Module 6

Monday, October 9, 2023 3:50 PM

- Establishment of Linear Independence

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \\ 3 & 1 \\ 0 & 4 \end{bmatrix} y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad x = 0$$

$$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Suppose we have three vectors

$$\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 6 \\ 1 \\ 7 \end{bmatrix}$$

and ask the question are there scalars α, β such that

$$\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}?$$

That is, can $\mathbf{w}$ be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$?

In-Class Exercise 1: How does one address the question for the above example?

The answer here lies in finding out whether the vectors u and v are linearly independent or not. We have already established that if two vectors are, when 3D is concerned, not on the same plane, then these vectors are enough to span the whole 3D space. Also, we should note that these vectors are not actually uniform basis vectors. For example, vector can be decomposed into the following.

$$u = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So the answer simply depends on whether we can establish that the two vectors are linearly independent or not.

$$\left[\begin{array}{cc|c} 2 & 0 & 6 \\ -1 & 1 & 1 \end{array}\right] \Rightarrow \begin{array}{c} \begin{array}{cc} \alpha & \beta \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 3 \\ -1 & 1 & 1 \end{array}\right] \end{array} \Rightarrow \alpha = 3$$

$$\left[\begin{array}{cc|c} 2 & 0 & 6 \\ -1 & 1 & 1 \\ 1 & 1 & 7 \end{array}\right] \xrightarrow{R_1/2} \begin{array}{c} \begin{array}{cc} \alpha & \beta \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 3 \\ -1 & 1 & 1 \\ 1 & 1 & 7 \end{array}\right] \end{array} \Rightarrow \alpha = 3$$

$$\boxed{\alpha + \beta = 7 \Rightarrow \beta = 4}$$

In-Class Exercise 2: Can **u** be expressed in terms of **v** and **w**?

Let's test this by calulating as well

$$\begin{array}{c} \begin{array}{cc} \alpha & \beta \end{array} \\ \left[\begin{array}{cc|c} 0 & 6 & 2 \\ 1 & 1 & -1 \\ 1 & 7 & 1 \end{array}\right] \end{array} \quad \beta = 1/3 \Rightarrow \alpha + \beta = -1$$

$$\alpha = -4/3$$

$$\boxed{\begin{array}{l} \alpha + 7\beta = 1 \\ -4/3 + 7/3 = 1 \end{array}}$$

Proven!

- On the other hand $\mathbf{z} = (6, 1, 8)$ cannot be expressed as a linear combination of **u** and **v**.

In-Class Exercise 3: Show that this is the case.

$$\left[\begin{array}{cc|c} 2 & 0 & 6 \\ -1 & 1 & 1 \\ 1 & 1 & 8 \end{array}\right] \xrightarrow{R_1/2} \begin{array}{c} \begin{array}{cc} \alpha & \beta \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 3 \\ -1 & 1 & 1 \\ 1 & 1 & 8 \end{array}\right] \end{array} \Rightarrow \alpha = 3$$

$$\alpha + \beta = 8 \Rightarrow \beta = 5$$

$-\alpha + \beta = 1$, However

$-\alpha+\beta=1$, However

$-3+5\neq 1$, Contradiction

$\mathbf{w}$ is *linearly dependent* on $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\mathbf{w} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$$

Since if 1 $\vec{v}$ is dependent on 2 others they are all dependent on each other, thus

- **Definition:** Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are *linearly independent* if the only solution to the equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

is $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

- The weirdness of the zero vector:
  - Suppose $\mathbf{v}_1 = \mathbf{0} = (0, 0, \ldots, 0)$.
  - Then, consider

$$\mathbf{v}_1 = \beta_2\mathbf{v}_2 + \ldots + \beta_n\mathbf{v}_n$$

  - Here, we can make $\beta_2 = \beta_3 = \ldots = \beta_n = 0$ so that

$$\mathbf{0} = 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_n$$

  - That is, the zero vector is can always be interpreted as "dependent" on any other collection of vectors.
  - If we were to use the definition, we would write:

$$(-1)\mathbf{0} + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_n = \mathbf{0}$$

  - Thus, *no collection of vectors that includes the zero vector can be linearly independent.*

This is because this coefficient, by definition, would have to be 0 to assume linear indep., but that is not a necessity due to the $\vec{0}$ vect,

- The interpretation in terms of the RREF gives us a *method* to determine whether a collection of vectors is linearly independent.
  - Put the vectors in a matrix.
  - Compute the RREF.
  - See if the RREF = **I**.

In-Class Exercise 5: Use the definition of linear independence to show both parts of Proposition 6.1 are true for this example, and then generalize to prove 6.1 for all RREFs.

Let's work on the following.

$$\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{bmatrix} 1 & 0 & -1 & 3 & 0 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\alpha_1 v_1 + \alpha_2 v_2 = 0$ only when $\alpha_1, \alpha_2 = 0$

$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 4 \end{bmatrix} = \vec{x}$ for the first 2 entries of $\vec{x}$ to be 0, the only possibility is for α_1 and α_2 to be 0.

v_3 and v_4 are 0 vect.

$\alpha_3 v_3 + \alpha v_4 = 0$ for any α_3 and α_4

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{c}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{c}_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{c}_4 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{c}_5 = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

- Proposition 6.2 says that vectors $\mathbf{c}_1, \mathbf{c}_2$ are independent.
- And that any of $\mathbf{c}_3, \mathbf{c}_3, \mathbf{c}_5$ can be expressed as a linear combination of $\mathbf{c}_1, \mathbf{c}_2$.

In-Class Exercise 6: Use the definition of linear independence to show both parts of Proposition 6.2 are true for this example, and then generalize to prove 6.2 for all RREFs.

For the same reasons as above, c_1 and c_2 are independent and we can easily find the coeff. with which c_3, c_4 & c_5 can be obtained using just c_1 and c_2

- Suppose we have the following vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_5 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

- Consider $\mathbf{v}_2$ and $\mathbf{v}_4$. Then, it's easy to see that
  - $\mathbf{v}_2$ and $\mathbf{v}_4$ are independent.
  - $\mathbf{v}_1 = \mathbf{v}_2 - \mathbf{v}_4$
  - $\mathbf{v}_3 = \mathbf{v}_2 + \mathbf{v}_4$
  - $\mathbf{v}_5 = 3\mathbf{v}_2 - \mathbf{v}_4$

  This suggests that we need at least two of these vectors to express the others.

- Similarly, it's easy to show that
  - $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent.
  - $\mathbf{v}_3 = -\mathbf{v}_1 + 2\mathbf{v}_2$
  - $\mathbf{v}_4 = -\mathbf{v}_1 + \mathbf{v}_2$
  - $\mathbf{v}_5 = \mathbf{v}_1 + 2\mathbf{v}_2$

In-Class Exercise 7: Show that the above is true given what was shown earlier for $\mathbf{v}_2$ and $\mathbf{v}_4$.

These vect are all on 3D since el. 4 is 0. V_2 and V_4 both are composed of unit vectors that can span the entire 3D space.

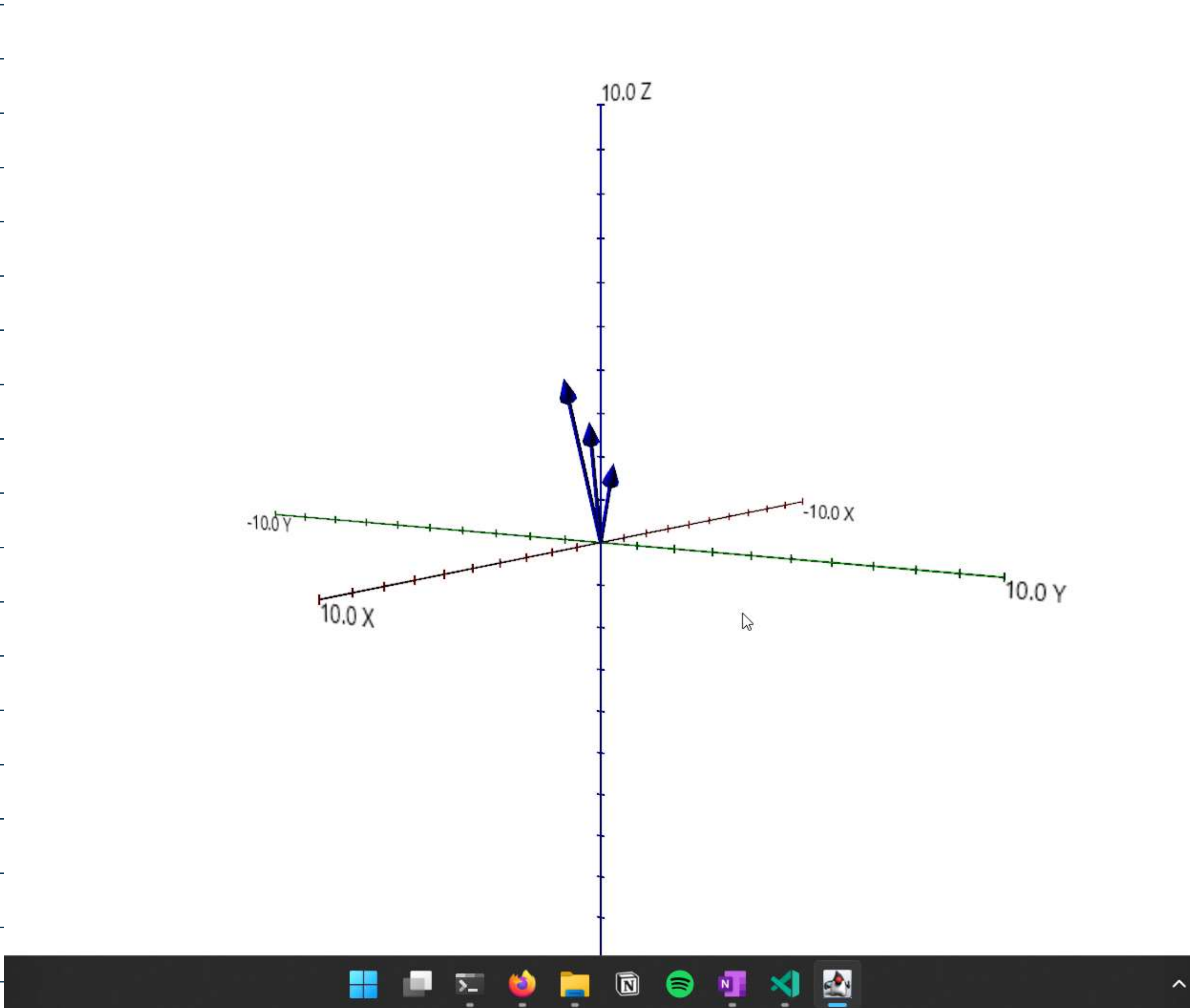
In-Class Exercise 8: Prove the following: if the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are independent, then so is any subset of these vectors.

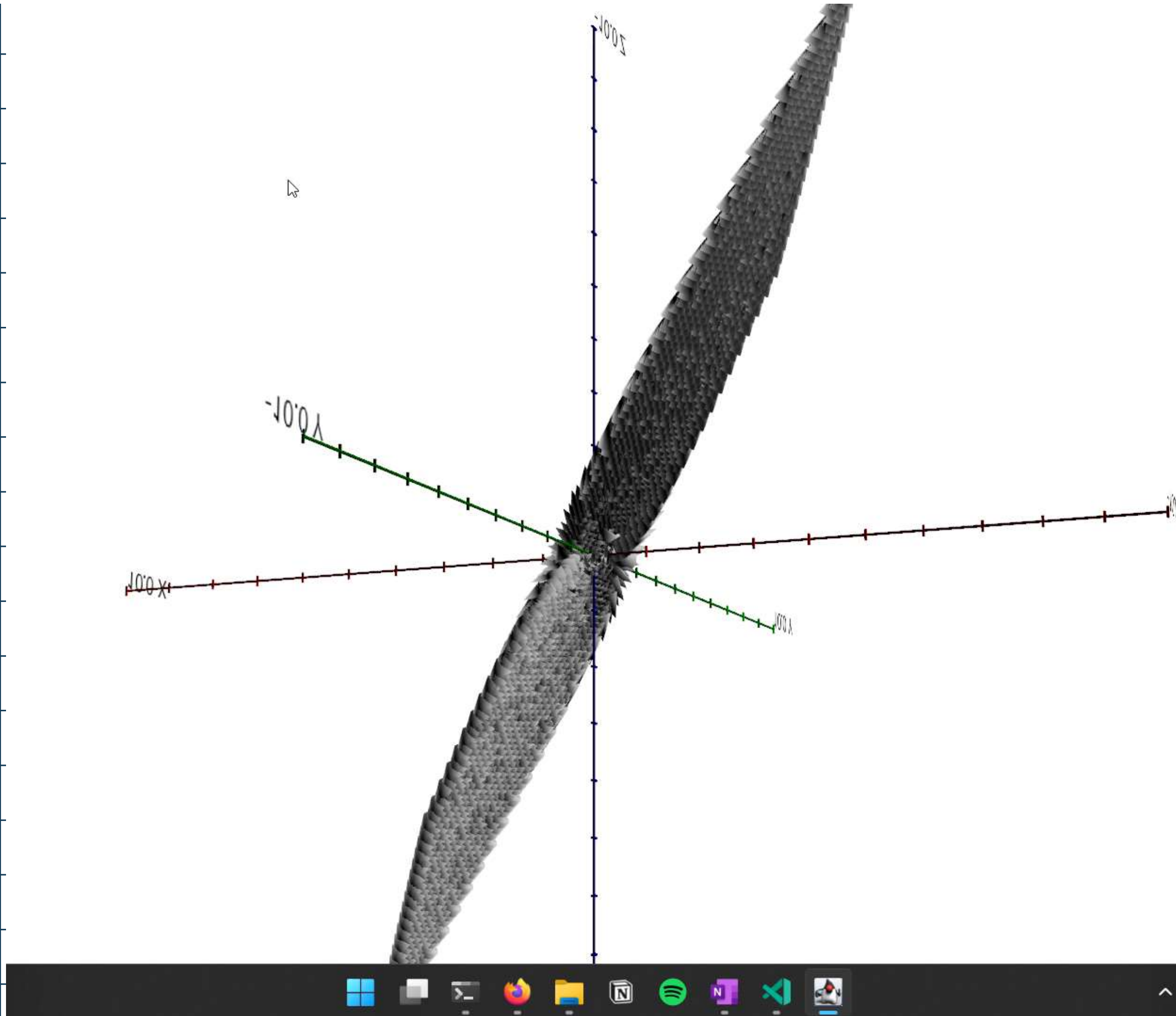
$\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n = 0$

$\alpha_1 v_1 + \alpha_2 v_2 = 0$

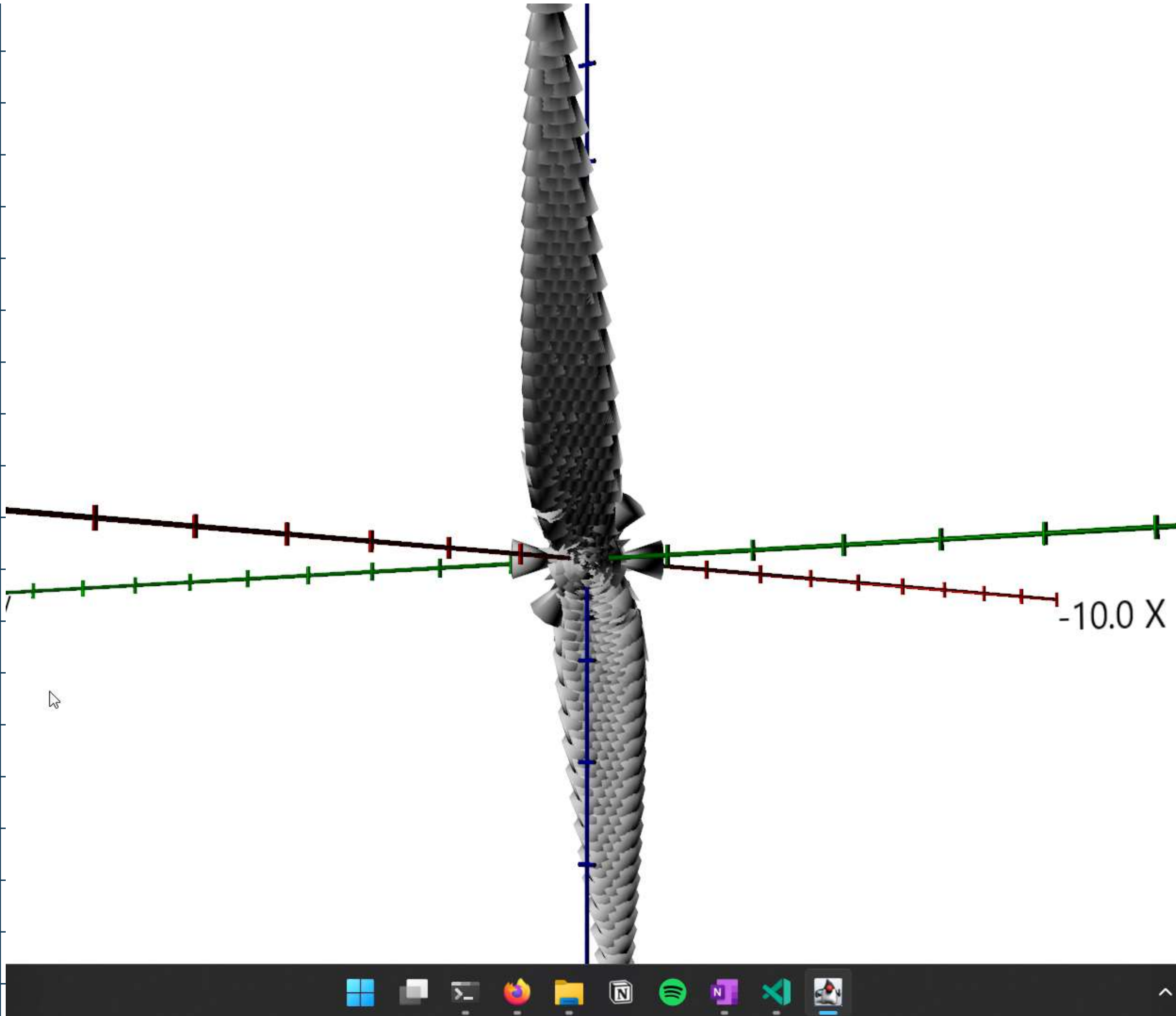
$-v_1 = \frac{\alpha_2 v_2}{\alpha_1}$

In-Class Exercise 9: In 3D, explore the span of $\mathbf{u} = (1, 1, 2)$, $\mathbf{v} = (2, 1, 3)$, $\mathbf{w} = (3, 1, 4)$ in Span3DExample.java. You will need your `MatrixTool.java` from earlier, or you can use your implementation of `LinTool` to compute scalar multiplication and addition of vectors.





In-Class Exercise 10: We'll now explore the span of just two of the above vectors: $\mathbf{u} = (1, 1, 2), \mathbf{v} = (2, 1, 3)$ in Span3DExample2.java. Is the span of the two the same as the span of the three? Try different bounds for the scalars. What is the span of just $\mathbf{u} = (1, 1, 2)$ all by itself, and is that the same as any of the other spans?



- Row operations do not change the rowspace, however the columnspace does change!

Thus, for a Matrix A and its RREF A'

$$\text{rowspace}(A) = \text{rowspace}(A')$$

$$\text{columnspace}(A) \neq \text{columnspace}(A')$$

- Basis vectors for a Matrix can be obtained by taking the RREF

- Basis vectors for a Matrix can be obtained by taking the RREF

In-Class Exercise 11: On paper, draw the vectors $\mathbf{u} = (1, 1, 0)$, $\mathbf{v} = (-1, 1, 0)$, $\mathbf{w} = (0, -1, 0)$. What is the set of vectors spanned by these three vectors? What two obvious vectors should one use to span the same set?

u, v and w span the vector space that can be described as below:

$$\alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

→ the basis vectors needed to span the same set.

This leads to the next important concept: *dimension of a subspace*

- **Definition:** The *dimension* of a subspace is the minimum *number* of vectors needed to generate the subspace via linear combinations.
- We saw earlier that for the subspace generated by $\mathbf{u} = (1, 1, 0), \mathbf{v} = (-1, 1, 0), \mathbf{w} = (0, -1, 0)$
  - $\mathbf{u} = (1, 1, 0), \mathbf{v} = (-1, 1, 0)$ are enough to generate the subspace
  - Alternatively, $\mathbf{e}_1 = (1, 0, 0), \mathbf{e}_2 = (0, 1, 0)$ are sufficient.
  - No single vector by itself can generate the (x,y)-plane.

In-Class Exercise 12: Why?

Because a single vector represents a line with its origin at the origin of the coord. sys. it belongs to. This means that any linear trans. will merely stretch this line (a 1D object) as opposed adding a new Dimension.

- Consider $\mathbf{W} = \text{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ where $\mathbf{u} = (1, 1, 0), \mathbf{v} = (-1, 1, 0), \mathbf{w} = (0, -1, 0)$.
  - We saw earlier that $\dim(\mathbf{W}) = 2$.
  - By Proposition 6.5, $\mathbf{u}, \mathbf{v}, \mathbf{w}$ should *not* be independent.
  - Which is indeed the case.

In-Class Exercise 13: Show that $\mathbf{u}, \mathbf{v}$ are independent and that $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 = (R_1 + R_2)/2} \begin{array}{c} \begin{array}{cc} \alpha & \beta \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & -0.5 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \boxed{\begin{array}{l} \alpha = -0.5 \\ \beta = 0.5 \end{array}}$$

In-Class Exercise 14: Prove Proposition 6.5. Hint: start by assuming that one vector among $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is dependent on the others. Can the span be generated by the others?

Let's assume v_1 is lin dep. on the others

$$= \quad v_1 = \beta_2 v_2 + \beta_3 v_3 + \ldots + \beta_n v_n$$

This means that v_1 lies in the subspace spanned by the other vectors!

$\Rightarrow$ Dim $W = n-1$, Contradiction

We will prove Theorem 6.6 in steps:

- First, recall the three row operations in creating an RREF:
  - Swapping two rows.
  - Scaling a row by a number.
  - Adding to a row a scalar multiple of another row.
- Now consider an $m \times n$ matrix $\mathbf{A}$ with rows $\mathbf{r}_1, \ldots, \mathbf{r}_m$ and its rowspace: $\text{span}(\mathbf{r}_1, \ldots, \mathbf{r}_m)$.
- Suppose we apply a row operation to $\mathbf{A}$ and $\mathbf{A}'$ is the resulting matrix.

In-Class Exercise 15: Show that the rowspace of $\mathbf{A}'$ is the same as the rowspace of $\mathbf{A}$. Do this for each of three types of row operations.

If we pay attention, these operations are all lin. trans,

$$\begin{bmatrix} - & r_1 & - \\ - & r_2 & - \\ - & r_3 & - \end{bmatrix} \Rightarrow \begin{bmatrix} - & r_2 & - \\ - & r_1 & - \\ - & r_3 & - \end{bmatrix} \rightarrow \text{no changes to the vectors.}$$

The scaling operation will not change the linear combination of the vectors either, so it'll not affect the resultant rowspace
When it comes to the addition of rows, this is simply a linear combination and can easily be reversed. Meaning that the resultant vectors can be decomposed into the original vectors and is still within the span of the 2 original vectors. This means that the rowspace remains unchanged.

In-Class Exercise 16: Why are the pivot rows independent?

the pivot rows assure that within the same colums all entries except the Diag. are 0s. This means that for a row vector whose i^{th} element has no non-zero counterpart in the other row vectors no scalar except for 0 can be used to get a zero-vector as a lin. comb. of all row vects

=> Upper Triangular Structure.

- Which proves Theorem 6.6.
- The proof of Proposition 6.4 is now trivial because we've proved something far stronger.
- Let's remind ourselves ...
  Proposition 6.4: The rank of the collection $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is the *number* of pivot columns of $RREF(\mathbf{A})$.

In-Class Exercise 17: Complete the proof of Proposition 6.4.

The rank of a matrix is defined as the maximum number of linearly independent column vectors in the matrix. This is also equal to the dimension of the column space of the matrix. When a matrix A is transformed into RREF, the pivot columns correspond to the positions of the leading 1's in the rows. These pivot positions, and thus the pivot columns, are linearly independent by definition. The column space of A and the column space of RREF(A) may not be the same because row operations can change the column space. However, the number of linearly independent columns (the rank) remains the same.

Rank = Num Lin. Indep. Cols = Num. Pivot Cols in RREF

In-Class Exercise 18: Prove or disprove the following. Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a collection of m-dimensional *linearly independent* vectors and $\mathbf{u} \in \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$. Then there is exactly one linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ that will produce $\mathbf{u}$. Consider both cases, $m = n$ and $m \neq n$.

Since these, supposed to be, row vectors are linearly independent and there are n of them in an n-dimensional space, they form a basis for this space R^n. And in this space any point will be unique and can only be expressed as a unique combination of the basis vectors

When n<m
If the vectors are in R^m but there are fewer than m of them (n<m), they cannot the whole.
However, they do span a subspace of R^m, R^n.
Within this subspace, any vector u can still be expressed as a unique linear combination of v1,v2,...,vn because the vectors are linearly independent

In-Class Exercise 19: What is the more natural basis (3D vectors) for the (x,y) plane?

$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

In-Class Exercise 20: Prove this result:

- Apply the definition of linear independence to $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
- Then compute the dot product with $\mathbf{v}_1$ on both sides.
- What does this imply for the value of α_1?
- Complete the proof.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\delta = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots \alpha_n v_n = 0$$

$$v_1 \delta = v_1 0$$

$$\alpha_1 \cdot |v_1|^2 = 0 = \alpha_1 = 0$$

We apply this recursively for $v_2 \ldots v_n$ and it becomes obv that they are all orthogonal as

$\alpha_2 = 0, \ldots \ldots \alpha_n = 0$