→ EARTH OBSERVATION SUMMER SCHOOL

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Recapitulation of Variational Methods

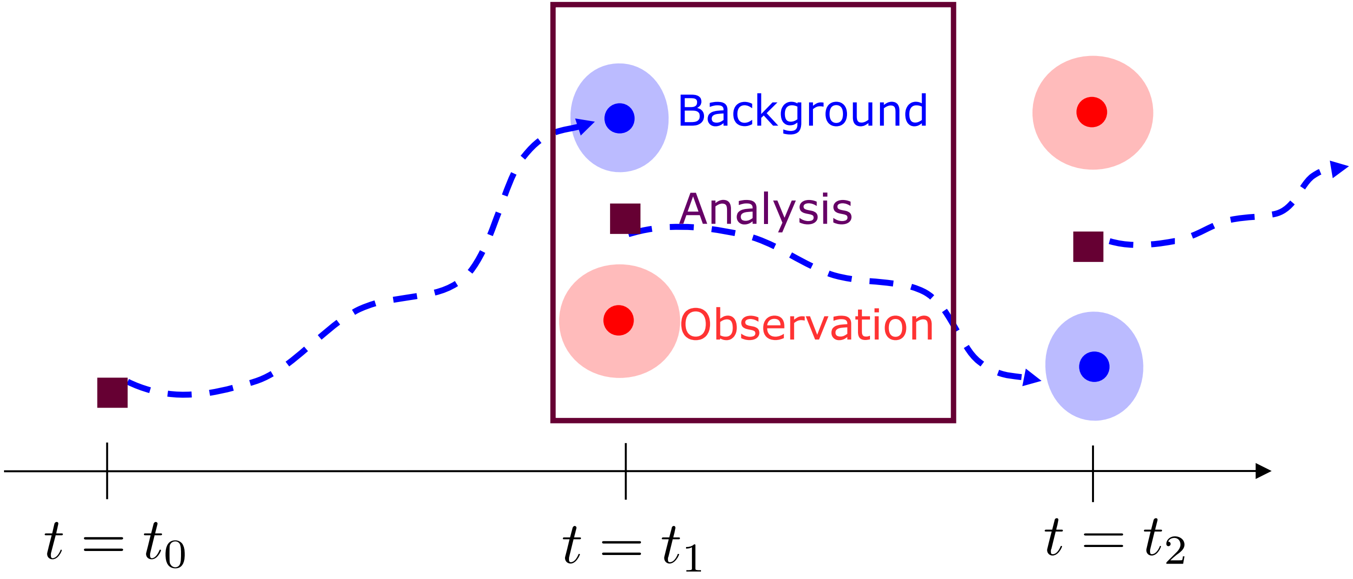
Javier Amezcua, Zackary Bell, Natalie Douglas, Ewan Pinnington

National Centre for Earth Observation
NATURAL ENVIRONMENT RESEARCH COUNCIL

University of Reading

UNIVERSITY OF SURREY

3DVar is a **variational method** that acts at a **single time**. It is a **maximum-a-posteriori** (MAP) estimator.



The **light-coloured ellipses** represent **covariances** (uncertainty). Recall that in **3DVar** this is **not updated** in the analysis step.

3DVar

3DVar needs to find the minimum of the following **cost-function**.

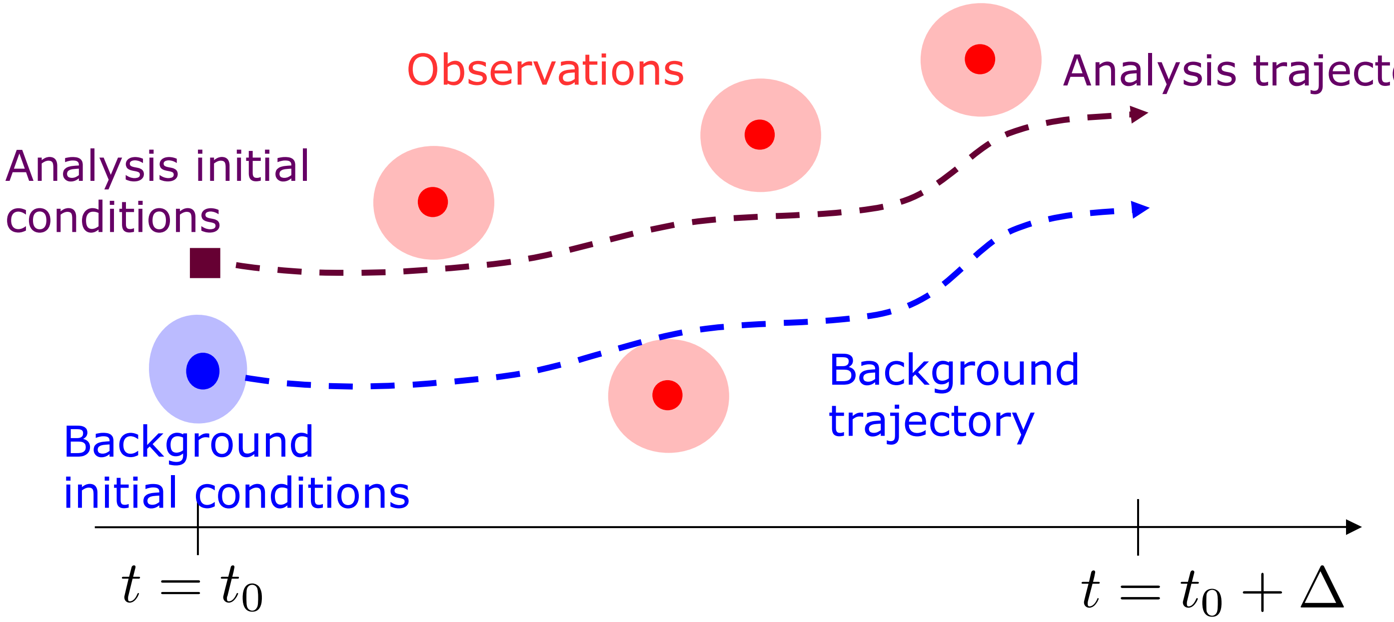
$$\mathbf{x}^a = \mathrm{argmin}_{\mathbf{x}}\ \mathcal{J}(\mathbf{x})$$

$$\mathcal{J}(\mathbf{x}) = \frac{1}{2}\left(\mathbf{x} - \mathbf{x}^b\right)^{\mathbf{T}} \mathbf{B}^{-1}\left(\mathbf{x} - \mathbf{x}^b\right) + \frac{1}{2}\left(\mathbf{y} - h(\mathbf{x})\right)^{\mathbf{T}} \mathbf{R}^{-1}\left(\mathbf{y} - h(\mathbf{x})\right)$$

- This function is **quadratic** for linear observation operator h=**H**. Quadratic functions have a **unique extremum**!
- Complications arise from: the structure of **B** and **R**, and the complexity of h.

SC-4DVar

4DVar is a **MAP** estimator over a time window. In the **strong-constraint** case (no model error), it reduces to the search for **'optimal' initial conditions**.



4DVar also uses **fixed covariances.** However, in the **optimisation process** within the **time window**, the background covariance is **evolved implicitly.**

SC-4DVar

The **cost-function** is more complicated since there is a **time component**.

$$\mathbf{x}_0^a = \mathrm{argmin}_{\mathbf{x}_0} \; \mathcal{J}\left(\mathbf{x}_0\right)$$

$$\mathcal{J}\left(\mathbf{x}_0\right) = \frac{1}{2}\left(\mathbf{x}_0 - \mathbf{x}_0^b\right)^{\mathbf{T}} \mathbf{B}^{-1}\left(\mathbf{x}_0 - \mathbf{x}_0^b\right) + \frac{1}{2}\sum_{l=1}^{L}\left(\mathbf{y}_l - h_l\left(m_{0\to l}\left(\mathbf{x}_0\right)\right)\right)^{\mathbf{T}} \mathbf{R}_l^{-1}\left(\mathbf{y}_l - h_l\left(m_{0\to l}\left(\mathbf{x}_0\right)\right)\right)$$

- **Observations** may **not occur** every **model time step**.
- The type of **observations** can be **different** at **different times** of the assimilation window. Both *h* and **R** can **depend on time**.
- For **linear evolution and observation operators**, this is a **quadratic function**. Otherwise it can become really complicated.
- One may need to solve this as **a sequence of linearised problems** (incremental form).