A ADDITIONAL PROOFS

Theorem 1. *Least Expected Regret Winner is equivalent to MEW.*

Proof. Let $\boldsymbol{P}^{\mathsf{M}}$ denote a general voting profile with $\Omega(\boldsymbol{P}^{\mathsf{M}}) = \{\boldsymbol{P}_1, \ldots, \boldsymbol{P}_z\}$. The expected regret of a candidate w can be rewritten as follows.

$$\begin{aligned}
&\mathbb{E}(\texttt{Regret}(w, \boldsymbol{P}^{\mathsf{M}})) \\
&= \sum_{i=1}^{z} \texttt{Regret}(c, \boldsymbol{P}_i) \cdot \Pr(\boldsymbol{P}_i \mid \boldsymbol{P}^{\mathsf{M}}) \\
&= \sum_{i=1}^{z} \left(\max_{c \in C} s(c, \boldsymbol{P}_i) - s(w, \boldsymbol{P}_i) \right) \cdot \Pr(\boldsymbol{P}_i \mid \boldsymbol{P}^{\mathsf{M}}) \\
&= \sum_{i=1}^{z} \max_{c \in C} s(c, \boldsymbol{P}_i) \cdot \Pr(\boldsymbol{P}_i \mid \boldsymbol{P}^{\mathsf{M}}) - \mathbb{E}(s(w, \boldsymbol{P}^{\mathsf{M}}))
\end{aligned}$$

The first term $\sum_{i=1}^{z} \max_{c \in C} s(c, \boldsymbol{P}_i) \cdot \Pr(\boldsymbol{P}_i \mid \boldsymbol{P}^{\mathsf{M}})$ is a constant value, when $\boldsymbol{P}^{\mathsf{M}}$ and the voting rule are fixed. Thus, $\mathbb{E}(\texttt{Regret}(w, \boldsymbol{P}^{\mathsf{M}}))$ is minimized by maximizing $\mathbb{E}(s(w, \boldsymbol{P}^{\mathsf{M}}))$, the expected score of the candidate w. □

Theorem 2. *Meta-Election Winner is equivalent to MEW.*

Proof. Let $\boldsymbol{P}^{\mathsf{M}}$ denote a general voting profile with $\Omega(\boldsymbol{P}^{\mathsf{M}}) = \{\boldsymbol{P}_1, \ldots, \boldsymbol{P}_z\}$, and $\boldsymbol{P}_{meta} = (\boldsymbol{P}_1, \ldots, \boldsymbol{P}_z)$ denote the large meta profile where rankings in $\boldsymbol{P}_i$ are weighted by $\Pr(\boldsymbol{P}_i \mid \boldsymbol{P}^{\mathsf{M}})$. According to the definition of the Meta-Election Winner, $s(w, \boldsymbol{P}_{meta}) = \max_{c \in C} s(c, \boldsymbol{P}_{meta})$. As a result, for any candidate c,

$$\mathbb{E}(s(c, \boldsymbol{P}^{\mathsf{M}})) = \sum_{\boldsymbol{P} \in \Omega(\boldsymbol{P}^{\mathsf{M}})} s(c, \boldsymbol{P}) \cdot \Pr(\boldsymbol{P} \mid \boldsymbol{P}^{\mathsf{M}}) = s(c, \boldsymbol{P}_{meta})$$

Her expected score in $\boldsymbol{P}^{\mathsf{M}}$ is precisely her score in $\boldsymbol{P}_{meta}$. The two winner definitions are optimizing the same metric. □

Theorem 3. *The FCP is #P-complete.*

Proof. First, we prove its membership in #P. The FCP is the counting version of the following decision problem: given a partial order $\boldsymbol{\nu}$, an item c, and an integer j, determine whether $\boldsymbol{\nu}$ has a linear extension $\boldsymbol{\tau} \in \Omega(\boldsymbol{\nu})$ where c is ranked at j. This decision problem is obviously in NP, meaning that the FCP is in #P.

Then, we prove that the FCP is #P-hard by reduction. Recall that counting $|\Omega(\boldsymbol{\nu})|$, the number of linear extensions of a partial order $\boldsymbol{\nu}$, is #P-complete [3]. This problem can be reduced to the FCP by $|\Omega(\boldsymbol{\nu})| = \sum_{j=1}^{m} N(c@j \mid \boldsymbol{\nu})$.

In conclusion, the FCP is #P-complete. □

Lemma 1. *If ranking model* M *is a partial order* $\boldsymbol{\nu}$ *of* m *items representing a uniform distribution of* $\Omega(\boldsymbol{\nu})$*, the REP-t is* $FP^{\#P}$*-complete.*

Proof. First, we prove that the REP-t is in $\text{FP}^{\#P}$. Recall that $\Omega(\boldsymbol{\nu})$ is the linear extensions of a partial order $\boldsymbol{\nu}$, and $N(c@1|\boldsymbol{\nu})$ is the number of linear extensions in $\Omega(\boldsymbol{\nu})$ where candidate c is at rank 1. Then $\Pr(c@1|\boldsymbol{\nu}) = N(c@1|\boldsymbol{\nu})/|\Omega(\boldsymbol{\nu})|$. Consider that counting $N(c@1 \mid \boldsymbol{\nu})$ is in #P (Theorem 3) and counting $|\Omega(\boldsymbol{\nu})|$ is #P-complete [3], so $\Pr(c@1|\boldsymbol{\nu})$ is in $\text{FP}^{\#P}$.

In the rest of this proof, we prove that the REP-t is #P-hard by reduction from the #P-complete problem of counting $|\Omega(\boldsymbol{\nu})|$.

Let c^* denote an item that has no parent in $\boldsymbol{\nu}$. Let $\boldsymbol{\nu}_{-c^*}$ denote the partial order of $\boldsymbol{\nu}$ with item c^* removed. If we are interested in the probability that c^* is placed at rank 1, we can write $\Pr(c^*@1|\boldsymbol{\nu}) = N(c^*@1 \mid \boldsymbol{\nu})/|\Omega(\boldsymbol{\nu})|$. The item c^* has been fixed at rank 1, so any placement of the rest items will definitely satisfy any relative order involving c^*. That is to say, the placement of the rest items just needs to satisfy $\boldsymbol{\nu}_{-c^*}$, which leads to $N(c^*@1 \mid \boldsymbol{\nu}) = |\Omega(\boldsymbol{\nu}_{-c^*})|$.

For example, let $\boldsymbol{\nu}' = \{c_1 \succ c_4, c_2 \succ c_4, c_3 \succ c_4\}$. Then $N(c_1@1 \mid \boldsymbol{\nu}') = |\Omega(\boldsymbol{\nu}'_{-c_1})| = |\Omega(\{c_2 \succ c_4, c_3 \succ c_4\})|$.

Then we re-write $\Pr(c^*@1|\boldsymbol{\nu}) = N(c^*@1|\boldsymbol{\nu})/|\Omega(\boldsymbol{\nu})| = |\Omega(\boldsymbol{\nu}_{-c^*})|/|\Omega(\boldsymbol{\nu})|$. The oracle for $\Pr(c^*@1|\boldsymbol{\nu})$ manages to reduce the size of the counting problem from $|\Omega(\boldsymbol{\nu})|$ to $|\Omega(\boldsymbol{\nu}_{-c^*})|$. This oracle should be as hard as counting $|\Omega(\boldsymbol{\nu})|$. Thus calculating $\Pr(c^*@1|\boldsymbol{\nu})$ is $\text{FP}^{\#P}$-hard.

In conclusion, the REP-t is $\text{FP}^{\#P}$-complete. □

Lemma 2. *If ranking model* M *is a partial order* $\boldsymbol{\nu}$ *of* m *items representing a uniform distribution of* $\Omega(\boldsymbol{\nu})$*, the REP-b is* $FP^{\#P}$*-complete.*

Proof. This proof adopts the same approach as the proof of Lemma 1.

Let m be the number of items in the ranking model M. For the membership proof that the REP-b is in $\text{FP}^{\#P}$, let $N(c@m|\boldsymbol{\nu})$ denote the number of linear extensions in $\Omega(\boldsymbol{\nu})$ where candidate c is at the bottom rank m. Then $\Pr(c@m|\boldsymbol{\nu}) = N(c@m|\boldsymbol{\nu})/|\Omega(\boldsymbol{\nu})|$. Consider that counting $N(c@m|\boldsymbol{\nu})$ is in #P (Theorem 3) and counting $|\Omega(\boldsymbol{\nu})|$ is #P-complete [3], so $\Pr(c@m \mid \boldsymbol{\nu})$ is in $\text{FP}^{\#P}$.

In the proof of Lemma 1, item c^* is an item with no parent in the partial order $\boldsymbol{\nu}$. In the current proof, item c^* is set to be an item with no child in $\boldsymbol{\nu}$. The $\boldsymbol{\nu}_{-c^*}$ still denotes the partial order of $\boldsymbol{\nu}$ but with item c^* removed. Then the probability that item c^* at the bottom rank m is $\Pr(c^*@m \mid \boldsymbol{\nu}) = N(c^*@m \mid \boldsymbol{\nu})/|\Omega(\boldsymbol{\nu})| = |\Omega(\boldsymbol{\nu}_{-c^*})|/|\Omega(\boldsymbol{\nu})|$. The oracle for $\Pr(c^*@m \mid \boldsymbol{\nu})$ manages to reduce the size of the counting problem again from $|\Omega(\boldsymbol{\nu})|$ to $|\Omega(\boldsymbol{\nu}_{-c^*})|$. Thus, this oracle is #P-hard, and calculating $\Pr(c^*@m \mid \boldsymbol{\nu})$ is $\text{FP}^{\#P}$-hard.

In conclusion, the REP-b is $\text{FP}^{\#P}$-complete. □

Theorem 4. *If ranking model* M *is a partial order* $\boldsymbol{\nu}$ *of* m *items representing a uniform distribution of* $\Omega(\boldsymbol{\nu})$*, the REP is* $FP^{\#P}$*-complete.*

Proof. First, we prove that the REP is in $\text{FP}^{\#P}$. Recall that $\Omega(\boldsymbol{\nu})$ is the linear extensions of a partial order $\boldsymbol{\nu}$, and $N(c@j|\boldsymbol{\nu})$ is the number of linear extensions in $\Omega(\boldsymbol{\nu})$ where candidate c is at rank j. Then $\Pr(c@j|\boldsymbol{\nu}) = N(c@j|\boldsymbol{\nu})/|\Omega(\boldsymbol{\nu})|$. Consider that counting $N(c@j|\boldsymbol{\nu})$ is #P-complete (Theorem 3) and counting $|\Omega(\boldsymbol{\nu})|$ is #P-complete [3] as well. So $\Pr(c@j|\boldsymbol{\nu})$ is in $\text{FP}^{\#P}$.

Lemma 1 demonstrates that REP-t, a special case of REP, is $\text{FP}^{\#P}$-hard. Thus REP is #P-hard as well.

In conclusion, REP is $\text{FP}^{\#P}$-complete. □

Theorem 5. *Given a general voting profile* $\boldsymbol{P}^{\mathsf{M}}$ *and a positional scoring rule* $\boldsymbol{r}_m$*, the ESC problem can be reduced to the REP.*

Proof. Recall that the MEW w maximizes the expected score, i.e.,

$$s(w, \boldsymbol{P}^{\mathsf{M}}) = \max_{c \in C} \mathbb{E}(s(c, \boldsymbol{P}^{\mathsf{M}}))$$

The voting profile $\boldsymbol{P}^{\mathsf{M}}$ contains n ranking distributions $\{\mathsf{M}_1, \ldots, \mathsf{M}_n\}$, so

$$\mathbb{E}(s(c, \boldsymbol{P}^{\mathsf{M}})) = \sum_{i=1}^{n} \mathbb{E}(s(c, \mathsf{M}_i))$$

where $\mathbb{E}(s(c, \mathsf{M}_i))$ is the expected score of c from voter v_i.

$$\mathbb{E}(s(c, \mathsf{M}_i)) = \sum_{j=1}^{m} Pr(c@j \mid \mathsf{M}_i) \cdot \boldsymbol{r}_m(j)$$

where $c@j$ denotes candidate c at rank j, and $\boldsymbol{r}_m(j)$ is the score of rank j.

Let $\mathbb{T}$ denote the complexity of calculating $Pr(c@j|\mathsf{M}_i)$. The original MEW problem can be solved by calculating $Pr(c@j|\mathsf{M}_i)$ for all m candidates, m ranks and n voters, which leads to the complexity of $O(n \cdot m^2 \cdot \mathbb{T})$. □

Theorem 6. *The REP for rank k is equivalent to the ESC problem over either one or both of the $(k-1)$-approval and k-approval rules.*

Proof. The ESC problem has been reduced to the REP (Theorem 5). This proof will focus on the other direction, i.e., reducing the REP to the ESC problem.

Let $\Pr(c@j \mid \mathsf{M})$ denote the probability of placing candidate c at rank j over a ranking distribution M. Let $\boldsymbol{P}^{\mathsf{M}}$ denote a single-voter profile consisting of only this ranking distribution M.

When $k = 1$, the REP can be reduced to solving the ESC problem under plurality or 1-approval rule.

$$\Pr(c@1 \mid \mathsf{M}) = \mathbb{E}(s(c \mid \boldsymbol{P}^{\mathsf{M}}, \text{1-approval}))$$

When $k = m$, the REP can be reduced to solving the ESC problem under veto or $(m-1)$-approval rule.

$$\Pr(c@m \mid \mathsf{M}) = 1 - \mathbb{E}(s(c \mid \boldsymbol{P}^{\mathsf{M}}, (m-1)\text{-approval}))$$

When $2 \le k \le m$, the REP can be reduced to solving the ESC problem twice under k-approval and $(k-1)$-approval rules.

$$\begin{aligned}\Pr(c@k \mid \mathsf{M}) &= \mathbb{E}(s(c \mid \boldsymbol{P}^{\mathsf{M}}, k\text{-approval})) \\ &\quad - \mathbb{E}(s(c \mid \boldsymbol{P}^{\mathsf{M}}, (k-1)\text{-approval}))\end{aligned}$$

□

Theorem 7. *Given a partial voting profile $\boldsymbol{P}^{PO}$, a distinguished candidate c, and plurality rule $\boldsymbol{r}_m$, the ESC problem of calculating $\mathbb{E}(s(c \mid \boldsymbol{P}^{PO}, \boldsymbol{r}_m))$ is $FP^{\#P}$-complete.*

Proof. Firstly, we prove the membership of the ESC problem as an $\text{FP}^{\#P}$ problem. Consider that the REP is $\text{FP}^{\#P}$-complete over partial orders (Theorem 4), and the ESC problem can be reduced to the REP (Theorem 5) So the ESC problem is in $\text{FP}^{\#P}$ for partial voting profiles.

Secondly, we prove that the ESC problem is $\text{FP}^{\#P}$-hard, even for plurality rule, by reduction from the REP-t that is $\text{FP}^{\#P}$-hard (Lemma 1).

Let $\boldsymbol{\nu}$ denote the partial order of the REP-t problem. Recall that the REP-t problem aims to calculate $\Pr(c@1 \mid \boldsymbol{\nu})$ for a given item c. Let $\boldsymbol{P}^{\boldsymbol{\nu}}$ denote a voting profile consisting of just this partial order $\boldsymbol{\nu}$. The answer to the REP-t problem is the same as the answer to the corresponding ESC problem, i.e., $\Pr(c@1 \mid \boldsymbol{\nu}) = \mathbb{E}(s(c \mid \boldsymbol{P}^{\boldsymbol{\nu}}, \text{plurality}))$. So the ESC problem is $\text{FP}^{\#P}$-hard, even for plurality voting rule.

In conclusion, the ESC problem is $\text{FP}^{\#P}$-complete, under plurality rule. □

Theorem 8. *Given a partial voting profile $\boldsymbol{P}^{PO}$, a distinguished candidate c, and veto rule $\boldsymbol{r}_m$, the ESC problem of calculating $\mathbb{E}(s(c \mid \boldsymbol{P}^{PO}, \boldsymbol{r}_m))$ is $FP^{\#P}$-complete.*

Proof. This proof adopts the same approach as the proof of Theorem 7.

Firstly, the membership proof that the ESC is in $\text{FP}^{\#P}$ is based on the conclusions that the REP is $\text{FP}^{\#P}$-complete over partial orders (Theorem 4), and that the ESC can be reduced to the REP (Theorem 5) So the ESC is in $\text{FP}^{\#P}$ for partial voting profiles.

Secondly, we prove that the ESC is $\text{FP}^{\#P}$-hard, under veto voting rule, by reduction from the REP-b that is $\text{FP}^{\#P}$-hard (Lemma 2).

Let $\boldsymbol{\nu}$ denote the partial order of the REP-b problem. Recall that the REP-b problem aims to calculate $\Pr(c@m \mid \boldsymbol{\nu})$ for a given item c. Let $\boldsymbol{P}^{\boldsymbol{\nu}}$ denote a voting profile consisting of just this partial order $\boldsymbol{\nu}$. The answer to the ESC indirectly solves the REP-b, i.e., $\Pr(c@m \mid \boldsymbol{\nu}) = 1 - \mathbb{E}(s(c \mid \boldsymbol{P}^{\boldsymbol{\nu}}, \text{veto}))$. So the ESC problem is $\text{FP}^{\#P}$-hard under veto rule.

In conclusion, the ESC is $\text{FP}^{\#P}$-complete, under veto rule. □

Theorem 9. *Given a partial voting profile $\boldsymbol{P}^{PO}$, a distinguished candidate c, and k-approval rule $\boldsymbol{r}_m$, the ESC problem of calculating $\mathbb{E}(s(c \mid \boldsymbol{P}^{PO}, \boldsymbol{r}_m))$ is $FP^{\#P}$-complete.*

Proof. Firstly, the proof that the Expected Score Computation (ESC) is in $\text{FP}^{\#P}$ is the same as the proof of Theorem 7. Now we prove that the ESC problem is $\text{FP}^{\#P}$-hard, under k-approval rule $\boldsymbol{r}_m$, by reduction from the REP-t problem that is $\text{FP}^{\#P}$-hard (Lemma 1).

Let $\boldsymbol{\nu}$ denote the partial order of the REP-t problem. Recall that the REP-t problem aims to calculate $\Pr(c@1 \mid \boldsymbol{\nu})$ for a given item c. Let $\boldsymbol{\nu}_+$ denote a new partial order by inserting $(k-1)$ ordered items $d_1 \succ \ldots \succ d_{k-1}$ into $\boldsymbol{\nu}$ such that item d_{k-1} is preferred to every item in $\boldsymbol{\nu}$. Such placement of items $\{d_1, \ldots, d_{k-1}\}$ is to guarantee that all linear extensions of $\boldsymbol{\nu}_+$ start with $d_1 \succ \ldots \succ d_{k-1}$ and these linear extensions will be precisely the linear extensions of $\boldsymbol{\nu}$ after removing $\{d_1, \ldots, d_{k-1}\}$.

Let $\boldsymbol{P}^{\boldsymbol{\nu}_+}$ denote a voting profile consisting of just this partial order $\boldsymbol{\nu}_+$. The answer to the ESC problem for item c is $\mathbb{E}(s(c \mid \boldsymbol{P}^{\boldsymbol{\nu}_+}, k\text{-approval}))$. Since there is only one partial order $\boldsymbol{\nu}_+$ in the voting profile, $\mathbb{E}(s(c \mid \boldsymbol{P}^{\boldsymbol{\nu}_+}, k\text{-approval})) = \sum_{j=1}^{k} \Pr(c@j \mid \boldsymbol{\nu}_+)$. Recall that any linear extension of $\boldsymbol{\nu}_+$ always starts with $d_1 \succ \ldots \succ d_{k-1}$, so $\forall 1 \le j \le (k-1), \Pr(c@j \mid \boldsymbol{\nu}_+) = 0$, which leads to $\mathbb{E}(s(c \mid \boldsymbol{P}^{\boldsymbol{\nu}_+}, k\text{-approval})) = \Pr(c@k \mid \boldsymbol{\nu}_+)$. Since $\boldsymbol{\nu}_+$ is constructed by inserting $(k-1)$ items before items in $\boldsymbol{\nu}$, $\Pr(c@k \mid \boldsymbol{\nu}_+) = \Pr(c@1 \mid \boldsymbol{\nu})$. So $\mathbb{E}(s(c \mid \boldsymbol{P}^{\boldsymbol{\nu}_+}, k\text{-approval})) = \Pr(c@1 \mid \boldsymbol{\nu})$. The answer to the REP-t problem has been reduced to the ESC problem. So the ESC problem is $\text{FP}^{\#P}$-hard, under k-approval rule.

In conclusion, the ESC problem is $\text{FP}^{\#P}$-complete, under k-approval rule. □

Theorem 10. *Given a positional scoring rule $\boldsymbol{r}_m$, a fully partitioned voting profile $\boldsymbol{P}^{FP} = (\boldsymbol{\nu}_1^{FP}, \ldots, \boldsymbol{\nu}_n^{FP})$, and candidate w, determining $w \in MEW(\boldsymbol{r}_m, \boldsymbol{P}^{FP})$ is in $O(nm^2)$.*

Proof. Any $\boldsymbol{\nu}^{\mathrm{FP}} \in \boldsymbol{P}^{\mathrm{FP}}$ defines a set of consecutive ranks in the linear extensions of $\boldsymbol{\nu}^{\mathrm{FP}}$ for each of its partitions of candidates. Any candidate is equally likely to be positioned at these ranks. So the REP can be solved in $O(1)$ for any candidate. Thus, the MEW problem can be solved in $O(nm^2)$ by calculating the expected scores of all candidates. □

Theorem 11. *Given a positional scoring rule $\boldsymbol{r}_m$, a partial chain voting profile $\boldsymbol{P}^{PC} = (\boldsymbol{\nu}_1^{PC}, \ldots, \boldsymbol{\nu}_n^{PC})$, and candidate w, determining $w \in MEW(\boldsymbol{r}_m, \boldsymbol{P}^{PC})$ is in $O(nm^2)$.*

Proof. For any $\boldsymbol{\nu}^{\mathrm{PC}} \in \boldsymbol{P}^{\mathrm{PC}}$ and any candidate c, the $\Pr(c \rightarrow j \mid \boldsymbol{\nu}^{\mathrm{PC}})$ is proportional to the degree of freedom to place the rest of the candidates, after fixing c at rank j.

- If $c \notin \boldsymbol{\nu}^{\mathrm{PC}}$, this is a trivial case where c is equally likely to be placed at any rank, thus $\forall 1 \leq j \leq m, \Pr(c \rightarrow j \mid \boldsymbol{\nu}^{\mathrm{PC}}) = 1/m$.
- If $c \in \boldsymbol{\nu}^{\mathrm{PC}}$, let $K_l = |\{c' \mid c' \succ_{\boldsymbol{\nu}^{\mathrm{PC}}} c\}|$ be the number of items preferred to c by $\boldsymbol{\nu}^{\mathrm{PC}}$ and $K_r = |\{c' \mid c \succ_{\boldsymbol{\nu}^{\mathrm{PC}}} c'\}|$ be the number of items less preferred to c by $\boldsymbol{\nu}^{\mathrm{PC}}$, then $\Pr(c \rightarrow j \mid \boldsymbol{\nu}^{\mathrm{PC}}) \propto \binom{j-1}{K_l} \cdot \binom{m-j}{K_r}$ where .

It takes $O(nm^2)$ to obtain the expected scores of all candidates and to determine whether w is a MEW. □

Theorem 12. *Given a positional scoring rule $\boldsymbol{r}_m$, a partially partitioned voting profile $\boldsymbol{P}^{PP} = (\boldsymbol{\nu}_1^{PP}, \ldots, \boldsymbol{\nu}_n^{PP})$, and candidate w, determining $w \in MEW(\boldsymbol{r}_m, \boldsymbol{P}^{PP})$ is in $O(nm^2)$.*

Proof. For any $\boldsymbol{\nu}^{\mathrm{PP}} \in \boldsymbol{P}^{\mathrm{PP}}$ and any candidate c, the $\Pr(c \rightarrow j \mid \boldsymbol{\nu}^{\mathrm{PP}})$ is proportional to the degree of freedom to place the rest of the candidates, after fixing c at rank j.

- If $c \notin \boldsymbol{\nu}^{\mathrm{PP}}$, this is a trivial case where c is equally likely to be placed at any rank, thus $\forall 1 \leq j \leq m, \Pr(c \rightarrow j \mid \boldsymbol{\nu}^{\mathrm{PC}}) = 1/m$.
- If $c \in \boldsymbol{\nu}^{\mathrm{PP}}$, let $K_l = |\{c' \mid c' \succ_{\boldsymbol{\nu}^{\mathrm{PP}}} c\}|$ be the number of items preferred to c by $\boldsymbol{\nu}^{\mathrm{PP}}$, $K_r = |\{c' \mid c \succ_{\boldsymbol{\nu}^{\mathrm{PP}}} c'\}|$ be the number of items less preferred to c by $\boldsymbol{\nu}^{\mathrm{PP}}$, and K_c be the number of items in the partition of c, then $\Pr(c \rightarrow j \mid \boldsymbol{\nu}^{\mathrm{PC}}) \propto \sum_{x=0}^{K_c-1} \binom{j-1}{K_l+x} \cdot \binom{m-j}{K_r+K_c-1-x}$ where x is the number of items from the same partition as c and placed to the left of c.

It takes $O(nm^2)$ to obtain the expected scores of all candidates and to determine whether w is a MEW. □

Theorem 13. *Given positional scoring rule $\boldsymbol{r}_m$, a RIM voting profile $\boldsymbol{P}^{\mathsf{RIM}} = (\mathsf{RIM}_1, \ldots, \mathsf{RIM}_n)$, and candidate w, determining $w \in MEW(\boldsymbol{r}_m, \boldsymbol{P}^{\mathsf{RIM}})$ is in $O(nm^4)$.*

Proof. Given any $\mathsf{RIM} \in \boldsymbol{P}^{\mathsf{RIM}}$ and any candidate c, the $Pr(c \rightarrow j \mid \mathsf{RIM})$ for $j = 1, \ldots, m$ can be calculated by Algorithm 3 in $O(m^3)$. Algorithm 3 is a variant of RIMDP [13]. RIMDP calculates the marginal probability of a partial order over RIM via Dynamic Programming (DP). Algorithm 3 is simplified RIMDP in the sense that Algorithm 3 only tracks a particular item c, while RIMDP tracks multiple items to calculate the insertion ranges of items that satisfy the partial order. Note that Algorithm 3 calculates all m different values of j simultaneously. So it takes $O(nm \cdot m^3) = O(nm^4)$ to obtain the expected scores of m candidates over n RIMs to determine MEW. □

Theorem 14. *Given a positional scoring rule $\boldsymbol{r}_m$, a voting profile $\boldsymbol{P}^{\mathsf{RIM}+TR} = \big((\mathsf{RIM}_1, \boldsymbol{\tau}_1^{(t_1,b_1)}), \ldots, (\mathsf{RIM}_n, \boldsymbol{\tau}_n^{(t_n,b_n)})\big)$, and candidate w, determining $w \in MEW(\boldsymbol{r}_m, \boldsymbol{P}^{\mathsf{RIM}+TR})$ is in $O(nm^4)$.*

Proof. Given any $(\mathsf{RIM}, \boldsymbol{\tau}^{(t,b)}) \in \boldsymbol{P}^{\mathsf{RIM+TR}}$, candidate c, and rank j, if c is in the top or bottom part of $\boldsymbol{\tau}^{(t,b)}$, its rank has been fixed, which is a trivial case; If c is in the middle part of $\boldsymbol{\tau}^{(t,b)}$, we just need to slightly modify Algorithm 3 to calculate $\Pr(c \rightarrow j \mid \mathsf{RIM}, \boldsymbol{\tau}^{(t,b)})$. Line 5 in Algorithm 3 enumerates values for j from 1 to i. The constraints made by $\boldsymbol{\tau}^{(t,b)}$ limits this insertion range of item $\boldsymbol{\sigma}(i)$. If $\boldsymbol{\sigma}(i)$ is in the top or bottom part of $\boldsymbol{\tau}^{(t,b)}$, its insertion position has been fixed by $\boldsymbol{\tau}^{(t,b)}$ and the inserted items of the top and bottom parts of $\boldsymbol{\tau}^{(t,b)}$ should be recorded as well by the state δ'; If $\boldsymbol{\sigma}(i)$ is in the middle part of $\boldsymbol{\tau}^{(t,b)}$, $\boldsymbol{\sigma}(i)$ can be inserted into any position between the inserted top and bottom items.

Theoretically, the algorithm needs to track as many as $(t+b+1)$ items. But $(t+b)$ items are fixed, which makes c the only item leading to multiple DP states. The complexity of calculating $\Pr(c \rightarrow j \mid \mathsf{RIM}, \boldsymbol{\tau}^{(t,b)})$ for all j values is $O(m^3)$. It takes $O(nm^4)$ to calculate the expected scores of all candidates across all voters to determine the MEW. □

Theorem 15. *Given a positional scoring rule $\boldsymbol{r}_m$, a voting profile $\boldsymbol{P}^{\mathsf{MAL}+FP} = \big((\mathsf{MAL}_1, \boldsymbol{\nu}_1^{FP}), \ldots, (\mathsf{MAL}_n, \boldsymbol{\nu}_n^{FP})\big)$, and candidate w, determining $w \in MEW(\boldsymbol{r}_m, \boldsymbol{P}^{\mathsf{MAL}+FP})$ is in $O(nm^4)$.*

Proof. Given any $(\mathsf{MAL}(\boldsymbol{\sigma}, \phi), \boldsymbol{\nu}^{\mathrm{FP}}) \in \boldsymbol{P}^{\mathsf{MAL+FP}}$, candidate c, and rank j, consider calculating $Pr(c \rightarrow j \mid \boldsymbol{\sigma}, \phi, \boldsymbol{\nu}^{\mathrm{FP}})$. Let C_P denote the set of candidates in the same partition with c in $\boldsymbol{\nu}^{\mathrm{FP}}$. The relative orders between c and items out of C_P are already determined by $\boldsymbol{\nu}^{\mathrm{FP}}$. That is to say, for a non-trivial j value, $Pr(c \rightarrow j \mid \boldsymbol{\sigma}, \phi, \boldsymbol{\nu}^{\mathrm{FP}})$ is proportional to the exponential of the number of disagreed pairs within C_P. So we can construct a new Mallows model $\mathsf{MAL}'(\boldsymbol{\sigma}', \phi)$ over C_P. It has the same ϕ as MAL and its reference ranking $\boldsymbol{\sigma}'$ is shorter than but consistent with $\boldsymbol{\sigma}$. The $Pr(c \rightarrow j \mid \mathsf{MAL}', \boldsymbol{\nu}^{\mathrm{FP}})$ for all non-trivial j values can be calculated in $O(|C_P|^3) < O(m^3)$ by Algorithm 3.

The MEW problem can be solved in $O(nm^4)$ by calculating the expected scores of all candidates across all voters to determine whether w is a MEW. □

B TRACTABILITY OVER RSM PROFILES

RSM [5] denoted by $\mathsf{RSM}(\boldsymbol{\sigma}, \Pi, p)$ is another generalization of the Mallows. It is parameterized by a reference ranking $\boldsymbol{\sigma}$, a probability function Π where $\Pi(i, j)$ is the probability of the j^{th} item selected at step i, and a probability function $p : \{1, ..., m-1\} \rightarrow [0, 1]$ where $p(i)$ is the probability that the i^{th} selected item preferred to the remaining items. In contrast to the RIM that randomizes the item insertion position, the RSM randomized the item insertion order. In this paper, we use RSM as a ranking model, i.e., $p \equiv 1$ such that it only outputs rankings. This ranking version is named rRSM and denoted by $\mathsf{rRSM}(\boldsymbol{\sigma}, \Pi)$.

Algorithm 5 REP solver for rRSM

Input: Item c, rank k, $\mathrm{rRSM}(\boldsymbol{\sigma}, \Pi)$
Output: $\Pr(c@k \mid \boldsymbol{\sigma}, \Pi)$

1: $\alpha_0 := |\{\sigma_i | \sigma_i \succ_{\boldsymbol{\sigma}} c\}|$, $\beta_0 := |\{\sigma_i \mid c \succ_{\boldsymbol{\sigma}} \sigma_i\}|$
2: $\mathcal{P}_0 := \{\langle \alpha_0, \beta_0 \rangle\}$ and $q_0(\langle \alpha_0, \beta_0 \rangle) := 1$
3: **for** $i = 1, \ldots, (k-1)$ **do**
4: $\quad \mathcal{P}_i := \{\}$
5: $\quad$ **for** $\langle \alpha, \beta \rangle \in \mathcal{P}_{i-1}$ **do**
6: $\quad\quad$ **if** $\alpha > 0$ **then**
7: $\quad\quad\quad$ Generate a new state $\langle \alpha', \beta' \rangle = \langle \alpha - 1, \beta \rangle$.
8: $\quad\quad\quad$ **if** $\langle \alpha', \beta' \rangle \notin \mathcal{P}_i$ **then**
9: $\quad\quad\quad\quad \mathcal{P}_i.add(\langle \alpha', \beta' \rangle)$
10: $\quad\quad\quad\quad q_i(\langle \alpha', \beta' \rangle) := 0$
11: $\quad\quad\quad$ **end if**
12: $\quad\quad\quad q_i(\langle \alpha', \beta' \rangle) \mathrel{+}= q_{i-1}(\langle \alpha, \beta \rangle) \cdot \sum_{j=1}^{\alpha} \Pi(i, j)$
13: $\quad\quad$ **end if**
14: $\quad\quad$ **if** $\beta > 0$ **then**
15: $\quad\quad\quad$ Generate a new state $\langle \alpha', \beta' \rangle = \langle \alpha, \beta - 1 \rangle$.
16: $\quad\quad\quad$ **if** $\langle \alpha', \beta' \rangle \notin \mathcal{P}_i$ **then**
17: $\quad\quad\quad\quad \mathcal{P}_i.add(\langle \alpha', \beta' \rangle)$
18: $\quad\quad\quad\quad q_i(\langle \alpha', \beta' \rangle) := 0$
19: $\quad\quad\quad$ **end if**
20: $\quad\quad\quad q_i(\langle \alpha', \beta' \rangle) \mathrel{+}= q_{i-1}(\langle \alpha, \beta \rangle) \cdot \sum_{j=\alpha+2}^{\alpha+1+\beta} \Pi(i, j)$
21: $\quad\quad$ **end if**
22: $\quad$ **end for**
23: **end for**
24: **return** $\sum_{\langle \alpha, \beta \rangle \in \mathcal{P}_{k-1}} q_{k-1}(\langle \alpha, \beta \rangle) \cdot \Pi(k, \alpha + 1)$

Example 10. *$\mathrm{rRSM}(\boldsymbol{\sigma}, \Pi)$ with $\boldsymbol{\sigma} = \langle a, b, c \rangle$ generates $\boldsymbol{\tau}=\langle c, a, b \rangle$ as follows. Initialize $\boldsymbol{\tau}_0=\langle \rangle$. When $i = 1$, $\boldsymbol{\tau}_1=\langle c \rangle$ by selecting c with probability $\Pi(1, 3)$, making the remaining $\boldsymbol{\sigma} = \langle a, b \rangle$. When $i = 2$, $\boldsymbol{\tau}_2=\langle c, a \rangle$ by selecting a with probability $\Pi(2, 1)$, making the remaining $\boldsymbol{\sigma} = \langle b \rangle$. When $i = 3$, $\boldsymbol{\tau}=\langle c, a, b \rangle$ by selecting b with probability $\Pi(3, 1)$. Overall, $\Pr(\boldsymbol{\tau} \mid \boldsymbol{\sigma}, \Pi)=\Pi(1, 3) \cdot \Pi(2, 1) \cdot \Pi(3, 1)$.*

Theorem 16. *Given a positional scoring rule $\boldsymbol{r}_m$, an RSM voting profile $\boldsymbol{P}^{\mathrm{rRSM}} = (\mathrm{rRSM}_1, \ldots, \mathrm{rRSM}_n)$, and candidate w, determining $w \in MEW(\boldsymbol{r}_m, \boldsymbol{P}^{\mathrm{rRSM}})$ is in $O(nm^4)$.*

Proof. Given any $\mathrm{rRSM} \in \boldsymbol{P}^{\mathrm{rRSM}}$, candidate c, and rank j, the $Pr(c@j \mid \mathrm{rRSM})$ is computed by Algorithm 5 in a fashion that is similar to Algorithm 3. This is also a Dynamic Programming (DP) approach. The states are in the form of $\langle \alpha, \beta \rangle$, where α is the number of items before c, and β is that after c in the remaining $\boldsymbol{\sigma}$. For state $\langle \alpha, \beta \rangle$, there are $(\alpha + 1 + \beta)$ items in the remaining $\boldsymbol{\sigma}$. Algorithm 5 only runs up to $i = (k-1)$ (in line 3), since item c must be selected at step k and the rest steps do not change the rank of c anymore. Each step i generates at most $(i + 1)$ states, corresponding to $[0, \ldots, i]$ items are selected from items before c in the original $\boldsymbol{\sigma}$. The complexity of Algorithm 5 is bounded by $O(m^2)$. It takes $O(nm^4)$ to obtain the expected scores of all candidates and to determine the MEW.

□

Example 11. *Let $\mathrm{rRSM}(\boldsymbol{\sigma}, \Pi)$ denote a RSM where $\boldsymbol{\sigma} = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle$, and $\Pi = [[0.1, 0.3, 0.4, 0.2], [0.2, 0.5, 0.3], [0.3, 0.7], [1]]$. Assume we are interested in $\Pr(\sigma_2@3 \mid \boldsymbol{\sigma}, \Pi)$, the probability of $\mathrm{rRSM}(\boldsymbol{\sigma}, \Pi)$ placing σ_2 at rank 3.*

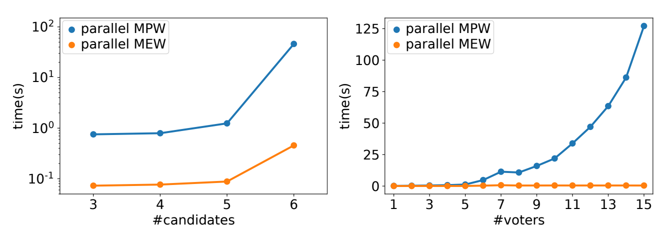
- *Before running RSM, there is $\alpha_0 = 1$ item before σ_2 and $\beta_0 = 2$ items after σ_2 in $\boldsymbol{\sigma}$. So the initial state is $\langle \alpha_0, \beta_0 \rangle = \langle 1, 2 \rangle$, and $q_0(\langle 1, 2 \rangle) = 1$.*
- *At step $i = 1$, the selected item can be either from $\{\sigma_1\}$ or $\{\sigma_3, \sigma_4\}$. So two new states are generated here.*
  - *The σ_1 is selected with probability $\Pi(1, 1) = 0.1$, which generates a new state $\langle 0, 2 \rangle$, and $q_1(\langle 0, 2 \rangle) = q_0(\langle 1, 2 \rangle) \cdot \Pi(1, 1) = 0.1$.*
  - *An item $\sigma \in \{\sigma_3, \sigma_4\}$ is selected with probability $\Pi(1, 3) + \Pi(1, 4) = 0.6$, which generates a new state $\langle 1, 1 \rangle$, and $q_1(\langle 1, 1 \rangle) = q_0(\langle 1, 2 \rangle) \cdot 0.6 = 0.6$.*

  So $\mathcal{P}_1=\{\langle 0, 2 \rangle, \langle 1, 1 \rangle\}$, $q_1=\{\langle 0, 2 \rangle \mapsto 0.1, \langle 1, 1 \rangle \mapsto 0.6\}$.
- *At step $i = 2$, iterate states in $\mathcal{P}_1$.*
  - *For state $\langle 0, 2 \rangle$, the selected item must be from the last two items in the remaining reference ranking. A new state $\langle 0, 1 \rangle$ is generated with probability $\Pi(2, 2) + \Pi(2, 3) = 0.8$.*
  - *For state $\langle 1, 1 \rangle$, the selected item is either the first or last item in remaining reference ranking. A new state $\langle 0, 1 \rangle$ is generated with probability $\Pi(2, 1) = 0.1$, and another state $\langle 1, 0 \rangle$ is generated with probability $\Pi(2, 3) = 0.3$.*

  So $\mathcal{P}_2 = \{\langle 0, 1 \rangle, \langle 1, 0 \rangle\}$ and
  - □ *$q_2(\langle 0, 1 \rangle) = q_1(\langle 0, 2 \rangle) \cdot 0.8 + q_1(\langle 1, 1 \rangle) \cdot 0.1 = 0.1 \cdot 0.8 + 0.6 \cdot 0.1 = 0.14$*
  - □ *$q_2(\langle 1, 0 \rangle) = q_1(\langle 1, 1 \rangle) \cdot 0.3 = 0.6 \cdot 0.3 = 0.18$*
- *At step $i = 3$, item σ_2 must be selected to meet the requirement. For each state $\langle \alpha, \beta \rangle \in \mathcal{P}_2$, the rank of σ_2 is $(\alpha+1)$ in the corresponding remaining ranking. So $\Pr(\sigma_2@3 | \boldsymbol{\sigma}, \Pi) = q_2(\langle 0, 1 \rangle) \cdot \Pi(3, 1) + q_2(\langle 1, 0 \rangle) \cdot \Pi(3, 2) = 0.14 \cdot 0.3 + 0.18 \cdot 0.7 = 0.168$.*

C ADDITIONAL EXPERIMENTS

Figure 15 in Section 10 has demonstrated that MEW is much more scalable than MPW under the plurality rule. Figure 16 presents results of a similar experiment under the Borda rule, with up to 6 candidates and up to 15 voters. We first fixed the number of voters to 5 and varied the number of candidates from 3 to 6, then fixed the number of candidates to 5 and varied the number of voters from 1 to 15. In this experiment, MEW is still much more scalable than MPW.



(a) 5 voters, varying #candidates **(b)** 5 candidates, varying #voters

Figure 16: Average time of parallel MPW and MEW, using 48 worker processes, under Borda, over partial voting profiles, fixing $\phi = 0.5$ and $p_{max} = 0.1$. MEW scales much better than MPW, with both #candidates and #voters.