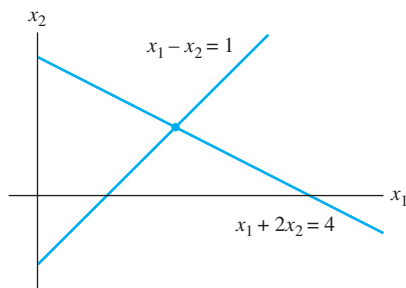


## 1.1 EXERCISES

Solve each system in Exercises 1–4 by using elementary row operations on the equations or on the augmented matrix. Follow the systematic elimination procedure described in this section.

1.  $x_1 + 5x_2 = 7$   
 $-2x_1 - 7x_2 = -5$
2.  $3x_1 + 6x_2 = -3$   
 $5x_1 + 7x_2 = 10$
3. Find the point  $(x_1, x_2)$  that lies on the line  $x_1 + 2x_2 = 4$  and on the line  $x_1 - x_2 = 1$ . See the figure.



4. Find the point of intersection of the lines  $x_1 + 2x_2 = -13$  and  $3x_1 - 2x_2 = 1$

Consider each matrix in Exercises 5 and 6 as the augmented matrix of a linear system. State in words the next two elementary row operations that should be performed in the process of solving the system.

$$5. \begin{bmatrix} 1 & -4 & -3 & 0 & 7 \\ 0 & 1 & 4 & 0 & 6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}$$

$$6. \begin{bmatrix} 1 & -6 & 4 & 0 & -1 \\ 0 & 2 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 4 & 1 & 2 \end{bmatrix}$$

In Exercises 7–10, the augmented matrix of a linear system has been reduced by row operations to the form shown. In each case, continue the appropriate row operations and describe the solution set of the original system.

$$7. \begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$8. \begin{bmatrix} 1 & -5 & 4 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

$$9. \begin{bmatrix} 1 & -1 & 0 & 0 & -5 \\ 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$10. \begin{bmatrix} 1 & 3 & 0 & -2 & -7 \\ 0 & 1 & 0 & 3 & 6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Solve the systems in Exercises 11–14.

$$11. \begin{aligned} x_2 + 5x_3 &= -4 \\ x_1 + 4x_2 + 3x_3 &= -2 \\ 2x_1 + 7x_2 + x_3 &= -2 \end{aligned}$$

$$12. \begin{aligned} x_1 - 5x_2 + 4x_3 &= -3 \\ 2x_1 - 7x_2 + 3x_3 &= -2 \\ -2x_1 + x_2 + 7x_3 &= -1 \end{aligned}$$

$$13. \begin{aligned} x_1 - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

$$14. \begin{aligned} 2x_1 - 6x_3 &= -8 \\ x_2 + 2x_3 &= 3 \\ 3x_1 + 6x_2 - 2x_3 &= -4 \end{aligned}$$

Determine if the systems in Exercises 15 and 16 are consistent. Do not completely solve the systems.

$$15. \begin{aligned} x_1 - 6x_2 &= 5 \\ x_2 - 4x_3 + x_4 &= 0 \\ -x_1 + 6x_2 + x_3 + 5x_4 &= 3 \\ -x_2 + 5x_3 + 4x_4 &= 0 \end{aligned}$$

$$16. \begin{aligned} 2x_1 - 4x_4 &= -10 \\ 3x_2 + 3x_3 &= 0 \\ x_3 + 4x_4 &= -1 \\ -3x_1 + 2x_2 + 3x_3 + x_4 &= 5 \end{aligned}$$

17. Do the three lines  $2x_1 + 3x_2 = -1$ ,  $6x_1 + 5x_2 = 0$ , and  $2x_1 - 5x_2 = 7$  have a common point of intersection? Explain.

18. Do the three planes  $2x_1 + 4x_2 + 4x_3 = 4$ ,  $x_2 - 2x_3 = -2$ , and  $2x_1 + 3x_2 = 0$  have at least one common point of intersection? Explain.

In Exercises 19–22, determine the value(s) of  $h$  such that the matrix is the augmented matrix of a consistent linear system.

$$19. \begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix}$$

$$20. \begin{bmatrix} 1 & h & -5 \\ 2 & -8 & 6 \end{bmatrix}$$

$$21. \begin{bmatrix} 1 & 4 & -2 \\ 3 & h & -6 \end{bmatrix}$$

$$22. \begin{bmatrix} -4 & 12 & h \\ 2 & -6 & -3 \end{bmatrix}$$

In Exercises 23 and 24, key statements from this section are either quoted directly, restated slightly (but still true), or altered in some way that makes them false in some cases. Mark each statement True or False, and *justify* your answer. (If true, give the

approximate location where a similar statement appears, or refer to a definition or theorem. If false, give the location of a statement that has been quoted or used incorrectly, or cite an example that shows the statement is not true in all cases.) Similar true/false questions will appear in many sections of the text.

23. a. Every elementary row operation is reversible.  
 b. A  $5 \times 6$  matrix has six rows.  
 c. The solution set of a linear system involving variables  $x_1, \dots, x_n$  is a list of numbers  $(s_1, \dots, s_n)$  that makes each equation in the system a true statement when the values  $s_1, \dots, s_n$  are substituted for  $x_1, \dots, x_n$ , respectively.  
 d. Two fundamental questions about a linear system involve existence and uniqueness.
24. a. Two matrices are row equivalent if they have the same number of rows.  
 b. Elementary row operations on an augmented matrix never change the solution set of the associated linear system.  
 c. Two equivalent linear systems can have different solution sets.  
 d. A consistent system of linear equations has one or more solutions.

25. Find an equation involving  $g$ ,  $h$ , and  $k$  that makes this augmented matrix correspond to a consistent system:

$$\left[ \begin{array}{ccc|c} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{array} \right]$$

26. Suppose the system below is consistent for all possible values of  $f$  and  $g$ . What can you say about the coefficients  $c$  and  $d$ ? Justify your answer.

$$2x_1 + 4x_2 = f$$

$$cx_1 + dx_2 = g$$

27. Suppose  $a$ ,  $b$ ,  $c$ , and  $d$  are constants such that  $a$  is not zero and the system below is consistent for all possible values of  $f$  and  $g$ . What can you say about the numbers  $a$ ,  $b$ ,  $c$ , and  $d$ ? Justify your answer.

$$ax_1 + bx_2 = f$$

$$cx_1 + dx_2 = g$$

28. Construct three different augmented matrices for linear systems whose solution set is  $x_1 = 3$ ,  $x_2 = -2$ ,  $x_3 = -1$ .

In Exercises 29–32, find the elementary row operation that transforms the first matrix into the second, and then find the reverse row operation that transforms the second matrix into the first.

29.  $\begin{bmatrix} 0 & -2 & 5 \\ 1 & 3 & -5 \\ 3 & -1 & 6 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 6 \\ 1 & 3 & -5 \\ 0 & -2 & 5 \end{bmatrix}$

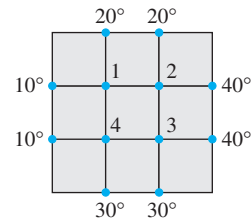
30.  $\begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & -5 & 10 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & 1 & -2 \end{bmatrix}$

31.  $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 4 & -1 & 3 & -6 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 8 \\ 0 & 7 & -1 & -6 \end{bmatrix}$

32.  $\begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 4 & -12 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 15 \end{bmatrix}$

An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin plate when the temperature around the boundary is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let  $T_1, \dots, T_4$  denote the temperatures at the four interior nodes of the mesh in the figure. The temperature at a node is approximately equal to the average of the four nearest nodes—to the left, above, to the right, and below.<sup>3</sup> For instance,

$$T_1 = (10 + 20 + T_2 + T_4)/4, \quad \text{or} \quad 4T_1 - T_2 - T_4 = 30$$



33. Write a system of four equations whose solution gives estimates for the temperatures  $T_1, \dots, T_4$ .

34. Solve the system of equations from Exercise 33. [Hint: To speed up the calculations, interchange rows 1 and 4 before starting “replace” operations.]

<sup>3</sup> See Frank M. White, *Heat and Mass Transfer* (Reading, MA: Addison-Wesley Publishing, 1991), pp. 145–149.

## SOLUTIONS TO PRACTICE PROBLEMS

1. a. For “hand computation,” the best choice is to interchange equations 3 and 4. Another possibility is to multiply equation 3 by  $1/5$ . Or, replace equation 4 by its sum with  $-1/5$  times row 3. (In any case, do not use the  $x_2$  in equation 2 to eliminate the  $4x_2$  in equation 1. Wait until a triangular form has been reached and the  $x_3$  terms and  $x_4$  terms have been eliminated from the first two equations.)

## THEOREM 2

## Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \quad \cdots \quad 0 \quad b] \quad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

The following procedure outlines how to find and describe all solutions of a linear system.

## USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

## PRACTICE PROBLEMS

1. Find the general solution of the linear system whose augmented matrix is

$$\begin{bmatrix} 1 & -3 & -5 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

2. Find the general solution of the system

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2 \end{aligned}$$

## 1.2 EXERCISES

In Exercises 1 and 2, determine which matrices are in reduced echelon form and which others are only in echelon form.

1. a.  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

b.  $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

c.  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

d.  $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

2. a.  $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  b.  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

c.  $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

d.  $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Row reduce the matrices in Exercises 3 and 4 to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

3.  $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$  4.  $\begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 2 \end{bmatrix}$

5. Describe the possible echelon forms of a nonzero  $2 \times 2$  matrix. Use the symbols ■, \*, and 0, as in the first part of Example 1.

6. Repeat Exercise 5 for a nonzero  $3 \times 2$  matrix.

Find the general solutions of the systems whose augmented matrices are given in Exercises 7–14.

7.  $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$  8.  $\begin{bmatrix} 1 & -3 & 0 & -5 \\ -3 & 7 & 0 & 9 \end{bmatrix}$

9.  $\begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & -3 & 4 & -6 \end{bmatrix}$  10.  $\begin{bmatrix} 1 & -2 & -1 & 4 \\ -2 & 4 & -5 & 6 \end{bmatrix}$

11.  $\begin{bmatrix} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{bmatrix}$  12.  $\begin{bmatrix} 1 & 0 & -9 & 0 & 4 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

13.  $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

14.  $\begin{bmatrix} 1 & 0 & -5 & 0 & -8 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Exercises 15 and 16 use the notation of Example 1 for matrices in echelon form. Suppose each matrix represents the augmented matrix for a system of linear equations. In each case, determine if the system is consistent. If the system is consistent, determine if the solution is unique.

15. a.  $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b.  $\begin{bmatrix} 0 & \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & 0 \end{bmatrix}$

16. a.  $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$

b.  $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$

In Exercises 17 and 18, determine the value(s) of  $h$  such that the matrix is the augmented matrix of a consistent linear system.

17.  $\begin{bmatrix} 1 & -1 & 4 \\ -2 & 3 & h \end{bmatrix}$  18.  $\begin{bmatrix} 1 & -3 & 1 \\ h & 6 & -2 \end{bmatrix}$

In Exercises 19 and 20, choose  $h$  and  $k$  such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

19.  $\begin{aligned} x_1 + hx_2 &= 2 \\ 4x_1 + 8x_2 &= k \end{aligned}$  20.  $\begin{aligned} x_1 - 3x_2 &= 1 \\ 2x_1 + hx_2 &= k \end{aligned}$

In Exercises 21 and 22, mark each statement True or False. Justify each answer.<sup>4</sup>

21. a. In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.  
b. The row reduction algorithm applies only to augmented matrices for a linear system.  
c. A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.  
d. Finding a parametric description of the solution set of a linear system is the same as *solving* the system.  
e. If one row in an echelon form of an augmented matrix is  $[0 \ 0 \ 0 \ 5 \ 0]$ , then the associated linear system is inconsistent.
22. a. The reduced echelon form of a matrix is unique.  
b. If every column of an augmented matrix contains a pivot, then the corresponding system is consistent.  
c. The pivot positions in a matrix depend on whether row interchanges are used in the row reduction process.  
d. A general solution of a system is an explicit description of all solutions of the system.  
e. Whenever a system has free variables, the solution set contains many solutions.
23. Suppose the coefficient matrix of a linear system of four equations in four variables has a pivot in each column. Explain why the system has a unique solution.
24. Suppose a system of linear equations has a  $3 \times 5$  augmented matrix whose fifth column is not a pivot column. Is the system consistent? Why (or why not)?

<sup>4</sup> True/false questions of this type will appear in many sections. Methods for justifying your answers were described before Exercises 23 and 24 in Section 1.1.

25. Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent.
26. Suppose a  $3 \times 5$  coefficient matrix for a system has three pivot columns. Is the system consistent? Why or why not?
27. Restate the last sentence in Theorem 2 using the concept of pivot columns: “If a linear system is consistent, then the solution is unique if and only if \_\_\_\_\_.”
28. What would you have to know about the pivot columns in an augmented matrix in order to know that the linear system is consistent and has a unique solution?
29. A system of linear equations with fewer equations than unknowns is sometimes called an *underdetermined system*. Can such a system have a unique solution? Explain.
30. Give an example of an inconsistent underdetermined system of two equations in three unknowns.
31. A system of linear equations with more equations than unknowns is sometimes called an *overdetermined system*. Can such a system be consistent? Illustrate your answer with a specific system of three equations in two unknowns.
32. Suppose an  $n \times (n + 1)$  matrix is row reduced to reduced echelon form. Approximately what fraction of the total number of operations (flops) is involved in the backward phase of the reduction when  $n = 20$ ? when  $n = 200$ ?

Suppose experimental data are represented by a set of points in the plane. An **interpolating polynomial** for the data is a polynomial whose graph passes through every point. In scientific work,

such a polynomial can be used, for example, to estimate values between the known data points. Another use is to create curves for graphical images on a computer screen. One method for finding an interpolating polynomial is to solve a system of linear equations.

#### WEB

33. Find the interpolating polynomial  $p(t) = a_0 + a_1t + a_2t^2$  for the data  $(1, 6)$ ,  $(2, 15)$ ,  $(3, 28)$ . That is, find  $a_0$ ,  $a_1$ , and  $a_2$  such that

$$a_0 + a_1(1) + a_2(1)^2 = 6$$

$$a_0 + a_1(2) + a_2(2)^2 = 15$$

$$a_0 + a_1(3) + a_2(3)^2 = 28$$

34. [M] In a wind tunnel experiment, the force on a projectile due to air resistance was measured at different velocities:

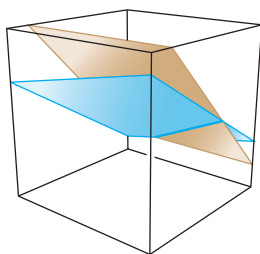
Velocity (100 ft/sec)    0    2    4    6    8    10

Force (100 lb)            0    2.90    14.8    39.6    74.3    119

Find an interpolating polynomial for these data and estimate the force on the projectile when the projectile is traveling at 750 ft/sec. Use  $p(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5$ . What happens if you try to use a polynomial of degree less than 5? (Try a cubic polynomial, for instance.)<sup>5</sup>

<sup>5</sup> Exercises marked with the symbol [M] are designed to be worked with the aid of a “Matrix program” (a computer program, such as MATLAB®, Maple™, Mathematica®, MathCad®, or Derive™, or a programmable calculator with matrix capabilities, such as those manufactured by Texas Instruments or Hewlett-Packard).

### SOLUTIONS TO PRACTICE PROBLEMS



The general solution of the system of equations is the line of intersection of the two planes.

1. The reduced echelon form of the augmented matrix and the corresponding system are

$$\begin{bmatrix} 1 & 0 & -2 & 9 \\ 0 & 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \begin{cases} x_1 - 2x_3 = 9 \\ x_2 + x_3 = 3 \end{cases}$$

The basic variables are  $x_1$  and  $x_2$ , and the general solution is

$$\begin{cases} x_1 = 9 + 2x_3 \\ x_2 = 3 - x_3 \\ x_3 \text{ is free} \end{cases}$$

*Note:* It is essential that the general solution describe each variable, with any parameters clearly identified. The following statement does *not* describe the solution:

$$\begin{cases} x_1 = 9 + 2x_3 \\ x_2 = 3 - x_3 \\ x_3 = 3 - x_2 \end{cases} \quad \text{Incorrect solution}$$

This description implies that  $x_2$  and  $x_3$  are *both* free, which certainly is not the case.

## 1.3 EXERCISES

In Exercises 1 and 2, compute  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{u} - 2\mathbf{v}$ .

1.  $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$       2.  $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

In Exercises 3 and 4, display the following vectors using arrows on an  $xy$ -graph:  $\mathbf{u}, \mathbf{v}, -\mathbf{v}, -2\mathbf{v}, \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}$ . Notice that  $\mathbf{u} - \mathbf{v}$  is the vertex of a parallelogram whose other vertices are  $\mathbf{u}, \mathbf{0}$ , and  $-\mathbf{v}$ .

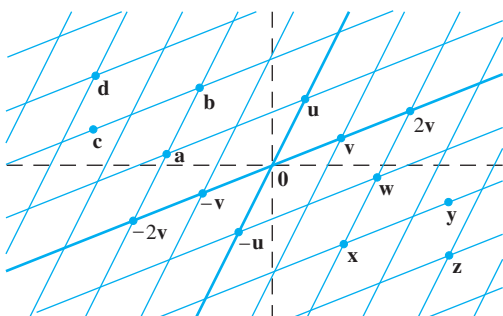
3.  $\mathbf{u}$  and  $\mathbf{v}$  as in Exercise 1      4.  $\mathbf{u}$  and  $\mathbf{v}$  as in Exercise 2

In Exercises 5 and 6, write a system of equations that is equivalent to the given vector equation.

5.  $x_1 \begin{bmatrix} 3 \\ -2 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 0 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}$

6.  $x_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Use the accompanying figure to write each vector listed in Exercises 7 and 8 as a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ . Is every vector in  $\mathbb{R}^2$  a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ ?



7. Vectors  $\mathbf{a}, \mathbf{b}, \mathbf{c}$ , and  $\mathbf{d}$

8. Vectors  $\mathbf{w}, \mathbf{x}, \mathbf{y}$ , and  $\mathbf{z}$

In Exercises 9 and 10, write a vector equation that is equivalent to the given system of equations.

9.  $x_2 + 5x_3 = 0$       10.  $3x_1 - 2x_2 + 4x_3 = 3$   
 $4x_1 + 6x_2 - x_3 = 0$        $-2x_1 - 7x_2 + 5x_3 = 1$   
 $-x_1 + 3x_2 - 8x_3 = 0$        $5x_1 + 4x_2 - 3x_3 = 2$

In Exercises 11 and 12, determine if  $\mathbf{b}$  is a linear combination of  $\mathbf{a}_1, \mathbf{a}_2$ , and  $\mathbf{a}_3$ .

11.  $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

12.  $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$

In Exercises 13 and 14, determine if  $\mathbf{b}$  is a linear combination of the vectors formed from the columns of the matrix  $A$ .

13.  $A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$

14.  $A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

15. Let  $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}$ , and  $\mathbf{b} = \begin{bmatrix} 3 \\ -5 \\ h \end{bmatrix}$ . For what value(s) of  $h$  is  $\mathbf{b}$  in the plane spanned by  $\mathbf{a}_1$  and  $\mathbf{a}_2$ ?

16. Let  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$ , and  $\mathbf{y} = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}$ . For what value(s) of  $h$  is  $\mathbf{y}$  in the plane generated by  $\mathbf{v}_1$  and  $\mathbf{v}_2$ ?

In Exercises 17 and 18, list five vectors in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ . For each vector, show the weights on  $\mathbf{v}_1$  and  $\mathbf{v}_2$  used to generate the vector and list the three entries of the vector. Do not make a sketch.

17.  $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$

18.  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$

19. Give a geometric description of  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$  for the vectors  $\mathbf{v}_1 = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}$  and  $\mathbf{v}_2 = \begin{bmatrix} 12 \\ 3 \\ -9 \end{bmatrix}$ .

20. Give a geometric description of  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$  for the vectors in Exercise 18.

21. Let  $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ . Show that  $\begin{bmatrix} h \\ k \end{bmatrix}$  is in  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$  for all  $h$  and  $k$ .

22. Construct a  $3 \times 3$  matrix  $A$ , with nonzero entries, and a vector  $\mathbf{b}$  in  $\mathbb{R}^3$  such that  $\mathbf{b}$  is not in the set spanned by the columns of  $A$ .

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. Another notation for the vector  $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$  is  $[-4 \ 3]$ .

b. The points in the plane corresponding to  $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$  and  $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$  lie on a line through the origin.

c. An example of a linear combination of vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  is the vector  $\frac{1}{2}\mathbf{v}_1$ .

- d. The solution set of the linear system whose augmented matrix is  $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}]$  is the same as the solution set of the equation  $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ .
- e. The set  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$  is always visualized as a plane through the origin.
24. a. When  $\mathbf{u}$  and  $\mathbf{v}$  are nonzero vectors,  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$  contains only the line through  $\mathbf{u}$  and the origin, and the line through  $\mathbf{v}$  and the origin.
- b. Any list of five real numbers is a vector in  $\mathbb{R}^5$ .
- c. Asking whether the linear system corresponding to an augmented matrix  $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}]$  has a solution amounts to asking whether  $\mathbf{b}$  is in  $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ .
- d. The vector  $\mathbf{v}$  results when a vector  $\mathbf{u} - \mathbf{v}$  is added to the vector  $\mathbf{v}$ .
- e. The weights  $c_1, \dots, c_p$  in a linear combination  $c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$  cannot all be zero.
25. Let  $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$  and  $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$ . Denote the columns of  $A$  by  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ , and let  $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ .
- a. Is  $\mathbf{b}$  in  $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ ? How many vectors are in  $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ ?
- b. Is  $\mathbf{b}$  in  $W$ ? How many vectors are in  $W$ ?
- c. Show that  $\mathbf{a}_1$  is in  $W$ . [Hint: Row operations are unnecessary.]
26. Let  $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$ , let  $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$ , and let  $W$  be the set of all linear combinations of the columns of  $A$ .
- a. Is  $\mathbf{b}$  in  $W$ ?
- b. Show that the second column of  $A$  is in  $W$ .
27. A mining company has two mines. One day's operation at mine #1 produces ore that contains 30 metric tons of copper and 600 kilograms of silver, while one day's operation at mine #2 produces ore that contains 40 metric tons of copper and 380 kilograms of silver. Let  $\mathbf{v}_1 = \begin{bmatrix} 30 \\ 600 \end{bmatrix}$  and  $\mathbf{v}_2 = \begin{bmatrix} 40 \\ 380 \end{bmatrix}$ . Then  $\mathbf{v}_1$  and  $\mathbf{v}_2$  represent the "output per day" of mine #1 and mine #2, respectively.
- a. What physical interpretation can be given to the vector  $5\mathbf{v}_1$ ?
- b. Suppose the company operates mine #1 for  $x_1$  days and mine #2 for  $x_2$  days. Write a vector equation whose solution gives the number of days each mine should operate in order to produce 240 tons of copper and 2824 kilograms of silver. Do not solve the equation.
- c. [M] Solve the equation in (b).
28. A steam plant burns two types of coal: anthracite (A) and bituminous (B). For each ton of A burned, the plant produces 27.6 million Btu of heat, 3100 grams (g) of sulfur dioxide, and 250 g of particulate matter (solid-particle pollutants). For

each ton of B burned, the plant produces 30.2 million Btu, 6400 g of sulfur dioxide, and 360 g of particulate matter.

- a. How much heat does the steam plant produce when it burns  $x_1$  tons of A and  $x_2$  tons of B?
- b. Suppose the output of the steam plant is described by a vector that lists the amounts of heat, sulfur dioxide, and particulate matter. Express this output as a linear combination of two vectors, assuming that the plant burns  $x_1$  tons of A and  $x_2$  tons of B.
- c. [M] Over a certain time period, the steam plant produced 162 million Btu of heat, 23,610 g of sulfur dioxide, and 1623 g of particulate matter. Determine how many tons of each type of coal the steam plant must have burned. Include a vector equation as part of your solution.
29. Let  $\mathbf{v}_1, \dots, \mathbf{v}_k$  be points in  $\mathbb{R}^3$  and suppose that for  $j = 1, \dots, k$  an object with mass  $m_j$  is located at point  $\mathbf{v}_j$ . Physicists call such objects *point masses*. The total mass of the system of point masses is

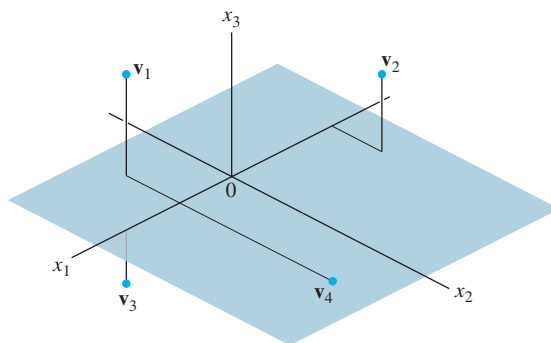
$$m = m_1 + \dots + m_k$$

The *center of gravity* (or *center of mass*) of the system is

$$\bar{\mathbf{v}} = \frac{1}{m} [m_1\mathbf{v}_1 + \dots + m_k\mathbf{v}_k]$$

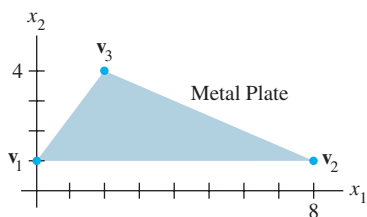
Compute the center of gravity of the system consisting of the following point masses (see the figure):

Point	Mass
$\mathbf{v}_1 = (2, -2, 4)$	4 g
$\mathbf{v}_2 = (-4, 2, 3)$	2 g
$\mathbf{v}_3 = (4, 0, -2)$	3 g
$\mathbf{v}_4 = (1, -6, 0)$	5 g



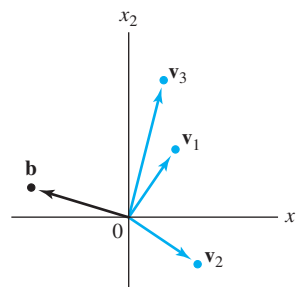
30. Let  $\mathbf{v}$  be the center of mass of a system of point masses located at  $\mathbf{v}_1, \dots, \mathbf{v}_k$  as in Exercise 29. Is  $\mathbf{v}$  in  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ ? Explain.

31. A thin triangular plate of uniform density and thickness has vertices at  $\mathbf{v}_1 = (0, 1)$ ,  $\mathbf{v}_2 = (8, 1)$ , and  $\mathbf{v}_3 = (2, 4)$ , as in the figure below, and the mass of the plate is 3 g.



- Find the  $(x, y)$ -coordinates of the center of mass of the plate. This “balance point” of the plate coincides with the center of mass of a system consisting of three 1-gram point masses located at the vertices of the plate.
  - Determine how to distribute an additional mass of 6 g at the three vertices of the plate to move the balance point of the plate to  $(2, 2)$ . [Hint: Let  $w_1$ ,  $w_2$ , and  $w_3$  denote the masses added at the three vertices, so that  $w_1 + w_2 + w_3 = 6$ .]
32. Consider the vectors  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ ,  $\mathbf{v}_3$ , and  $\mathbf{b}$  in  $\mathbb{R}^2$ , shown in the figure. Does the equation  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$  have a

solution? Is the solution unique? Use the figure to explain your answers.



33. Use the vectors  $\mathbf{u} = (u_1, \dots, u_n)$ ,  $\mathbf{v} = (v_1, \dots, v_n)$ , and  $\mathbf{w} = (w_1, \dots, w_n)$  to verify the following algebraic properties of  $\mathbb{R}^n$ .
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  - $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$  for each scalar  $c$
34. Use the vector  $\mathbf{u} = (u_1, \dots, u_n)$  to verify the following algebraic properties of  $\mathbb{R}^n$ .
- $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$
  - $c(d\mathbf{u}) = (cd)\mathbf{u}$  for all scalars  $c$  and  $d$

### SOLUTIONS TO PRACTICE PROBLEMS

1. Take arbitrary vectors  $\mathbf{u} = (u_1, \dots, u_n)$  and  $\mathbf{v} = (v_1, \dots, v_n)$  in  $\mathbb{R}^n$ , and compute

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1 + v_1, \dots, u_n + v_n) && \text{Definition of vector addition} \\ &= (v_1 + u_1, \dots, v_n + u_n) && \text{Commutativity of addition in } \mathbb{R} \\ &= \mathbf{v} + \mathbf{u} && \text{Definition of vector addition}\end{aligned}$$

2. The vector  $\mathbf{y}$  belongs to  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  if and only if there exist scalars  $x_1, x_2, x_3$  such that

$$x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

This vector equation is equivalent to a system of three linear equations in three unknowns. If you row reduce the augmented matrix for this system, you find that

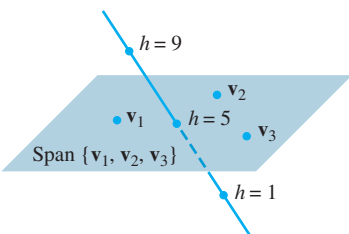
$$\left[ \begin{array}{cccc} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array} \right] \sim \left[ \begin{array}{cccc} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{array} \right] \sim \left[ \begin{array}{cccc} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{array} \right]$$

The system is consistent if and only if there is no pivot in the fourth column. That is,  $h - 5$  must be 0. So  $\mathbf{y}$  is in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  if and only if  $h = 5$ .

**Remember:** The presence of a free variable in a system does not guarantee that the system is consistent.

## 1.4 THE MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

A fundamental idea in linear algebra is to view a linear combination of vectors as the product of a matrix and a vector. The following definition permits us to rephrase some of the concepts of Section 1.3 in new ways.



The points  $\begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$  lie on a line that intersects the plane when  $h = 5$ .



If statement (d) is true, then each row of  $U$  contains a pivot position and there can be no pivot in the augmented column. So  $A\mathbf{x} = \mathbf{b}$  has a solution for any  $\mathbf{b}$ , and (a) is true. If (d) is false, the last row of  $U$  is all zeros. Let  $\mathbf{d}$  be any vector with a 1 in its last entry. Then  $[U \ \mathbf{d}]$  represents an *inconsistent* system. Since row operations are reversible,  $[U \ \mathbf{d}]$  can be transformed into the form  $[A \ \mathbf{b}]$ . The new system  $A\mathbf{x} = \mathbf{b}$  is also inconsistent, and (a) is false. ■

## PRACTICE PROBLEMS

1. Let  $A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}$ ,  $\mathbf{p} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix}$ , and  $\mathbf{b} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$ . It can be shown that  $\mathbf{p}$  is a solution of  $A\mathbf{x} = \mathbf{b}$ . Use this fact to exhibit  $\mathbf{b}$  as a specific linear combination of the columns of  $A$ .
2. Let  $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$ ,  $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ , and  $\mathbf{v} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ . Verify Theorem 5(a) in this case by computing  $A(\mathbf{u} + \mathbf{v})$  and  $A\mathbf{u} + A\mathbf{v}$ .

## 1.4 EXERCISES

Compute the products in Exercises 1–4 using (a) the definition, as in Example 1, and (b) the row–vector rule for computing  $A\mathbf{x}$ . If a product is undefined, explain why.

1.  $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$       2.  $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$
3.  $\begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$       4.  $\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

In Exercises 5–8, use the definition of  $A\mathbf{x}$  to write the matrix equation as a vector equation, or vice versa.

5.  $\begin{bmatrix} 1 & 2 & -3 & 1 \\ -2 & -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$
6.  $\begin{bmatrix} 2 & -3 \\ 3 & 2 \\ 8 & -5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ -49 \\ 11 \end{bmatrix}$
7.  $x_1 \begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$
8.  $z_1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + z_2 \begin{bmatrix} -1 \\ 5 \end{bmatrix} + z_3 \begin{bmatrix} -4 \\ 3 \end{bmatrix} + z_4 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

In Exercises 9 and 10, write the system first as a vector equation and then as a matrix equation.

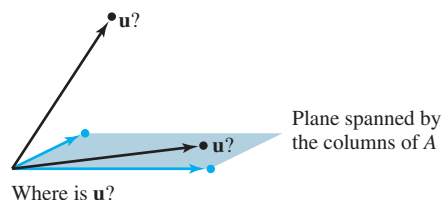
9.  $5x_1 + x_2 - 3x_3 = 8$   
 $2x_2 + 4x_3 = 0$
10.  $4x_1 - x_2 = 8$   
 $5x_1 + 3x_2 = 2$   
 $3x_1 - x_2 = 1$

Given  $A$  and  $\mathbf{b}$  in Exercises 11 and 12, write the augmented matrix for the linear system that corresponds to the matrix equation  $A\mathbf{x} = \mathbf{b}$ . Then solve the system and write the solution as a vector.

11.  $A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & 2 \\ -3 & -7 & 6 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 12 \end{bmatrix}$

12.  $A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

13. Let  $\mathbf{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$  and  $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$ . Is  $\mathbf{u}$  in the plane in  $\mathbb{R}^3$  spanned by the columns of  $A$ ? (See the figure.) Why or why not?



14. Let  $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$  and  $A = \begin{bmatrix} 2 & 5 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$ . Is  $\mathbf{u}$  in the subset of  $\mathbb{R}^3$  spanned by the columns of  $A$ ? Why or why not?

15. Let  $A = \begin{bmatrix} 3 & -1 \\ -9 & 3 \end{bmatrix}$  and  $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ . Show that the equation  $A\mathbf{x} = \mathbf{b}$  does not have a solution for all possible  $\mathbf{b}$ , and describe the set of all  $\mathbf{b}$  for which  $A\mathbf{x} = \mathbf{b}$  does have a solution.

16. Repeat the requests from Exercise 15 with

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Exercises 17–20 refer to the matrices  $A$  and  $B$  below. Make appropriate calculations that justify your answers and mention an appropriate theorem.

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix}$$

17. How many rows of  $A$  contain a pivot position? Does the equation  $A\mathbf{x} = \mathbf{b}$  have a solution for each  $\mathbf{b}$  in  $\mathbb{R}^4$ ?
18. Can every vector in  $\mathbb{R}^4$  be written as a linear combination of the columns of the matrix  $B$  above? Do the columns of  $B$  span  $\mathbb{R}^3$ ?
19. Can each vector in  $\mathbb{R}^4$  be written as a linear combination of the columns of the matrix  $A$  above? Do the columns of  $A$  span  $\mathbb{R}^4$ ?
20. Do the columns of  $B$  span  $\mathbb{R}^4$ ? Does the equation  $B\mathbf{x} = \mathbf{y}$  have a solution for each  $\mathbf{y}$  in  $\mathbb{R}^4$ ?

21. Let  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ ,  $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ . Does  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  span  $\mathbb{R}^4$ ? Why or why not?

22. Let  $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}$ ,  $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$ . Does  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  span  $\mathbb{R}^3$ ? Why or why not?

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. The equation  $A\mathbf{x} = \mathbf{b}$  is referred to as a *vector equation*.  
 b. A vector  $\mathbf{b}$  is a linear combination of the columns of a matrix  $A$  if and only if the equation  $A\mathbf{x} = \mathbf{b}$  has at least one solution.  
 c. The equation  $A\mathbf{x} = \mathbf{b}$  is consistent if the augmented matrix  $[A \ \mathbf{b}]$  has a pivot position in every row.  
 d. The first entry in the product  $A\mathbf{x}$  is a sum of products.  
 e. If the columns of an  $m \times n$  matrix  $A$  span  $\mathbb{R}^m$ , then the equation  $A\mathbf{x} = \mathbf{b}$  is consistent for each  $\mathbf{b}$  in  $\mathbb{R}^m$ .  
 f. If  $A$  is an  $m \times n$  matrix and if the equation  $A\mathbf{x} = \mathbf{b}$  is inconsistent for some  $\mathbf{b}$  in  $\mathbb{R}^m$ , then  $A$  cannot have a pivot position in every row.

24. a. Every matrix equation  $A\mathbf{x} = \mathbf{b}$  corresponds to a vector equation with the same solution set.  
 b. If the equation  $A\mathbf{x} = \mathbf{b}$  is consistent, then  $\mathbf{b}$  is in the set spanned by the columns of  $A$ .  
 c. Any linear combination of vectors can always be written in the form  $A\mathbf{x}$  for a suitable matrix  $A$  and vector  $\mathbf{x}$ .  
 d. If the coefficient matrix  $A$  has a pivot position in every row, then the equation  $A\mathbf{x} = \mathbf{b}$  is inconsistent.  
 e. The solution set of a linear system whose augmented matrix is  $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}]$  is the same as the solution set of  $A\mathbf{x} = \mathbf{b}$ , if  $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$ .  
 f. If  $A$  is an  $m \times n$  matrix whose columns do not span  $\mathbb{R}^m$ , then the equation  $A\mathbf{x} = \mathbf{b}$  is consistent for every  $\mathbf{b}$  in  $\mathbb{R}^m$ .

25. Note that  $\begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$ . Use this fact (and no row operations) to find scalars  $c_1, c_2, c_3$  such that  $\begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$ .

26. Let  $\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$ ,  $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$ , and  $\mathbf{w} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$ . It can be shown that  $2\mathbf{u} - 3\mathbf{v} - \mathbf{w} = \mathbf{0}$ . Use this fact (and no row operations) to find  $x_1$  and  $x_2$  that satisfy the equation  $\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$ .

27. Rewrite the (numerical) matrix equation below in symbolic form as a vector equation, using symbols  $\mathbf{v}_1, \mathbf{v}_2, \dots$  for the vectors and  $c_1, c_2, \dots$  for scalars. Define what each symbol represents, using the data given in the matrix equation.

$$\begin{bmatrix} -3 & 5 & -4 & 9 & 7 \\ 5 & 8 & 1 & -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ -11 \end{bmatrix}$$

28. Let  $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ , and  $\mathbf{v}$  represent vectors in  $\mathbb{R}^5$ , and let  $x_1, x_2$ , and  $x_3$  denote scalars. Write the following vector equation as a matrix equation. Identify any symbols you choose to use.  
 $x_1\mathbf{q}_1 + x_2\mathbf{q}_2 + x_3\mathbf{q}_3 = \mathbf{v}$
29. Construct a  $3 \times 3$  matrix, not in echelon form, whose columns span  $\mathbb{R}^3$ . Show that the matrix you construct has the desired property.
30. Construct a  $3 \times 3$  matrix, not in echelon form, whose columns do *not* span  $\mathbb{R}^3$ . Show that the matrix you construct has the desired property.
31. Let  $A$  be a  $3 \times 2$  matrix. Explain why the equation  $A\mathbf{x} = \mathbf{b}$  cannot be consistent for all  $\mathbf{b}$  in  $\mathbb{R}^3$ . Generalize your argument to the case of an arbitrary  $A$  with more rows than columns.

32. Could a set of three vectors in  $\mathbb{R}^4$  span all of  $\mathbb{R}^4$ ? Explain. What about  $n$  vectors in  $\mathbb{R}^m$  when  $n$  is less than  $m$ ?
33. Suppose  $A$  is a  $4 \times 3$  matrix and  $\mathbf{b}$  is a vector in  $\mathbb{R}^4$  with the property that  $A\mathbf{x} = \mathbf{b}$  has a unique solution. What can you say about the reduced echelon form of  $A$ ? Justify your answer.
34. Let  $A$  be a  $3 \times 4$  matrix, let  $\mathbf{v}_1$  and  $\mathbf{v}_2$  be vectors in  $\mathbb{R}^3$ , and let  $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$ . Suppose  $\mathbf{v}_1 = A\mathbf{u}_1$  and  $\mathbf{v}_2 = A\mathbf{u}_2$  for some vectors  $\mathbf{u}_1$  and  $\mathbf{u}_2$  in  $\mathbb{R}^4$ . What fact allows you to conclude that the system  $A\mathbf{x} = \mathbf{w}$  is consistent? (Note:  $\mathbf{u}_1$  and  $\mathbf{u}_2$  denote vectors, not scalar entries in vectors.)
35. Let  $A$  be a  $5 \times 3$  matrix, let  $\mathbf{y}$  be a vector in  $\mathbb{R}^3$ , and let  $\mathbf{z}$  be a vector in  $\mathbb{R}^5$ . Suppose  $A\mathbf{y} = \mathbf{z}$ . What fact allows you to conclude that the system  $A\mathbf{x} = 5\mathbf{z}$  is consistent?
36. Suppose  $A$  is a  $4 \times 4$  matrix and  $\mathbf{b}$  is a vector in  $\mathbb{R}^4$  with the property that  $A\mathbf{x} = \mathbf{b}$  has a unique solution. Explain why the columns of  $A$  must span  $\mathbb{R}^4$ .

[M] In Exercises 37–40, determine if the columns of the matrix span  $\mathbb{R}^4$ .

$$37. \begin{bmatrix} 7 & 2 & -5 & 8 \\ -5 & -3 & 4 & -9 \\ 6 & 10 & -2 & 7 \\ -7 & 9 & 2 & 15 \end{bmatrix} \quad 38. \begin{bmatrix} 4 & -5 & -1 & 8 \\ 3 & -7 & -4 & 2 \\ 5 & -6 & -1 & 4 \\ 9 & 1 & 10 & 7 \end{bmatrix}$$

$$39. \begin{bmatrix} 10 & -7 & 1 & 4 & 6 \\ -8 & 4 & -6 & -10 & -3 \\ -7 & 11 & -5 & -1 & -8 \\ 3 & -1 & 10 & 12 & 12 \end{bmatrix}$$

$$40. \begin{bmatrix} 5 & 11 & -6 & -7 & 12 \\ -7 & -3 & -4 & 6 & -9 \\ 11 & 5 & 6 & -9 & -3 \\ -3 & 4 & -7 & 2 & 7 \end{bmatrix}$$

41. [M] Find a column of the matrix in Exercise 39 that can be deleted and yet have the remaining matrix columns still span  $\mathbb{R}^4$ .
42. [M] Find a column of the matrix in Exercise 40 that can be deleted and yet have the remaining matrix columns still span  $\mathbb{R}^4$ . Can you delete more than one column?

SG

Mastering Linear Algebra Concepts: Span 1–18

WEB

## SOLUTIONS TO PRACTICE PROBLEMS

### 1. The matrix equation

$$\begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$$

is equivalent to the vector equation

$$3 \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} - 2 \begin{bmatrix} 5 \\ 1 \\ -8 \end{bmatrix} + 0 \begin{bmatrix} -2 \\ 9 \\ -1 \end{bmatrix} - 4 \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$$

which expresses  $\mathbf{b}$  as a linear combination of the columns of  $A$ .

$$\begin{aligned} 2. \quad \mathbf{u} + \mathbf{v} &= \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\ A(\mathbf{u} + \mathbf{v}) &= \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 + 20 \\ 3 + 4 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \end{bmatrix} \\ A\mathbf{u} + A\mathbf{v} &= \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 11 \end{bmatrix} + \begin{bmatrix} 19 \\ -4 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \end{bmatrix} \end{aligned}$$

## 1.5 EXERCISES

In Exercises 1–4, determine if the system has a nontrivial solution. Try to use as few row operations as possible.

$$\begin{array}{ll} 1. & \begin{array}{l} 2x_1 - 5x_2 + 8x_3 = 0 \\ -2x_1 - 7x_2 + x_3 = 0 \\ 4x_1 + 2x_2 + 7x_3 = 0 \end{array} \\ 2. & \begin{array}{l} x_1 - 2x_2 + 3x_3 = 0 \\ -2x_1 - 3x_2 - 4x_3 = 0 \\ 2x_1 - 4x_2 + 9x_3 = 0 \end{array} \end{array}$$

$$\begin{array}{ll} 3. & \begin{array}{l} -3x_1 + 4x_2 - 8x_3 = 0 \\ -2x_1 + 5x_2 + 4x_3 = 0 \end{array} \\ 4. & \begin{array}{l} 5x_1 - 3x_2 + 2x_3 = 0 \\ -3x_1 - 4x_2 + 2x_3 = 0 \end{array} \end{array}$$

In Exercises 5 and 6, follow the method of Examples 1 and 2 to write the solution set of the given homogeneous system in parametric vector form.

$$\begin{array}{ll} 5. & \begin{array}{l} 2x_1 + 2x_2 + 4x_3 = 0 \\ -4x_1 - 4x_2 - 8x_3 = 0 \\ -3x_2 - 3x_3 = 0 \end{array} \\ 6. & \begin{array}{l} x_1 + 2x_2 - 3x_3 = 0 \\ 2x_1 + x_2 - 3x_3 = 0 \\ -1x_1 + x_2 = 0 \end{array} \end{array}$$

In Exercises 7–12, describe all solutions of  $A\mathbf{x} = \mathbf{0}$  in parametric vector form, where  $A$  is row equivalent to the given matrix.

$$7. \begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix} \quad 8. \begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

$$9. \begin{bmatrix} 3 & -6 & 6 \\ -2 & 4 & -2 \end{bmatrix} \quad 10. \begin{bmatrix} -1 & -4 & 0 & -4 \\ 2 & -8 & 0 & 8 \end{bmatrix}$$

$$11. \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$12. \begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

13. Suppose the solution set of a certain system of linear equations can be described as  $x_1 = 5 + 4x_3$ ,  $x_2 = -2 - 7x_3$ , with  $x_3$  free. Use vectors to describe this set as a line in  $\mathbb{R}^3$ .
14. Suppose the solution set of a certain system of linear equations can be described as  $x_1 = 5x_4$ ,  $x_2 = 3 - 2x_4$ ,  $x_3 = 2 + 5x_4$ , with  $x_4$  free. Use vectors to describe this set as a “line” in  $\mathbb{R}^4$ .
15. Describe and compare the solution sets of  $x_1 + 5x_2 - 3x_3 = 0$  and  $x_1 + 5x_2 - 3x_3 = -2$ .
16. Describe and compare the solution sets of  $x_1 - 2x_2 + 3x_3 = 0$  and  $x_1 - 2x_2 + 3x_3 = 4$ .
17. Follow the method of Example 3 to describe the solutions of the following system in parametric vector form. Also, give a geometric description of the solution set and compare it to that in Exercise 5.

$$\begin{array}{rcl} 2x_1 + 2x_2 + 4x_3 & = & 8 \\ -4x_1 - 4x_2 - 8x_3 & = & -16 \\ -3x_2 - 3x_3 & = & 12 \end{array}$$

18. As in Exercise 17, describe the solutions of the following system in parametric vector form, and provide a geometric comparison with the solution set in Exercise 6.

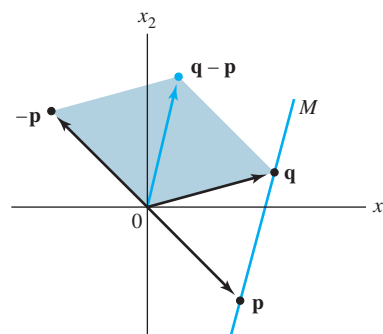
$$\begin{array}{rcl} x_1 + 2x_2 - 3x_3 & = & 5 \\ 2x_1 + x_2 - 3x_3 & = & 13 \\ -x_1 + x_2 & = & -8 \end{array}$$

In Exercises 19 and 20, find the parametric equation of the line through  $\mathbf{a}$  parallel to  $\mathbf{b}$ .

$$19. \mathbf{a} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ 3 \end{bmatrix} \quad 20. \mathbf{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -7 \\ 6 \end{bmatrix}$$

In Exercises 21 and 22, find a parametric equation of the line  $M$  through  $\mathbf{p}$  and  $\mathbf{q}$ . [Hint:  $M$  is parallel to the vector  $\mathbf{q} - \mathbf{p}$ . See the figure below.]

$$21. \mathbf{p} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad 22. \mathbf{p} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$



In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23.
  - a. A homogeneous equation is always consistent.
  - b. The equation  $A\mathbf{x} = \mathbf{0}$  gives an explicit description of its solution set.
  - c. The homogeneous equation  $A\mathbf{x} = \mathbf{0}$  has the trivial solution if and only if the equation has at least one free variable.
  - d. The equation  $\mathbf{x} = \mathbf{p} + t\mathbf{v}$  describes a line through  $\mathbf{v}$  parallel to  $\mathbf{p}$ .
  - e. The solution set of  $A\mathbf{x} = \mathbf{b}$  is the set of all vectors of the form  $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ , where  $\mathbf{v}_h$  is any solution of the equation  $A\mathbf{x} = \mathbf{0}$ .
24.
  - a. A homogeneous system of equations can be inconsistent.
  - b. If  $\mathbf{x}$  is a nontrivial solution of  $A\mathbf{x} = \mathbf{0}$ , then every entry in  $\mathbf{x}$  is nonzero.
  - c. The effect of adding  $\mathbf{p}$  to a vector is to move the vector in a direction parallel to  $\mathbf{p}$ .
  - d. The equation  $A\mathbf{x} = \mathbf{b}$  is homogeneous if the zero vector is a solution.

- e. If  $A\mathbf{x} = \mathbf{b}$  is consistent, then the solution set of  $A\mathbf{x} = \mathbf{b}$  is obtained by translating the solution set of  $A\mathbf{x} = \mathbf{0}$ .
25. Prove Theorem 6:
- Suppose  $\mathbf{p}$  is a solution of  $A\mathbf{x} = \mathbf{b}$ , so that  $A\mathbf{p} = \mathbf{b}$ . Let  $\mathbf{v}_h$  be any solution of the homogeneous equation  $A\mathbf{x} = \mathbf{0}$ , and let  $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ . Show that  $\mathbf{w}$  is a solution of  $A\mathbf{x} = \mathbf{b}$ .
  - Let  $\mathbf{w}$  be any solution of  $A\mathbf{x} = \mathbf{b}$ , and define  $\mathbf{v}_h = \mathbf{w} - \mathbf{p}$ . Show that  $\mathbf{v}_h$  is a solution of  $A\mathbf{x} = \mathbf{0}$ . This shows that every solution of  $A\mathbf{x} = \mathbf{b}$  has the form  $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ , with  $\mathbf{p}$  a particular solution of  $A\mathbf{x} = \mathbf{b}$  and  $\mathbf{v}_h$  a solution of  $A\mathbf{x} = \mathbf{0}$ .
26. Suppose  $A$  is the  $3 \times 3$  zero matrix (with all zero entries). Describe the solution set of the equation  $A\mathbf{x} = \mathbf{0}$ .
27. Suppose  $A\mathbf{x} = \mathbf{b}$  has a solution. Explain why the solution is unique precisely when  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.
- In Exercises 28–31, (a) does the equation  $A\mathbf{x} = \mathbf{0}$  have a nontrivial solution and (b) does the equation  $A\mathbf{x} = \mathbf{b}$  have at least one solution for every possible  $\mathbf{b}$ ?
- $A$  is a  $3 \times 3$  matrix with three pivot positions.
  - $A$  is a  $4 \times 4$  matrix with three pivot positions.
  - $A$  is a  $2 \times 5$  matrix with two pivot positions.
  - $A$  is a  $3 \times 2$  matrix with two pivot positions.
32. If  $\mathbf{b} \neq \mathbf{0}$ , can the solution set of  $A\mathbf{x} = \mathbf{b}$  be a plane through the origin? Explain.
33. Construct a  $3 \times 3$  nonzero matrix  $A$  such that the vector  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  is a solution of  $A\mathbf{x} = \mathbf{0}$ .
34. Construct a  $3 \times 3$  nonzero matrix  $A$  such that the vector  $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$  is a solution of  $A\mathbf{x} = \mathbf{0}$ .
35. Given  $A = \begin{bmatrix} -1 & -3 \\ 7 & 21 \\ -2 & -6 \end{bmatrix}$ , find one nontrivial solution of  $A\mathbf{x} = \mathbf{0}$  by inspection. [Hint: Think of the equation  $A\mathbf{x} = \mathbf{0}$  written as a vector equation.]
36. Given  $A = \begin{bmatrix} 3 & -2 \\ -6 & 4 \\ 12 & -8 \end{bmatrix}$ , find one nontrivial solution of  $A\mathbf{x} = \mathbf{0}$  by inspection.
37. Construct a  $2 \times 2$  matrix  $A$  such that the solution set of the equation  $A\mathbf{x} = \mathbf{0}$  is the line in  $\mathbb{R}^2$  through  $(4, 1)$  and the origin. Then, find a vector  $\mathbf{b}$  in  $\mathbb{R}^2$  such that the solution set of  $A\mathbf{x} = \mathbf{b}$  is *not* a line in  $\mathbb{R}^2$  parallel to the solution set of  $A\mathbf{x} = \mathbf{0}$ . Why does this *not* contradict Theorem 6?
38. Let  $A$  be an  $m \times n$  matrix and let  $\mathbf{w}$  be a vector in  $\mathbb{R}^n$  that satisfies the equation  $A\mathbf{x} = \mathbf{0}$ . Show that for any scalar  $c$ , the vector  $c\mathbf{w}$  also satisfies  $A\mathbf{x} = \mathbf{0}$ . [That is, show that  $A(c\mathbf{w}) = \mathbf{0}$ .]
39. Let  $A$  be an  $m \times n$  matrix, and let  $\mathbf{v}$  and  $\mathbf{w}$  be vectors in  $\mathbb{R}^n$  with the property that  $A\mathbf{v} = \mathbf{0}$  and  $A\mathbf{w} = \mathbf{0}$ . Explain why  $A(\mathbf{v} + \mathbf{w})$  must be the zero vector. Then explain why  $A(c\mathbf{v} + d\mathbf{w}) = \mathbf{0}$  for each pair of scalars  $c$  and  $d$ .
40. Suppose  $A$  is a  $3 \times 3$  matrix and  $\mathbf{b}$  is a vector in  $\mathbb{R}^3$  such that the equation  $A\mathbf{x} = \mathbf{b}$  does *not* have a solution. Does there exist a vector  $\mathbf{y}$  in  $\mathbb{R}^3$  such that the equation  $A\mathbf{x} = \mathbf{y}$  has a unique solution? Discuss.

### SOLUTIONS TO PRACTICE PROBLEMS

1. Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -5 & 0 \\ 0 & -9 & 18 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

$$\begin{aligned} x_1 + 3x_3 &= 4 \\ x_2 - 2x_3 &= -1 \end{aligned}$$

Thus  $x_1 = 4 - 3x_3$ ,  $x_2 = -1 + 2x_3$ , with  $x_3$  free. The general solution in parametric vector form is

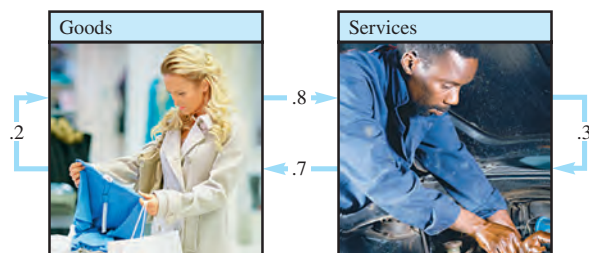
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 - 3x_3 \\ -1 + 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$\uparrow \qquad \qquad \uparrow$   
 $\mathbf{p} \qquad \qquad \mathbf{v}$

The intersection of the two planes is the line through  $\mathbf{p}$  in the direction of  $\mathbf{v}$ .

## 1.6 EXERCISES

1. Suppose an economy has only two sectors: Goods and Services. Each year, Goods sells 80% of its output to Services and keeps the rest, while Services sells 70% of its output to Goods and retains the rest. Find equilibrium prices for the annual outputs of the Goods and Services sectors that make each sector's income match its expenditures.



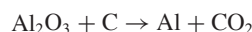
2. Find another set of equilibrium prices for the economy in Example 1. Suppose the same economy used Japanese yen instead of dollars to measure the values of the various sectors' outputs. Would this change the problem in any way? Discuss.
3. Consider an economy with three sectors: Fuels and Power, Manufacturing, and Services. Fuels and Power sells 80% of its output to Manufacturing, 10% to Services, and retains the rest. Manufacturing sells 10% of its output to Fuels and Power, 80% to Services, and retains the rest. Services sells 20% to Fuels and Power, 40% to Manufacturing, and retains the rest.
- Construct the exchange table for this economy.
  - Develop a system of equations that leads to prices at which each sector's income matches its expenses. Then write the augmented matrix that can be row reduced to find these prices.
  - [M] Find a set of equilibrium prices when the price for the Services output is 100 units.
4. Suppose an economy has four sectors: Mining, Lumber, Energy, and Transportation. Mining sells 10% of its output to Lumber, 60% to Energy, and retains the rest. Lumber sells 15% of its output to Mining, 50% to Energy, 20% to Transportation, and retains the rest. Energy sells 20% of its output to Mining, 15% to Lumber, 20% to Transportation, and retains the rest. Transportation sells 20% of its output to Mining, 10% to Lumber, 50% to Energy, and retains the rest.
- Construct the exchange table for this economy.
  - [M] Find a set of equilibrium prices for the economy.
5. An economy has four sectors: Agriculture, Manufacturing, Services, and Transportation. Agriculture sells 20% of its output to Manufacturing, 30% to Services, 30% to Transportation, and retains the rest. Manufacturing sells 35% of its output to Agriculture, 35% to Services, 20% to Transportation, and retains the rest. Services sells 10% of its output to Agriculture, 20% to Manufacturing, 20% to Transportation,

and retains the rest. Transportation sells 20% of its output to Agriculture, 30% to Manufacturing, 20% to Services, and retains the rest.

- Construct the exchange table for this economy.
- [M] Find a set of equilibrium prices for the economy if the value of Transportation is \$10.00 per unit.
- The Services sector launches a successful "eat farm fresh" campaign, and increases its share of the output from the Agricultural sector to 40%, whereas the share of Agricultural production going to Manufacturing falls to 10%. Construct the exchange table for this new economy.
- [M] Find a set of equilibrium prices for this new economy if the value of Transportation is still \$10.00 per unit. What effect has the "eat farm fresh" campaign had on the equilibrium prices for the sectors in this economy?

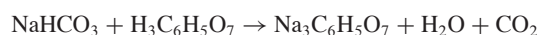
Balance the chemical equations in Exercises 6–11 using the vector equation approach discussed in this section.

6. Aluminum oxide and carbon react to create elemental aluminum and carbon dioxide:

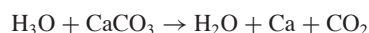


[For each compound, construct a vector that lists the numbers of atoms of aluminum, oxygen, and carbon.]

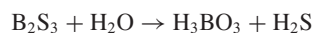
7. Alka-Seltzer contains sodium bicarbonate ( $\text{NaHCO}_3$ ) and citric acid ( $\text{H}_3\text{C}_6\text{H}_5\text{O}_7$ ). When a tablet is dissolved in water, the following reaction produces sodium citrate, water, and carbon dioxide (gas):



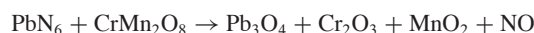
8. Limestone,  $\text{CaCO}_3$ , neutralizes the acid,  $\text{H}_3\text{O}$ , in acid rain by the following unbalanced equation:



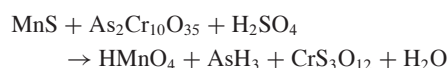
9. Boron sulfide reacts violently with water to form boric acid and hydrogen sulfide gas (the smell of rotten eggs). The unbalanced equation is



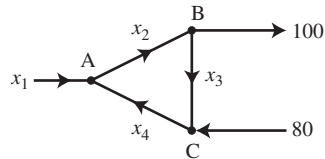
10. [M] If possible, use exact arithmetic or a rational format for calculations in balancing the following chemical reaction:



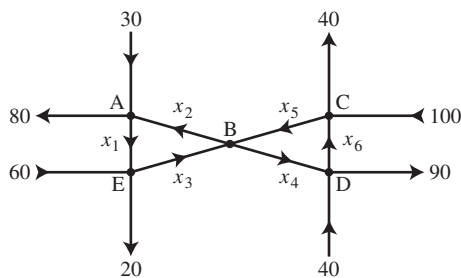
11. [M] The chemical reaction below can be used in some industrial processes, such as the production of arsene ( $\text{AsH}_3$ ). Use exact arithmetic or a rational format for calculations to balance this equation.



12. Find the general flow pattern of the network shown in the figure. Assuming that the flows are all nonnegative, what is the smallest possible value for  $x_4$ ?



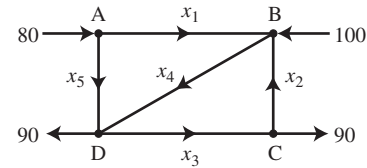
13. a. Find the general flow pattern of the network shown in the figure.  
b. Assuming that the flow must be in the directions indicated, find the minimum flows in the branches denoted by  $x_2$ ,  $x_3$ ,  $x_4$ , and  $x_5$ .



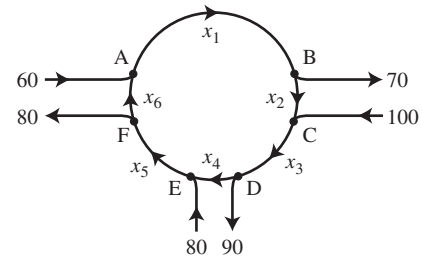
14. a. Find the general traffic pattern of the freeway network

shown in the figure. (Flow rates are in cars/minute.)

- b. Describe the general traffic pattern when the road whose flow is  $x_5$  is closed.  
c. When  $x_5 = 0$ , what is the minimum value of  $x_4$ ?



15. Intersections in England are often constructed as one-way "roundabouts," such as the one shown in the figure. Assume that traffic must travel in the directions shown. Find the general solution of the network flow. Find the smallest possible value for  $x_6$ .



### SOLUTIONS TO PRACTICE PROBLEMS

1. Write the percentages as decimals. Since all output must be taken into account, each column must sum to 1. This fact helps to fill in any missing entries.

Distribution of Output from:			
Agriculture	Mining	Manufacturing	Purchased by:
.65	.20	.20	Agriculture
.05	.10	.30	Mining
.30	.70	.50	Manufacturing

2. Since  $x_5 \leq 500$ , the equations D and A for  $x_1$  and  $x_2$  imply that  $x_1 \geq 100$  and  $x_2 \leq 700$ . The fact that  $x_5 \geq 0$  implies that  $x_1 \leq 600$  and  $x_2 \geq 200$ . So,  $100 \leq x_1 \leq 600$ , and  $200 \leq x_2 \leq 700$ .

## 1.7 LINEAR INDEPENDENCE

The homogeneous equations in Section 1.5 can be studied from a different perspective by writing them as vector equations. In this way, the focus shifts from the unknown solutions of  $A\mathbf{x} = \mathbf{0}$  to the vectors that appear in the vector equations.

## SG

Mastering: Linear  
Independence 1-31

In general, you should read a section thoroughly *several* times to absorb an important concept such as linear independence. The notes in the *Study Guide* for this section will help you learn to form mental images of key ideas in linear algebra. For instance, the following proof is worth reading carefully because it shows how the definition of linear independence can be *used*.

**PROOF OF THEOREM 7 (Characterization of Linearly Dependent Sets)**

If some  $\mathbf{v}_j$  in  $S$  equals a linear combination of the other vectors, then  $\mathbf{v}_j$  can be subtracted from both sides of the equation, producing a linear dependence relation with a nonzero weight  $(-1)$  on  $\mathbf{v}_j$ . [For instance, if  $\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ , then  $\mathbf{0} = (-1)\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + 0\mathbf{v}_4 + \cdots + 0\mathbf{v}_p$ .] Thus  $S$  is linearly dependent.

Conversely, suppose  $S$  is linearly dependent. If  $\mathbf{v}_1$  is zero, then it is a (trivial) linear combination of the other vectors in  $S$ . Otherwise,  $\mathbf{v}_1 \neq \mathbf{0}$ , and there exist weights  $c_1, \dots, c_p$ , not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

Let  $j$  be the largest subscript for which  $c_j \neq 0$ . If  $j = 1$ , then  $c_1\mathbf{v}_1 = \mathbf{0}$ , which is impossible because  $\mathbf{v}_1 \neq \mathbf{0}$ . So  $j > 1$ , and

$$c_1\mathbf{v}_1 + \cdots + c_j\mathbf{v}_j + 0\mathbf{v}_{j+1} + \cdots + 0\mathbf{v}_p = \mathbf{0}$$

$$c_j\mathbf{v}_j = -c_1\mathbf{v}_1 - \cdots - c_{j-1}\mathbf{v}_{j-1}$$

$$\mathbf{v}_j = \left(-\frac{c_1}{c_j}\right)\mathbf{v}_1 + \cdots + \left(-\frac{c_{j-1}}{c_j}\right)\mathbf{v}_{j-1} \quad \blacksquare$$

**PRACTICE PROBLEMS**

Let  $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$ ,  $\mathbf{v} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}$ ,  $\mathbf{w} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ , and  $\mathbf{z} = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$ .

1. Are the sets  $\{\mathbf{u}, \mathbf{v}\}$ ,  $\{\mathbf{u}, \mathbf{w}\}$ ,  $\{\mathbf{u}, \mathbf{z}\}$ ,  $\{\mathbf{v}, \mathbf{w}\}$ ,  $\{\mathbf{v}, \mathbf{z}\}$ , and  $\{\mathbf{w}, \mathbf{z}\}$  each linearly independent? Why or why not?
2. Does the answer to Problem 1 imply that  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$  is linearly independent?
3. To determine if  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$  is linearly dependent, is it wise to check if, say,  $\mathbf{w}$  is a linear combination of  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{z}$ ?
4. Is  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$  linearly dependent?

**1.7 EXERCISES**

In Exercises 1–4, determine if the vectors are linearly independent. Justify each answer.

1.  $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}$ ,  $\begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$       2.  $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 0 \\ -8 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$

3.  $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ ,  $\begin{bmatrix} -4 \\ 6 \end{bmatrix}$       4.  $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ ,  $\begin{bmatrix} -3 \\ -9 \end{bmatrix}$

In Exercises 5–8, determine if the columns of the matrix form a linearly independent set. Justify each answer.

5.  $\begin{bmatrix} 0 & -3 & 9 \\ 2 & 1 & -7 \\ -1 & 4 & -5 \\ 1 & -4 & -2 \end{bmatrix}$

6.  $\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 5 \\ 1 & 1 & -5 \\ 2 & 1 & -10 \end{bmatrix}$

7.  $\begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{bmatrix}$

8.  $\begin{bmatrix} 1 & -2 & 3 & 2 \\ -2 & 4 & -6 & 2 \\ 0 & 1 & -1 & 3 \end{bmatrix}$

In Exercises 9 and 10, (a) for what values of  $h$  is  $\mathbf{v}_3$  in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ , and (b) for what values of  $h$  is  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  linearly dependent? Justify each answer.



$$9. \mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$$

$$10. \mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ 15 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -5 \\ h \end{bmatrix}$$

In Exercises 11–14, find the value(s) of  $h$  for which the vectors are linearly dependent. Justify each answer.

$$11. \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ h \end{bmatrix} \quad 12. \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}$$

$$13. \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix} \quad 14. \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix}$$

Determine by inspection whether the vectors in Exercises 15–20 are linearly independent. Justify each answer.

$$15. \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix} \quad 16. \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -12 \end{bmatrix}$$

$$17. \begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix} \quad 18. \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$19. \begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} \quad 20. \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In Exercises 21 and 22, mark each statement True or False. Justify each answer on the basis of a careful reading of the text.

21. a. The columns of a matrix  $A$  are linearly independent if the equation  $A\mathbf{x} = \mathbf{0}$  has the trivial solution.  
 b. If  $S$  is a linearly dependent set, then each vector is a linear combination of the other vectors in  $S$ .  
 c. The columns of any  $4 \times 5$  matrix are linearly dependent.  
 d. If  $\mathbf{x}$  and  $\mathbf{y}$  are linearly independent, and if  $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$  is linearly dependent, then  $\mathbf{z}$  is in  $\text{Span}\{\mathbf{x}, \mathbf{y}\}$ .
22. a. If  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent, and if  $\mathbf{w}$  is in  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ , then  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  is linearly dependent.  
 b. If three vectors in  $\mathbb{R}^3$  lie in the same plane in  $\mathbb{R}^3$ , then they are linearly dependent.  
 c. If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.  
 d. If a set in  $\mathbb{R}^n$  is linearly dependent, then the set contains more than  $n$  vectors.

In Exercises 23–26, describe the possible echelon forms of the matrix. Use the notation of Example 1 in Section 1.2.

23.  $A$  is a  $2 \times 2$  matrix with linearly dependent columns.  
 24.  $A$  is a  $3 \times 3$  matrix with linearly independent columns.

25.  $A$  is a  $4 \times 2$  matrix,  $A = [\mathbf{a}_1 \ \mathbf{a}_2]$ , and  $\mathbf{a}_2$  is not a multiple of  $\mathbf{a}_1$ .  
 26.  $A$  is a  $4 \times 3$  matrix,  $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$ , such that  $\{\mathbf{a}_1, \mathbf{a}_2\}$  is linearly independent and  $\mathbf{a}_3$  is not in  $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ .  
 27. How many pivot columns must a  $6 \times 4$  matrix have if its columns are linearly independent? Why?  
 28. How many pivot columns must a  $4 \times 6$  matrix have if its columns span  $\mathbb{R}^4$ ? Why?  
 29. Construct  $3 \times 2$  matrices  $A$  and  $B$  such that  $A\mathbf{x} = \mathbf{0}$  has a nontrivial solution, but  $B\mathbf{x} = \mathbf{0}$  has only the trivial solution.  
 30. a. Fill in the blank in the following statement: “If  $A$  is an  $m \times n$  matrix, then the columns of  $A$  are linearly independent if and only if  $A$  has \_\_\_\_\_ pivot columns.”  
 b. Explain why the statement in (a) is true.

Exercises 31 and 32 should be solved *without performing row operations*. [Hint: Write  $A\mathbf{x} = \mathbf{0}$  as a vector equation.]

$$31. \text{ Given } A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}, \text{ observe that the third column}$$

is the sum of the first two columns. Find a nontrivial solution of  $A\mathbf{x} = \mathbf{0}$ .

$$32. \text{ Given } A = \begin{bmatrix} 4 & 3 & -5 \\ -2 & -2 & 4 \\ -2 & -3 & 7 \end{bmatrix}, \text{ observe that the first column}$$

minus three times the second column equals the third column. Find a nontrivial solution of  $A\mathbf{x} = \mathbf{0}$ .

Each statement in Exercises 33–38 is either true (in all cases) or false (for at least one example). If false, construct a specific example to show that the statement is not always true. Such an example is called a *counterexample* to the statement. If a statement is true, give a justification. (One specific example cannot explain why a statement is always true. You will have to do more work here than in Exercises 21 and 22.)

33. If  $\mathbf{v}_1, \dots, \mathbf{v}_4$  are in  $\mathbb{R}^4$  and  $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$ , then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  is linearly dependent.  
 34. If  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are in  $\mathbb{R}^4$  and  $\mathbf{v}_2$  is not a scalar multiple of  $\mathbf{v}_1$ , then  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is linearly independent.  
 35. If  $\mathbf{v}_1, \dots, \mathbf{v}_5$  are in  $\mathbb{R}^5$  and  $\mathbf{v}_3 = \mathbf{0}$ , then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$  is linearly dependent.  
 36. If  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are in  $\mathbb{R}^3$  and  $\mathbf{v}_3$  is *not* a linear combination of  $\mathbf{v}_1, \mathbf{v}_2$ , then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly independent.  
 37. If  $\mathbf{v}_1, \dots, \mathbf{v}_4$  are in  $\mathbb{R}^4$  and  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent, then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  is also linearly dependent.  
 38. If  $\{\mathbf{v}_1, \dots, \mathbf{v}_4\}$  is a linearly independent set of vectors in  $\mathbb{R}^4$ , then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is also linearly independent. [Hint: Think about  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0 \cdot \mathbf{v}_4 = \mathbf{0}$ .]  
 39. Suppose  $A$  is an  $m \times n$  matrix with the property that for all  $\mathbf{b}$  in  $\mathbb{R}^m$  the equation  $A\mathbf{x} = \mathbf{b}$  has at most one solution. Use the

definition of linear independence to explain why the columns of  $A$  must be linearly independent.

40. Suppose an  $m \times n$  matrix  $A$  has  $n$  pivot columns. Explain why for each  $\mathbf{b}$  in  $\mathbb{R}^m$  the equation  $A\mathbf{x} = \mathbf{b}$  has at most one solution. [Hint: Explain why  $A\mathbf{x} = \mathbf{b}$  cannot have infinitely many solutions.]

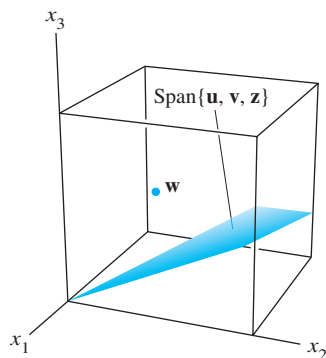
[M] In Exercises 41 and 42, use as many columns of  $A$  as possible to construct a matrix  $B$  with the property that the equation  $B\mathbf{x} = \mathbf{0}$  has only the trivial solution. Solve  $B\mathbf{x} = \mathbf{0}$  to verify your work.

$$41. A = \begin{bmatrix} 3 & -4 & 10 & 7 & -4 \\ -5 & -3 & -7 & -11 & 15 \\ 4 & 3 & 5 & 2 & 1 \\ 8 & -7 & 23 & 4 & 15 \end{bmatrix}$$

$$42. A = \begin{bmatrix} 12 & 10 & -6 & 8 & 4 & -14 \\ -7 & -6 & 4 & -5 & -7 & 9 \\ 9 & 9 & -9 & 9 & 9 & -18 \\ -4 & -3 & -1 & 0 & -8 & 1 \\ 8 & 7 & -5 & 6 & 1 & -11 \end{bmatrix}$$

43. [M] With  $A$  and  $B$  as in Exercise 41, select a column  $\mathbf{v}$  of  $A$  that was not used in the construction of  $B$  and determine if  $\mathbf{v}$  is in the set spanned by the columns of  $B$ . (Describe your calculations.)

44. [M] Repeat Exercise 43 with the matrices  $A$  and  $B$  from Exercise 42. Then give an explanation for what you discover, assuming that  $B$  was constructed as specified.



### SOLUTIONS TO PRACTICE PROBLEMS

1. Yes. In each case, neither vector is a multiple of the other. Thus each set is linearly independent.
2. No. The observation in Practice Problem 1, by itself, says nothing about the linear independence of  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ .
3. No. When testing for linear independence, it is usually a poor idea to check if one selected vector is a linear combination of the others. It may happen that the selected vector is not a linear combination of the others and yet the whole set of vectors is linearly dependent. In this practice problem,  $\mathbf{w}$  is not a linear combination of  $\mathbf{u}, \mathbf{v}$ , and  $\mathbf{z}$ .
4. Yes, by Theorem 8. There are more vectors (four) than entries (three) in them.

## 1.8 INTRODUCTION TO LINEAR TRANSFORMATIONS

The difference between a matrix equation  $A\mathbf{x} = \mathbf{b}$  and the associated vector equation  $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$  is merely a matter of notation. However, a matrix equation  $A\mathbf{x} = \mathbf{b}$  can arise in linear algebra (and in applications such as computer graphics and signal processing) in a way that is not directly connected with linear combinations of vectors. This happens when we think of the matrix  $A$  as an object that “acts” on a vector  $\mathbf{x}$  by multiplication to produce a new vector called  $A\mathbf{x}$ .

For instance, the equations

$$\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\uparrow$                        $\uparrow$                        $\uparrow$                        $\uparrow$                        $\uparrow$   
 $A$                        $\mathbf{x}$                        $\mathbf{b}$                        $A$                        $\mathbf{u}$                        $\mathbf{0}$

say that multiplication by  $A$  transforms  $\mathbf{x}$  into  $\mathbf{b}$  and transforms  $\mathbf{u}$  into the zero vector. See Fig. 1.

2. If  $\mathbf{x}$  and  $\mathbf{y}$  are production vectors, then the total cost vector associated with the combined production  $\mathbf{x} + \mathbf{y}$  is precisely the sum of the cost vectors  $T(\mathbf{x})$  and  $T(\mathbf{y})$ . ■

## PRACTICE PROBLEMS

1. Suppose  $T : \mathbb{R}^5 \rightarrow \mathbb{R}^2$  and  $T(\mathbf{x}) = A\mathbf{x}$  for some matrix  $A$  and for each  $\mathbf{x}$  in  $\mathbb{R}^5$ . How many rows and columns does  $A$  have?
2. Let  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ . Give a geometric description of the transformation  $\mathbf{x} \mapsto A\mathbf{x}$ .
3. The line segment from  $\mathbf{0}$  to a vector  $\mathbf{u}$  is the set of points of the form  $t\mathbf{u}$ , where  $0 \leq t \leq 1$ . Show that a linear transformation  $T$  maps this segment into the segment between  $\mathbf{0}$  and  $T(\mathbf{u})$ .

## 1.8 EXERCISES

1. Let  $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ , and define  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by  $T(\mathbf{x}) = A\mathbf{x}$ .

Find the images under  $T$  of  $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ .

2. Let  $A = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ ,  $\mathbf{u} = \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix}$ , and  $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ .

Define  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  by  $T(\mathbf{x}) = A\mathbf{x}$ . Find  $T(\mathbf{u})$  and  $T(\mathbf{v})$ .

In Exercises 3–6, with  $T$  defined by  $T(\mathbf{x}) = A\mathbf{x}$ , find a vector  $\mathbf{x}$  whose image under  $T$  is  $\mathbf{b}$ , and determine whether  $\mathbf{x}$  is unique.

3.  $A = \begin{bmatrix} 1 & 0 & -3 \\ -3 & 1 & 6 \\ 2 & -2 & -1 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$

4.  $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 2 & -5 & 6 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} -6 \\ -4 \\ -5 \end{bmatrix}$

5.  $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

6.  $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -8 & 8 \\ 0 & 1 & 2 \\ 1 & 0 & 8 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} 1 \\ 6 \\ 3 \\ 10 \end{bmatrix}$

7. Let  $A$  be a  $6 \times 5$  matrix. What must  $a$  and  $b$  be in order to define  $T : \mathbb{R}^a \rightarrow \mathbb{R}^b$  by  $T(\mathbf{x}) = A\mathbf{x}$ ?

8. How many rows and columns must a matrix  $A$  have in order to define a mapping from  $\mathbb{R}^5$  into  $\mathbb{R}^7$  by the rule  $T(\mathbf{x}) = A\mathbf{x}$ ?

For Exercises 9 and 10, find all  $\mathbf{x}$  in  $\mathbb{R}^4$  that are mapped into the zero vector by the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  for the given matrix  $A$ .

9.  $A = \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 2 & -4 & 4 & -4 \end{bmatrix}$

10.  $A = \begin{bmatrix} 3 & 2 & 10 & -6 \\ 1 & 0 & 2 & -4 \\ 0 & 1 & 2 & 3 \\ 1 & 4 & 10 & 8 \end{bmatrix}$

11. Let  $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ , and let  $A$  be the matrix in Exercise 9. Is  $\mathbf{b}$  in the range of the linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$ ? Why or why not?

12. Let  $\mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$ , and let  $A$  be the matrix in Exercise 10. Is  $\mathbf{b}$  in the range of the linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$ ? Why or why not?

In Exercises 13–16, use a rectangular coordinate system to plot  $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ ,  $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ , and their images under the given transformation  $T$ . (Make a separate and reasonably large sketch for each exercise.) Describe geometrically what  $T$  does to each vector  $\mathbf{x}$  in  $\mathbb{R}^2$ .

13.  $T(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

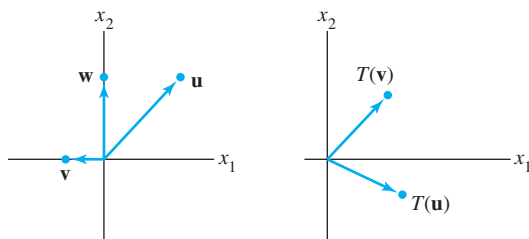
14.  $T(\mathbf{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

15.  $T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

16.  $T(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

17. Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear transformation that maps  $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$  into  $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$  and maps  $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$  into  $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ . Use the fact that  $T$  is linear to find the images under  $T$  of  $2\mathbf{u}$ ,  $3\mathbf{v}$ , and  $2\mathbf{u} + 3\mathbf{v}$ .

18. The figure shows vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$ , along with the images  $T(\mathbf{u})$  and  $T(\mathbf{v})$  under the action of a linear transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . Copy this figure carefully, and draw the image  $T(\mathbf{w})$  as accurately as possible. [Hint: First, write  $\mathbf{w}$  as a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ .]



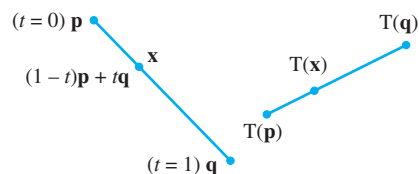
19. Let  $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ , and  $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ , and let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear transformation that maps  $\mathbf{e}_1$  into  $\mathbf{y}_1$  and maps  $\mathbf{e}_2$  into  $\mathbf{y}_2$ . Find the images of  $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$  and  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ .
20. Let  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ,  $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ , and  $\mathbf{v}_2 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$ , and let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear transformation that maps  $\mathbf{x}$  into  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$ . Find a matrix  $A$  such that  $T(\mathbf{x}) = A\mathbf{x}$  for each  $\mathbf{x}$ .

In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. A linear transformation is a special type of function.  
 b. If  $A$  is a  $3 \times 5$  matrix and  $T$  is a transformation defined by  $T(\mathbf{x}) = A\mathbf{x}$ , then the domain of  $T$  is  $\mathbb{R}^3$ .  
 c. If  $A$  is an  $m \times n$  matrix, then the range of the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is  $\mathbb{R}^m$ .  
 d. Every linear transformation is a matrix transformation.  
 e. A transformation  $T$  is linear if and only if  

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$$
for all  $\mathbf{v}_1$  and  $\mathbf{v}_2$  in the domain of  $T$  and for all scalars  $c_1$  and  $c_2$ .
22. a. The range of the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is the set of all linear combinations of the columns of  $A$ .  
 b. Every matrix transformation is a linear transformation.  
 c. If  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear transformation and if  $\mathbf{c}$  is in  $\mathbb{R}^m$ , then a uniqueness question is “Is  $\mathbf{c}$  in the range of  $T$ ?”  
 d. A linear transformation preserves the operations of vector addition and scalar multiplication.  
 e. A linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  always maps the origin of  $\mathbb{R}^n$  to the origin of  $\mathbb{R}^m$ .
23. Define  $f: \mathbb{R} \rightarrow \mathbb{R}$  by  $f(x) = mx + b$ .  
 a. Show that  $f$  is a linear transformation when  $b = 0$ .  
 b. Find a property of a linear transformation that is violated when  $b \neq 0$ .  
 c. Why is  $f$  called a linear function?

24. An affine transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  has the form  $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ , with  $A$  an  $m \times n$  matrix and  $\mathbf{b}$  in  $\mathbb{R}^m$ . Show that  $T$  is not a linear transformation when  $\mathbf{b} \neq \mathbf{0}$ . (Affine transformations are important in computer graphics.)
25. Given  $\mathbf{v} \neq \mathbf{0}$  and  $\mathbf{p}$  in  $\mathbb{R}^n$ , the line through  $\mathbf{p}$  in the direction of  $\mathbf{v}$  has the parametric equation  $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ . Show that a linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  maps this line onto another line or onto a single point (a degenerate line).
26. a. Show that the line through vectors  $\mathbf{p}$  and  $\mathbf{q}$  in  $\mathbb{R}^n$  may be written in the parametric form  $\mathbf{x} = (1-t)\mathbf{p} + t\mathbf{q}$ . (Refer to the figure with Exercises 21 and 22 in Section 1.5.)  
 b. The line segment from  $\mathbf{p}$  to  $\mathbf{q}$  is the set of points of the form  $(1-t)\mathbf{p} + t\mathbf{q}$  for  $0 \leq t \leq 1$  (as shown in the figure below). Show that a linear transformation  $T$  maps this line segment onto a line segment or onto a single point.



27. Let  $\mathbf{u}$  and  $\mathbf{v}$  be linearly independent vectors in  $\mathbb{R}^3$ , and let  $P$  be the plane through  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{0}$ . The parametric equation of  $P$  is  $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$  (with  $s, t$  in  $\mathbb{R}$ ). Show that a linear transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  maps  $P$  onto a plane through  $\mathbf{0}$ , or onto a line through  $\mathbf{0}$ , or onto just the origin in  $\mathbb{R}^3$ . What must be true about  $T(\mathbf{u})$  and  $T(\mathbf{v})$  in order for the image of the plane  $P$  to be a plane?
28. Let  $\mathbf{u}$  and  $\mathbf{v}$  be vectors in  $\mathbb{R}^n$ . It can be shown that the set  $P$  of all points in the parallelogram determined by  $\mathbf{u}$  and  $\mathbf{v}$  has the form  $a\mathbf{u} + b\mathbf{v}$ , for  $0 \leq a \leq 1$ ,  $0 \leq b \leq 1$ . Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. Explain why the image of a point in  $P$  under the transformation  $T$  lies in the parallelogram determined by  $T(\mathbf{u})$  and  $T(\mathbf{v})$ .
29. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation that reflects each point through the  $x_2$ -axis. Make two sketches similar to Fig. 6 that illustrate properties (i) and (ii) of a linear transformation.
30. Suppose vectors  $\mathbf{v}_1, \dots, \mathbf{v}_p$  span  $\mathbb{R}^n$ , and let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation. Suppose  $T(\mathbf{v}_i) = \mathbf{0}$  for  $i = 1, \dots, p$ . Show that  $T$  is the zero transformation. That is, show that if  $\mathbf{x}$  is any vector in  $\mathbb{R}^n$ , then  $T(\mathbf{x}) = \mathbf{0}$ .
31. Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation, and let  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  be a linearly dependent set in  $\mathbb{R}^n$ . Explain why the set  $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$  is linearly dependent.

In Exercises 32–36, column vectors are written as rows, such as  $\mathbf{x} = (x_1, x_2)$ , and  $T(\mathbf{x})$  is written as  $T(x_1, x_2)$ .

32. Show that the transformation  $T$  defined by  $T(x_1, x_2) = (x_1 - 2|x_2|, x_1 - 4x_2)$  is not linear.
33. Show that the transformation  $T$  defined by  $T(x_1, x_2) = (x_1 - 2x_2, x_1 - 3, 2x_1 - 5x_2)$  is not linear.

34. Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the transformation that reflects each vector  $\mathbf{x} = (x_1, x_2, x_3)$  through the plane  $x_3 = 0$  onto  $T(\mathbf{x}) = (x_1, x_2, -x_3)$ . Show that  $T$  is a linear transformation. [See Example 4 for ideas.]

35. Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the transformation that projects each vector  $\mathbf{x} = (x_1, x_2, x_3)$  onto the plane  $x_2 = 0$ , so  $T(\mathbf{x}) = (x_1, 0, x_3)$ . Show that  $T$  is a linear transformation.

36. Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. Suppose  $\{\mathbf{u}, \mathbf{v}\}$  is a linearly independent set, but  $\{T(\mathbf{u}), T(\mathbf{v})\}$  is a linearly dependent set. Show that  $T(\mathbf{x}) = \mathbf{0}$  has a nontrivial solution. [Hint: Use the fact that  $c_1 T(\mathbf{u}) + c_2 T(\mathbf{v}) = \mathbf{0}$  for some weights  $c_1$  and  $c_2$ , not both zero.]

[M] In Exercises 37 and 38, the given matrix determines a linear transformation  $T$ . Find all  $\mathbf{x}$  such that  $T(\mathbf{x}) = \mathbf{0}$ .

$$37. \begin{bmatrix} 2 & 3 & 5 & -5 \\ -7 & 7 & 0 & 0 \\ -3 & 4 & 1 & 3 \\ -9 & 3 & -6 & -4 \end{bmatrix}$$

$$38. \begin{bmatrix} 3 & 4 & -7 & 0 \\ 5 & -8 & 7 & 4 \\ 6 & -8 & 6 & 4 \\ 9 & -7 & -2 & 0 \end{bmatrix}$$

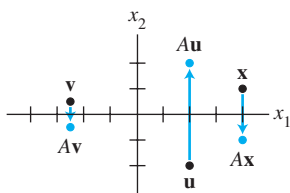
$$39. \text{ [M] Let } \mathbf{b} = \begin{bmatrix} 8 \\ 7 \\ 5 \\ -3 \end{bmatrix} \text{ and let } A \text{ be the matrix in Exercise 37.}$$

Is  $\mathbf{b}$  in the range of the transformation  $\mathbf{x} \mapsto A\mathbf{x}$ ? If so, find an  $\mathbf{x}$  whose image under the transformation is  $\mathbf{b}$ .

$$40. \text{ [M] Let } \mathbf{b} = \begin{bmatrix} 4 \\ -4 \\ -4 \\ -7 \end{bmatrix} \text{ and let } A \text{ be the matrix in Exercise 38.}$$

Is  $\mathbf{b}$  in the range of the transformation  $\mathbf{x} \mapsto A\mathbf{x}$ ? If so, find an  $\mathbf{x}$  whose image under the transformation is  $\mathbf{b}$ .

**SG** Mastering: Linear Transformations 1–34



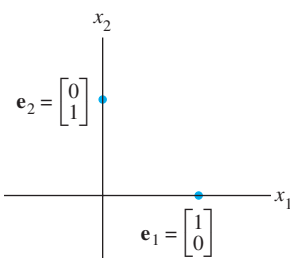
The transformation  $\mathbf{x} \mapsto A\mathbf{x}$ .

### SOLUTIONS TO PRACTICE PROBLEMS

1.  $A$  must have five columns for  $A\mathbf{x}$  to be defined.  $A$  must have two rows for the codomain of  $T$  to be  $\mathbb{R}^2$ .
2. Plot some random points (vectors) on graph paper to see what happens. A point such as  $(4, 1)$  maps into  $(4, -1)$ . The transformation  $\mathbf{x} \mapsto A\mathbf{x}$  reflects points through the  $x$ -axis (or  $x_1$ -axis).
3. Let  $\mathbf{x} = t\mathbf{u}$  for some  $t$  such that  $0 \leq t \leq 1$ . Since  $T$  is linear,  $T(t\mathbf{u}) = tT(\mathbf{u})$ , which is a point on the line segment between  $\mathbf{0}$  and  $T(\mathbf{u})$ .

## 1.9 THE MATRIX OF A LINEAR TRANSFORMATION

Whenever a linear transformation  $T$  arises geometrically or is described in words, we usually want a “formula” for  $T(\mathbf{x})$ . The discussion that follows shows that every linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is actually a matrix transformation  $\mathbf{x} \mapsto A\mathbf{x}$  and that important properties of  $T$  are intimately related to familiar properties of  $A$ . The key to finding  $A$  is to observe that  $T$  is completely determined by what it does to the columns of the  $n \times n$  identity matrix  $I_n$ .



**EXAMPLE 1** The columns of  $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  are  $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . Suppose  $T$  is a linear transformation from  $\mathbb{R}^2$  into  $\mathbb{R}^3$  such that

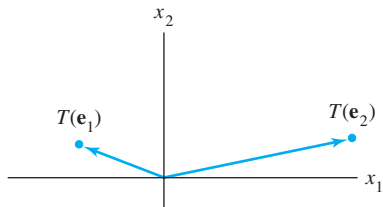
$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

With no additional information, find a formula for the image of an arbitrary  $\mathbf{x}$  in  $\mathbb{R}^2$ .

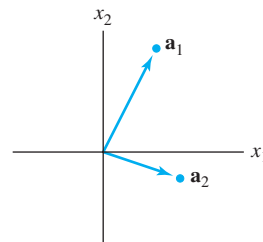
## 1.9 EXERCISES

In Exercises 1–10, assume that  $T$  is a linear transformation. Find the standard matrix of  $T$ .

- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^4$ ,  $T(\mathbf{e}_1) = (3, 1, 3, 1)$ , and  $T(\mathbf{e}_2) = (-5, 2, 0, 0)$ , where  $\mathbf{e}_1 = (1, 0)$  and  $\mathbf{e}_2 = (0, 1)$ .
- $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ ,  $T(\mathbf{e}_1) = (1, 4)$ ,  $T(\mathbf{e}_2) = (-2, 9)$ , and  $T(\mathbf{e}_3) = (3, -8)$ , where  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$  are the columns of the  $3 \times 3$  identity matrix.
- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a vertical shear transformation that maps  $\mathbf{e}_1$  into  $\mathbf{e}_1 - 3\mathbf{e}_2$ , but leaves  $\mathbf{e}_2$  unchanged.
- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a horizontal shear transformation that leaves  $\mathbf{e}_1$  unchanged and maps  $\mathbf{e}_2$  into  $\mathbf{e}_2 + 2\mathbf{e}_1$ .
- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  rotates points (about the origin) through  $\pi/2$  radians (counterclockwise).
- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  rotates points (about the origin) through  $-3\pi/2$  radians (clockwise).
- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  first rotates points through  $-3\pi/4$  radians (clockwise) and then reflects points through the horizontal  $x_1$ -axis. [Hint:  $T(\mathbf{e}_1) = (-1/\sqrt{2}, 1/\sqrt{2})$ .]
- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  first performs a horizontal shear that transforms  $\mathbf{e}_2$  into  $\mathbf{e}_2 + 2\mathbf{e}_1$  (leaving  $\mathbf{e}_1$  unchanged) and then reflects points through the line  $x_2 = -x_1$ .
- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  first reflects points through the horizontal  $x_1$ -axis and then rotates points  $-\pi/2$  radians.
- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  first reflects points through the horizontal  $x_1$ -axis and then reflects points through the line  $x_2 = x_1$ .
- A linear transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  first reflects points through the  $x_1$ -axis and then reflects points through the  $x_2$ -axis. Show that  $T$  can also be described as a linear transformation that rotates points about the origin. What is the angle of that rotation?
- Show that the transformation in Exercise 10 is merely a rotation about the origin. What is the angle of the rotation?
- Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation such that  $T(\mathbf{e}_1)$  and  $T(\mathbf{e}_2)$  are the vectors shown in the figure. Using the figure, sketch the vector  $T(2, 1)$ .



- Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear transformation with standard matrix  $A = [\mathbf{a}_1 \ \mathbf{a}_2]$ , where  $\mathbf{a}_1$  and  $\mathbf{a}_2$  are shown in the figure at the top of column 2. Using the figure, draw the image of  $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$  under the transformation  $T$ .



In Exercises 15 and 16, fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables.

$$15. \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 4x_2 \\ x_1 - x_3 \\ -x_2 + 3x_3 \end{bmatrix}$$

$$16. \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_2 \\ x_1 + 4x_2 \\ x_2 \end{bmatrix}$$

In Exercises 17–20, show that  $T$  is a linear transformation by finding a matrix that implements the mapping. Note that  $x_1, x_2, \dots$  are not vectors but are entries in vectors.

- $T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2, 0, 2x_2 + x_4, x_2 - x_4)$
- $T(x_1, x_2) = (x_1 + 4x_2, 0, x_1 - 3x_2, x_1)$
- $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$
- $T(x_1, x_2, x_3, x_4) = 3x_1 + 4x_3 - 2x_4$  (Notice:  $T: \mathbb{R}^4 \rightarrow \mathbb{R}$ )
- Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear transformation such that  $T(x_1, x_2) = (x_1 + x_2, 4x_1 + 5x_2)$ . Find  $\mathbf{x}$  such that  $T(\mathbf{x}) = (3, 8)$ .
- Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a linear transformation with  $T(x_1, x_2) = (2x_1 - x_2, -3x_1 + x_2, 2x_1 - 3x_2)$ . Find  $\mathbf{x}$  such that  $T(\mathbf{x}) = (0, -1, -4)$ .

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

- A linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is completely determined by its effect on the columns of the  $n \times n$  identity matrix.
  - If  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  rotates vectors about the origin through an angle  $\varphi$ , then  $T$  is a linear transformation.
  - When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.
  - A mapping  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is onto  $\mathbb{R}^m$  if every vector  $\mathbf{x}$  in  $\mathbb{R}^n$  maps onto some vector in  $\mathbb{R}^m$ .
  - If  $A$  is a  $3 \times 2$  matrix, then the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  cannot be one-to-one.
- If  $A$  is a  $4 \times 3$  matrix, then the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  maps  $\mathbb{R}^3$  onto  $\mathbb{R}^4$ .

- b. Every linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is a matrix transformation.
- c. The columns of the standard matrix for a linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  are the images of the columns of the  $n \times n$  identity matrix under  $T$ .
- d. A mapping  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is one-to-one if each vector in  $\mathbb{R}^n$  maps onto a unique vector in  $\mathbb{R}^m$ .
- e. The standard matrix of a horizontal shear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  has the form  $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ , where  $a$  and  $d$  are  $\pm 1$ .

In Exercises 25–28, determine if the specified linear transformation is (a) one-to-one and (b) onto. Justify each answer.

25. The transformation in Exercise 17

26. The transformation in Exercise 2

27. The transformation in Exercise 19

28. The transformation in Exercise 14

In Exercises 29 and 30, describe the possible echelon forms of the standard matrix for a linear transformation  $T$ . Use the notation of Example 1 in Section 1.2.

29.  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  is one-to-one.      30.  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  is onto.

31. Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation, with  $A$  its standard matrix. Complete the following statement to make it true: “ $T$  is one-to-one if and only if  $A$  has \_\_\_\_ pivot columns.” Explain why the statement is true. [Hint: Look in the exercises for Section 1.7.]

32. Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation, with  $A$  its standard matrix. Complete the following statement to make it true: “ $T$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^m$  if and only if  $A$  has \_\_\_\_ pivot columns.” Find some theorems that explain why the statement is true.

33. Verify the uniqueness of  $A$  in Theorem 10. Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation such that  $T(\mathbf{x}) = B\mathbf{x}$  for some  $m \times n$  matrix  $B$ . Show that if  $A$  is the standard matrix for  $T$ , then  $A = B$ . [Hint: Show that  $A$  and  $B$  have the same columns.]

34. Let  $S : \mathbb{R}^p \rightarrow \mathbb{R}^n$  and  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be linear transformations. Show that the mapping  $\mathbf{x} \mapsto T(S(\mathbf{x}))$  is a linear transformation (from  $\mathbb{R}^p$  to  $\mathbb{R}^m$ ). [Hint: Compute  $T(S(c\mathbf{u} + d\mathbf{v}))$  for  $\mathbf{u}, \mathbf{v}$  in  $\mathbb{R}^p$  and scalars  $c$  and  $d$ . Justify each step of the computation, and explain why this computation gives the desired conclusion.]

35. If a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^m$ , can you give a relation between  $m$  and  $n$ ? If  $T$  is one-to-one, what can you say about  $m$  and  $n$ ?

36. Why is the question “Is the linear transformation  $T$  onto?” an existence question?

[M] In Exercises 37–40, let  $T$  be the linear transformation whose standard matrix is given. In Exercises 37 and 38, decide if  $T$  is a one-to-one mapping. In Exercises 39 and 40, decide if  $T$  maps  $\mathbb{R}^5$  onto  $\mathbb{R}^5$ . Justify your answers.

37.  $\begin{bmatrix} -5 & 6 & -5 & -6 \\ 8 & 3 & -3 & 8 \\ 2 & 9 & 5 & -12 \\ -3 & 2 & 7 & -12 \end{bmatrix}$       38.  $\begin{bmatrix} 7 & 5 & 9 & -9 \\ 5 & 6 & 4 & -4 \\ 4 & 8 & 0 & 7 \\ -6 & -6 & 6 & 5 \end{bmatrix}$

39.  $\begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & -9 & 14 \\ 3 & -5 & 4 & 2 & -6 \\ -5 & 6 & -6 & -7 & 3 \end{bmatrix}$

40.  $\begin{bmatrix} 9 & 43 & 5 & 6 & -1 \\ 14 & 15 & -7 & -5 & 4 \\ -8 & -6 & 12 & -5 & -9 \\ -5 & -6 & -4 & 9 & 8 \\ 13 & 14 & 15 & 3 & 11 \end{bmatrix}$

### SOLUTION TO PRACTICE PROBLEM

WEB

Follow what happens to  $\mathbf{e}_1$  and  $\mathbf{e}_2$ . See Fig. 5. First,  $\mathbf{e}_1$  is unaffected by the shear and then is reflected into  $-\mathbf{e}_1$ . So  $T(\mathbf{e}_1) = -\mathbf{e}_1$ . Second,  $\mathbf{e}_2$  goes to  $\mathbf{e}_2 + .5\mathbf{e}_1$  by the shear

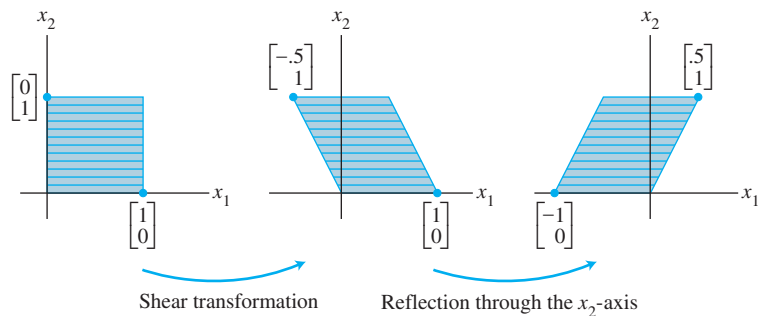


FIGURE 5 The composition of two transformations.



## 1.10 EXERCISES

- The container of a breakfast cereal usually lists the number of calories and the amounts of protein, carbohydrate, and fat contained in one serving of the cereal. The amounts for two common cereals are given below. Suppose a mixture of these two cereals is to be prepared that contains exactly 295 calories, 9 g of protein, 48 g of carbohydrate, and 8 g of fat.
  - Set up a vector equation for this problem. Include a statement of what the variables in your equation represent.
  - Write an equivalent matrix equation, and then determine if the desired mixture of the two cereals can be prepared.

Nutrition Information per Serving

Nutrient	General Mills Cheerios®	Quaker® 100% Natural Cereal
Calories	110	130
Protein (g)	4	3
Carbohydrate (g)	20	18
Fat (g)	2	5

- One serving of Shredded Wheat supplies 160 calories, 5 g of protein, 6 g of fiber, and 1 g of fat. One serving of Crispix® supplies 110 calories, 2 g of protein, .1 g of fiber, and .4 g of fat.
  - Set up a matrix  $B$  and a vector  $\mathbf{u}$  such that  $B\mathbf{u}$  gives the amounts of calories, protein, fiber, and fat contained in a mixture of three servings of Shredded Wheat and two servings of Crispix.
  - [M] Suppose that you want a cereal with more fiber than Crispix but fewer calories than Shredded Wheat. Is it possible for a mixture of the two cereals to supply 130 calories, 3.20 g of protein, 2.46 g of fiber, and .64 g of fat? If so, what is the mixture?

- After taking a nutrition class, a big Annie's® Mac and Cheese fan decides to improve the levels of protein and fiber in her favorite lunch by adding broccoli and canned chicken. The nutritional information for the foods referred to in this exercise are given in the table below.

Nutrition Information per Serving

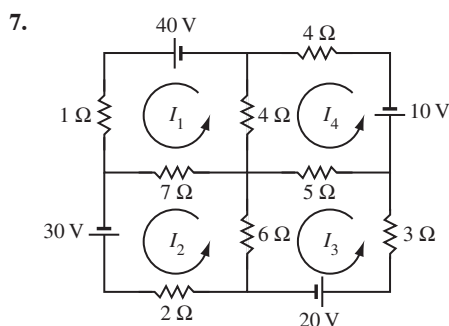
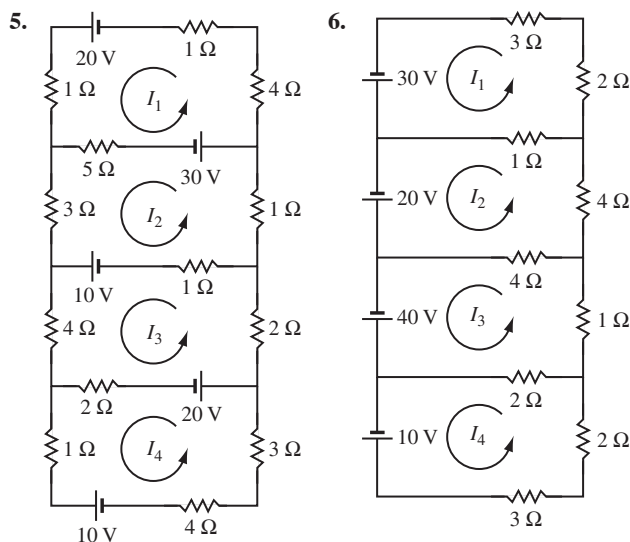
Nutrient	Mac and Cheese	Broccoli	Chicken	Shells
Calories	270	51	70	260
Protein (g)	10	5.4	15	9
Fiber (g)	2	5.2	0	5

- [M] If she wants to limit her lunch to 400 calories but get 30 g of protein and 10 g of fiber, what proportions of servings of Mac and Cheese, broccoli, and chicken should she use?
- [M] She found that there was too much broccoli in the proportions from part (a), so she decided to switch from

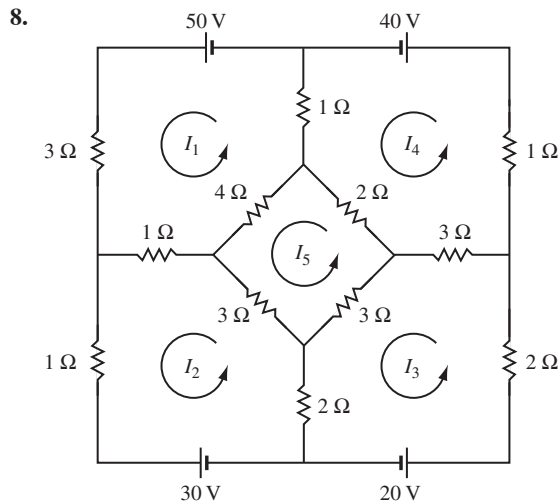
classical Mac and Cheese to Annie's® Whole Wheat Shells and White Cheddar. What proportions of servings of each food should she use to meet the same goals as in part (a)?

- The Cambridge Diet supplies .8 g of calcium per day, in addition to the nutrients listed in the Table 1 for Example 1. The amounts of calcium per unit (100 g) supplied by the three ingredients in the Cambridge Diet are as follows: 1.26 g from nonfat milk, .19 g from soy flour, and .8 g from whey. Another ingredient in the diet mixture is isolated soy protein, which provides the following nutrients in each unit: 80 g of protein, 0 g of carbohydrate, 3.4 g of fat, and .18 g of calcium.
  - Set up a matrix equation whose solution determines the amounts of nonfat milk, soy flour, whey, and isolated soy protein necessary to supply the precise amounts of protein, carbohydrate, fat, and calcium in the Cambridge Diet. State what the variables in the equation represent.
  - [M] Solve the equation in (a) and discuss your answer.

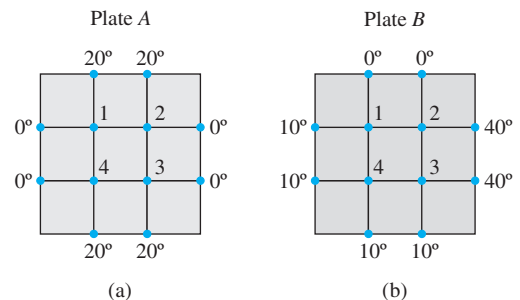
In Exercises 5–8, write a matrix equation that determines the loop currents. [M] If MATLAB or another matrix program is available, solve the system for the loop currents.







9. In a certain region, about 7% of a city's population moves to the surrounding suburbs each year, and about 5% of the suburban population moves into the city. In 2010, there were 800,000 residents in the city and 500,000 in the suburbs. Set up a difference equation that describes this situation, where  $\mathbf{x}_0$  is the initial population in 2010. Then estimate the populations in the city and in the suburbs two years later, in 2012. (Ignore other factors that might influence the population sizes.)
10. In a certain region, about 6% of a city's population moves to the surrounding suburbs each year, and about 4% of the suburban population moves into the city. In 2010, there were 10,000,000 residents in the city and 800,000 in the suburbs. Set up a difference equation that describes this situation, where  $\mathbf{x}_0$  is the initial population in 2010. Then estimate the populations in the city and in the suburbs two years later, in 2012.
11. In 1994, the population of California was 31,524,000, and the population living in the United States but *outside* California was 228,680,000. During the year, it is estimated that 516,100 persons moved from California to elsewhere in the United States, while 381,262 persons moved to California from elsewhere in the United States.<sup>4</sup>
- Set up the migration matrix for this situation, using five decimal places for the migration rates into and out of California. Let your work show how you produced the migration matrix.
  - [M] Compute the projected populations in the year 2000 for California and elsewhere in the United States, assuming that the migration rates did not change during the 6-year period. (These calculations do not take into account births, deaths, or the substantial migration of persons into California and elsewhere in the United States from other countries.)
12. [M] Budget<sup>®</sup> Rent A Car in Wichita, Kansas has a fleet of about 500 cars, at three locations. A car rented at one location may be returned to any of the three locations. The various fractions of cars returned to the three locations are shown in the matrix below. Suppose that on Monday there are 295 cars at the airport (or rented from there), 55 cars at the east side office, and 150 cars at the west side office. What will be the approximate distribution of cars on Wednesday?
- Cars Rented From:
- | Airport | East | West | Returned To: |
|---------|------|------|--------------|
| .97     | .05  | .10  | Airport      |
| .00     | .90  | .05  | East         |
| .03     | .05  | .85  | West         |
13. [M] Let  $M$  and  $\mathbf{x}_0$  be as in Example 3.
- Compute the population vectors  $\mathbf{x}_k$  for  $k = 1, \dots, 20$ . Discuss what you find.
  - Repeat part (a) with an initial population of 350,000 in the city and 650,000 in the suburbs. What do you find?
14. [M] Study how changes in boundary temperatures on a steel plate affect the temperatures at interior points on the plate.
- Begin by estimating the temperatures  $T_1, T_2, T_3, T_4$  at each of the sets of four points on the steel plate shown in the figure. In each case, the value of  $T_k$  is approximated by the average of the temperatures at the four closest points. See Exercises 33 and 34 in Section 1.1, where the values (in degrees) turn out to be (20, 27.5, 30, 22.5). How is this list of values related to your results for the points in set (a) and set (b)?
  - Without making any computations, guess the interior temperatures in (a) when the boundary temperatures are all multiplied by 3. Check your guess.
  - Finally, make a general conjecture about the correspondence from the list of eight boundary temperatures to the list of four interior temperatures.



<sup>4</sup>Migration data supplied by the Demographic Research Unit of the California State Department of Finance.

## 2 Matrix Algebra

### INTRODUCTORY EXAMPLE

#### Computer Models in Aircraft Design

To design the next generation of commercial and military aircraft, engineers at Boeing's Phantom Works use 3D modeling and computational fluid dynamics (CFD). They study the airflow around a virtual airplane to answer important design questions before physical models are created. This has drastically reduced design cycle times and cost—and linear algebra plays a crucial role in the process.

The virtual airplane begins as a mathematical “wire-frame” model that exists only in computer memory and on graphics display terminals. (A model of a Boeing 777 is shown.) This mathematical model organizes and influences each step of the design and manufacture of the airplane—both the exterior and interior. The CFD analysis concerns the exterior surface.

Although the finished skin of a plane may seem smooth, the geometry of the surface is complicated. In addition to wings and a fuselage, an aircraft has nacelles, stabilizers, slats, flaps, and ailerons. The way air flows around these structures determines how the plane moves through the sky. Equations that describe the airflow are complicated, and they must account for engine intake, engine exhaust, and the wakes left by the wings of the plane. To study the airflow, engineers need a highly refined description of the plane's surface.

A computer creates a model of the surface by first superimposing a three-dimensional grid of “boxes” on the

original wire-frame model. Boxes in this grid lie either completely inside or completely outside the plane, or they intersect the surface of the plane. The computer selects the boxes that intersect the surface and subdivides them, retaining only the smaller boxes that still intersect the surface. The subdividing process is repeated until the grid is extremely fine. A typical grid can include over 400,000 boxes.

The process for finding the airflow around the plane involves repeatedly solving a system of linear equations  $A\mathbf{x} = \mathbf{b}$  that may involve up to 2 million equations and variables. The vector  $\mathbf{b}$  changes each time, based on data from the grid and solutions of previous equations. Using the fastest computers available commercially, a Phantom Works team can spend from a few hours to several days setting up and solving a single airflow problem. After the team analyzes the solution, they may make small changes to the airplane surface and begin the whole process again. Thousands of CFD runs may be required.

This chapter presents two important concepts that assist in the solution of such massive systems of equations:

- *Partitioned matrices:* A typical CFD system of equations has a “sparse” coefficient matrix with mostly zero entries. Grouping the variables correctly leads to a partitioned matrix with many zero blocks. Section 2.4 introduces such matrices and describes some of their applications.



## NUMERICAL NOTES

1. The fastest way to obtain  $AB$  on a computer depends on the way in which the computer stores matrices in its memory. The standard high-performance algorithms, such as in LAPACK, calculate  $AB$  by columns, as in our definition of the product. (A version of LAPACK written in C++ calculates  $AB$  by rows.)
2. The definition of  $AB$  lends itself well to parallel processing on a computer. The columns of  $B$  are assigned individually or in groups to different processors, which independently and hence simultaneously compute the corresponding columns of  $AB$ .

## PRACTICE PROBLEMS

1. Since vectors in  $\mathbb{R}^n$  may be regarded as  $n \times 1$  matrices, the properties of transposes in Theorem 3 apply to vectors, too. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Compute  $(A\mathbf{x})^T$ ,  $\mathbf{x}^T A^T$ ,  $\mathbf{x}\mathbf{x}^T$ , and  $\mathbf{x}^T \mathbf{x}$ . Is  $A^T \mathbf{x}^T$  defined?

2. Let  $A$  be a  $4 \times 4$  matrix and let  $\mathbf{x}$  be a vector in  $\mathbb{R}^4$ . What is the fastest way to compute  $A^2 \mathbf{x}$ ? Count the multiplications.

## 2.1 EXERCISES

In Exercises 1 and 2, compute each matrix sum or product if it is defined. If an expression is undefined, explain why. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

1.  $-2A$ ,  $B - 2A$ ,  $AC$ ,  $CD$
2.  $A + 3B$ ,  $2C - 3E$ ,  $DB$ ,  $EC$

In the rest of this exercise set and in those to follow, assume that each matrix expression is defined. That is, the sizes of the matrices (and vectors) involved “match” appropriately.

3. Let  $A = \begin{bmatrix} 2 & -5 \\ 3 & -2 \end{bmatrix}$ . Compute  $3I_2 - A$  and  $(3I_2)A$ .

4. Compute  $A - 5I_3$  and  $(5I_3)A$ , where

$$A = \begin{bmatrix} 5 & -1 & 3 \\ -4 & 3 & -6 \\ -3 & 1 & 2 \end{bmatrix}.$$

In Exercises 5 and 6, compute the product  $AB$  in two ways: (a) by the definition, where  $A\mathbf{b}_1$  and  $A\mathbf{b}_2$  are computed separately, and (b) by the row-column rule for computing  $AB$ .

$$5. \quad A = \begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix}$$

$$6. \quad A = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$$

7. If a matrix  $A$  is  $5 \times 3$  and the product  $AB$  is  $5 \times 7$ , what is the size of  $B$ ?
8. How many rows does  $B$  have if  $BC$  is a  $5 \times 4$  matrix?
9. Let  $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix}$ . What value(s) of  $k$ , if any, will make  $AB = BA$ ?
10. Let  $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix}$ , and  $C = \begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix}$ . Verify that  $AB = AC$  and yet  $B \neq C$ .
11. Let  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$  and  $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ . Compute  $AD$  and  $DA$ . Explain how the columns or rows of  $A$  change when  $A$  is multiplied by  $D$  on the right or on the left. Find a  $3 \times 3$  matrix  $B$ , not the identity matrix or the zero matrix, such that  $AB = BA$ .
12. Let  $A = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$ . Construct a  $2 \times 2$  matrix  $B$  such that  $AB$  is the zero matrix. Use two different nonzero columns for  $B$ .

13. Let  $\mathbf{r}_1, \dots, \mathbf{r}_p$  be vectors in  $\mathbb{R}^n$ , and let  $Q$  be an  $m \times n$  matrix. Write the matrix  $[Q\mathbf{r}_1 \ \cdots \ Q\mathbf{r}_p]$  as a *product* of two matrices (neither of which is an identity matrix).
14. Let  $U$  be the  $3 \times 2$  cost matrix described in Example 6 in Section 1.8. The first column of  $U$  lists the costs per dollar of output for manufacturing product  $B$ , and the second column lists the costs per dollar of output for product  $C$ . (The costs are categorized as materials, labor, and overhead.) Let  $\mathbf{q}_1$  be a vector in  $\mathbb{R}^2$  that lists the output (measured in dollars) of products  $B$  and  $C$  manufactured during the first quarter of the year, and let  $\mathbf{q}_2, \mathbf{q}_3$ , and  $\mathbf{q}_4$  be the analogous vectors that list the amounts of products  $B$  and  $C$  manufactured in the second, third, and fourth quarters, respectively. Give an economic description of the data in the matrix  $UQ$ , where  $Q = [\mathbf{q}_1 \ \mathbf{q}_2 \ \mathbf{q}_3 \ \mathbf{q}_4]$ .
22. Show that if the columns of  $B$  are linearly dependent, then so are the columns of  $AB$ .
23. Suppose  $CA = I_n$  (the  $n \times n$  identity matrix). Show that the equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution. Explain why  $A$  cannot have more columns than rows.
24. Suppose  $A$  is a  $3 \times n$  matrix whose columns span  $\mathbb{R}^3$ . Explain how to construct an  $n \times 3$  matrix  $D$  such that  $AD = I_3$ .
25. Suppose  $A$  is an  $m \times n$  matrix and there exist  $n \times m$  matrices  $C$  and  $D$  such that  $CA = I_n$  and  $AD = I_m$ . Prove that  $m = n$  and  $C = D$ . [Hint: Think about the product  $CAD$ .]
26. Suppose  $AD = I_m$  (the  $m \times m$  identity matrix). Show that for any  $\mathbf{b}$  in  $\mathbb{R}^m$ , the equation  $A\mathbf{x} = \mathbf{b}$  has a solution. [Hint: Think about the equation  $AD\mathbf{b} = \mathbf{b}$ .] Explain why  $A$  cannot have more rows than columns.

Exercises 15 and 16 concern arbitrary matrices  $A$ ,  $B$ , and  $C$  for which the indicated sums and products are defined. Mark each statement True or False. Justify each answer.

15. a. If  $A$  and  $B$  are  $2 \times 2$  matrices with columns  $\mathbf{a}_1, \mathbf{a}_2$ , and  $\mathbf{b}_1, \mathbf{b}_2$ , respectively, then  $AB = [\mathbf{a}_1\mathbf{b}_1 \ \mathbf{a}_2\mathbf{b}_2]$ .  
 b. Each column of  $AB$  is a linear combination of the columns of  $B$  using weights from the corresponding column of  $A$ .  
 c.  $AB + AC = A(B + C)$   
 d.  $A^T + B^T = (A + B)^T$   
 e. The transpose of a product of matrices equals the product of their transposes in the same order.
16. a. The first row of  $AB$  is the first row of  $A$  multiplied on the right by  $B$ .  
 b. If  $A$  and  $B$  are  $3 \times 3$  matrices and  $B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3]$ , then  $AB = [A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3]$ .  
 c. If  $A$  is an  $n \times n$  matrix, then  $(A^2)^T = (A^T)^2$ .  
 d.  $(ABC)^T = C^T A^T B^T$   
 e. The transpose of a sum of matrices equals the sum of their transposes.
17. If  $A = \begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}$  and  $AB = \begin{bmatrix} -3 & -11 \\ 1 & 17 \end{bmatrix}$ , determine the first and second columns of  $B$ .
18. Suppose the third column of  $B$  is all zeros. What can be said about the third column of  $AB$ ?
19. Suppose the third column of  $B$  is the sum of the first two columns. What can be said about the third column of  $AB$ ? Why?
20. Suppose the first two columns,  $\mathbf{b}_1$  and  $\mathbf{b}_2$ , of  $B$  are equal. What can be said about the columns of  $AB$ ? Why?
21. Suppose the last column of  $AB$  is entirely zeros but  $B$  itself has no column of zeros. What can be said about the columns of  $A$ ?
27. Let  $\mathbf{u} = \begin{bmatrix} -3 \\ 2 \\ -5 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ . Compute  $\mathbf{u}^T \mathbf{v}$ ,  $\mathbf{v}^T \mathbf{u}$ ,  $\mathbf{u}\mathbf{v}^T$ , and  $\mathbf{v}\mathbf{u}^T$ .
28. If  $\mathbf{u}$  and  $\mathbf{v}$  are in  $\mathbb{R}^n$ , how are  $\mathbf{u}^T \mathbf{v}$  and  $\mathbf{v}^T \mathbf{u}$  related? How are  $\mathbf{u}\mathbf{v}^T$  and  $\mathbf{v}\mathbf{u}^T$  related?
29. Prove Theorem 2(b) and 2(c). Use the row-column rule. The  $(i, j)$ -entry in  $A(B + C)$  can be written as  $a_{i1}(b_{1j} + c_{1j}) + \cdots + a_{in}(b_{nj} + c_{nj})$  or  $\sum_{k=1}^n a_{ik}(b_{kj} + c_{kj})$
30. Prove Theorem 2(d). [Hint: The  $(i, j)$ -entry in  $(rA)B$  is  $(ra_{i1})b_{1j} + \cdots + (ra_{in})b_{nj}$ .]
31. Show that  $I_m A = A$  where  $A$  is an  $m \times n$  matrix. Assume  $I_m \mathbf{x} = \mathbf{x}$  for all  $\mathbf{x}$  in  $\mathbb{R}^m$ .
32. Show that  $AI_n = A$  when  $A$  is an  $m \times n$  matrix. [Hint: Use the (column) definition of  $AI_n$ .]
33. Prove Theorem 3(d). [Hint: Consider the  $j$ th row of  $(AB)^T$ .]
34. Give a formula for  $(AB\mathbf{x})^T$ , where  $\mathbf{x}$  is a vector and  $A$  and  $B$  are matrices of appropriate sizes.
35. [M] Read the documentation for your matrix program, and write the commands that will produce the following matrices (without keying in each entry of the matrix).  
 a. A  $4 \times 5$  matrix of zeros  
 b. A  $5 \times 3$  matrix of ones  
 c. The  $5 \times 5$  identity matrix  
 d. A  $4 \times 4$  diagonal matrix, with diagonal entries 3, 4, 2, 5

A useful way to test new ideas in matrix algebra, or to make conjectures, is to make calculations with matrices selected at random. Checking a property for a few matrices does not prove that the property holds in general, but it makes the property more believable. Also, if the property is actually false, making a few calculations may help to discover this.

36. [M] Write the command(s) that will create a  $5 \times 6$  matrix with random entries. In what range of numbers do the entries lie? Tell how to create a  $4 \times 4$  matrix with random integer entries between  $-9$  and  $9$ . [Hint: If  $x$  is a random number such that  $0 < x < 1$ , then  $-9.5 < 19(x - .5) < 9.5$ .]
37. [M] Construct random  $4 \times 4$  matrices  $A$  and  $B$  to test whether  $AB = BA$ . The best way to do this is to compute  $AB - BA$  and check whether this difference is the zero matrix. Then test  $AB - BA$  for three more pairs of random  $4 \times 4$  matrices. Report your conclusions.
38. [M] Construct a random  $5 \times 5$  matrix  $A$  and test whether  $(A + I)(A - I) = A^2 - I$ . The best way to do this is to compute  $(A + I)(A - I) - (A^2 - I)$  and verify that this difference is the zero matrix. Do this for three random matrices. Then test  $(A + B)(A - B) = A^2 - B^2$  the same

way for three pairs of random  $4 \times 4$  matrices. Report your conclusions.

39. [M] Use at least three pairs of random  $4 \times 4$  matrices  $A$  and  $B$  to test the equalities  $(A + B)^T = A^T + B^T$  and  $(AB)^T = B^T A^T$ , as well as  $(AB)^T = A^T B^T$ . (See Exercise 37.) Report your conclusions. [Note: Most matrix programs use  $A'$  for  $A^T$ .]

40. [M] Let

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Compute  $S^k$  for  $k = 2, \dots, 6$ .

41. [M] Describe in words what happens when  $A^5$ ,  $A^{10}$ ,  $A^{20}$ , and  $A^{30}$  are computed for

$$A = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1/3 & 1/6 \\ 1/4 & 1/6 & 7/12 \end{bmatrix}$$

### SOLUTIONS TO PRACTICE PROBLEMS

1.  $A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$ . So  $(A\mathbf{x})^T = \begin{bmatrix} -4 & 2 \end{bmatrix}$ . Also,

$$\mathbf{x}^T A^T = \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 2 \end{bmatrix}.$$

The quantities  $(A\mathbf{x})^T$  and  $\mathbf{x}^T A^T$  are equal, by Theorem 3(d). Next,

$$\mathbf{x}\mathbf{x}^T = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \begin{bmatrix} 5 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 15 \\ 15 & 9 \end{bmatrix}$$

$$\mathbf{x}^T \mathbf{x} = \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = [25 + 9] = 34$$

A  $1 \times 1$  matrix such as  $\mathbf{x}^T \mathbf{x}$  is usually written without the brackets. Finally,  $A^T \mathbf{x}^T$  is not defined, because  $\mathbf{x}^T$  does not have two rows to match the two columns of  $A^T$ .

2. The fastest way to compute  $A^2 \mathbf{x}$  is to compute  $A(A\mathbf{x})$ . The product  $A\mathbf{x}$  requires 16 multiplications, 4 for each entry, and  $A(A\mathbf{x})$  requires 16 more. In contrast, the product  $A^2$  requires 64 multiplications, 4 for each of the 16 entries in  $A^2$ . After that,  $A^2 \mathbf{x}$  takes 16 more multiplications, for a total of 80.

## 2.2 THE INVERSE OF A MATRIX

Matrix algebra provides tools for manipulating matrix equations and creating various useful formulas in ways similar to doing ordinary algebra with real numbers. This section investigates the matrix analogue of the reciprocal, or multiplicative inverse, of a nonzero number.

## WEB

## NUMERICAL NOTE

In practical work,  $A^{-1}$  is seldom computed, unless the entries of  $A^{-1}$  are needed. Computing both  $A^{-1}$  and  $A^{-1}\mathbf{b}$  takes about three times as many arithmetic operations as solving  $A\mathbf{x} = \mathbf{b}$  by row reduction, and row reduction may be more accurate.

## PRACTICE PROBLEMS

1. Use determinants to determine which of the following matrices are invertible.

a.  $\begin{bmatrix} 3 & -9 \\ 2 & 6 \end{bmatrix}$       b.  $\begin{bmatrix} 4 & -9 \\ 0 & 5 \end{bmatrix}$       c.  $\begin{bmatrix} 6 & -9 \\ -4 & 6 \end{bmatrix}$

2. Find the inverse of the matrix  $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$ , if it exists.

## 2.2 EXERCISES

Find the inverses of the matrices in Exercises 1–4.

1.  $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$       2.  $\begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$

3.  $\begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix}$       4.  $\begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix}$

5. Use the inverse found in Exercise 1 to solve the system

$$8x_1 + 6x_2 = -9$$

$$5x_1 + 4x_2 = -1$$

6. Use the inverse found in Exercise 3 to solve the system

$$7x_1 + 3x_2 = -9$$

$$-6x_1 - 3x_2 = 4$$

7. Let  $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$ ,  $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ ,  $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ ,  $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ , and  $\mathbf{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ .

- a. Find  $A^{-1}$ , and use it to solve the four equations

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \quad A\mathbf{x} = \mathbf{b}_3, \quad A\mathbf{x} = \mathbf{b}_4$$

- b. The four equations in part (a) can be solved by the *same* set of row operations, since the coefficient matrix is the same in each case. Solve the four equations in part (a) by row reducing the augmented matrix  $[A \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4]$ .

8. Suppose  $P$  is invertible and  $A = PBP^{-1}$ . Solve for  $B$  in terms of  $A$ .

In Exercises 9 and 10, mark each statement True or False. Justify each answer.

9. a. In order for a matrix  $B$  to be the inverse of  $A$ , the equations  $AB = I$  and  $BA = I$  must both be true.  
b. If  $A$  and  $B$  are  $n \times n$  and invertible, then  $A^{-1}B^{-1}$  is the inverse of  $AB$ .  
c. If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $ab - cd \neq 0$ , then  $A$  is invertible.  
d. If  $A$  is an invertible  $n \times n$  matrix, then the equation  $A\mathbf{x} = \mathbf{b}$  is consistent for *each*  $\mathbf{b}$  in  $\mathbb{R}^n$ .  
e. Each elementary matrix is invertible.
10. a. If  $A$  is invertible, then elementary row operations that reduce  $A$  to the identity  $I_n$  also reduce  $A^{-1}$  to  $I_n$ .  
b. If  $A$  is invertible, then the inverse of  $A^{-1}$  is  $A$  itself.  
c. A product of invertible  $n \times n$  matrices is invertible, and the inverse of the product is the product of their inverses in the same order.  
d. If  $A$  is an  $n \times n$  matrix and  $A\mathbf{x} = \mathbf{e}_j$  is consistent for every  $j \in \{1, 2, \dots, n\}$ , then  $A$  is invertible. Note:  $\mathbf{e}_1, \dots, \mathbf{e}_n$  represent the columns of the identity matrix.  
e. If  $A$  can be row reduced to the identity matrix, then  $A$  must be invertible.
11. Let  $A$  be an invertible  $n \times n$  matrix, and let  $B$  be an  $n \times p$  matrix. Show that the equation  $AX = B$  has a unique solution  $A^{-1}B$ .
12. Use matrix algebra to show that if  $A$  is invertible and  $D$  satisfies  $AD = I$ , then  $D = A^{-1}$ .
13. Suppose  $AB = AC$ , where  $B$  and  $C$  are  $n \times p$  matrices and  $A$  is invertible. Show that  $B = C$ . Is this true, in general, when  $A$  is not invertible?



14. Suppose  $(B - C)D = 0$ , where  $B$  and  $C$  are  $m \times n$  matrices and  $D$  is invertible. Show that  $B = C$ .

15. Let  $A$  be an invertible  $n \times n$  matrix, and let  $B$  be an  $n \times p$  matrix. Explain why  $A^{-1}B$  can be computed by row reduction:

$$\text{If } [A \ B] \sim \cdots \sim [I \ X], \text{ then } X = A^{-1}B.$$

If  $A$  is larger than  $2 \times 2$ , then row reduction of  $[A \ B]$  is much faster than computing both  $A^{-1}$  and  $A^{-1}B$ .

16. Suppose  $A$  and  $B$  are  $n \times n$  matrices,  $B$  is invertible, and  $AB$  is invertible. Show that  $A$  is invertible. [Hint: Let  $C = AB$ , and solve this equation for  $A$ .]

17. Suppose  $A$ ,  $B$ , and  $C$  are invertible  $n \times n$  matrices. Show that  $ABC$  is also invertible by producing a matrix  $D$  such that  $(ABC)D = I$  and  $D(ABC) = I$ .

18. Solve the equation  $AB = BC$  for  $A$ , assuming that  $A$ ,  $B$ , and  $C$  are square and  $B$  is invertible.

19. If  $A$ ,  $B$ , and  $C$  are  $n \times n$  invertible matrices, does the equation  $C^{-1}(A + X)B^{-1} = I_n$  have a solution,  $X$ ? If so, find it.

20. Suppose  $A$ ,  $B$ , and  $X$  are  $n \times n$  matrices with  $A$ ,  $X$ , and  $A - AX$  invertible, and suppose

$$(A - AX)^{-1} = X^{-1}B \quad (3)$$

- Explain why  $B$  is invertible.
- Solve equation (3) for  $X$ . If a matrix needs to be inverted, explain why that matrix is invertible.

21. Explain why the columns of an  $n \times n$  matrix  $A$  are linearly independent when  $A$  is invertible.

22. Explain why the columns of an  $n \times n$  matrix  $A$  span  $\mathbb{R}^n$  when  $A$  is invertible. [Hint: Review Theorem 4 in Section 1.4.]

23. Suppose  $A$  is  $n \times n$  and the equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution. Explain why  $A$  has  $n$  pivot columns and  $A$  is row equivalent to  $I_n$ . By Theorem 7, this shows that  $A$  must be invertible. (This exercise and Exercise 24 will be cited in Section 2.3.)

24. Suppose  $A$  is  $n \times n$  and the equation  $A\mathbf{x} = \mathbf{b}$  has a solution for each  $\mathbf{b}$  in  $\mathbb{R}^n$ . Explain why  $A$  must be invertible. [Hint: Is  $A$  row equivalent to  $I_n$ ?]

Exercises 25 and 26 prove Theorem 4 for  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

25. Show that if  $ad - bc = 0$ , then the equation  $A\mathbf{x} = \mathbf{0}$  has more than one solution. Why does this imply that  $A$  is not invertible? [Hint: First, consider  $a = b = 0$ . Then, if  $a$  and  $b$  are not both zero, consider the vector  $\mathbf{x} = \begin{bmatrix} -b \\ a \end{bmatrix}$ .]

26. Show that if  $ad - bc \neq 0$ , the formula for  $A^{-1}$  works.

Exercises 27 and 28 prove special cases of the facts about elementary matrices stated in the box following Example 5. Here  $A$  is a  $3 \times 3$  matrix and  $I = I_3$ . (A general proof would require slightly more notation.)

27. Let  $A$  be a  $3 \times 3$  matrix.

- Use equation (2) from Section 2.1 to show that  $\text{row}_i(A) = \text{row}_i(I) \cdot A$ , for  $i = 1, 2, 3$ .
- Show that if rows 1 and 2 of  $A$  are interchanged, then the result may be written as  $EA$ , where  $E$  is an elementary matrix formed by interchanging rows 1 and 2 of  $I$ .
- Show that if row 3 of  $A$  is multiplied by 5, then the result may be written as  $EA$ , where  $E$  is formed by multiplying row 3 of  $I$  by 5.

28. Suppose row 2 of  $A$  is replaced by  $\text{row}_2(A) - 3 \cdot \text{row}_1(A)$ . Show that the result is  $EA$ , where  $E$  is formed from  $I$  by replacing  $\text{row}_2(I)$  by  $\text{row}_2(I) - 3 \cdot \text{row}_1(I)$ .

Find the inverses of the matrices in Exercises 29–32, if they exist. Use the algorithm introduced in this section.

29.  $\begin{bmatrix} 1 & -3 \\ 4 & -9 \end{bmatrix}$

30.  $\begin{bmatrix} 3 & 6 \\ 4 & 7 \end{bmatrix}$

31.  $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

32.  $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}$

33. Use the algorithm from this section to find the inverses of

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let  $A$  be the corresponding  $n \times n$  matrix, and let  $B$  be its inverse. Guess the form of  $B$ , and then show that  $AB = I$ .

34. Repeat the strategy of Exercise 33 to guess the inverse  $B$  of

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2 & 2 & 0 & & 0 \\ 3 & 3 & 3 & & 0 \\ \vdots & & & \ddots & \vdots \\ n & n & n & \cdots & n \end{bmatrix}.$$

Show that  $AB = I$ .

35. Let  $A = \begin{bmatrix} -1 & -7 & -3 \\ 2 & 15 & 6 \\ 1 & 3 & 2 \end{bmatrix}$ . Find the third column of  $A^{-1}$  without computing the other columns.

36. [M] Let  $A = \begin{bmatrix} -25 & -9 & -27 \\ 536 & 185 & 537 \\ 154 & 52 & 143 \end{bmatrix}$ . Find the second and third columns of  $A^{-1}$  without computing the first column.

37. Let  $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}$ . Construct a  $2 \times 3$  matrix  $C$  (by trial and error) using only 1,  $-1$ , and 0 as entries, such that  $CA = I_2$ . Compute  $AC$  and note that  $AC \neq I_3$ .

38. Let  $A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$ . Construct a  $4 \times 2$  matrix

$D$  using only 1 and 0 as entries, such that  $AD = I_2$ . Is it possible that  $CA = I_4$  for some  $4 \times 2$  matrix  $C$ ? Why or why not?

39. [M] Let

$$D = \begin{bmatrix} .011 & .003 & .001 \\ .003 & .009 & .003 \\ .001 & .003 & .011 \end{bmatrix}$$

be a flexibility matrix, with flexibility measured in inches per pound. Suppose that forces of 40, 50, and 30 lb are applied at points 1, 2, and 3, respectively, in Fig. 1 of Example 3. Find the corresponding deflections.

40. [M] Compute the stiffness matrix  $D^{-1}$  for  $D$  in Exercise 39. List the forces needed to produce a deflection of .04 in. at point 3, with zero deflections at the other points.

41. [M] Let

$$D = \begin{bmatrix} .0130 & .0050 & .0020 & .0010 \\ .0050 & .0100 & .0040 & .0020 \\ .0020 & .0040 & .0100 & .0050 \\ .0010 & .0020 & .0050 & .0130 \end{bmatrix}$$

be a flexibility matrix for an elastic beam such as the one in Example 3, with four points at which force is applied. Units are centimeters per newton of force. Measurements at the four points show deflections of .07, .12, .16, and .12 cm. Determine the forces at the four points.

42. [M] With  $D$  as in Exercise 41, determine the forces that produce a deflection of .22 cm at the second point on the beam, with zero deflections at the other three points. How is the answer related to the entries in  $D^{-1}$ ? [Hint: First answer the question when the deflection is 1 cm at the second point.]

### SOLUTIONS TO PRACTICE PROBLEMS

1. a.  $\det \begin{bmatrix} 3 & -9 \\ 2 & 6 \end{bmatrix} = 3 \cdot 6 - (-9) \cdot 2 = 18 + 18 = 36$ . The determinant is nonzero, so the matrix is invertible.

b.  $\det \begin{bmatrix} 4 & -9 \\ 0 & 5 \end{bmatrix} = 4 \cdot 5 - (-9) \cdot 0 = 20 \neq 0$ . The matrix is invertible.

c.  $\det \begin{bmatrix} 6 & -9 \\ -4 & 6 \end{bmatrix} = 6 \cdot 6 - (-9)(-4) = 36 - 36 = 0$ . The matrix is not invertible.

$$\begin{aligned} 2. [A \ I] &\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{bmatrix} \end{aligned}$$

So  $[A \ I]$  is row equivalent to a matrix of the form  $[B \ D]$ , where  $B$  is square and has a row of zeros. Further row operations will not transform  $B$  into  $I$ , so we stop.  $A$  does not have an inverse.

## 2.3 CHARACTERIZATIONS OF INVERTIBLE MATRICES

This section provides a review of most of the concepts introduced in Chapter 1, in relation to systems of  $n$  linear equations in  $n$  unknowns and to *square* matrices. The main result is Theorem 8.



## 2.3 EXERCISES

Unless otherwise specified, assume that all matrices in these exercises are  $n \times n$ . Determine which of the matrices in Exercises 1–10 are invertible. Use as few calculations as possible. Justify your answers.

1.  $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$
2.  $\begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix}$
3.  $\begin{bmatrix} 3 & 0 & 0 \\ -3 & -4 & 0 \\ 8 & 5 & -3 \end{bmatrix}$
4.  $\begin{bmatrix} -5 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix}$
5.  $\begin{bmatrix} 3 & 0 & -3 \\ 2 & 0 & 4 \\ -4 & 0 & 7 \end{bmatrix}$
6.  $\begin{bmatrix} 1 & -3 & -6 \\ 0 & 4 & 3 \\ -3 & 6 & 0 \end{bmatrix}$
7.  $\begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$
8.  $\begin{bmatrix} 3 & 4 & 7 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
9. [M]  $\begin{bmatrix} 4 & 0 & -3 & -7 \\ -6 & 9 & 9 & 9 \\ 7 & -5 & 10 & 19 \\ -1 & 2 & 4 & -1 \end{bmatrix}$
10. [M]  $\begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 6 & 4 & 2 & 8 & -8 \\ 7 & 5 & 3 & 10 & 9 \\ 9 & 6 & 4 & -9 & -5 \\ 8 & 5 & 2 & 11 & 4 \end{bmatrix}$

In Exercises 11 and 12, the matrices are all  $n \times n$ . Each part of the exercises is an *implication* of the form “If (statement 1), then (statement 2).” Mark an implication as True if the truth of (statement 2) *always* follows whenever (statement 1) happens to be true. An implication is False if there is an instance in which (statement 2) is false but (statement 1) is true. Justify each answer.

11. a. If the equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution, then  $A$  is row equivalent to the  $n \times n$  identity matrix.  
 b. If the columns of  $A$  span  $\mathbb{R}^n$ , then the columns are linearly independent.  
 c. If  $A$  is an  $n \times n$  matrix, then the equation  $A\mathbf{x} = \mathbf{b}$  has at least one solution for each  $\mathbf{b}$  in  $\mathbb{R}^n$ .  
 d. If the equation  $A\mathbf{x} = \mathbf{0}$  has a nontrivial solution, then  $A$  has fewer than  $n$  pivot positions.  
 e. If  $A^T$  is not invertible, then  $A$  is not invertible.
12. a. If there is an  $n \times n$  matrix  $D$  such that  $AD = I$ , then  $DA = I$ .  
 b. If the linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  maps  $\mathbb{R}^n$  into  $\mathbb{R}^n$ , then the row reduced echelon form of  $A$  is  $I$ .  
 c. If the columns of  $A$  are linearly independent, then the columns of  $A$  span  $\mathbb{R}^n$ .  
 d. If the equation  $A\mathbf{x} = \mathbf{b}$  has at least one solution for each  $\mathbf{b}$  in  $\mathbb{R}^n$ , then the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is not one-to-one.  
 e. If there is a  $\mathbf{b}$  in  $\mathbb{R}^n$  such that the equation  $A\mathbf{x} = \mathbf{b}$  is consistent, then the solution is unique.
13. An  $m \times n$  **upper triangular matrix** is one whose entries *below* the main diagonal are 0's (as in Exercise 8). When is a square upper triangular matrix invertible? Justify your answer.
14. An  $m \times n$  **lower triangular matrix** is one whose entries *above* the main diagonal are 0's (as in Exercise 3). When is a square lower triangular matrix invertible? Justify your answer.
15. Is it possible for a  $4 \times 4$  matrix to be invertible when its columns do not span  $\mathbb{R}^4$ ? Why or why not?
16. If an  $n \times n$  matrix  $A$  is invertible, then the columns of  $A^T$  are linearly independent. Explain why.
17. Can a square matrix with two identical columns be invertible? Why or why not?
18. Can a square matrix with two identical rows be invertible? Why or why not?
19. If the columns of a  $7 \times 7$  matrix  $D$  are linearly independent, what can be said about the solutions of  $D\mathbf{x} = \mathbf{b}$ ? Why?
20. If  $A$  is a  $5 \times 5$  matrix and the equation  $A\mathbf{x} = \mathbf{b}$  is consistent for every  $\mathbf{b}$  in  $\mathbb{R}^5$ , is it possible that for some  $\mathbf{b}$ , the equation  $A\mathbf{x} = \mathbf{b}$  has more than one solution? Why or why not?
21. If the equation  $C\mathbf{u} = \mathbf{v}$  has more than one solution for some  $\mathbf{v}$  in  $\mathbb{R}^n$ , can the columns of the  $n \times n$  matrix  $C$  span  $\mathbb{R}^n$ ? Why or why not?
22. If  $n \times n$  matrices  $E$  and  $F$  have the property that  $EF = I$ , then  $E$  and  $F$  commute. Explain why.
23. Assume that  $F$  is an  $n \times n$  matrix. If the equation  $F\mathbf{x} = \mathbf{y}$  is inconsistent for some  $\mathbf{y}$  in  $\mathbb{R}^n$ , what can you say about the equation  $F\mathbf{x} = \mathbf{0}$ ? Why?
24. If an  $n \times n$  matrix  $G$  cannot be row reduced to  $I_n$ , what can you say about the columns of  $G$ ? Why?
25. Verify the boxed statement preceding Example 1.
26. Explain why the columns of  $A^2$  span  $\mathbb{R}^n$  whenever the columns of an  $n \times n$  matrix  $A$  are linearly independent.
27. Let  $A$  and  $B$  be  $n \times n$  matrices. Show that if  $AB$  is invertible, so is  $A$ . You cannot use Theorem 6(b), because you cannot *assume* that  $A$  and  $B$  are invertible. [Hint: There is a matrix  $W$  such that  $ABW = I$ . Why?]
28. Let  $A$  and  $B$  be  $n \times n$  matrices. Show that if  $AB$  is invertible, so is  $B$ .
29. If  $A$  is an  $n \times n$  matrix and the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is one-to-one, what else can you say about this transformation? Justify your answer.

30. If  $A$  is an  $n \times n$  matrix and the equation  $A\mathbf{x} = \mathbf{b}$  has more than one solution for some  $\mathbf{b}$ , then the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is not one-to-one. What else can you say about this transformation? Justify your answer.
31. Suppose  $A$  is an  $n \times n$  matrix with the property that the equation  $A\mathbf{x} = \mathbf{b}$  has at least one solution for each  $\mathbf{b}$  in  $\mathbb{R}^n$ . Without using Theorems 5 or 8, explain why each equation  $A\mathbf{x} = \mathbf{b}$  has in fact exactly one solution.
32. Suppose  $A$  is an  $n \times n$  matrix with the property that the equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution. Without using the Invertible Matrix Theorem, explain directly why the equation  $A\mathbf{x} = \mathbf{b}$  must have a solution for each  $\mathbf{b}$  in  $\mathbb{R}^n$ .

In Exercises 33 and 34,  $T$  is a linear transformation from  $\mathbb{R}^2$  into  $\mathbb{R}^2$ . Show that  $T$  is invertible and find a formula for  $T^{-1}$ .

33.  $T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2)$
34.  $T(x_1, x_2) = (2x_1 - 8x_2, -2x_1 + 7x_2)$
35. Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an invertible linear transformation. Explain why  $T$  is both one-to-one and onto  $\mathbb{R}^n$ . Use equations (1) and (2). Then give a second explanation using one or more theorems.
36. Suppose a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  has the property that  $T(\mathbf{u}) = T(\mathbf{v})$  for some pair of distinct vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^n$ . Can  $T$  map  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ ? Why or why not?
37. Suppose  $T$  and  $U$  are linear transformations from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  such that  $T(U(\mathbf{x})) = \mathbf{x}$  for all  $\mathbf{x}$  in  $\mathbb{R}^n$ . Is it true that  $U(T(\mathbf{x})) = \mathbf{x}$  for all  $\mathbf{x}$  in  $\mathbb{R}^n$ ? Why or why not?
38. Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be an invertible linear transformation, and let  $S$  and  $U$  be functions from  $\mathbb{R}^n$  into  $\mathbb{R}^n$  such that  $S(T(\mathbf{x})) = \mathbf{x}$  and  $U(T(\mathbf{x})) = \mathbf{x}$  for all  $\mathbf{x}$  in  $\mathbb{R}^n$ . Show that  $U(\mathbf{v}) = S(\mathbf{v})$  for all  $\mathbf{v}$  in  $\mathbb{R}^n$ . This will show that  $T$  has a unique inverse, as asserted in Theorem 9. [Hint: Given any  $\mathbf{v}$  in  $\mathbb{R}^n$ , we can write  $\mathbf{v} = T(\mathbf{x})$  for some  $\mathbf{x}$ . Why? Compute  $S(\mathbf{v})$  and  $U(\mathbf{v})$ .]
39. Let  $T$  be a linear transformation that maps  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ . Show that  $T^{-1}$  exists and maps  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ . Is  $T^{-1}$  also one-to-one?
40. Suppose  $T$  and  $S$  satisfy the invertibility equations (1) and (2), where  $T$  is a linear transformation. Show directly that  $S$  is a linear transformation. [Hint: Given  $\mathbf{u}, \mathbf{v}$  in  $\mathbb{R}^n$ , let  $\mathbf{x} = S(\mathbf{u})$ ,  $\mathbf{y} = S(\mathbf{v})$ . Then  $T(\mathbf{x}) = \mathbf{u}$ ,  $T(\mathbf{y}) = \mathbf{v}$ . Why? Apply  $S$  to both sides of the equation  $T(\mathbf{x}) + T(\mathbf{y}) = T(\mathbf{x} + \mathbf{y})$ . Also, consider  $T(c\mathbf{x}) = cT(\mathbf{x})$ .]

41. [M] Suppose an experiment leads to the following system of equations:

$$\begin{aligned} 4.5x_1 + 3.1x_2 &= 19.249 \\ 1.6x_1 + 1.1x_2 &= 6.843 \end{aligned} \quad (3)$$

- a. Solve system (3), and then solve system (4), below, in which the data on the right have been rounded to two decimal places. In each case, find the *exact* solution.

$$\begin{aligned} 4.5x_1 + 3.1x_2 &= 19.25 \\ 1.6x_1 + 1.1x_2 &= 6.84 \end{aligned} \quad (4)$$

- b. The entries in system (4) differ from those in system (3) by less than .05%. Find the percentage error when using the solution of (4) as an approximation for the solution of (3).
- c. Use a matrix program to produce the condition number of the coefficient matrix in (3).

Exercises 42–44 show how to use the condition number of a matrix  $A$  to estimate the accuracy of a computed solution of  $A\mathbf{x} = \mathbf{b}$ . If the entries of  $A$  and  $\mathbf{b}$  are accurate to about  $r$  significant digits and if the condition number of  $A$  is approximately  $10^k$  (with  $k$  a positive integer), then the computed solution of  $A\mathbf{x} = \mathbf{b}$  should usually be accurate to at least  $r - k$  significant digits.

42. [M] Let  $A$  be the matrix in Exercise 9. Find the condition number of  $A$ . Construct a random vector  $\mathbf{x}$  in  $\mathbb{R}^4$  and compute  $\mathbf{b} = A\mathbf{x}$ . Then use a matrix program to compute the solution  $\mathbf{x}_1$  of  $A\mathbf{x} = \mathbf{b}$ . To how many digits do  $\mathbf{x}$  and  $\mathbf{x}_1$  agree? Find out the number of digits the matrix program stores accurately, and report how many digits of accuracy are lost when  $\mathbf{x}_1$  is used in place of the exact solution  $\mathbf{x}$ .
43. [M] Repeat Exercise 42 for the matrix in Exercise 10.
44. [M] Solve an equation  $A\mathbf{x} = \mathbf{b}$  for a suitable  $\mathbf{b}$  to find the last column of the inverse of the *fifth-order Hilbert matrix*

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{bmatrix}$$

How many digits in each entry of  $\mathbf{x}$  do you expect to be correct? Explain. [Note: The exact solution is (630, -12600, 56700, -88200, 44100).]

45. [M] Some matrix programs, such as MATLAB, have a command to create Hilbert matrices of various sizes. If possible, use an inverse command to compute the inverse of a twelfth-order or larger Hilbert matrix,  $A$ . Compute  $AA^{-1}$ . Report what you find.

# 3

## Determinants

### WEB

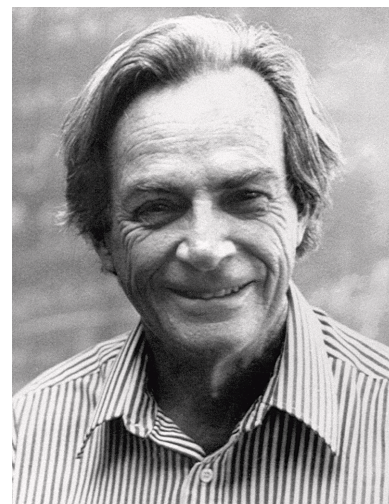
#### INTRODUCTORY EXAMPLE

### Random Paths and Distortion

In his autobiographical book “Surely You’re Joking, Mr. Feynman,” the Nobel Prize–winning physicist Richard Feynman tells of observing ants in his Princeton graduate school apartment. He studied the ants’ behavior by providing paper ferries to sugar suspended on a string where the ants would not accidentally find it. When an ant would step onto a paper ferry, Feynman would transport the ant to the food and then back. After the ants learned to use the ferry, he relocated the return landing. The colony soon confused the outbound and return ferry landings, indicating that their “learning” consisted of creating and following trails. Feynman confirmed this conjecture by laying glass slides on the floor. Once the ants established trails on the glass slides, he rearranged the slides and therefore the trails on them. The ants followed the repositioned trails and Feynman could direct the ants where he wished.

Suppose Feynman had decided to conduct additional investigations using a globe built of wire mesh on which an ant must follow individual wires and choose between going left and right at each intersection. If several ants and an equal number of food sources are placed on the globe, how likely is it that each ant would find its own food source rather than encountering another ant’s trail and following it to a shared resource?<sup>1</sup>

<sup>1</sup> The solution to the ant-path problem (and two other applications) can be found in a June 2005, *Mathematical Monthly* article by Arthur Benjamin and Naomi Cameron.



In order to record the actual routes of the ants and to communicate the results to others, it is convenient to use a rectangular map of the globe. There are many ways to create such maps. One simple way is to use the longitude and latitude on the globe as  $x$  and  $y$  coordinates on the map. As is the case with all maps, the result is not a faithful representation of the globe. Features near the “equator” look much the same on the globe and the map, but regions near the “poles” of the globe are distorted. Images of polar regions are much larger than the images of similar sized regions near the equator. To fit in with its surroundings on the map, the image of an ant near one of the poles should be larger than one near the equator. How much larger?

Surprisingly, both the ant-path and the area distortion problems are best answered through the use of the determinant, the subject of this chapter. Indeed, the determinant has so many uses that a summary of the applications known in the early 1900’s filled a four volume treatise by Thomas Muir. With changes in emphasis and the greatly increased sizes of the matrices used in modern applications, many uses that were important then are no longer critical today. Nevertheless, the determinant still plays an important role.

Henceforth we will omit the zero terms in the cofactor expansion. Next, expand this  $4 \times 4$  determinant down the first column, in order to take advantage of the zeros there. We have

$$\det A = 3 \cdot 2 \cdot \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

This  $3 \times 3$  determinant was computed in Example 1 and found to equal  $-2$ . Hence  $\det A = 3 \cdot 2 \cdot (-2) = -12$ . ■

The matrix in Example 3 was nearly triangular. The method in that example is easily adapted to prove the following theorem.

## THEOREM 2

If  $A$  is a triangular matrix, then  $\det A$  is the product of the entries on the main diagonal of  $A$ .

The strategy in Example 3 of looking for zeros works extremely well when an entire row or column consists of zeros. In such a case, the cofactor expansion along such a row or column is a sum of zeros! So the determinant is zero. Unfortunately, most cofactor expansions are not so quickly evaluated.

### NUMERICAL NOTE

By today's standards, a  $25 \times 25$  matrix is small. Yet it would be impossible to calculate a  $25 \times 25$  determinant by cofactor expansion. In general, a cofactor expansion requires over  $n!$  multiplications, and  $25!$  is approximately  $1.5 \times 10^{25}$ .

If a computer performs one trillion multiplications per second, it would have to run for over 500,000 years to compute a  $25 \times 25$  determinant by this method. Fortunately, there are faster methods, as we'll soon discover.

Exercises 19–38 explore important properties of determinants, mostly for the  $2 \times 2$  case. The results from Exercises 33–36 will be used in the next section to derive the analogous properties for  $n \times n$  matrices.

### PRACTICE PROBLEM

Compute  $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$ .

## 3.1 EXERCISES

Compute the determinants in Exercises 1–8 using a cofactor expansion across the first row. In Exercises 1–4, also compute the determinant by a cofactor expansion down the second column.

1.  $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$

2.  $\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}$

3.  $\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix}$

5.  $\begin{vmatrix} 2 & 3 & -4 \\ 4 & 0 & 5 \\ 5 & 1 & 6 \end{vmatrix}$

4.  $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix}$

6.  $\begin{vmatrix} 5 & -2 & 4 \\ 0 & 3 & -5 \\ 2 & -4 & 7 \end{vmatrix}$

$$7. \begin{vmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{vmatrix}$$

$$8. \begin{vmatrix} 8 & 1 & 6 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{vmatrix}$$

Compute the determinants in Exercises 9–14 by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

$$9. \begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix}$$

$$10. \begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix}$$

$$11. \begin{vmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

$$12. \begin{vmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 5 & -8 & 4 & -3 \end{vmatrix}$$

$$13. \begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$$

$$14. \begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$$

The expansion of a  $3 \times 3$  determinant can be remembered by the following device. Write a second copy of the first two columns to the right of the matrix, and compute the determinant by multiplying entries on six diagonals:

Add the downward diagonal products and subtract the upward products. Use this method to compute the determinants in Exercises 15–18. **Warning:** This trick does not generalize in any reasonable way to  $4 \times 4$  or larger matrices.

$$15. \begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$$

$$16. \begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}$$

$$17. \begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix}$$

$$18. \begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix}$$

In Exercises 19–24, explore the effect of an elementary row operation on the determinant of a matrix. In each case, state the row operation and describe how it affects the determinant.

$$19. \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$20. \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$$

$$21. \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 5+3k & 6+4k \end{bmatrix}$$

$$22. \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$$

$$23. \begin{bmatrix} 1 & 1 & 1 \\ -3 & 8 & -4 \\ 2 & -3 & 2 \end{bmatrix}, \begin{bmatrix} k & k & k \\ -3 & 8 & -4 \\ 2 & -3 & 2 \end{bmatrix}$$

$$24. \begin{bmatrix} a & b & c \\ 3 & 2 & 2 \\ 6 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 2 \\ a & b & c \\ 6 & 5 & 6 \end{bmatrix}$$

Compute the determinants of the elementary matrices given in Exercises 25–30. (See Section 2.2.)

$$25. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$$

$$26. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$$

$$27. \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$28. \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$29. \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$30. \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Use Exercises 25–28 to answer the questions in Exercises 31 and 32. Give reasons for your answers.

31. What is the determinant of an elementary row replacement matrix?

32. What is the determinant of an elementary scaling matrix with  $k$  on the diagonal?

In Exercises 33–36, verify that  $\det EA = (\det E)(\det A)$ , where  $E$  is the elementary matrix shown and  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

$$33. \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$34. \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

$$35. \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

$$36. \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

37. Let  $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ . Write  $5A$ . Is  $\det 5A = 5 \det A$ ?

38. Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and let  $k$  be a scalar. Find a formula that relates  $\det kA$  to  $k$  and  $\det A$ .

In Exercises 39 and 40,  $A$  is an  $n \times n$  matrix. Mark each statement True or False. Justify each answer.

39. a. An  $n \times n$  determinant is defined by determinants of  $(n-1) \times (n-1)$  submatrices.

b. The  $(i, j)$ -cofactor of a matrix  $A$  is the matrix  $A_{ij}$  obtained by deleting from  $A$  its  $i$ th row and  $j$ th column.

40. a. The cofactor expansion of  $\det A$  down a column is the negative of the cofactor expansion along a row.

- b. The determinant of a triangular matrix is the sum of the entries on the main diagonal.
41. Let  $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . Compute the area of the parallelogram determined by  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{u} + \mathbf{v}$ , and  $\mathbf{0}$ , and compute the determinant of  $[\mathbf{u} \ \mathbf{v}]$ . How do they compare? Replace the first entry of  $\mathbf{v}$  by an arbitrary number  $x$ , and repeat the problem. Draw a picture and explain what you find.
42. Let  $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} c \\ 0 \end{bmatrix}$ , where  $a, b, c$  are positive (for simplicity). Compute the area of the parallelogram determined by  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{u} + \mathbf{v}$ , and  $\mathbf{0}$ , and compute the determinants of the matrices  $[\mathbf{u} \ \mathbf{v}]$  and  $[\mathbf{v} \ \mathbf{u}]$ . Draw a picture and explain what you find.
43. [M] Is it true that  $\det(A + B) = \det A + \det B$ ? To find out, generate random  $5 \times 5$  matrices  $A$  and  $B$ , and compute  $\det(A + B) - \det A - \det B$ . (Refer to Exercise 37 in Section 2.1.) Repeat the calculations for three other pairs of  $n \times n$  matrices, for various values of  $n$ . Report your results.
44. [M] Is it true that  $\det AB = (\det A)(\det B)$ ? Experiment with four pairs of random matrices as in Exercise 43, and make a conjecture.
45. [M] Construct a random  $4 \times 4$  matrix  $A$  with integer entries between  $-9$  and  $9$ , and compare  $\det A$  with  $\det A^T$ ,  $\det(-A)$ ,  $\det(2A)$ , and  $\det(10A)$ . Repeat with two other random  $4 \times 4$  integer matrices, and make conjectures about how these determinants are related. (Refer to Exercise 36 in Section 2.1.) Then check your conjectures with several random  $5 \times 5$  and  $6 \times 6$  integer matrices. Modify your conjectures, if necessary, and report your results.
46. [M] How is  $\det A^{-1}$  related to  $\det A$ ? Experiment with random  $n \times n$  integer matrices for  $n = 4, 5$ , and  $6$ , and make a conjecture. *Note:* In the unlikely event that you encounter a matrix with a zero determinant, reduce it to echelon form and discuss what you find.

### SOLUTION TO PRACTICE PROBLEM

Take advantage of the zeros. Begin with a cofactor expansion down the third column to obtain a  $3 \times 3$  matrix, which may be evaluated by an expansion down its first column.

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix} = (-1)^{1+3} 2 \begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix} \\ = 2 \cdot (-1)^{2+1} (-5) \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 20$$

The  $(-1)^{2+1}$  in the next-to-last calculation came from the  $(2, 1)$ -position of the  $-5$  in the  $3 \times 3$  determinant.

## 3.2 PROPERTIES OF DETERMINANTS

The secret of determinants lies in how they change when row operations are performed. The following theorem generalizes the results of Exercises 19–24 in Section 3.1. The proof is at the end of this section.

### THEOREM 3

#### Row Operations

Let  $A$  be a square matrix.

- If a multiple of one row of  $A$  is added to another row to produce a matrix  $B$ , then  $\det B = \det A$ .
- If two rows of  $A$  are interchanged to produce  $B$ , then  $\det B = -\det A$ .
- If one row of  $A$  is multiplied by  $k$  to produce  $B$ , then  $\det B = k \cdot \det A$ .

The following examples show how to use Theorem 3 to find determinants efficiently.

2. Use a determinant to decide if  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly independent, when

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$$

## 3.2 EXERCISES

Each equation in Exercises 1–4 illustrates a property of determinants. State the property.

$$1. \begin{vmatrix} 0 & 5 & -2 \\ 1 & -3 & 6 \\ 4 & -1 & 8 \end{vmatrix} = - \begin{vmatrix} 1 & -3 & 6 \\ 0 & 5 & -2 \\ 4 & -1 & 8 \end{vmatrix}$$

$$2. \begin{vmatrix} 2 & -6 & 4 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix} = 2 \begin{vmatrix} 1 & -3 & 2 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix}$$

$$3. \begin{vmatrix} 1 & 3 & -4 \\ 2 & 0 & -3 \\ 5 & -4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -4 \\ 0 & -6 & 5 \\ 5 & -4 & 7 \end{vmatrix}$$

$$4. \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 3 & 7 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 0 & 1 & -5 \end{vmatrix}$$

Find the determinants in Exercises 5–10 by row reduction to echelon form.

$$5. \begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$$

$$6. \begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix}$$

$$7. \begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix}$$

$$8. \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{vmatrix}$$

$$9. \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}$$

$$10. \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{vmatrix}$$

Combine the methods of row reduction and cofactor expansion to compute the determinants in Exercises 11–14.

$$11. \begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{vmatrix}$$

$$12. \begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix}$$

$$13. \begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$$

$$14. \begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix}$$

Find the determinants in Exercises 15–20, where

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

$$15. \begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}$$

$$16. \begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix}$$

$$17. \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$$

$$18. \begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$$

$$19. \begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$$

$$20. \begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix}$$

In Exercises 21–23, use determinants to find out if the matrix is invertible.

$$21. \begin{bmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}$$

$$22. \begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$$

$$23. \begin{bmatrix} 2 & 0 & 0 & 8 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}$$

In Exercises 24–26, use determinants to decide if the set of vectors is linearly independent.

$$24. \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}$$

$$25. \begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$$

$$26. \begin{bmatrix} 3 \\ 5 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \end{bmatrix}$$

In Exercises 27 and 28,  $A$  and  $B$  are  $n \times n$  matrices. Mark each statement True or False. Justify each answer.

27. a. A row replacement operation does not affect the determinant of a matrix.

b. The determinant of  $A$  is the product of the pivots in any echelon form  $U$  of  $A$ , multiplied by  $(-1)^r$ , where  $r$  is the number of row interchanges made during row reduction from  $A$  to  $U$ .



- c. If the columns of  $A$  are linearly dependent, then  $\det A = 0$ .
- d.  $\det(A + B) = \det A + \det B$ .
28. a. If two row interchanges are made in succession, then the new determinant equals the old determinant.
- b. The determinant of  $A$  is the product of the diagonal entries in  $A$ .
- c. If  $\det A$  is zero, then two rows or two columns are the same, or a row or a column is zero.
- d.  $\det A^T = (-1) \det A$ .
29. Compute  $\det B^5$ , where  $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ .
30. Use Theorem 3 (but not Theorem 4) to show that if two rows of a square matrix  $A$  are equal, then  $\det A = 0$ . The same is true for two columns. Why?

In Exercises 31–36, mention an appropriate theorem in your explanation.

31. Show that if  $A$  is invertible, then  $\det A^{-1} = \frac{1}{\det A}$ .
32. Find a formula for  $\det(rA)$  when  $A$  is an  $n \times n$  matrix.
33. Let  $A$  and  $B$  be square matrices. Show that even though  $AB$  and  $BA$  may not be equal, it is always true that  $\det AB = \det BA$ .
34. Let  $A$  and  $P$  be square matrices, with  $P$  invertible. Show that  $\det(PAP^{-1}) = \det A$ .
35. Let  $U$  be a square matrix such that  $U^T U = I$ . Show that  $\det U = \pm 1$ .
36. Suppose that  $A$  is a square matrix such that  $\det A^4 = 0$ . Explain why  $A$  cannot be invertible.

Verify that  $\det AB = (\det A)(\det B)$  for the matrices in Exercises 37 and 38. (Do not use Theorem 6.)

37.  $A = \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix}$
38.  $A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 4 & 2 \\ -1 & -1 \end{bmatrix}$
39. Let  $A$  and  $B$  be  $3 \times 3$  matrices, with  $\det A = 4$  and  $\det B = -3$ . Use properties of determinants (in the text and

in the exercises above) to compute:

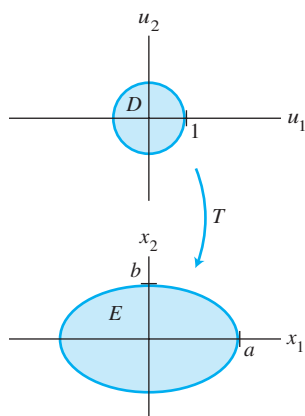
- a.  $\det AB$       b.  $\det 5A$       c.  $\det B^T$
- d.  $\det A^{-1}$       e.  $\det A^3$
40. Let  $A$  and  $B$  be  $4 \times 4$  matrices, with  $\det A = -1$  and  $\det B = 2$ . Compute:
- a.  $\det AB$       b.  $\det B^5$       c.  $\det 2A$
- d.  $\det A^T A$       e.  $\det B^{-1} AB$
41. Verify that  $\det A = \det B + \det C$ , where
- $$A = \begin{bmatrix} a+e & b+f \\ c & d \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, C = \begin{bmatrix} e & f \\ c & d \end{bmatrix}$$
42. Let  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Show that  $\det(A + B) = \det A + \det B$  if and only if  $a + d = 0$ .
43. Verify that  $\det A = \det B + \det C$ , where
- $$A = \begin{bmatrix} a_{11} & a_{12} & u_1 + v_1 \\ a_{21} & a_{22} & u_2 + v_2 \\ a_{31} & a_{32} & u_3 + v_3 \end{bmatrix},$$
- $$B = \begin{bmatrix} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ a_{31} & a_{32} & u_3 \end{bmatrix}, C = \begin{bmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{bmatrix}$$
- Note, however, that  $A$  is *not* the same as  $B + C$ .
44. Right-multiplication by an elementary matrix  $E$  affects the *columns* of  $A$  in the same way that left-multiplication affects the *rows*. Use Theorems 5 and 3 and the obvious fact that  $E^T$  is another elementary matrix to show that  $\det AE = (\det E)(\det A)$ .
- Do not use Theorem 6.
45. [M] Compute  $\det A^T A$  and  $\det AA^T$  for several random  $4 \times 5$  matrices and several random  $5 \times 6$  matrices. What can you say about  $A^T A$  and  $AA^T$  when  $A$  has more columns than rows?
46. [M] If  $\det A$  is close to zero, is the matrix  $A$  nearly singular? Experiment with the nearly singular  $4 \times 4$  matrix  $A$  in Exercise 9 of Section 2.3. Compute the determinants of  $A$ ,  $10A$ , and  $0.1A$ . In contrast, compute the condition numbers of these matrices. Repeat these calculations when  $A$  is the  $4 \times 4$  identity matrix. Discuss your results.

### SOLUTIONS TO PRACTICE PROBLEMS

1. Perform row replacements to create zeros in the first column and then create a row of zeros.

$$\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ 0 & 1 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$





**SOLUTION** We claim that  $E$  is the image of the unit disk  $D$  under the linear transformation  $T$  determined by the matrix  $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ , because if  $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ ,  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ , and  $\mathbf{x} = A\mathbf{u}$ , then

$$u_1 = \frac{x_1}{a} \quad \text{and} \quad u_2 = \frac{x_2}{b}$$

It follows that  $\mathbf{u}$  is in the unit disk, with  $u_1^2 + u_2^2 \leq 1$ , if and only if  $\mathbf{x}$  is in  $E$ , with  $(x_1/a)^2 + (x_2/b)^2 \leq 1$ . By the generalization of Theorem 10,

$$\begin{aligned} \{\text{area of ellipse}\} &= \{\text{area of } T(D)\} \\ &= |\det A| \cdot \{\text{area of } D\} \\ &= ab \cdot \pi(1)^2 = \pi ab \end{aligned}$$

### PRACTICE PROBLEM

Let  $S$  be the parallelogram determined by the vectors  $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  and  $\mathbf{b}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ , and let  $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ . Compute the area of the image of  $S$  under the mapping  $\mathbf{x} \mapsto A\mathbf{x}$ .

## 3.3 EXERCISES

Use Cramer's rule to compute the solutions of the systems in Exercises 1–6.

1.  $5x_1 + 7x_2 = 3$   
 $2x_1 + 4x_2 = 1$
2.  $4x_1 + x_2 = 6$   
 $5x_1 + 2x_2 = 7$
3.  $3x_1 - 2x_2 = 7$   
 $-5x_1 + 6x_2 = -5$
4.  $-5x_1 + 3x_2 = 9$   
 $3x_1 - x_2 = -5$
5.  $2x_1 + x_2 = 7$   
 $-3x_1 + x_3 = -8$   
 $x_2 + 2x_3 = -3$
6.  $2x_1 + x_2 + x_3 = 4$   
 $-x_1 + 2x_3 = 2$   
 $3x_1 + x_2 + 3x_3 = -2$

In Exercises 7–10, determine the values of the parameter  $s$  for which the system has a unique solution, and describe the solution.

7.  $6sx_1 + 4x_2 = 5$   
 $9x_1 + 2sx_2 = -2$
8.  $3sx_1 - 5x_2 = 3$   
 $9x_1 + 5sx_2 = 2$
9.  $sx_1 - 2sx_2 = -1$   
 $3x_1 + 6sx_2 = 4$
10.  $2sx_1 + x_2 = 1$   
 $3sx_1 + 6sx_2 = 2$

In Exercises 11–16, compute the adjugate of the given matrix, and then use Theorem 8 to give the inverse of the matrix.

11.  $\begin{bmatrix} 0 & -2 & -1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$
12.  $\begin{bmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
13.  $\begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$
14.  $\begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

$$15. \begin{bmatrix} 3 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 2 \end{bmatrix} \quad 16. \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

17. Show that if  $A$  is  $2 \times 2$ , then Theorem 8 gives the same formula for  $A^{-1}$  as that given by Theorem 4 in Section 2.2.
18. Suppose that all the entries in  $A$  are integers and  $\det A = 1$ . Explain why all the entries in  $A^{-1}$  are integers.

In Exercises 19–22, find the area of the parallelogram whose vertices are listed.

19.  $(0, 0)$ ,  $(5, 2)$ ,  $(6, 4)$ ,  $(11, 6)$
20.  $(0, 0)$ ,  $(-1, 3)$ ,  $(4, -5)$ ,  $(3, -2)$
21.  $(-1, 0)$ ,  $(0, 5)$ ,  $(1, -4)$ ,  $(2, 1)$
22.  $(0, -2)$ ,  $(6, -1)$ ,  $(-3, 1)$ ,  $(3, 2)$

23. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at  $(1, 0, -2)$ ,  $(1, 2, 4)$ , and  $(7, 1, 0)$ .

24. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at  $(1, 4, 0)$ ,  $(-2, -5, 2)$ , and  $(-1, 2, -1)$ .

25. Use the concept of volume to explain why the determinant of a  $3 \times 3$  matrix  $A$  is zero if and only if  $A$  is not invertible. Do not appeal to Theorem 4 in Section 3.2. [Hint: Think about the columns of  $A$ .]

26. Let  $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a linear transformation, and let  $\mathbf{p}$  be a vector and  $S$  a set in  $\mathbb{R}^m$ . Show that the image of  $\mathbf{p} + S$  under  $T$  is the translated set  $T(\mathbf{p}) + T(S)$  in  $\mathbb{R}^n$ .

27. Let  $S$  be the parallelogram determined by the vectors  $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$  and  $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ , and let  $A = \begin{bmatrix} 6 & -2 \\ -3 & 2 \end{bmatrix}$ . Compute the area of the image of  $S$  under the mapping  $\mathbf{x} \mapsto A\mathbf{x}$ .

28. Repeat Exercise 27 with  $\mathbf{b}_1 = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$ ,  $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , and  $A = \begin{bmatrix} 7 & 2 \\ 1 & 1 \end{bmatrix}$ .

29. Find a formula for the area of the triangle whose vertices are  $\mathbf{0}$ ,  $\mathbf{v}_1$ , and  $\mathbf{v}_2$  in  $\mathbb{R}^2$ .

30. Let  $R$  be the triangle with vertices at  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$ . Show that

$$\{\text{area of triangle}\} = \frac{1}{2} \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

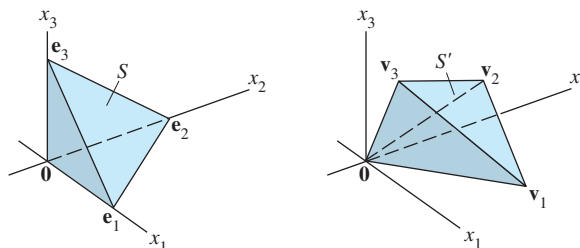
[Hint: Translate  $R$  to the origin by subtracting one of the vertices, and use Exercise 29.]

31. Let  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation determined by the matrix  $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ , where  $a$ ,  $b$ , and  $c$  are

positive numbers. Let  $S$  be the unit ball, whose bounding surface has the equation  $x_1^2 + x_2^2 + x_3^2 = 1$ .

- Show that  $T(S)$  is bounded by the ellipsoid with the equation  $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$ .
- Use the fact that the volume of the unit ball is  $4\pi/3$  to determine the volume of the region bounded by the ellipsoid in part (a).

32. Let  $S$  be the tetrahedron in  $\mathbb{R}^3$  with vertices at the vectors  $\mathbf{0}$ ,  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$ , and let  $S'$  be the tetrahedron with vertices at vectors  $\mathbf{0}$ ,  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$ . See the figure.



- Describe a linear transformation that maps  $S$  onto  $S'$ .
- Find a formula for the volume of the tetrahedron  $S'$  using the fact that

$$\{\text{volume of } S\} = (1/3)\{\text{area of base}\} \cdot \{\text{height}\}$$

33. [M] Test the inverse formula of Theorem 8 for a random  $4 \times 4$  matrix  $A$ . Use your matrix program to compute the cofactors of the  $3 \times 3$  submatrices, construct the adjugate, and set  $B = (\text{adj } A)/(\det A)$ . Then compute  $B - \text{inv}(A)$ , where  $\text{inv}(A)$  is the inverse of  $A$  as computed by the matrix program. Use floating point arithmetic with the maximum possible number of decimal places. Report your results.

34. [M] Test Cramer's rule for a random  $4 \times 4$  matrix  $A$  and a random  $4 \times 1$  vector  $\mathbf{b}$ . Compute each entry in the solution of  $A\mathbf{x} = \mathbf{b}$ , and compare these entries with the entries in  $A^{-1}\mathbf{b}$ . Write the command (or keystrokes) for your matrix program that uses Cramer's rule to produce the second entry of  $\mathbf{x}$ .

35. [M] If your version of MATLAB has the `flops` command, use it to count the number of floating point operations to compute  $A^{-1}$  for a random  $30 \times 30$  matrix. Compare this number with the number of flops needed to form  $(\text{adj } A)/(\det A)$ .

### SOLUTION TO PRACTICE PROBLEM

The area of  $S$  is  $\left| \det \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix} \right| = 14$ , and  $\det A = 2$ . By Theorem 10, the area of the image of  $S$  under the mapping  $\mathbf{x} \mapsto A\mathbf{x}$  is

$$|\det A| \cdot \{\text{area of } S\} = 2 \cdot 14 = 28$$

## CHAPTER 3 SUPPLEMENTARY EXERCISES

- Mark each statement True or False. Justify each answer. Assume that all matrices here are square.
  - If  $A$  is a  $2 \times 2$  matrix with a zero determinant, then one column of  $A$  is a multiple of the other.
  - If two rows of a  $3 \times 3$  matrix  $A$  are the same, then  $\det A = 0$ .
  - If  $A$  is a  $3 \times 3$  matrix, then  $\det 5A = 5 \det A$ .
  - If  $A$  and  $B$  are  $n \times n$  matrices, with  $\det A = 2$  and  $\det B = 3$ , then  $\det(A + B) = 5$ .
  - If  $A$  is  $n \times n$  and  $\det A = 2$ , then  $\det A^3 = 6$ .
  - If  $B$  is produced by interchanging two rows of  $A$ , then  $\det B = \det A$ .
  - If  $B$  is produced by multiplying row 3 of  $A$  by 5, then  $\det B = 5 \cdot \det A$ .

# 5

## Eigenvalues and Eigenvectors

### INTRODUCTORY EXAMPLE

### Dynamical Systems and Spotted Owls

In 1990, the northern spotted owl became the center of a nationwide controversy over the use and misuse of the majestic forests in the Pacific Northwest. Environmentalists convinced the federal government that the owl was threatened with extinction if logging continued in the old-growth forests (with trees over 200 years old), where the owls prefer to live. The timber industry, anticipating the loss of 30,000 to 100,000 jobs as a result of new government restrictions on logging, argued that the owl should not be classified as a “threatened species” and cited a number of published scientific reports to support its case.<sup>1</sup>

Caught in the crossfire of the two lobbying groups, mathematical ecologists intensified their drive to understand the population dynamics of the spotted owl. The life cycle of a spotted owl divides naturally into three stages: juvenile (up to 1 year old), subadult (1 to 2 years), and adult (over 2 years). The owls mate for life during the subadult and adult stages, begin to breed as adults, and live for up to 20 years. Each owl pair requires about 1000 hectares (4 square miles) for its own home territory. A critical time in the life cycle is when the juveniles leave the nest. To survive and become a subadult, a juvenile must successfully find a new home range (and usually a mate).



A first step in studying the population dynamics is to model the population at yearly intervals, at times denoted by  $k = 0, 1, 2, \dots$ . Usually, one assumes that there is a 1:1 ratio of males to females in each life stage and counts only the females. The population at year  $k$  can be described by a vector  $\mathbf{x}_k = (j_k, s_k, a_k)$ , where  $j_k$ ,  $s_k$ , and  $a_k$  are the numbers of females in the juvenile, subadult, and adult stages, respectively.

Using actual field data from demographic studies, R. Lamberson and co-workers considered the following *stage-matrix model*:<sup>2</sup>

$$\begin{bmatrix} j_{k+1} \\ s_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & .33 \\ .18 & 0 & 0 \\ 0 & .71 & .94 \end{bmatrix} \begin{bmatrix} j_k \\ s_k \\ a_k \end{bmatrix}$$

Here the number of new juvenile females in year  $k + 1$  is .33 times the number of adult females in year  $k$  (based on the average birth rate per owl pair). Also, 18% of the juveniles survive to become subadults, and 71% of the subadults and 94% of the adults survive to be counted as adults.

The stage-matrix model is a difference equation of the form  $\mathbf{x}_{k+1} = A\mathbf{x}_k$ . Such an equation is often called a

<sup>1</sup> “The Great Spotted Owl War,” *Reader’s Digest*, November 1992, pp. 91–95.

<sup>2</sup> R. H. Lamberson, R. McKelvey, B. R. Noon, and C. Voss, “A Dynamic Analysis of the Viability of the Northern Spotted Owl in a Fragmented Forest Environment,” *Conservation Biology* 6 (1992), 505–512. Also, a private communication from Professor Lamberson, 1993.

## Eigenvectors and Difference Equations

This section concludes by showing how to construct solutions of the first-order difference equation discussed in the chapter introductory example:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad (k = 0, 1, 2, \dots) \quad (8)$$

If  $A$  is an  $n \times n$  matrix, then (8) is a *recursive* description of a sequence  $\{\mathbf{x}_k\}$  in  $\mathbb{R}^n$ . A **solution** of (8) is an explicit description of  $\{\mathbf{x}_k\}$  whose formula for each  $\mathbf{x}_k$  does not depend directly on  $A$  or on the preceding terms in the sequence other than the initial term  $\mathbf{x}_0$ .

The simplest way to build a solution of (8) is to take an eigenvector  $\mathbf{x}_0$  and its corresponding eigenvalue  $\lambda$  and let

$$\mathbf{x}_k = \lambda^k \mathbf{x}_0 \quad (k = 1, 2, \dots) \quad (9)$$

This sequence is a solution because

$$A\mathbf{x}_k = A(\lambda^k \mathbf{x}_0) = \lambda^k (A\mathbf{x}_0) = \lambda^k (\lambda \mathbf{x}_0) = \lambda^{k+1} \mathbf{x}_0 = \mathbf{x}_{k+1}$$

Linear combinations of solutions in the form of equation (9) are solutions, too! See Exercise 33.

### PRACTICE PROBLEMS

1. Is 5 an eigenvalue of  $A = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}$ ?
2. If  $\mathbf{x}$  is an eigenvector of  $A$  corresponding to  $\lambda$ , what is  $A^3\mathbf{x}$ ?
3. Suppose that  $\mathbf{b}_1$  and  $\mathbf{b}_2$  are eigenvectors corresponding to distinct eigenvalues  $\lambda_1$  and  $\lambda_2$ , respectively, and suppose that  $\mathbf{b}_3$  and  $\mathbf{b}_4$  are linearly independent eigenvectors corresponding to a third distinct eigenvalue  $\lambda_3$ . Does it necessarily follow that  $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$  is a linearly independent set? [Hint: Consider the equation  $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + (c_3\mathbf{b}_3 + c_4\mathbf{b}_4) = \mathbf{0}$ .]

## 5.1 EXERCISES

1. Is  $\lambda = 2$  an eigenvalue of  $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$ ? Why or why not?
  2. Is  $\lambda = -3$  an eigenvalue of  $\begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix}$ ? Why or why not?
  3. Is  $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$  an eigenvector of  $\begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}$ ? If so, find the eigenvalue.
  4. Is  $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$  an eigenvector of  $\begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$ ? If so, find the eigenvalue.
  5. Is  $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$  an eigenvector of  $\begin{bmatrix} -4 & 3 & 3 \\ 2 & -3 & -2 \\ -1 & 0 & -2 \end{bmatrix}$ ? If so, find the eigenvalue.
  6. Is  $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$  an eigenvector of  $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$ ? If so, find the eigenvalue.
  7. Is  $\lambda = 4$  an eigenvalue of  $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$ ? If so, find one corresponding eigenvector.
  8. Is  $\lambda = 1$  an eigenvalue of  $\begin{bmatrix} 4 & -2 & 3 \\ 0 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix}$ ? If so, find one corresponding eigenvector.
- In Exercises 9–16, find a basis for the eigenspace corresponding to each listed eigenvalue.
9.  $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, \lambda = 1, 3$

10.  $A = \begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix}, \lambda = -5$

11.  $A = \begin{bmatrix} 1 & -3 \\ -4 & 5 \end{bmatrix}, \lambda = -1, 7$

12.  $A = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}, \lambda = 3, 7$

13.  $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \lambda = 1, 2, 3$

14.  $A = \begin{bmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix}, \lambda = 3$

15.  $A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}, \lambda = -5$

16.  $A = \begin{bmatrix} 5 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ 4 & -2 & -2 & 4 \end{bmatrix}, \lambda = 4$

Find the eigenvalues of the matrices in Exercises 17 and 18.

17.  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix}$

18.  $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

19. For  $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ , find one eigenvalue, with no calculation. Justify your answer.

20. Without calculation, find one eigenvalue and two linearly independent eigenvectors of  $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ . Justify your answer.

In Exercises 21 and 22,  $A$  is an  $n \times n$  matrix. Mark each statement True or False. Justify each answer.

21. a. If  $A\mathbf{x} = \lambda\mathbf{x}$  for some vector  $\mathbf{x}$ , then  $\lambda$  is an eigenvalue of  $A$ .  
 b. A matrix  $A$  is not invertible if and only if 0 is an eigenvalue of  $A$ .  
 c. A number  $c$  is an eigenvalue of  $A$  if and only if the equation  $(A - cI)\mathbf{x} = \mathbf{0}$  has a nontrivial solution.  
 d. Finding an eigenvector of  $A$  may be difficult, but checking whether a given vector is in fact an eigenvector is easy.  
 e. To find the eigenvalues of  $A$ , reduce  $A$  to echelon form.
22. a. If  $A\mathbf{x} = \lambda\mathbf{x}$  for some scalar  $\lambda$ , then  $\mathbf{x}$  is an eigenvector of  $A$ .  
 b. If  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

c. A steady-state vector for a stochastic matrix is actually an eigenvector.

d. The eigenvalues of a matrix are on its main diagonal.

e. An eigenspace of  $A$  is a null space of a certain matrix.

23. Explain why a  $2 \times 2$  matrix can have at most two distinct eigenvalues. Explain why an  $n \times n$  matrix can have at most  $n$  distinct eigenvalues.

24. Construct an example of a  $2 \times 2$  matrix with only one distinct eigenvalue.

25. Let  $\lambda$  be an eigenvalue of an invertible matrix  $A$ . Show that  $\lambda^{-1}$  is an eigenvalue of  $A^{-1}$ . [Hint: Suppose a nonzero  $\mathbf{x}$  satisfies  $A\mathbf{x} = \lambda\mathbf{x}$ .]

26. Show that if  $A^2$  is the zero matrix, then the only eigenvalue of  $A$  is 0.

27. Show that  $\lambda$  is an eigenvalue of  $A$  if and only if  $\lambda$  is an eigenvalue of  $A^T$ . [Hint: Find out how  $A - \lambda I$  and  $A^T - \lambda I$  are related.]

28. Use Exercise 27 to complete the proof of Theorem 1 for the case in which  $A$  is lower triangular.

29. Consider an  $n \times n$  matrix  $A$  with the property that the row sums all equal the same number  $s$ . Show that  $s$  is an eigenvalue of  $A$ . [Hint: Find an eigenvector.]

30. Consider an  $n \times n$  matrix  $A$  with the property that the column sums all equal the same number  $s$ . Show that  $s$  is an eigenvalue of  $A$ . [Hint: Use Exercises 27 and 29.]

In Exercises 31 and 32, let  $A$  be the matrix of the linear transformation  $T$ . Without writing  $A$ , find an eigenvalue of  $A$  and describe the eigenspace.

31.  $T$  is the transformation on  $\mathbb{R}^2$  that reflects points across some line through the origin.

32.  $T$  is the transformation on  $\mathbb{R}^3$  that rotates points about some line through the origin.

33. Let  $\mathbf{u}$  and  $\mathbf{v}$  be eigenvectors of a matrix  $A$ , with corresponding eigenvalues  $\lambda$  and  $\mu$ , and let  $c_1$  and  $c_2$  be scalars. Define

$$\mathbf{x}_k = c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v} \quad (k = 0, 1, 2, \dots)$$

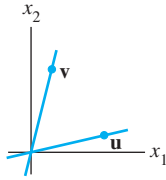
a. What is  $\mathbf{x}_{k+1}$ , by definition?

b. Compute  $A\mathbf{x}_k$  from the formula for  $\mathbf{x}_k$ , and show that  $A\mathbf{x}_k = \mathbf{x}_{k+1}$ . This calculation will prove that the sequence  $\{\mathbf{x}_k\}$  defined above satisfies the difference equation  $\mathbf{x}_{k+1} = A\mathbf{x}_k$  ( $k = 0, 1, 2, \dots$ ).

34. Describe how you might try to build a solution of a difference equation  $\mathbf{x}_{k+1} = A\mathbf{x}_k$  ( $k = 0, 1, 2, \dots$ ) if you were given the initial  $\mathbf{x}_0$  and this vector did not happen to be an eigenvector of  $A$ . [Hint: How might you relate  $\mathbf{x}_0$  to eigenvectors of  $A$ ?]

35. Let  $\mathbf{u}$  and  $\mathbf{v}$  be the vectors shown in the figure, and suppose  $\mathbf{u}$  and  $\mathbf{v}$  are eigenvectors of a  $2 \times 2$  matrix  $A$  that correspond to eigenvalues 2 and 3, respectively. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation given by  $T(\mathbf{x}) = A\mathbf{x}$  for each  $\mathbf{x}$  in  $\mathbb{R}^2$ , and let  $\mathbf{w} = \mathbf{u} + \mathbf{v}$ . Make a copy of the figure, and on

the same coordinate system, carefully plot the vectors  $T(\mathbf{u})$ ,  $T(\mathbf{v})$ , and  $T(\mathbf{w})$ .



36. Repeat Exercise 35, assuming  $\mathbf{u}$  and  $\mathbf{v}$  are eigenvectors of  $A$  that correspond to eigenvalues  $-1$  and  $3$ , respectively.

[M] In Exercises 37–40, use a matrix program to find the eigenvalues of the matrix. Then use the method of Example 4 with a row reduction routine to produce a basis for each eigenspace.

$$37. \begin{bmatrix} 12 & 1 & 4 \\ 2 & 11 & 4 \\ 1 & 3 & 7 \end{bmatrix} \quad 38. \begin{bmatrix} 5 & -2 & 2 & -4 \\ 7 & -4 & 2 & -4 \\ 4 & -4 & 2 & 0 \\ 3 & -1 & 1 & -3 \end{bmatrix}$$

$$39. \begin{bmatrix} 12 & -90 & 30 & 30 & 30 \\ 8 & -49 & 15 & 15 & 15 \\ 16 & -52 & 12 & 0 & 20 \\ 0 & -30 & 10 & 22 & 10 \\ 8 & -41 & 15 & 15 & 7 \end{bmatrix}$$

$$40. \begin{bmatrix} -23 & 57 & -9 & -15 & -59 \\ -10 & 12 & -10 & 2 & -22 \\ 11 & 5 & -3 & -19 & -15 \\ -27 & 31 & -27 & 25 & -37 \\ -5 & -15 & -5 & 1 & 31 \end{bmatrix}$$

### SOLUTIONS TO PRACTICE PROBLEMS

1. The number 5 is an eigenvalue of  $A$  if and only if the equation  $(A - 5I)\mathbf{x} = \mathbf{0}$  has a nontrivial solution. Form

$$A - 5I = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ 3 & -5 & 5 \\ 2 & 2 & 1 \end{bmatrix}$$

and row reduce the augmented matrix:

$$\left[ \begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 3 & -5 & 5 & 0 \\ 2 & 2 & 1 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 8 & -1 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & -5 & 0 \end{array} \right]$$

At this point, it is clear that the homogeneous system has no free variables. Thus  $A - 5I$  is an invertible matrix, which means that 5 is *not* an eigenvalue of  $A$ .

2. If  $\mathbf{x}$  is an eigenvector of  $A$  corresponding to  $\lambda$ , then  $A\mathbf{x} = \lambda\mathbf{x}$  and so

$$A^2\mathbf{x} = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$$

Again,  $A^3\mathbf{x} = A(A^2\mathbf{x}) = A(\lambda^2\mathbf{x}) = \lambda^2 A\mathbf{x} = \lambda^3\mathbf{x}$ . The general pattern,  $A^k\mathbf{x} = \lambda^k\mathbf{x}$ , is proved by induction.

3. Yes. Suppose  $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 + c_4\mathbf{b}_4 = \mathbf{0}$ . Since any linear combination of eigenvectors from the same eigenvalue is again an eigenvector for that eigenvalue,  $c_3\mathbf{b}_3 + c_4\mathbf{b}_4$  is an eigenvector for  $\lambda_3$ . By Theorem 2, the vectors  $\mathbf{b}_1$ ,  $\mathbf{b}_2$ , and  $c_3\mathbf{b}_3 + c_4\mathbf{b}_4$  are linearly independent, so

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + (c_3\mathbf{b}_3 + c_4\mathbf{b}_4) = \mathbf{0}$$

implies  $c_1 = c_2 = 0$ . But then,  $c_3$  and  $c_4$  must also be zero since  $\mathbf{b}_3$  and  $\mathbf{b}_4$  are linearly independent. Hence all the coefficients in the original equation must be zero, and the vectors  $\mathbf{b}_1$ ,  $\mathbf{b}_2$ ,  $\mathbf{b}_3$ , and  $\mathbf{b}_4$  are linearly independent.

## 5.2 THE CHARACTERISTIC EQUATION

Useful information about the eigenvalues of a square matrix  $A$  is encoded in a special scalar equation called the characteristic equation of  $A$ . A simple example will lead to the general case.

This explicit formula for  $\mathbf{x}_k$  gives the solution of the difference equation  $\mathbf{x}_{k+1} = A\mathbf{x}_k$ . As  $k \rightarrow \infty$ ,  $(.92)^k$  tends to zero and  $\mathbf{x}_k$  tends to  $\begin{bmatrix} .375 \\ .625 \end{bmatrix} = .125\mathbf{v}_1$ . ■

The calculations in Example 5 have an interesting application to a Markov chain discussed in Section 4.9. Those who read that section may recognize that matrix  $A$  in Example 5 above is the same as the migration matrix  $M$  in Section 4.9,  $\mathbf{x}_0$  is the initial population distribution between city and suburbs, and  $\mathbf{x}_k$  represents the population distribution after  $k$  years.

Theorem 18 in Section 4.9 stated that for a matrix such as  $A$ , the sequence  $\mathbf{x}_k$  tends to a steady-state vector. Now we know *why* the  $\mathbf{x}_k$  behave this way, at least for the migration matrix. The steady-state vector is  $.125\mathbf{v}_1$ , a multiple of the eigenvector  $\mathbf{v}_1$ , and formula (5) for  $\mathbf{x}_k$  shows precisely why  $\mathbf{x}_k \rightarrow .125\mathbf{v}_1$ .

### NUMERICAL NOTES

1. Computer software such as Mathematica and Maple can use symbolic calculations to find the characteristic polynomial of a moderate-sized matrix. But there is no formula or finite algorithm to solve the characteristic equation of a general  $n \times n$  matrix for  $n \geq 5$ .
2. The best numerical methods for finding eigenvalues avoid the characteristic polynomial entirely. In fact, MATLAB finds the characteristic polynomial of a matrix  $A$  by first computing the eigenvalues  $\lambda_1, \dots, \lambda_n$  of  $A$  and then expanding the product  $(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$ .
3. Several common algorithms for estimating the eigenvalues of a matrix  $A$  are based on Theorem 4. The powerful *QR algorithm* is discussed in the exercises. Another technique, called *Jacobi's method*, works when  $A = A^T$  and computes a sequence of matrices of the form

$$A_1 = A \quad \text{and} \quad A_{k+1} = P_k^{-1} A_k P_k \quad (k = 1, 2, \dots)$$

Each matrix in the sequence is similar to  $A$  and so has the same eigenvalues as  $A$ . The nondiagonal entries of  $A_{k+1}$  tend to zero as  $k$  increases, and the diagonal entries tend to approach the eigenvalues of  $A$ .

4. Other methods of estimating eigenvalues are discussed in Section 5.8.

### PRACTICE PROBLEM

Find the characteristic equation and eigenvalues of  $A = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$ .

## 5.2 EXERCISES

Find the characteristic polynomial and the real eigenvalues of the matrices in Exercises 1–8.

1.  $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

2.  $\begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$

3.  $\begin{bmatrix} -4 & 2 \\ 6 & 7 \end{bmatrix}$

4.  $\begin{bmatrix} 8 & 2 \\ 3 & 3 \end{bmatrix}$

5.  $\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$

6.  $\begin{bmatrix} 9 & -2 \\ 2 & 5 \end{bmatrix}$

7.  $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$

8.  $\begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix}$

Exercises 9–14 require techniques from Section 3.1. Find the characteristic polynomial of each matrix, using either a cofactor expansion or the special formula for  $3 \times 3$  determinants described



prior to Exercises 15–18 in Section 3.1. [Note: Finding the characteristic polynomial of a  $3 \times 3$  matrix is not easy to do with just row operations, because the variable  $\lambda$  is involved.]

$$\begin{array}{ll} 9. \begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix} & 10. \begin{bmatrix} 3 & 1 & 1 \\ 0 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix} \\ 11. \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 4 \\ 1 & 0 & 4 \end{bmatrix} & 12. \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \\ 13. \begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix} & 14. \begin{bmatrix} 4 & 0 & -1 \\ -1 & 0 & 4 \\ 0 & 2 & 3 \end{bmatrix} \end{array}$$

For the matrices in Exercises 15–17, list the real eigenvalues, repeated according to their multiplicities.

$$15. \begin{bmatrix} 5 & 5 & 0 & 2 \\ 0 & 2 & -3 & 6 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad 16. \begin{bmatrix} 3 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 0 & 3 & 6 & 0 \\ 2 & 3 & 3 & -5 \end{bmatrix}$$

$$17. \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 0 & 0 \\ 0 & -7 & 2 & 1 & 0 \\ -4 & 1 & 9 & -2 & 3 \end{bmatrix}$$

18. It can be shown that the algebraic multiplicity of an eigenvalue  $\lambda$  is always greater than or equal to the dimension of the eigenspace corresponding to  $\lambda$ . Find  $h$  in the matrix  $A$  below such that the eigenspace for  $\lambda = 4$  is two-dimensional:

$$A = \begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

19. Let  $A$  be an  $n \times n$  matrix, and suppose  $A$  has  $n$  real eigenvalues,  $\lambda_1, \dots, \lambda_n$ , repeated according to multiplicities, so that

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

Explain why  $\det A$  is the product of the  $n$  eigenvalues of  $A$ . (This result is true for any square matrix when complex eigenvalues are considered.)

20. Use a property of determinants to show that  $A$  and  $A^T$  have the same characteristic polynomial.

In Exercises 21 and 22,  $A$  and  $B$  are  $n \times n$  matrices. Mark each statement True or False. Justify each answer.

21. a. The determinant of  $A$  is the product of the diagonal entries in  $A$ .  
b. An elementary row operation on  $A$  does not change the determinant.  
c.  $(\det A)(\det B) = \det AB$   
d. If  $\lambda + 5$  is a factor of the characteristic polynomial of  $A$ , then 5 is an eigenvalue of  $A$ .

22. a. If  $A$  is  $3 \times 3$ , with columns  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ , then  $\det A$  equals the volume of the parallelepiped determined by  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ .  
b.  $\det A^T = (-1) \det A$ .  
c. The multiplicity of a root  $r$  of the characteristic equation of  $A$  is called the algebraic multiplicity of  $r$  as an eigenvalue of  $A$ .  
d. A row replacement operation on  $A$  does not change the eigenvalues.

A widely used method for estimating eigenvalues of a general matrix  $A$  is the *QR algorithm*. Under suitable conditions, this algorithm produces a sequence of matrices, all similar to  $A$ , that become almost upper triangular, with diagonal entries that approach the eigenvalues of  $A$ . The main idea is to factor  $A$  (or another matrix similar to  $A$ ) in the form  $A = Q_1 R_1$ , where  $Q_1^T = Q_1^{-1}$  and  $R_1$  is upper triangular. The factors are interchanged to form  $A_1 = R_1 Q_1$ , which is again factored as  $A_1 = Q_2 R_2$ ; then to form  $A_2 = R_2 Q_2$ , and so on. The similarity of  $A, A_1, \dots$  follows from the more general result in Exercise 23.

23. Show that if  $A = QR$  with  $Q$  invertible, then  $A$  is similar to  $A_1 = RQ$ .

24. Show that if  $A$  and  $B$  are similar, then  $\det A = \det B$ .

25. Let  $A = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}$ ,  $\mathbf{v}_1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$ , and  $\mathbf{x}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$ . [Note:  $A$  is the stochastic matrix studied in Example 5 in Section 4.9.]

- a. Find a basis for  $\mathbb{R}^2$  consisting of  $\mathbf{v}_1$  and another eigenvector  $\mathbf{v}_2$  of  $A$ .  
b. Verify that  $\mathbf{x}_0$  may be written in the form  $\mathbf{x}_0 = \mathbf{v}_1 + c\mathbf{v}_2$ .  
c. For  $k = 1, 2, \dots$ , define  $\mathbf{x}_k = A^k \mathbf{x}_0$ . Compute  $\mathbf{x}_1$  and  $\mathbf{x}_2$ , and write a formula for  $\mathbf{x}_k$ . Then show that  $\mathbf{x}_k \rightarrow \mathbf{v}_1$  as  $k$  increases.

26. Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Use formula (1) for a determinant (given before Example 2) to show that  $\det A = ad - bc$ . Consider two cases:  $a \neq 0$  and  $a = 0$ .

$$27. \text{ Let } A = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix},$$

$$\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

- a. Show that  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are eigenvectors of  $A$ . [Note:  $A$  is the stochastic matrix studied in Example 3 of Section 4.9.]  
b. Let  $\mathbf{x}_0$  be any vector in  $\mathbb{R}^3$  with nonnegative entries whose sum is 1. (In Section 4.9,  $\mathbf{x}_0$  was called a probability vector.) Explain why there are constants  $c_1, c_2, c_3$  such that  $\mathbf{x}_0 = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3$ . Compute  $\mathbf{w}^T \mathbf{x}_0$ , and deduce that  $c_1 = 1$ .  
c. For  $k = 1, 2, \dots$ , define  $\mathbf{x}_k = A^k \mathbf{x}_0$ , with  $\mathbf{x}_0$  as in part (b). Show that  $\mathbf{x}_k \rightarrow \mathbf{v}_1$  as  $k$  increases.

28. [M] Construct a random integer-valued  $4 \times 4$  matrix  $A$ , and verify that  $A$  and  $A^T$  have the same characteristic polynomial (the same eigenvalues with the same multiplicities). Do  $A$  and  $A^T$  have the same eigenvectors? Make the same analysis of a  $5 \times 5$  matrix. Report the matrices and your conclusions.
29. [M] Construct a random integer-valued  $4 \times 4$  matrix  $A$ .
- Reduce  $A$  to echelon form  $U$  with no row scaling, and use  $U$  in formula (1) (before Example 2) to compute  $\det A$ . (If  $A$  happens to be singular, start over with a new random matrix.)
  - Compute the eigenvalues of  $A$  and the product of these eigenvalues (as accurately as possible).
- c. List the matrix  $A$ , and, to four decimal places, list the pivots in  $U$  and the eigenvalues of  $A$ . Compute  $\det A$  with your matrix program, and compare it with the products you found in (a) and (b).
30. [M] Let  $A = \begin{bmatrix} -6 & 28 & 21 \\ 4 & -15 & -12 \\ -8 & a & 25 \end{bmatrix}$ . For each value of  $a$  in the set  $\{32, 31.9, 31.8, 32.1, 32.2\}$ , compute the characteristic polynomial of  $A$  and the eigenvalues. In each case, create a graph of the characteristic polynomial  $p(t) = \det(A - tI)$  for  $0 \leq t \leq 3$ . If possible, construct all graphs on one coordinate system. Describe how the graphs reveal the changes in the eigenvalues as  $a$  changes.

### SOLUTION TO PRACTICE PROBLEM

The characteristic equation is

$$\begin{aligned} 0 &= \det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & -4 \\ 4 & 2 - \lambda \end{bmatrix} \\ &= (1 - \lambda)(2 - \lambda) - (-4)(4) = \lambda^2 - 3\lambda + 18 \end{aligned}$$

From the quadratic formula,

$$\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4(18)}}{2} = \frac{3 \pm \sqrt{-63}}{2}$$

It is clear that the characteristic equation has no real solutions, so  $A$  has no real eigenvalues. The matrix  $A$  is acting on the real vector space  $\mathbb{R}^2$ , and there is no nonzero vector  $\mathbf{v}$  in  $\mathbb{R}^2$  such that  $A\mathbf{v} = \lambda\mathbf{v}$  for some scalar  $\lambda$ .

## 5.3 DIAGONALIZATION

In many cases, the eigenvalue–eigenvector information contained within a matrix  $A$  can be displayed in a useful factorization of the form  $A = PDP^{-1}$  where  $D$  is a diagonal matrix. In this section, the factorization enables us to compute  $A^k$  quickly for large values of  $k$ , a fundamental idea in several applications of linear algebra. Later, in Sections 5.6 and 5.7, the factorization will be used to analyze (and *decouple*) dynamical systems.

The following example illustrates that powers of a diagonal matrix are easy to compute.

**EXAMPLE 1** If  $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ , then  $D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$  and

$$D^3 = DD^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}$$

In general,

$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \quad \text{for } k \geq 1$$

If  $A = PDP^{-1}$  for some invertible  $P$  and diagonal  $D$ , then  $A^k$  is also easy to compute, as the next example shows.

**SOLUTION** Since  $A$  is a triangular matrix, the eigenvalues are 5 and  $-3$ , each with multiplicity 2. Using the method in Section 5.1, we find a basis for each eigenspace.

$$\text{Basis for } \lambda = 5: \mathbf{v}_1 = \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis for } \lambda = -3: \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_4\}$  is linearly independent, by Theorem 7. So the matrix  $P = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_4]$  is invertible, and  $A = PDP^{-1}$ , where

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

### PRACTICE PROBLEMS

1. Compute  $A^8$ , where  $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$ .
2. Let  $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$ ,  $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ , and  $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ . Suppose you are told that  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are eigenvectors of  $A$ . Use this information to diagonalize  $A$ .
3. Let  $A$  be a  $4 \times 4$  matrix with eigenvalues 5, 3, and  $-2$ , and suppose you know that the eigenspace for  $\lambda = 3$  is two-dimensional. Do you have enough information to determine if  $A$  is diagonalizable?

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## 5.3 EXERCISES

In Exercises 1 and 2, let  $A = PDP^{-1}$  and compute  $A^4$ .

$$1. \ P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2. \ P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

In Exercises 3 and 4, use the factorization  $A = PDP^{-1}$  to compute  $A^k$ , where  $k$  represents an arbitrary positive integer.

$$3. \ \begin{bmatrix} a & 0 \\ 2(a-b) & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$4. \ \begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

In Exercises 5 and 6, the matrix  $A$  is factored in the form  $PDP^{-1}$ . Use the Diagonalization Theorem to find the eigenvalues of  $A$  and a basis for each eigenspace.

$$5. \ A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$6. \ A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -3 & 1 & 9 \\ -1 & 0 & 3 \end{bmatrix}$$

Diagonalize the matrices in Exercises 7–20, if possible. The real eigenvalues for Exercises 11–16 and 18 are included below the matrix.

$$7. \ \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

$$8. \ \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$$

9.  $\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$
10.  $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$
11.  $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{bmatrix}$
12.  $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$
- $\lambda = -1, 5$
- $\lambda = 2, 5$
13.  $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$
14.  $\begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$
- $\lambda = 1, 5$
- $\lambda = 2, 3$
15.  $\begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$
16.  $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{bmatrix}$
- $\lambda = 0, 1$
- $\lambda = 0$
17.  $\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$
18.  $\begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix}$
- $\lambda = -2, -1, 0$
19.  $\begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$
20.  $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$

In Exercises 21 and 22,  $A$ ,  $B$ ,  $P$ , and  $D$  are  $n \times n$  matrices. Mark each statement True or False. Justify each answer. (Study Theorems 5 and 6 and the examples in this section carefully before you try these exercises.)

21. a.  $A$  is diagonalizable if  $A = PDP^{-1}$  for some matrix  $D$  and some invertible matrix  $P$ .  
 b. If  $\mathbb{R}^n$  has a basis of eigenvectors of  $A$ , then  $A$  is diagonalizable.  
 c.  $A$  is diagonalizable if and only if  $A$  has  $n$  eigenvalues, counting multiplicities.  
 d. If  $A$  is diagonalizable, then  $A$  is invertible.
22. a.  $A$  is diagonalizable if  $A$  has  $n$  eigenvectors.  
 b. If  $A$  is diagonalizable, then  $A$  has  $n$  distinct eigenvalues.  
 c. If  $AP = PD$ , with  $D$  diagonal, then the nonzero columns of  $P$  must be eigenvectors of  $A$ .  
 d. If  $A$  is invertible, then  $A$  is diagonalizable.
23.  $A$  is a  $5 \times 5$  matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is  $A$  diagonalizable? Why?
24.  $A$  is a  $3 \times 3$  matrix with two eigenvalues. Each eigenspace is one-dimensional. Is  $A$  diagonalizable? Why?

25.  $A$  is a  $4 \times 4$  matrix with three eigenvalues. One eigenspace is one-dimensional, and one of the other eigenspaces is two-dimensional. Is it possible that  $A$  is *not* diagonalizable? Justify your answer.
26.  $A$  is a  $7 \times 7$  matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspaces is three-dimensional. Is it possible that  $A$  is *not* diagonalizable? Justify your answer.
27. Show that if  $A$  is both diagonalizable and invertible, then so is  $A^{-1}$ .
28. Show that if  $A$  has  $n$  linearly independent eigenvectors, then so does  $A^T$ . [Hint: Use the Diagonalization Theorem.]
29. A factorization  $A = PDP^{-1}$  is not unique. Demonstrate this for the matrix  $A$  in Example 2. With  $D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ , use the information in Example 2 to find a matrix  $P_1$  such that  $A = P_1 D_1 P_1^{-1}$ .
30. With  $A$  and  $D$  as in Example 2, find an invertible  $P_2$  unequal to the  $P$  in Example 2, such that  $A = P_2 D P_2^{-1}$ .
31. Construct a nonzero  $2 \times 2$  matrix that is invertible but not diagonalizable.
32. Construct a nondiagonal  $2 \times 2$  matrix that is diagonalizable but not invertible.

[M] Diagonalize the matrices in Exercises 33–36. Use your matrix program's eigenvalue command to find the eigenvalues, and then compute bases for the eigenspaces as in Section 5.1.

33.  $\begin{bmatrix} 9 & -4 & -2 & -4 \\ -56 & 32 & -28 & 44 \\ -14 & -14 & 6 & -14 \\ 42 & -33 & 21 & -45 \end{bmatrix}$
34.  $\begin{bmatrix} 4 & -9 & -7 & 8 & 2 \\ -7 & -9 & 0 & 7 & 14 \\ 5 & 10 & 5 & -5 & -10 \\ -2 & 3 & 7 & 0 & 4 \\ -3 & -13 & -7 & 10 & 11 \end{bmatrix}$
35.  $\begin{bmatrix} 13 & -12 & 9 & -15 & 9 \\ 6 & -5 & 9 & -15 & 9 \\ 6 & -12 & -5 & 6 & 9 \\ 6 & -12 & 9 & -8 & 9 \\ -6 & 12 & 12 & -6 & -2 \end{bmatrix}$
36.  $\begin{bmatrix} 24 & -6 & 2 & 6 & 2 \\ 72 & 51 & 9 & -99 & 9 \\ 0 & -63 & 15 & 63 & 63 \\ 72 & 15 & 9 & -63 & 9 \\ 0 & 63 & 21 & -63 & -27 \end{bmatrix}$

## SOLUTIONS TO PRACTICE PROBLEMS

1.  $\det(A - \lambda I) = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$ . The eigenvalues are 2 and 1, and the corresponding eigenvectors are  $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$  and  $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Next, form

$$P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Since  $A = PDP^{-1}$ ,

$$\begin{aligned} A^8 &= PD^8P^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^8 & 0 \\ 0 & 1^8 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 256 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix} \end{aligned}$$

2. Compute  $A\mathbf{v}_1 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1 \cdot \mathbf{v}_1$ , and

$$A\mathbf{v}_2 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \cdot \mathbf{v}_2$$

So,  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are eigenvectors for the eigenvalues 1 and 3, respectively. Thus

$$A = PDP^{-1}, \quad \text{where} \quad P = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

3. Yes,  $A$  is diagonalizable. There is a basis  $\{\mathbf{v}_1, \mathbf{v}_2\}$  for the eigenspace corresponding to  $\lambda = 3$ . In addition, there will be at least one eigenvector for  $\lambda = 5$  and one for  $\lambda = -2$ . Call them  $\mathbf{v}_3$  and  $\mathbf{v}_4$ . Then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  is linearly independent by Theorem 2 and Practice Problem 3 in Section 5.1. There can be no additional eigenvectors that are linearly independent from  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ , because the vectors are all in  $\mathbb{R}^4$ . Hence the eigenspaces for  $\lambda = 5$  and  $\lambda = -2$  are both one-dimensional. It follows that  $A$  is diagonalizable by Theorem 7(b).

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## 5.4 EIGENVECTORS AND LINEAR TRANSFORMATIONS

The goal of this section is to understand the matrix factorization  $A = PDP^{-1}$  as a statement about linear transformations. We shall see that the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is essentially the same as the very simple mapping  $\mathbf{u} \mapsto D\mathbf{u}$ , when viewed from the proper perspective. A similar interpretation will apply to  $A$  and  $D$  even when  $D$  is not a diagonal matrix.

Recall from Section 1.9 that any linear transformation  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  can be implemented via left-multiplication by a matrix  $A$ , called the *standard matrix* of  $T$ . Now we need the same sort of representation for any linear transformation between two finite-dimensional vector spaces.