Modelling with Matlab Assignment 3

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1 Question 1

The appropriate two equations in Truscott and Brindley are:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - R_m Z \frac{P^2}{\alpha^2 + P^2}$$

$$\frac{dZ}{dt} = \gamma R_m Z \frac{P^2}{\alpha^2 + P^2} - \mu Z$$

Parameter values were taken from eqn(6) and can be seen in the model (matlab files A3Q1.m, A3Q1b.m, submitted with the question. As always A3Q1b.m is the file you run, which calls A3Q1.m, which must be in the same path). 10 randomised trails were done for 1000 days (roughly 3 years) in order to be sure of the fixed point. Fig1 shows the full behaviour, Fig2 shows a zoom in to smaller scale on the y axis so the behaviour of the zooplankton can be seen. Calling y in the command window gives the fixed point values as: phytoplankton=y(1)=4.1167, zooplankton=y(2)=4.9505. Units would be in $\mu gN/l$.

For "an appropriate range of positive initial conditions", I've initially chosen for the simulation modest initial populations taken from uniform random variables between 0 and 1. The large oscillations seen near the beginning are due to low zooplankton concentration; there are commented lines in A3Q1b.m where the initial values are 2.95 plus a (uniform) random increment between 1 and 4 and 3.95 plus the same increment, and in the former large initial oscillations are still seen whereas in the latter they are generally not, despite significantly increasing the random added value (and therefore the upper limit of the initial condition). The fact that occasionally you do see an occasional big trajectory suggests some threshold is being crossed at the upper limit of this range similar to the "superthreshold perturbations" discussed by Truscott and Brindley at the bottom of page 983 of the paper.

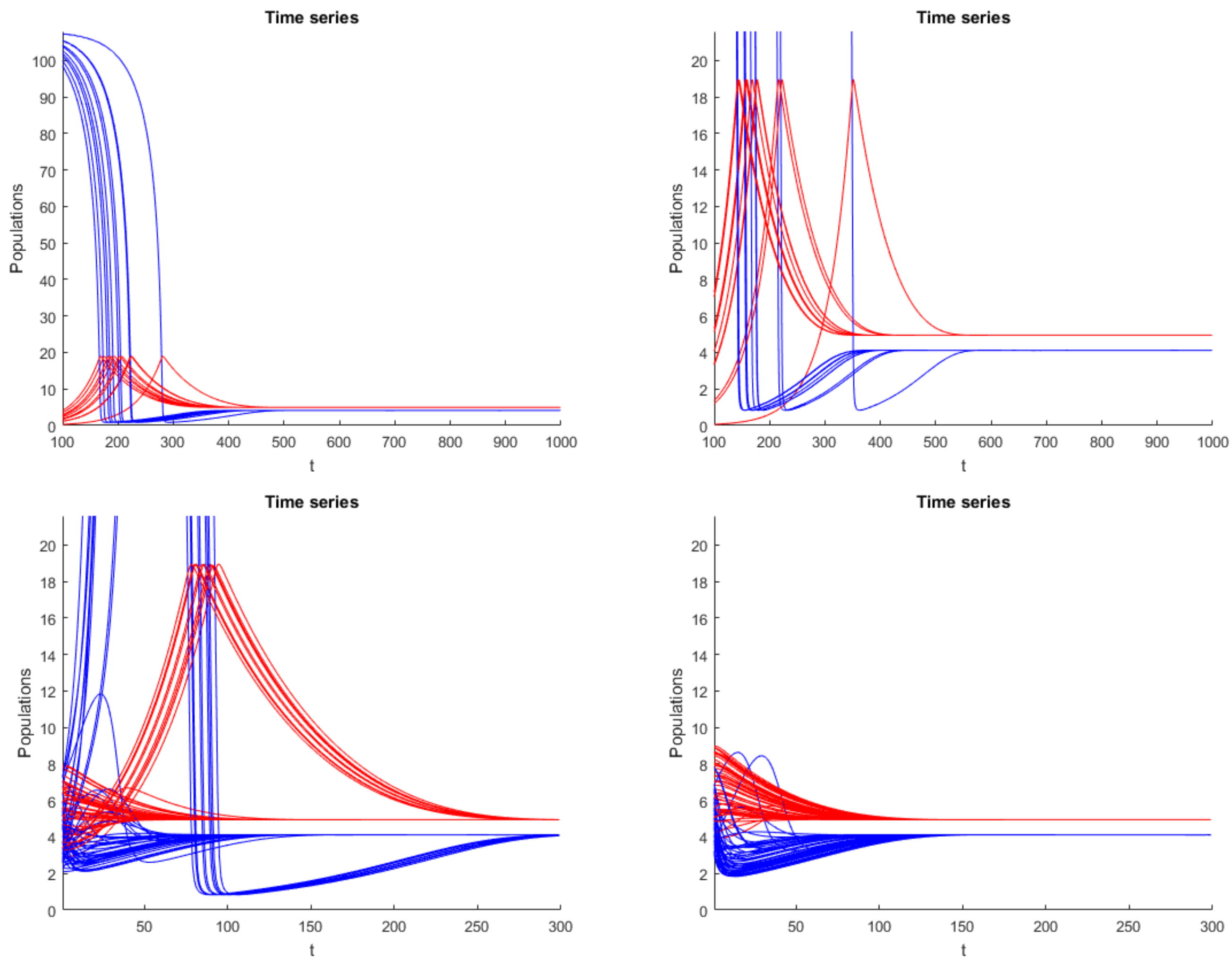


Figure 1: Top left: NPZ system plotted for 1000 days to show convergence to fixed point values at approxiately t=500 days. Top Right: same system, smaller y scale. Bottom Left: starting values closer to initial conditions, low zooplankton. Note the shorter time scale of only 300 days. Bottom Right: Same system, zooplankton now start at a minimum of 3.95 but can be more than twice the fixed point value. Blue lines are phytoplankton, red lines are zooplankton.

2 Question 2

The forcing function for $r(t)$ is

$$r(t) = 0.3 - 0.1\cos(\frac{2\pi t}{365})$$

The required subplot will be of the function

$$R(t) = \frac{1}{r(t)}\frac{dr}{dt} = \frac{1}{0.3 - 0.1\cos(2\pi t/365)}\frac{0.2\pi}{365}\sin(\frac{2\pi t}{365})$$

Refer to the Matlab file A3Q2.m, which runs without any other files. The figure below is the required plot: The question asks for a plot over 365 days only, and asks for no comment further than the script/file and the plot. (I overthought this at first and thought I was supposed to begin implementing question 3 here.) It's easy to change the scale of the plot and line5, which is commented out, does exactly this; giving more of a comparison of the periodicity and long term behaviour of the modified $R(t)$ as compared to the simple Cosine curve.

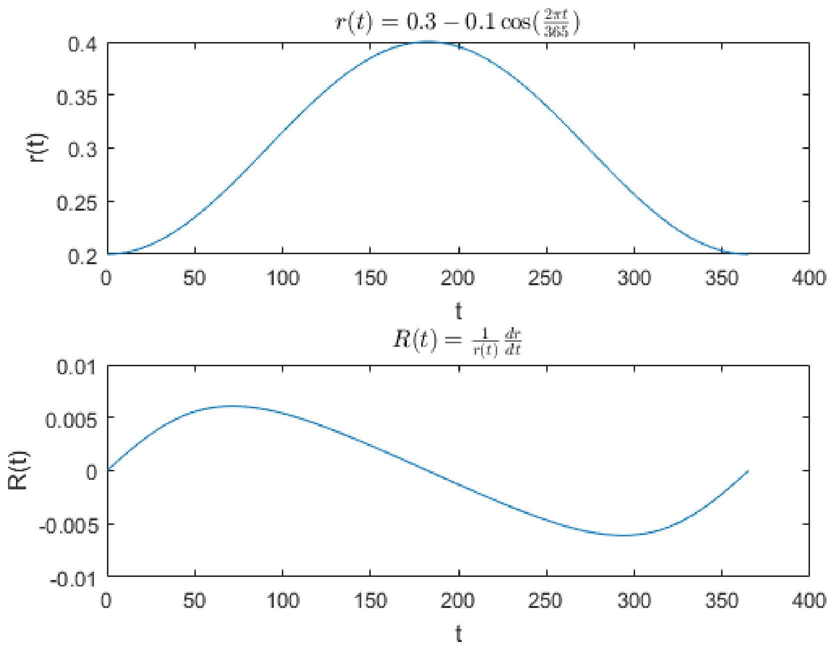
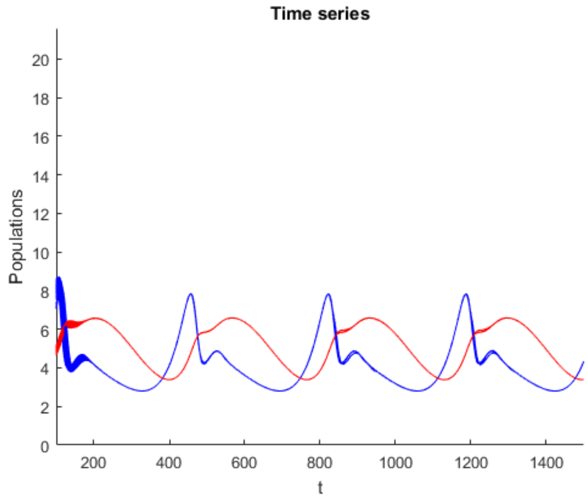


Figure 2: Plots of the forcing function designed to replace r in equation (2) as well as a modified variant

3 Question 3

As a starting point, observe what happens with our existing model when we implement $r(t)$ as it was originally given, with $A_0 = 0.1$, changing the initial conditions to be much closer to the fixed point: See fig3, below. 50 simulations with initial starting values at the fixed peak values from Q1 $\pm unif(0, 1)$ were used. We can see small peaks in the (blue) phytoplankton, roughly twice the fixed point value from Q1. This doesn't seem much like blooming behaviour. By contrast we also show what the plot looks like if $A_0 = 0.2$, where we see peaks over 20 times the fixed point population lasting approximately 100 days (slightly over 100 days, so closer to 4 months than 3).

A bloom can be defined in one of three ways: *increase* in population (or concentration or biomass) relative to some standard level (we would use the fixed point value of 4.1), *duration* of the increase, and *suddenness* with which the increase occurs (i.e. the steepness of the curve). For example in the right hand side plot in fig3 the increase is very sudden, to a very high value relative to the norm, and persists for over 100 days. We increase A_0



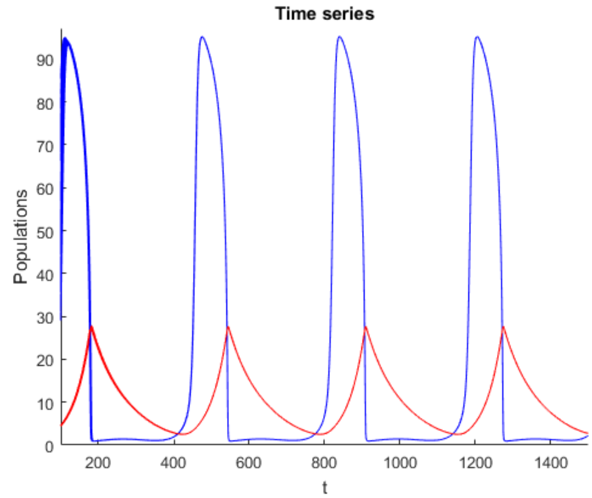


Figure 3: Left: Truscott and Brindley with periodic forcing exactly as in Q2. Right: A_0 increased to 0.2

from 1.000 to 2.000 in increments of 0.001, keeping in mind the following ad-hoc definition of a bloom:

Definition: We define a bloom as a periodic increase in the population of phytoplankton to a value 10 or more times the steady state population described in question (1), and persisting for a duration of at least 60 days as measured by the base of event on the plot, i.e. the point where the event starts to the point where it ends.

It is also worthwhile to distinguish between a single or multiple blooms as an event in the simulation as opposed to sustained blooms. We will give the value where the first transition into bloom-like behaviour occurs as well as the value that marks the beginning of sustained blooms; for this we need to run the simulation over a longer period, 3700 days. We will also ignore the initial behaviour of the system (first 200 days of the plot) as not being truly indicative, since the system has to adjust to the varying initial conditions. So if the first peak is very large it will be ignored unless repeated. This gives us the series of plots in fig4.

The answer to the question, as can be seen from the figures, is slightly complicated. The strict answer to the question, i.e. the value of A_0 required to trigger a *single* bloom, would be $A_0 = 0.1067$.
However, I think it's significant to note that the actual behaviour, while less impactful at first, starts slightly earlier at $A_0 = 0.1062$. And further only becomes large enough to permanently ensure sustained bloom events at $A_0 = 0.1076$ (This is where the bifurcation occurs and the limit cycle is formed).

4 Question 4

The required plots for part (a) were pulled directly from the same file used to stochastically force μ for Truscott and Brindley's system (equations (2) and (3) from question1). See fig.5, below. The file is A3Q4.m, the lines

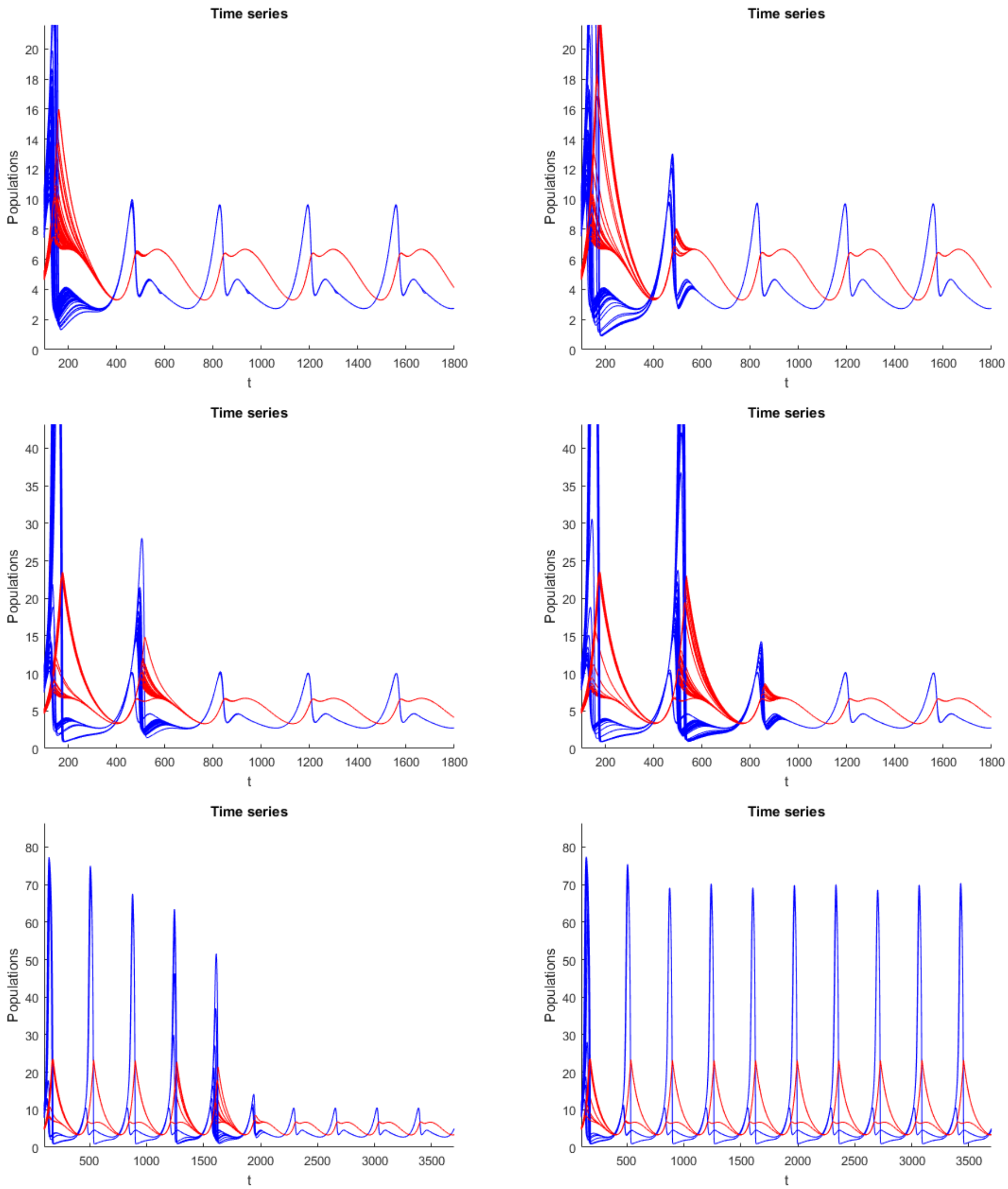


Figure 4: Top left, $A_0 = 0.1062$ initial conditions are perturbed but quickly settle into a regular pattern. Top right: $A_0 = 0.1064$, blooming behaviour threshold; this is not a bloom by our definition, but the first peak after the initial conditions is definitely affected. Middle left: $A_0 = 1.069$ the first value at which the first peak is significantly increased over the fixed point value from Q1. Middle Right: $A_0 = 0.1070$, first single event bloom. Bottom left: $A_0 = 0.1075$, we have several individual blooming events but the pattern is not persistent. Bottom right: $A_0 = 0.1076$, bloom events are large and persistent over long time scales.

used to generate the plots are lines 35 and 36, currently commented out. To reproduce the plots uncomment these two lines and instead comment out lines 31-33.

File A3Q4.m calls A3Q3.m to run. In A3Q3.m, which is the same file used in Question(3), now set A_0 to zero, so that r is again constant. in line 33 of A3Q4.m comment out line 20 (which sets random initial conditions) and uncomment line 21 and use it instead. This will reproduce a figure similar to fig.6 below. Note that 3 plots only

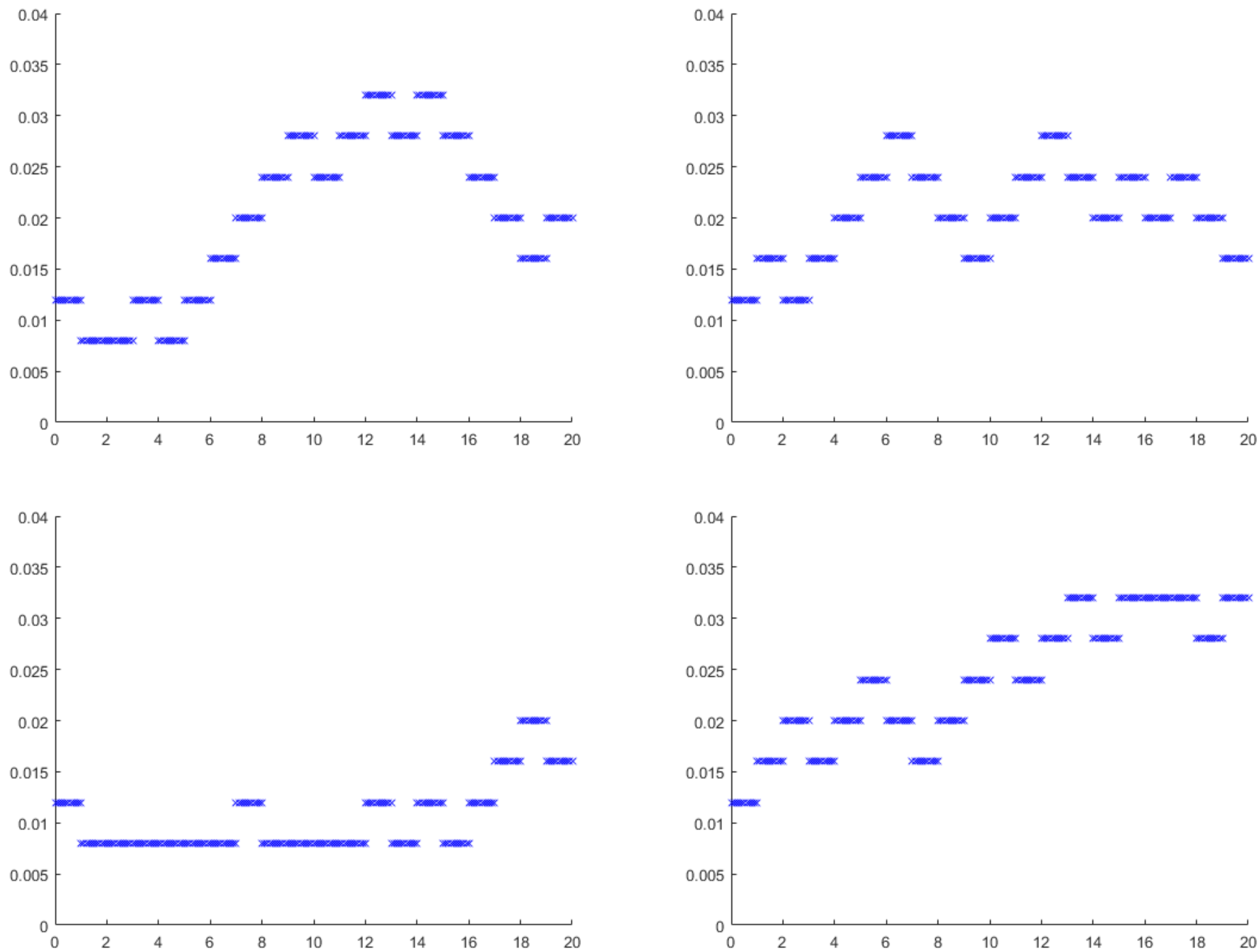


Figure 5: four plots of mu following a random walk. the period of 20 days was chosen to be long enough to show the randomness and the lower (0.0080) and upper (0.032) bounds of the function without making the plot too busy.

were produced, enough to see that despite all equations starting from exactly the same values the outcomes were very different, checking stochasticity.

In order to estimate the expected time until a bloom is triggered, I thought to plot 1,000 instances of the simulation and then take advantage of the central limit theorem, since the start times for blooms would now be normally distributed and the expected start time could be read from the centre of the range of times a first bloom occurred.

In order to cut down the runtime the simulation was plotted only for the first 150 days, as it is already obvious from figure 6 that most blooms created by forcing mu in this way end before that time. While Matlab did plot this figure, (i) some blooms seemed to start past the 150 day mark, (ii)the file would not save without crashing. So I used a smaller sample size (n=100) and a longer range (200 days). This still crashed matlab, but I was able to save a screencapture which resulted in figure 7 above. If we define "trigger a bloom" as being the time when the population rises above 40, then discounting outliers the range of 80 percent or so of the blooms are triggered in the t=40-100 range (t in days). If we again assume these are normally distributed the centre of the bell curve

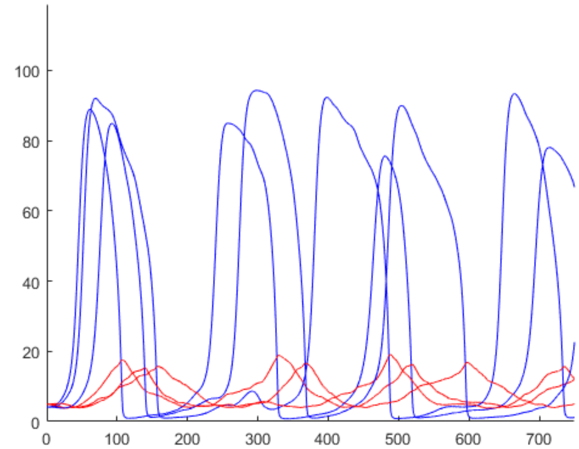


Figure 6: 3 independent plots of the Truscott and Brindley system with stochastically forced mu, plotted for 750 days (just over two years). blue = phytoplankton, red = zooplankton.

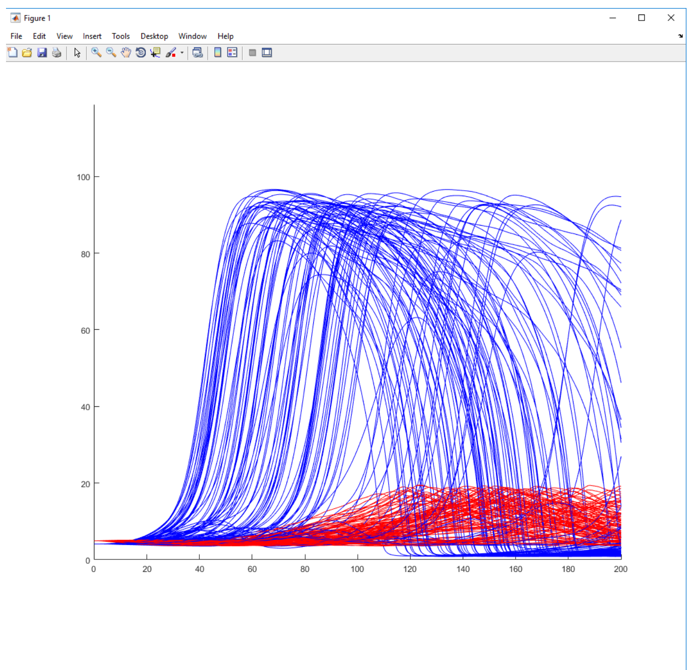


Figure 7: 100 simulations of stochastic forcing of mu over 200 days.

would be at 60 days. Finally, If we allow for the fact that there are some outliers to the right of our range but none to the left, we would suggest the expected time for a first bloom was somewhere in the 60-65 day range.

5 Question 5

Section 7.1 of the 2015/2016 lecture notes for Stochastic Processes taught by Gunther Delius define a stochastic differential equation as an equation of the form

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t)$$

If we for a moment ignore the presence of the variable Z in the rewritten equation(2), or imagine it as being fixed for some appropriate time scale, then the rewritten equation is of almost exactly this form. The solution to a Stochastic differential equation is a diffusion process with drift rate μ and diffusion rate σ^2; so in this case this that would mean we have replaced the variable for the phytoplankton population with such a diffusion process in a similar way. The Euler Maruyama method attempts to solve this problem numerically.

The only file with this question is A3Q5em.m, but it calls on the most general form of Truscott and Brindley's model implemented which is A3Q3.m so that needs to be in the path for it to run. The first two plots are single runs of this file:

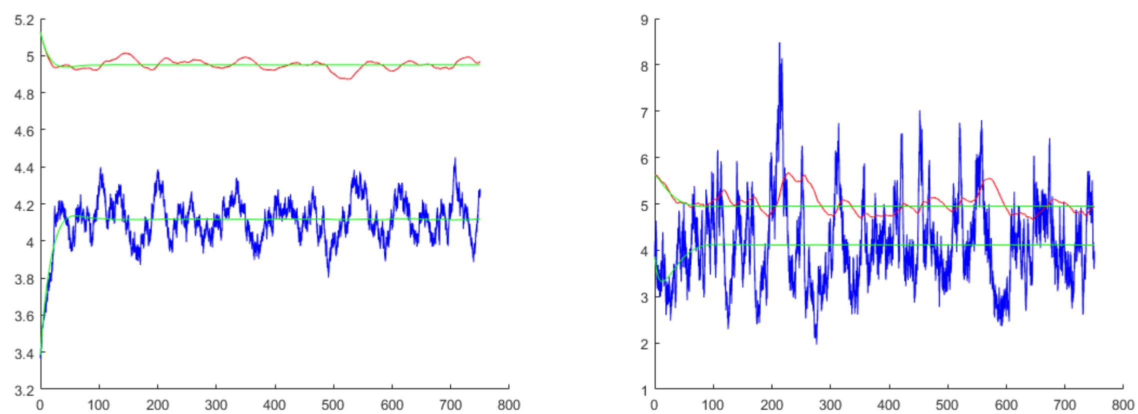


Figure 8: In the Left hand column, $\epsilon = 0.01$, in the Right hand column, $\epsilon = 0.1$. No parameter forcing of any kind was used.

Because the system is now stochastic, each run of the simulation could be very different and so multiple trials are needed to ensure the results are not just outliers. In each of the following cases, therefore, and for the required values $\epsilon = 0.01$ and $\epsilon = 0.1$, five trials were run for $t = 750$ days and the results plotted with initial conditions set at the fixed point values from question1 plus or minus a uniform random variable between zero and one. While we lose some clarity doing this, we gain a sense of the general behaviour of the system.

The first set of trials were a repeat of the above trails for $A_0 = 0$, representing zero parameter forcing as in question 1;

We can see already that while the stochastic variations for $\epsilon = 0.01$ were fairly modest, extreme noise was observed for $\epsilon = 0.1$. At times the noise spiked to over twice the steady state value.

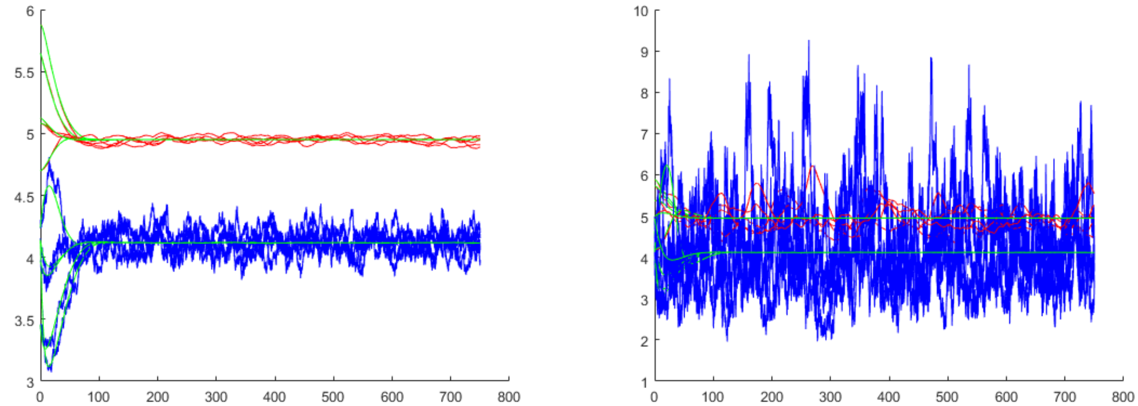


Figure 9: n=5 runs with random starting values close to the fixed point with $A_0 = 0$, i.e. no parameter forcing. In the Left hand column, $\epsilon = 0.01$, in the Right hand column, $\epsilon = 0.1$. Blue is the stochastic Phytoplankton P, Red is the resulting zooplankton Z, Green are the deterministic values as seen in previous questions.

Next we repeated the simulations for small amounts of R forcing, i.e. for $A_0 = 0.1000$, below the threshold limit of 1.060. At this point the Euler-Maruyama code seemed unable to closely track the deterministic solutions (mildly troubling). Otherwise, similar phenomena was observed as before; for $\epsilon = 0.01$ the solutions track a steady state, for $\epsilon = 0.1$ behaviour was much more erratic and now we see what look like bloom events:

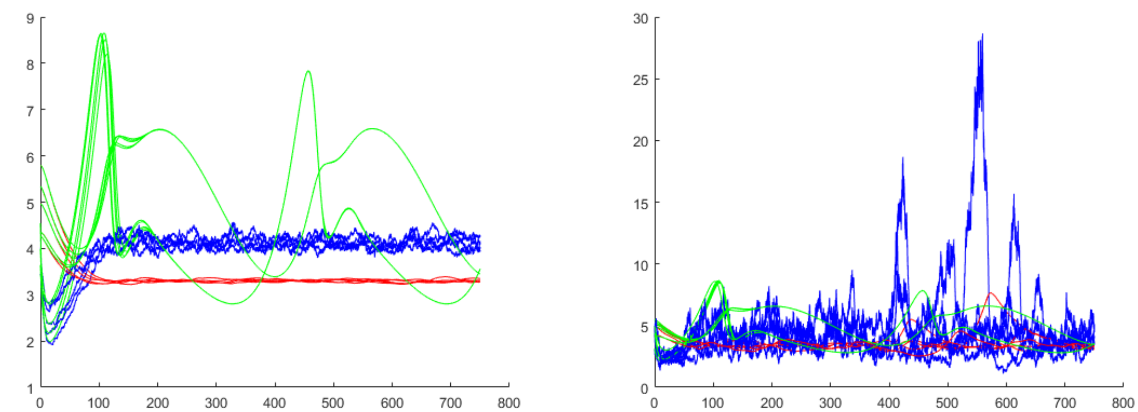


Figure 10: In the Left hand column, $\epsilon = 0.01$, in the Right hand column, $\epsilon = 0.1$. Subcritical forcing of R ($A_0 = 0.1000$). Blue is the stochastic Phytoplankton P, Red is the resulting zooplankton Z, Green are the deterministic values. 5 trial runs presented.

If we increase the forcing past the critical value of $A_0 = 0.1070$ seen in question3, the bahaviour is even more dramatic. Once again, for $\epsilon = 0.01$ the stochastic solution tracks the deterministic one only until it reaches the fixed point value, then seems to stay there. But now for $\epsilon = 0.1$ we see huge spikes that have the shape of bloom events, are usually larger than the deterministic bloom events and last about as long:

The common theme with all these simulations is that for small values of ϵ the solutions gravitate rapidly towards fixed point values and for large values of epsilon behaviour similar to the forcing we have seen elsewhere develops. As a last experiment therefore we look at what happens when epsilon is increased beyond the two

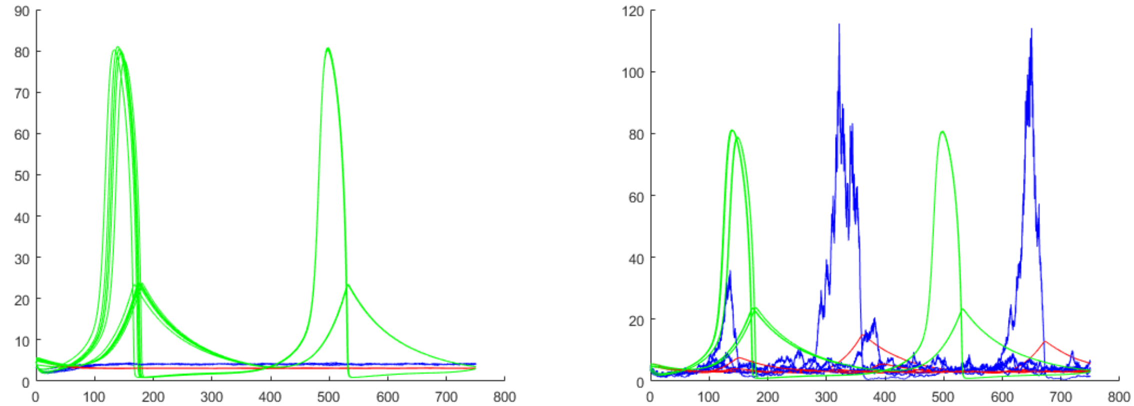


Figure 11: In the Left hand column, $\epsilon = 0.01$, in the Right hand column, $\epsilon = 0.1$. Supercritical forcing causing blooms. Blue is the stochastic Phytoplankton P, Red is the resulting zooplankton Z, Green are the deterministic values as seen in previous questions.

values suggested in the question.

The last figure is what happens when ϵ is increased to 0.2, with no other forcing ($A_0 = 0.000$)and we provide a single run and a run of five simulations:

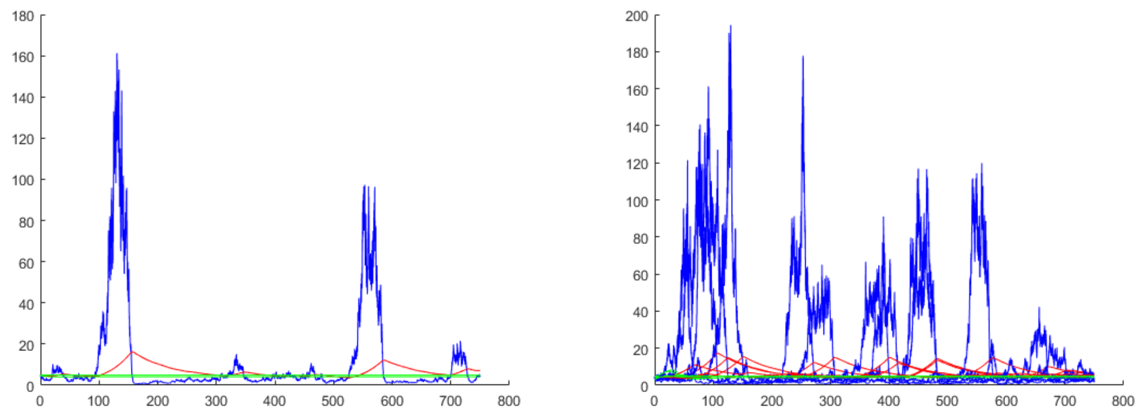


Figure 12: $\epsilon = 0.2$. Left, one single run: Right, five runs. Note the stochastic P does not merely spike but actually stays at the elevated value, simulating bloom behaviour.

The previous questions have focused on what happens when the system described by Truscott and Brindley experiences periodic parameter forcing in various different ways, but here we no longer force parts of the equation but replace P itself with a diffusion process $P = P(t, W(t))$, subjecting it to random stochastic fluctuations. What we have seen is that this, too, can induce bloom-like behaviour in the model, as could probably be predicted from the phase plane behaviour explained in figure 1 of section 3 page 983 of their original paper. Any perturbation, whether due to forcing parameters R, μ or by allowing white noise to affect P significantly, seems to push the solutions in a way that Truscott and Brindley describe as a "superthreshold perturbation", and *however* this happens it results in some kind of bloom. The results over the course of all the questions in this assignment seem to confirm this.