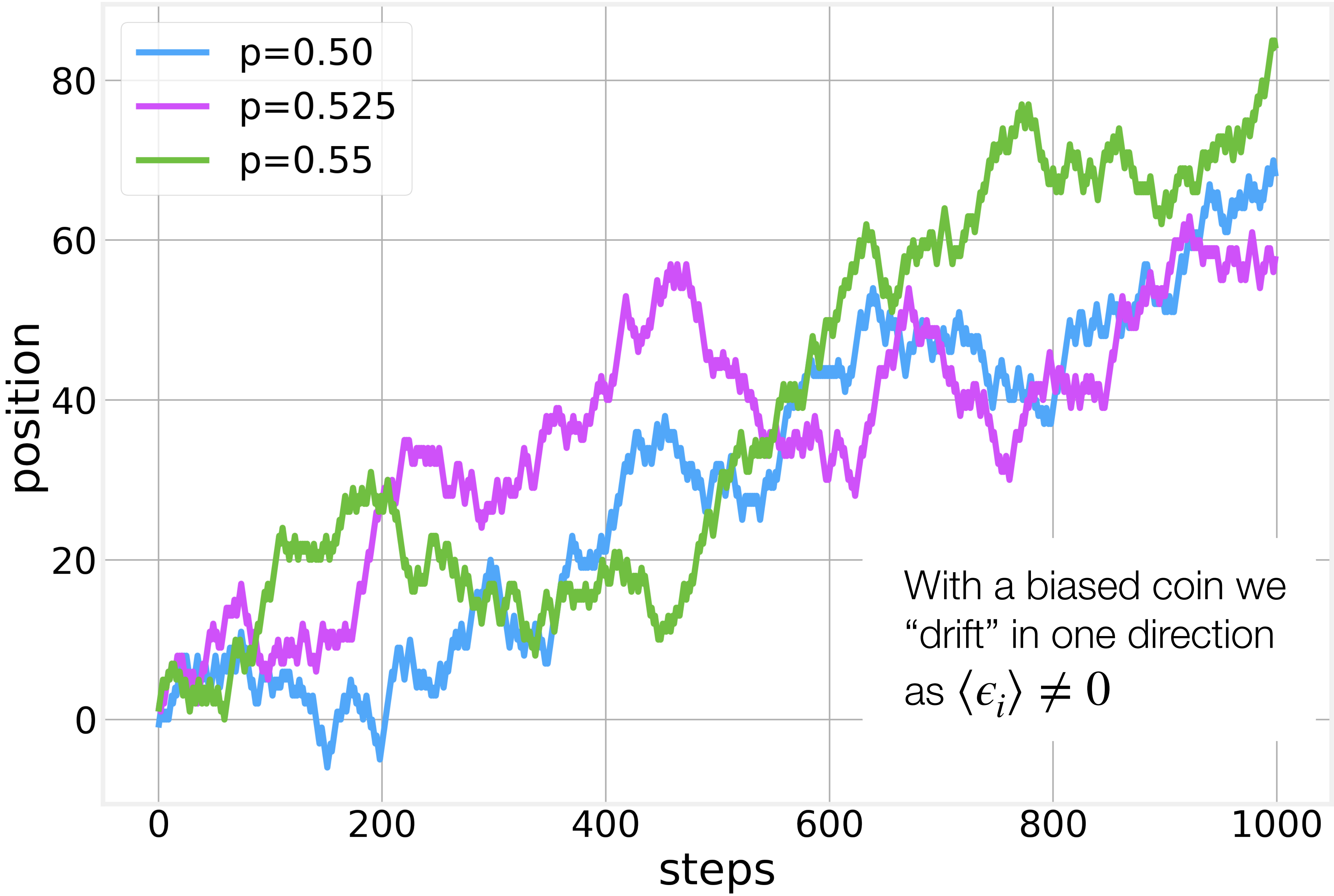
- which is not constant. So even the simple random walk is not a stationary process.

Random Walks



Deviations from Zero can be significant because $\sigma\left(\epsilon_i\right) \neq 0$ and $\sigma\left(x_t\right)$ grows linearly in time

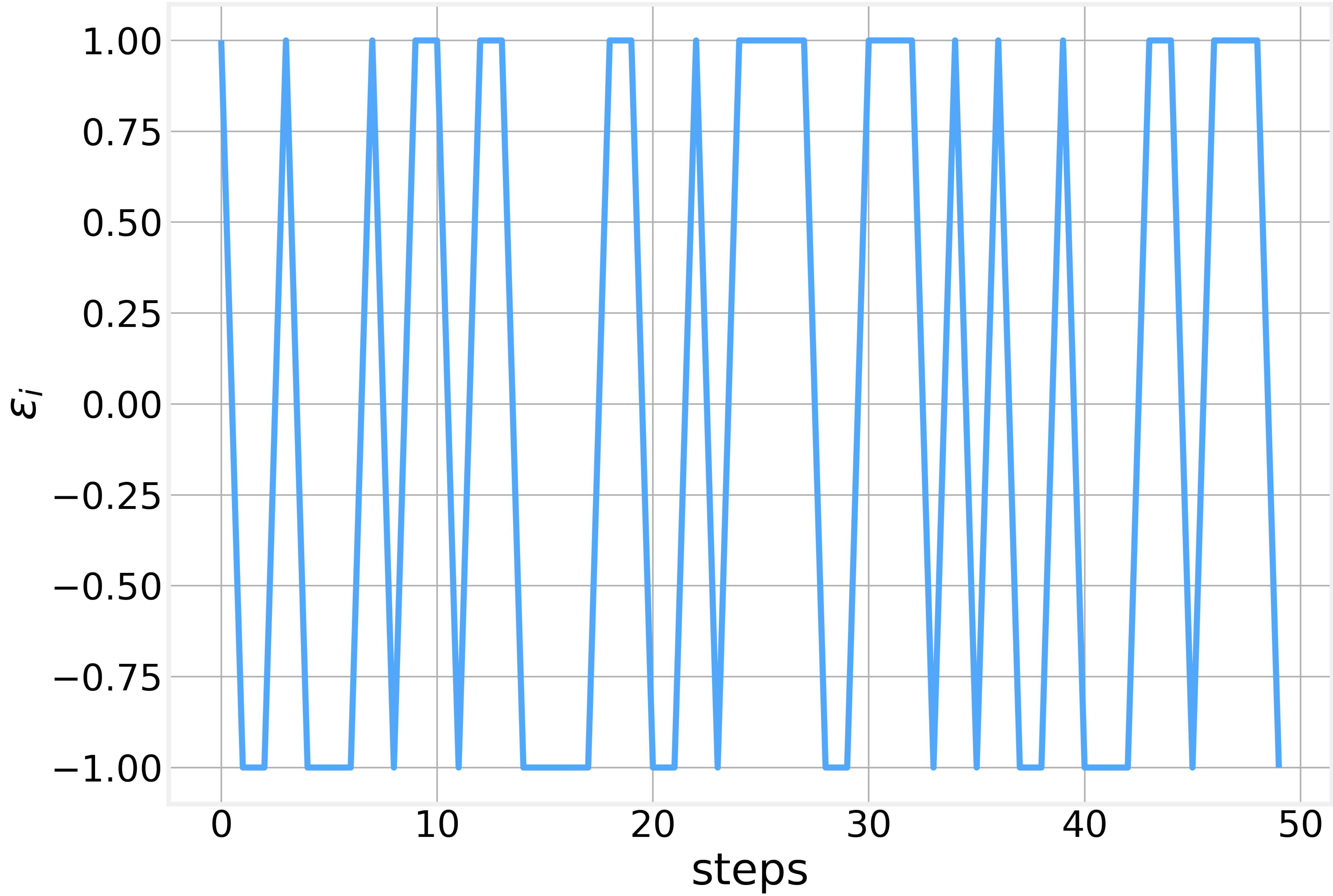
Random Walk with a drift



With a biased coin we "drift" in one direction as $\langle \epsilon_i \rangle \neq 0$

White noise

Let's take a deep look at our stochastic variables (the outcomes of the "coin flips")



Stationary vs Non-Stationary

- We can also describe this process in the same mathematical framework as:

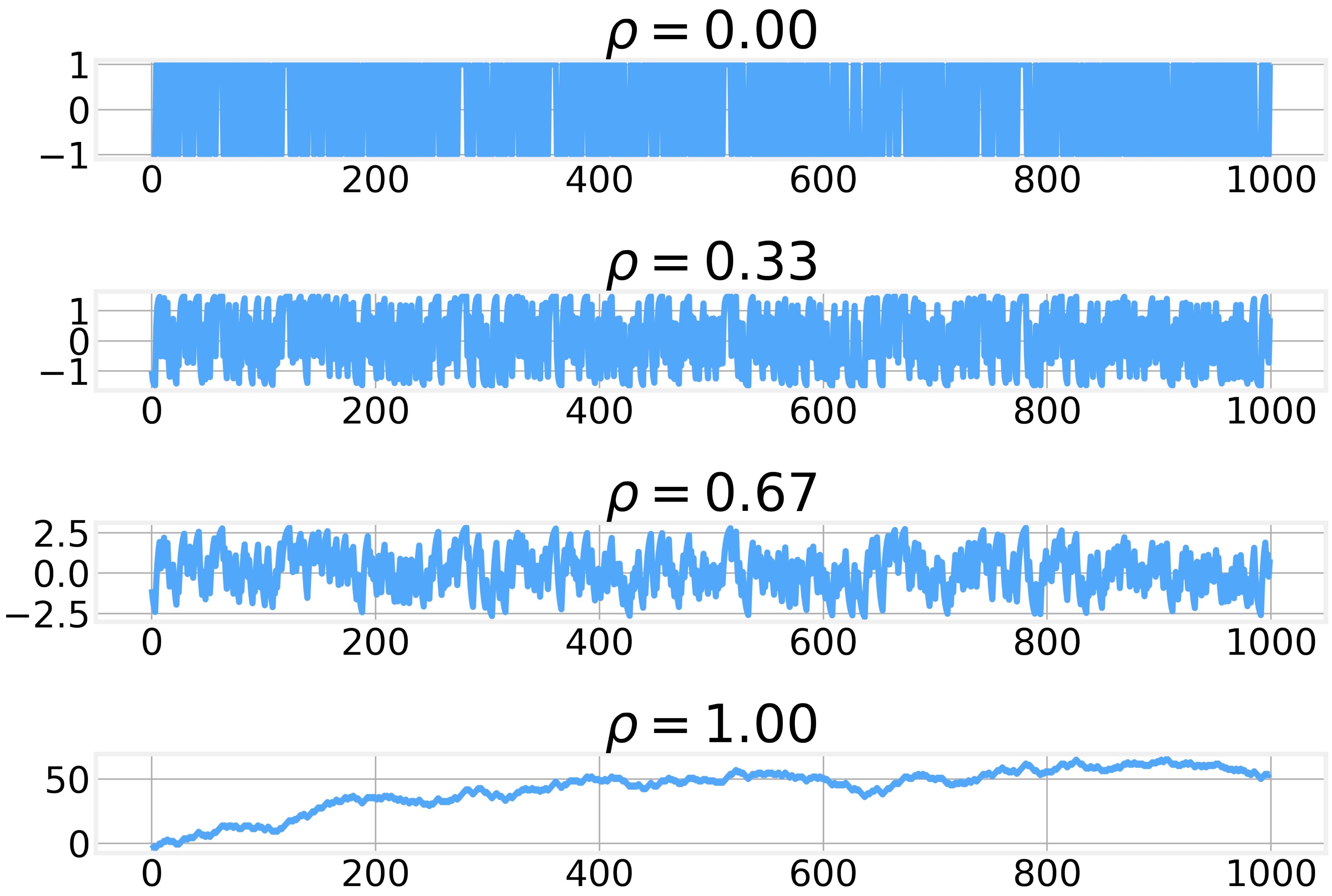
$$x_t = \epsilon_i$$

- which is clearly a stationary process.
- We can combine both expressions into a single one:

$$x_t = \rho x_{t-1} + \epsilon_i$$

- Where ρ gives us a "knob" to interpolate between the two extremes, between a stationary and a non-stationary process.

Stationary vs Non-Stationary



Dickey-Fuller Test

- The Dickey-Fuller Test is a test of stationarity inspired by a simple: Can we show that

$$\rho \neq 1$$

- with some degree of certainty?
- Numerically, we can express this as a linear regression fit

$$x_t - x_{t-1} = \gamma \; x_{t-1} + \epsilon_t$$

- where $\gamma = \rho - 1$
- The slope of the regression is then our expected value for $\rho - 1$.
- If the process is non-stationary then we expect $\gamma \neq 0$.

Dickey-Fuller Test

- From the residuals of γ we compute the Dickey-Fuller statistic:

$$DF = \frac{\hat{\gamma}}{SE(\gamma)}$$

- The value of this statistic is then compared with a critical values table.
- In general, the more negative it is, the more certain we can be that we can reject the null hypothesis
- The Dickey-Fuller Test has many variants.
- The most common one is the known as the Augmented-Dickey-Fuller test and is able to account for multiple lags, trends, etc.

Critical values for Dickey–Fuller *t*-distribution.

	Without trend		With trend	
Sample size	1%	5%	1%	5%
T = 25	–3.75	–3.00	–4.38	–3.60
T = 50	–3.58	–2.93	–4.15	–3.50
T = 100	–3.51	–2.89	–4.04	–3.45
T = 250	–3.46	–2.88	–3.99	–3.43
T = 500	–3.44	–2.87	–3.98	–3.42
T = ∞	–3.43	–2.86	–3.96	–3.41

Source[2]:373

Code - Random Walks

https://github.com/DataForScience/Timeseries

Lesson VI: ARIMA Models

Moving Average (MA) Model

- We start our exploration of the ARIMA family of models by considering the Moving Average model.
- The simplest moving average model can be written as:

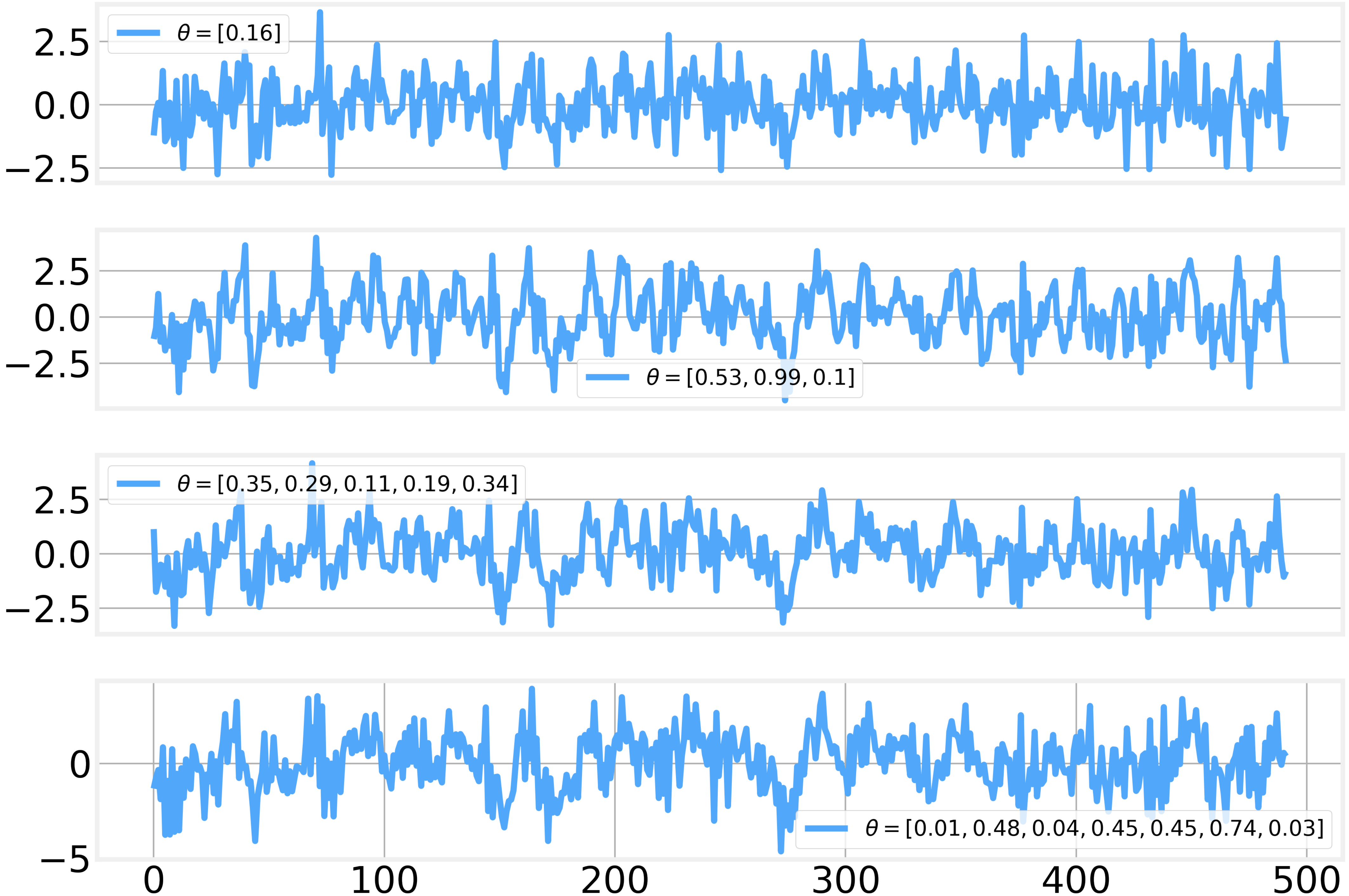
$$x_t = \epsilon_t$$

- Which we already saw in our discussion of random walks.
- A more general case can be written as:

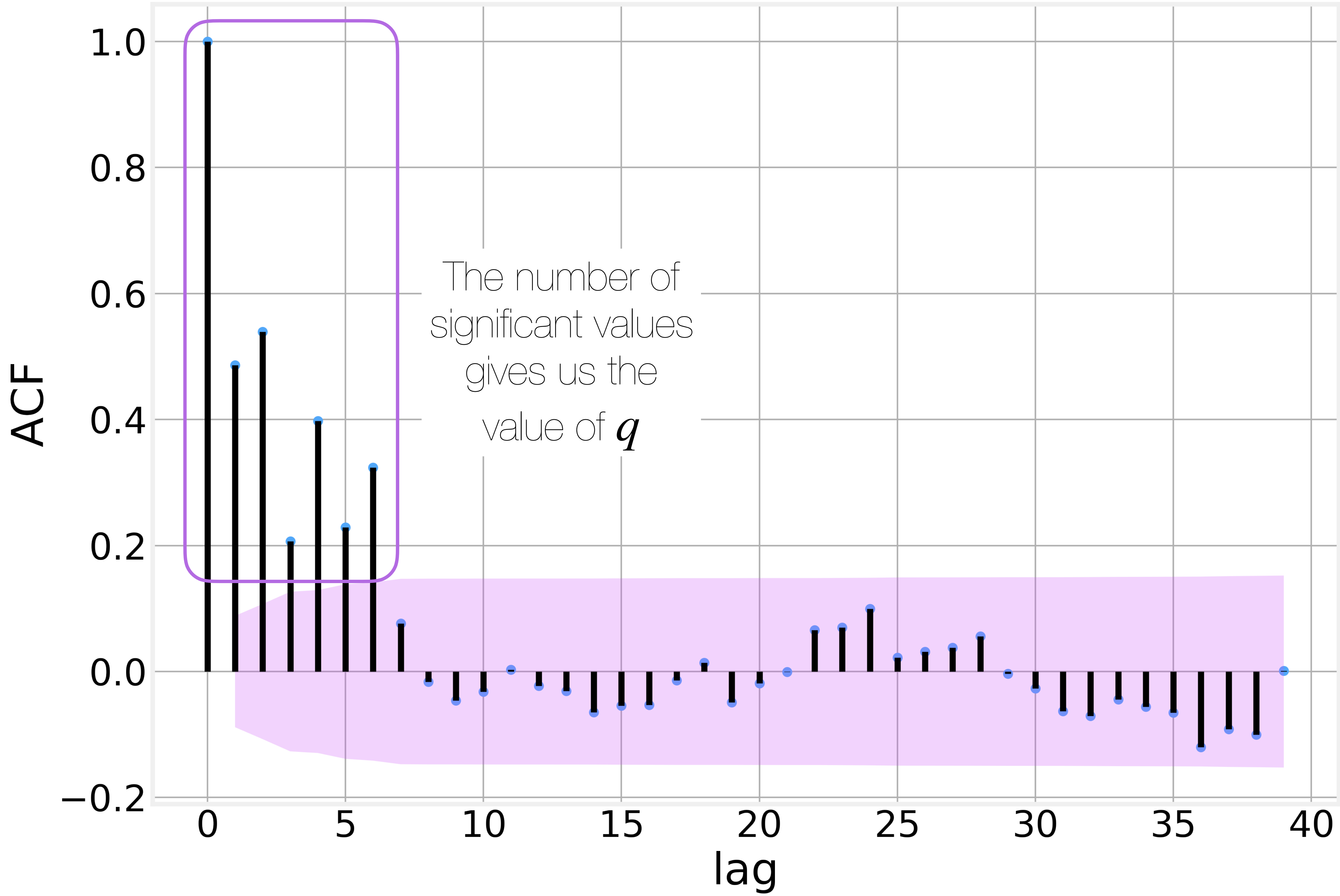
$$x_t = \beta + \sum_{l=0}^{q} \omega_l \, \epsilon_{t-l}$$

- where β is a constant offset, ω_l are the weights for the values at lag l and q is the moving window size. $\omega_0 \equiv 1$.
- The ϵ_t values are stochastic variables (often referred to as "errors") rather than the actual observed values x_t
- The generated x_t is uncorrelated with itself for any lag $l > q$

Moving Average (MA) Model



Moving Average (MA) Model



Auto-Regressive (AR) Models

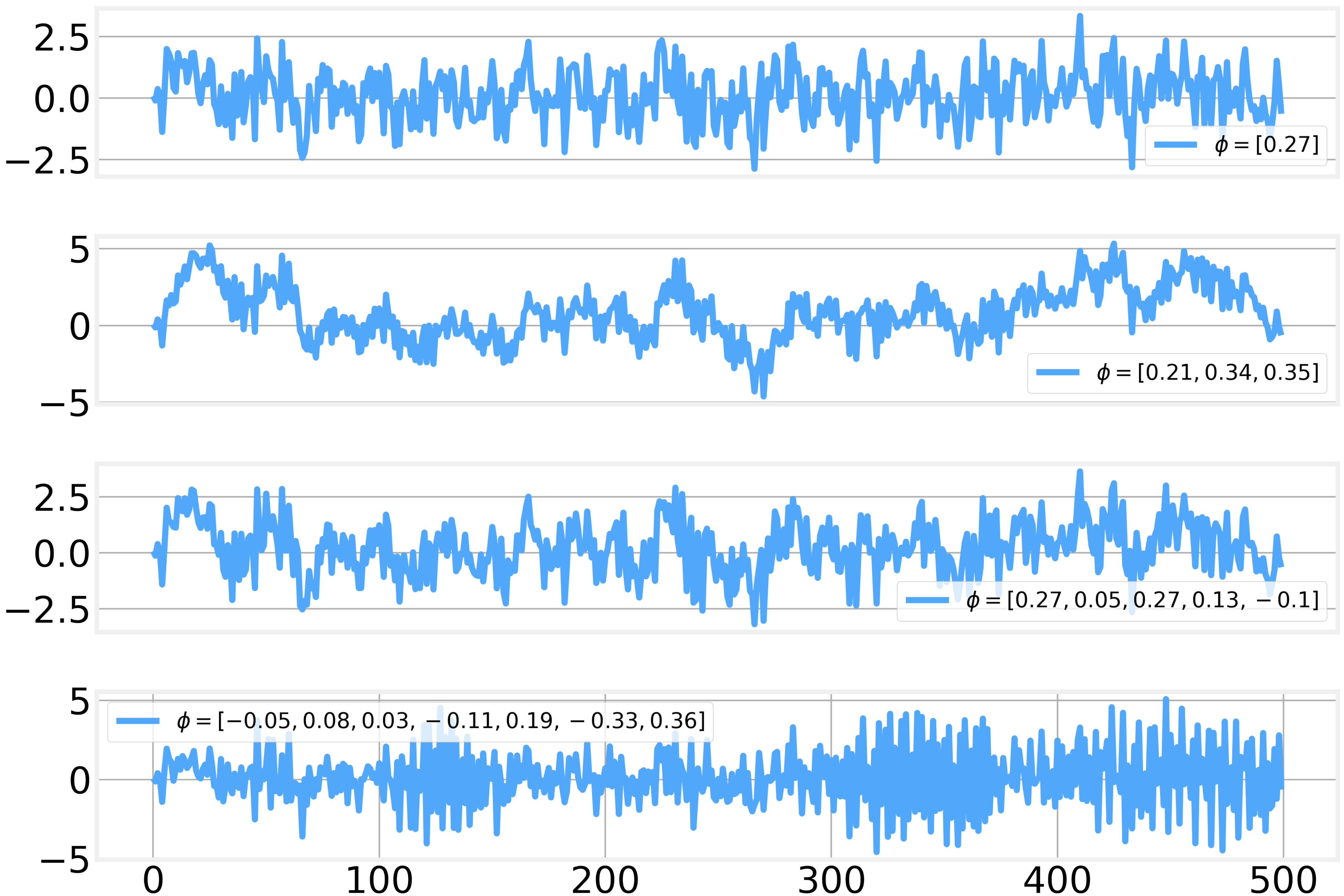
- Auto-Regressive models rely on the fact that in stationary models any deviation from the mean must be compensated. The series must "revert to the mean".

- AR models can be defined as:

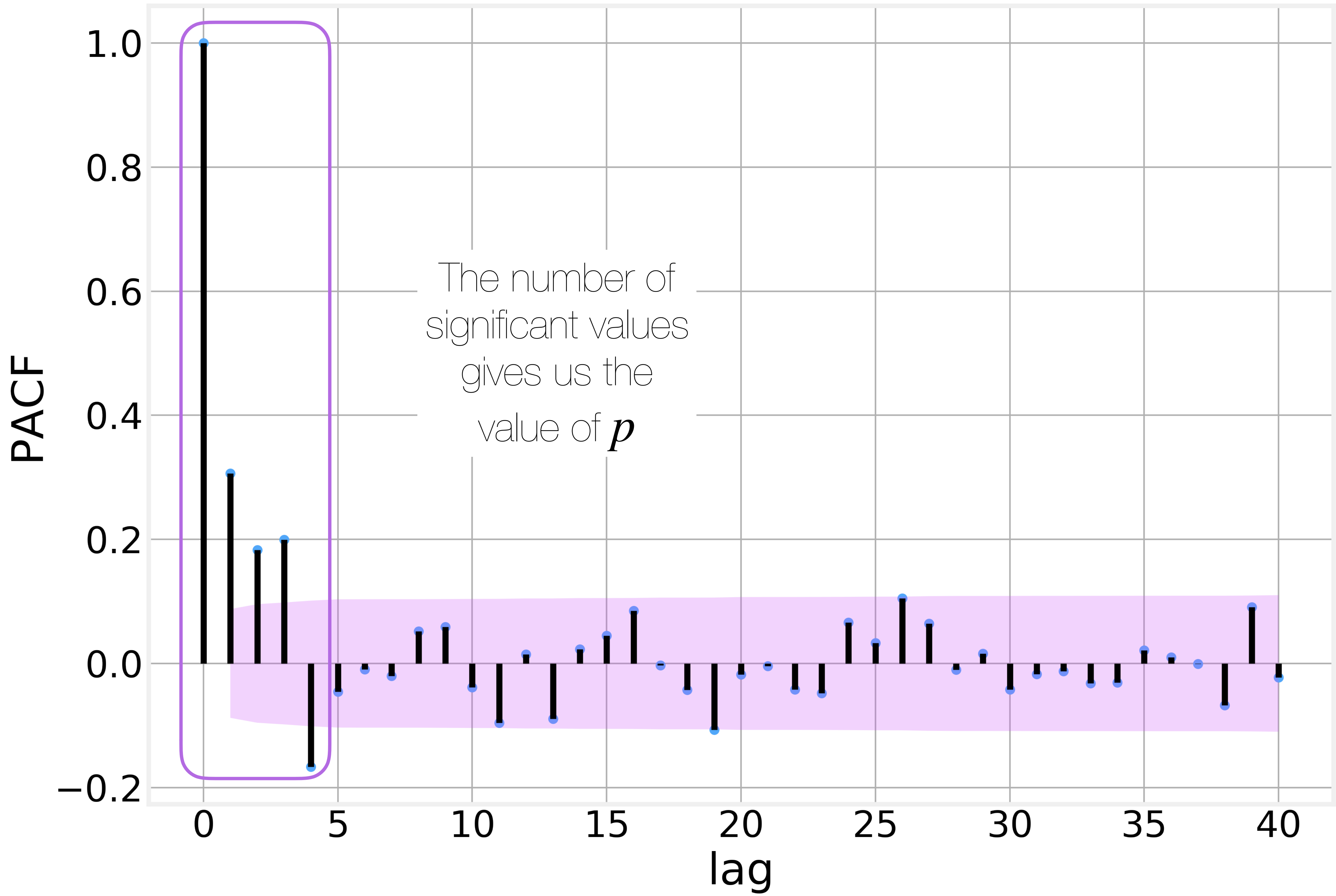
$$x_t = \alpha + \epsilon_t \sum_{l=1}^{p} \phi_l \ x_{t-l}$$

- Where the constant c represents the process' average value and the x_{t-l} are the observed values at a given lag l and ϕ_l are the corresponding weights.

Auto-Regressive (AR) Models



Auto-Regressive (AR) Models



Integrative (I) “model”

- We already saw that we can take differences to “stationarize" the time series.
- To recover the original values, we must then integrate
- While not a model by itself, it is often an important first step in modeling time series

ARIMA model

- The three classes of models we described above can be integrated into a single model:

Auto
Regressive
Integrated
Moving
Average

$$x_t = c + \sum_i^p \phi_i x_{t-i} + \epsilon_t$$

$$x_t = \mu + \epsilon_t + \sum_i^q \theta_i \epsilon_i$$

- The complete model can be written as:

$$\hat{x}_t = c + \mu + \sum_i^p \phi_i x_{t-i} + \sum_j^q \theta_i \epsilon_{t-i} + \epsilon_t$$

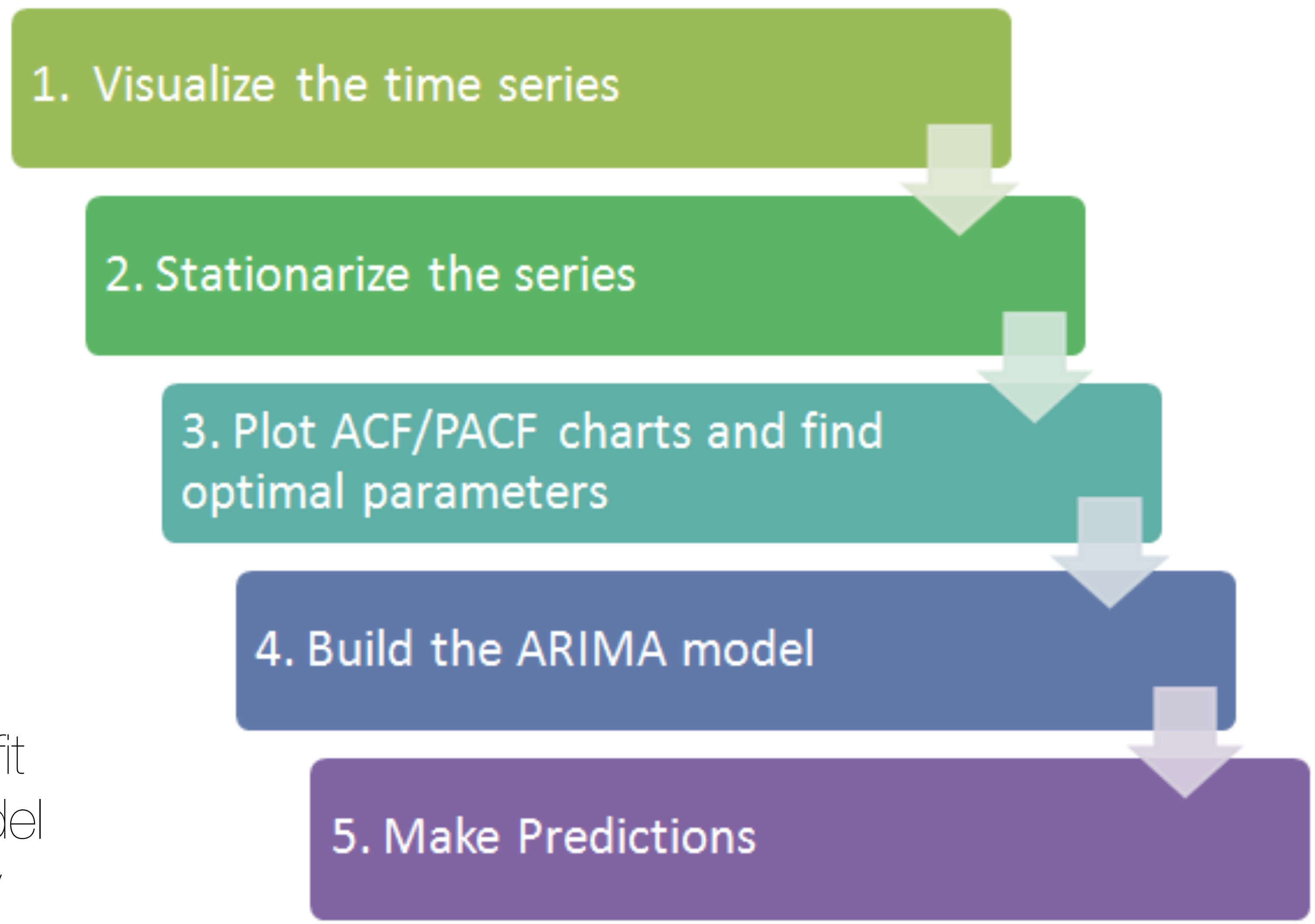
- where $\hat{x}_t$ is the properly differentiated time series

ARIMA model

- From this simple definition we can easily recover several interesting special cases:
- $ARIMA\ (0,1,0)$ - Random Walk (with or without drift)
  - $x_t - x_{t-1} = c + \epsilon_t$
- $ARIMA\ (0,0,0)$ - White noise (the sequence of stochastic variables)
  - $x_t = \epsilon_t$
- $ARIMA\ (0,1,1)$ - Exponential Smoothing
  - $x_t - x_{t-1} = \epsilon_t + \theta_1 \epsilon_{t-1}$
- $ARIMA\ (0,2,2)$ - Double exponential Smoothing
  - $x_t - 2x_{t-1} + x_{t-2} = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}$

Fitting ARIMA models

https://www.analyticsvidhya.com/blog/2015/12/complete-tutorial-time-series-modeling/



- d - degree of differencing
- p - number of lag observations included in the model ($PACF$)
- q - size of the moving average window (ACF)

The general procedure to fit an ARIMA model was originally proposal by Box-Jenkins

Code - ARIMA

https://github.com/DataForScience/Timeseries

Events

www.data4sci.com/newsletter

Applied Probability Theory for Everyone

Jan 27, 2020 - 5am-9am (PST)

Natural Language Processing (NLP) for Everyone

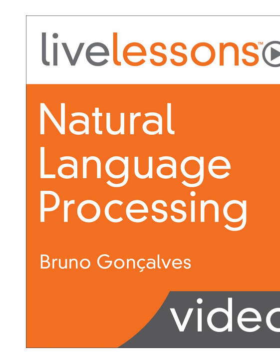
Feb 10, 2020 - 5am-9am (PST)

Graphs and Network Algorithms for Everyone

Feb 28, 2020 - 5am-9am (PST)

Natural Language Processing (NLP) from Scratch

http://bit.ly/LiveLessonNLP - On Demand



Time series modeling:

ML and deep learning approaches

http://bit.ly/StrataSJ20