Module 8

Sunday, November 5, 2023 8:42 PM

In-Class Exercise 2: Use the RREF to write down the corresponding equations with the basic variables on the left and free variables on the right. Express x_1 as a linear combination of x_3, x_4, x_5. Then express x_2 as a linear combination of x_3, x_4, x_5. What are the solutions when:

1. $x_3 = 1, x_4 = 0, x_5 = 0$?
2. $x_3 = 0, x_4 = 1, x_5 = 0$?
3. $x_3 = 0, x_4 = 0, x_5 = 1$?

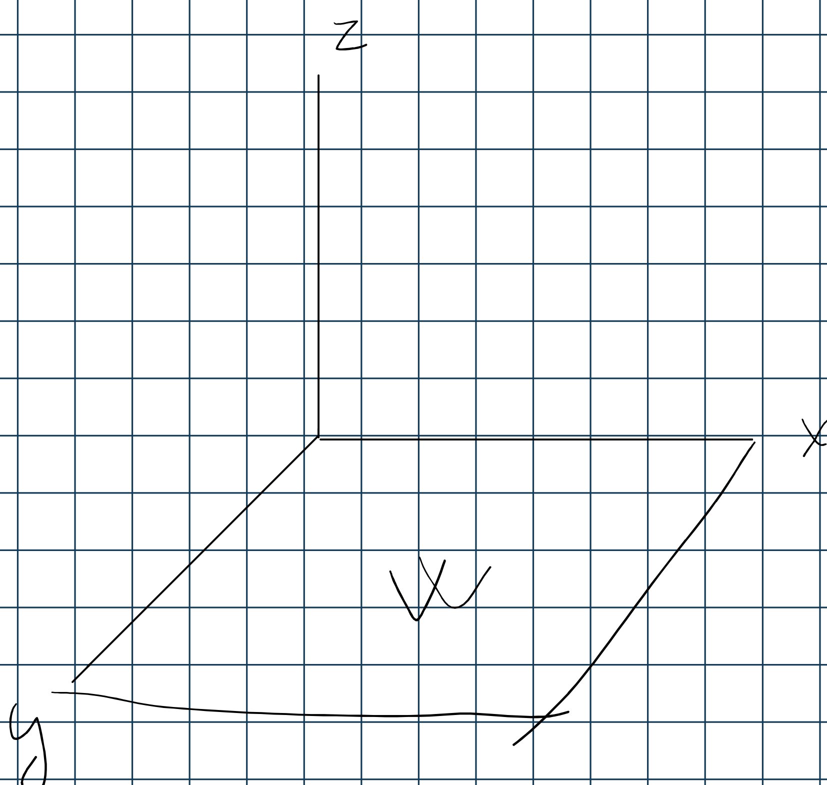
$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} \mathbf{1} & 0 & 1 & 1 & 2 & 0 \\ 0 & \mathbf{1} & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 = -(x_3 + x_4 + 2x_5)$

$x_2 = -(x_3 - 2x_4 - x_5)$

1. $x_1 = -1$ $\quad$ $x_2 = -1$
2. $x_1 = -1$ $\quad$ $x_2 = 2$
3. $x_1 = -2$ $\quad$ $x_2 = 1$

In-Class Exercise 3: Before we get to the proof, draw an example in 3D with **W** = the (x,y)-plane.



In-Class Exercise 4: How do we know that $\mathbf{W}^{\perp}$ has dimension $n - r$?

If V is a finite-dimensional vector space and W is a subspace of V, then

dim(W)+dim(W⊥)=dim(V)

In this context, VV is Rn, the space of all real n-tuples, and W is a subspace of V.

Given that dim(W)=r, and we are working in Rn, we have:

r+dim(W⊥)=n
Solving for dim(W⊥) gives us:

dim(W⊥)=n−r

- Proposition 8.4 says that any $\mathbf{x} \in \mathbb{R}^n$ can be written uniquely as $\mathbf{x} = \mathbf{y} + \mathbf{z}$ where

$$\mathbf{y} \in \text{rowspace}(\mathbf{A})$$
$$\mathbf{z} \in \text{nullspace}(\mathbf{A})$$

- Then,

$$\begin{aligned}\mathbf{b} &= \mathbf{Ax}\\ &= \mathbf{A}(\mathbf{y}+\mathbf{z})\\ &= \mathbf{Ay}+\mathbf{Az}\\ &= \mathbf{Ay}\end{aligned}$$

In-Class Exercise 5: Why is $\mathbf{Az} = \mathbf{0}$?

Z is the nullspace of A, which means it is the vector that results in a 0 vector when multiplied by A

- **Proposition 8.6:** For any matrix $\mathbf{A}_{m\times n}$

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T\mathbf{A})$$

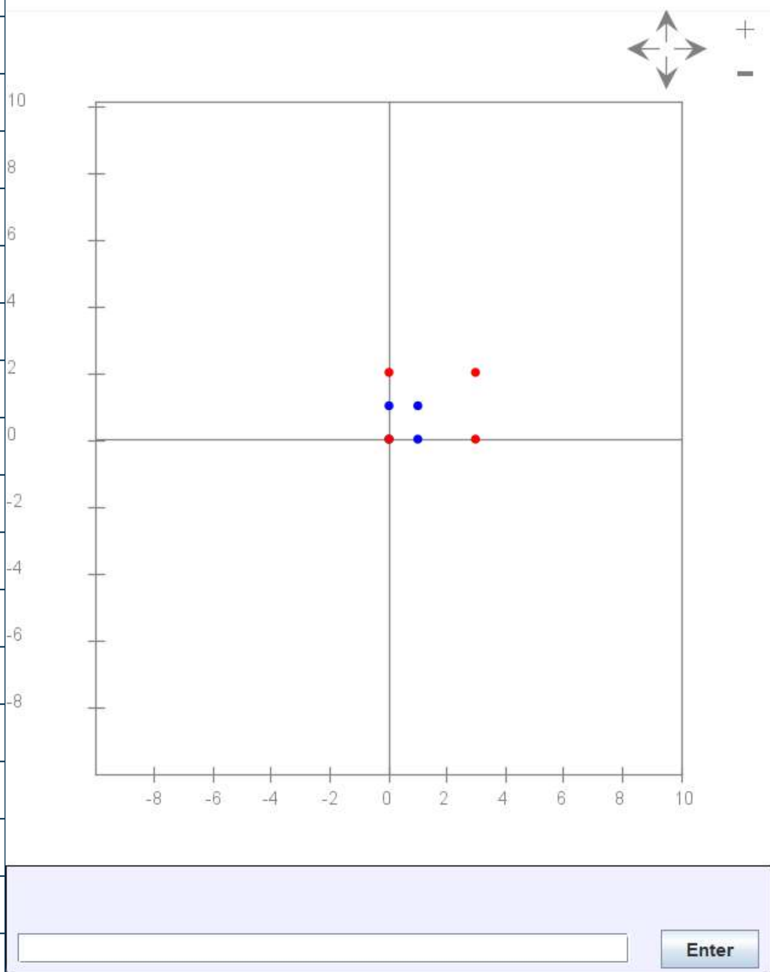
In-Class Exercise 7: Prove this result using the following steps:

1. How many columns does $\mathbf{A}^T\mathbf{A}$ have?
2. If the rank of $\mathbf{A}^T\mathbf{A}$ is r, what is the dimension of the nullspace of $\mathbf{A}^T\mathbf{A}$?
3. How are the nullities of $\mathbf{A}$ and $\mathbf{A}^T\mathbf{A}$ related?

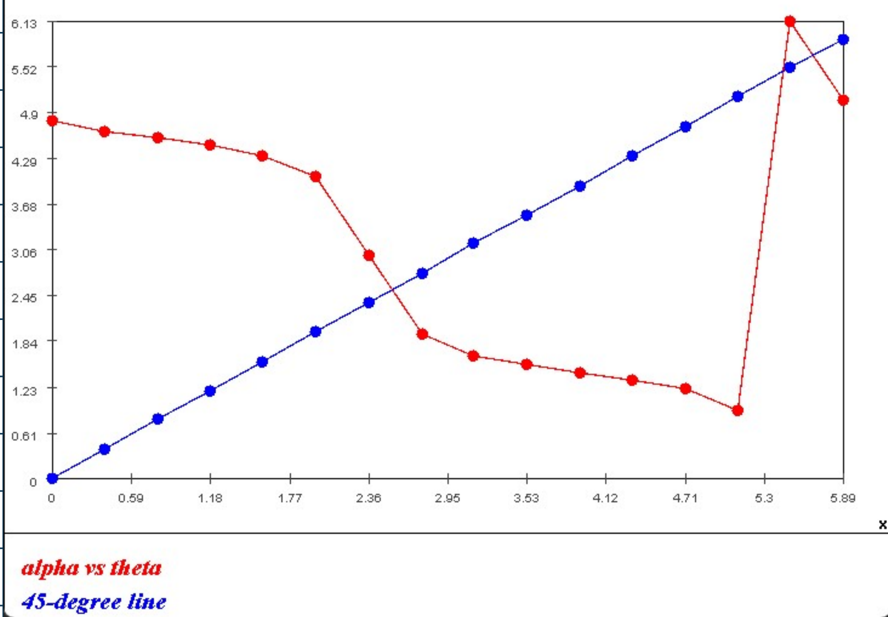
1) $A^T = \mathbb{R}^{n \times m}$ $A^T A = \mathbb{R}^{n \times n}$ -> n cols

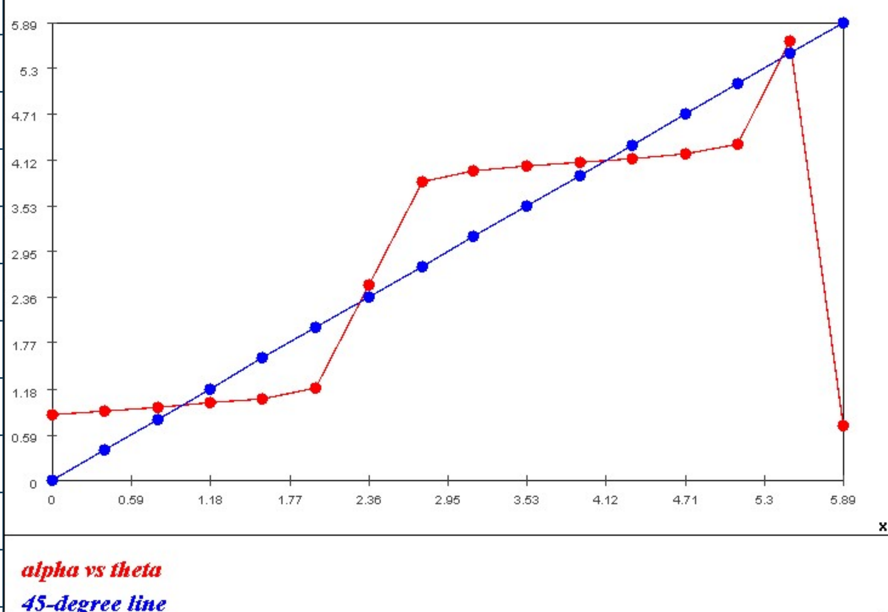
2) 2x1

In-Class Exercise 8: Download DiagonalExample.java, compile and execute for different values of `k`. You will also need your `MatrixTool`, and DrawTool.java.



In-Class Exercise 10: Download SVDExplore.java and compare the effect of transformation by a random $\mathbf{A}$ and then its $\mathbf{U}$ and $\mathbf{V}$ matrices when $\mathbf{A} = \mathbf{U\Sigma V}^T$. In all three cases, a graph of α_i vs. θ_i is plotted. Execute a few different times for each case: each execution creates a new random matrix. You will also need `lintool`, UniformRandom.java, Function.java, and SimplePlotPanel.java.





In-Class Exercise 11: Create a 3x3 matrix with rank=1. Download SVDRank.java and enter your matrix to confirm the rank. Observe that the remainder of the code generates random matrices to compute the average rank. Increase the number of such matrices. What is the intuition for why it's hard to generate a random 3x3 matrix with rank less than 3? [Hint: think about the columns, their independence, and use a geometric interpretation].

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \to \text{Rank } 1$$

```
(base) C:\Users\togru\ADA\cla\linAlg\module8>java --module-path %PATH_TO_FX% --add-modules javafx.controls SVDRank.java
rank=1
Avg rank: 3.0
```

Linear Independence: In three-dimensional space, R3, a set of three vectors is linearly independent if no vector in the set can be written as a linear combination of the others. Geometrically, this means that the vectors do not all lie in the same plane through the origin – instead, they span the entire space.

Random Vectors: When you choose vectors at random, the probability that any two of them will be perfectly aligned (i.e., scalar multiples of each other) is essentially zero. This is because there are infinitely many directions in which a vector can point, and choosing exactly the same direction or exactly opposite direction randomly is infinitely improbable.

Three-Dimensional Space: In R3, for a matrix to have rank less than 3, at least two of its columns must be linearly dependent. This would mean that, in geometric terms, two of the vectors must lie on the same line or, if one column is a zero vector, at least two vectors must lie in a plane. When choosing vectors randomly, it's highly unlikely that you would pick two vectors that are scalar multiples of each other or that three random vectors would all lie in the same plane.

Consider what it means to transform a vector:

- So far, the only way we know is through multiplication by a matrix.
- That is, we start with some vector $\mathbf{x}$, and multiply by a matrix $\mathbf{A}$ to get a vector $\mathbf{y}$:

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

In-Class Exercise 12: What is the relationship between the dimension of $\mathbf{x}$ and the dimension of $\mathbf{y}$?

For $A = \mathbb{R}^{m \times n}$ and $x = \mathbb{R}^{n \times 1}$ $\Rightarrow y = \mathbb{R}^{m \times 1}$

This means the dimension of yy corresponds to the number of rows in AA, which is independent of the number of columns in AA (the dimension of xx)

In-Class Exercise 15: Show that:

1. Matrix-vector multiplication $\mathbf{Ax}$ is a linear transformation.
2. If T is a linear transformation then $T(\mathbf{0}) = \mathbf{0}$.
3. Use the above result to show that a geometric translation cannot be a linear transformation.

(Remember, in the second case, both the argument and result are the zero *vector*).

To show that matrix-vector multiplication Ax is a linear transformation, we need to verify that it satisfies the two properties of a linear transformation for any vectors x,y and any scalars α,β:

Additivity: A(x+y)=Ax+Ay
Homogeneity: A(αx)=αAx

$$A = \begin{bmatrix} - a_1 - \\ - a_2 - \end{bmatrix}$$

$$A \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} a_1(x_1 + y_1) \\ a_2(x_2 + y_2) \end{bmatrix} =$$

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_1 x_1 + a_1 y_1 \\ a_2 x_2 + a_2 y_2 \end{bmatrix}$$

In-Class Exercise 16: What are the standard vectors for $n = 4$?

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Define the vectors

$$\mathbf{c}_1 = T(\mathbf{e}_1), \quad \mathbf{c}_2 = T(\mathbf{e}_2), \quad \ldots, \quad \mathbf{c}_n = T(\mathbf{e}_n),$$

In-Class Exercise 17: What is the dimension of each $T(\mathbf{e}_i)$?

$c_1 = \mathbb{R}^{m \times 1}$

- Now apply the transformation T:

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n) \\ &= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + \ldots + x_n T(\mathbf{e}_n) \\ &= x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \ldots + x_n\mathbf{c}_n \end{aligned}$$

In-Class Exercise 19: The third step is from the definition of the $\mathbf{c}_i$'s but what let's us conclude the second step from the first?

Additivity allows us to apply T to a sum of vectors by applying it to each vector in the sum individually.

Homogeneity allows us to pull out scalar multiples when applying T, thus preserving scalar multiplication within the transformation process.