Q1. (1.)

$$A=\begin{bmatrix}1&0&1\\0&2&3\\0&0&3\end{bmatrix}, \quad \det(A)=1\cdot2\cdot3+0\cdot3\cdot0+1\cdot0\cdot0-1\cdot2\cdot0-1\cdot3\cdot0-0\cdot0\cdot3$$
$$=6$$

$\det(A)\neq0$ 이기 때문에 행렬 A는 가역이라고 할 수 있습니다.

(2.) Eigen value equation:

$Ax=\lambda x,\quad Ax-\lambda x=0,\quad (A-\lambda I)x=0,\quad x\neq0,$ hence $\det(A-\lambda I)=0,$

$$\det(A-\lambda I)=\det\left(\begin{bmatrix}1&0&1\\0&2&3\\0&0&3\end{bmatrix}-\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}\right)=\det\left(\begin{bmatrix}1-\lambda&0&1\\0&2-\lambda&3\\0&0&3-\lambda\end{bmatrix}\right)=0$$

$$=(1-\lambda)\cdot(2-\lambda)\cdot(3-\lambda)=0,\quad \lambda=1,2,3$$

a.) $\lambda=1,$

$$A-\lambda I=\begin{bmatrix}0&0&1\\0&1&3\\0&0&2\end{bmatrix}\underset{-3r_1+r_2\to r_2}{\overset{-2r_1+r_3\to r_3}{\sim}}\begin{bmatrix}0&0&1\\0&1&0\\0&0&0\end{bmatrix}\overset{r_1\leftrightarrow r_2}{\sim}\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

$$v_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

b.) $\lambda=2,$

$$A-\lambda I=\begin{bmatrix}-1&0&1\\0&0&3\\0&0&1\end{bmatrix}\underset{-r_3+r_1\to r_1}{\overset{-3r_3+r_2\to r_2}{\sim}}\begin{bmatrix}-1&0&0\\0&0&0\\0&0&1\end{bmatrix}\overset{-r_1\to r_1}{\sim}\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

$$V_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

c.) $\lambda=3,$

$$A-\lambda I=\begin{bmatrix}-2&0&1\\0&-1&3\\0&0&0\end{bmatrix}\underset{-r_2\to r_2}{\overset{-\frac{1}{2}r_1\to r_1}{\sim}}\begin{bmatrix}1&0&-\frac{1}{2}\\0&1&3\\0&0&0\end{bmatrix}$$

$$v_3=\begin{bmatrix}\frac{1}{2}\\-3\\1\end{bmatrix}$$

Eigen values: $\lambda=1,2,3$

$$\text{Eigenvector: }\begin{bmatrix}v_1&V_2&V_3\end{bmatrix}=\begin{bmatrix}1&0&\frac{1}{2}\\0&1&-3\\0&0&1\end{bmatrix}$$

Q2. $B=\begin{bmatrix}0&0&0\\0&0&0\\3&0&1\end{bmatrix}$,

$$\det(B-\lambda I)=\det\left(\begin{bmatrix}-\lambda&0&0\\0&-\lambda&0\\3&0&1-\lambda\end{bmatrix}\right)=\lambda^2\cdot(1-\lambda)=0,\quad \lambda=0,1$$

a.) $\lambda=0$,

$$B-\lambda I=\begin{bmatrix}0&0&0\\0&0&0\\3&0&1\end{bmatrix}\underset{r_1\leftrightarrow r_3}{\overset{\frac{1}{3}r_3\to r_3}{\sim}}\begin{bmatrix}1&0&\frac{1}{3}\\0&0&0\\0&0&0\end{bmatrix}$$

$$V_1=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad V_2=\begin{bmatrix}-\frac{1}{3}\\0\\1\end{bmatrix}$$

b.) $\lambda=1$,

$$B-\lambda I=\begin{bmatrix}-1&0&0\\0&-1&0\\3&0&0\end{bmatrix}\overset{\substack{3r_1+r_3\to r_3\\-r_1\to r_1\\-r_3\to r_3}}{\sim}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},$$

$$V_3=\begin{bmatrix}0\\0\\1\end{bmatrix},$$

$$P=[V_1\ \ V_2\ \ V_3]=\begin{bmatrix}0&-\frac{1}{3}&0\\1&0&0\\0&1&1\end{bmatrix},\quad P^{-1}=\begin{bmatrix}0&1&0\\-3&0&0\\3&0&1\end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc}0&-\frac{1}{3}&0&1&0&0\\1&0&0&0&1&0\\0&1&1&0&0&1\end{array}\right]\overset{\substack{3\cdot r_1+r_3\to r_3\\-3r_1\to r_1\\r_1\leftrightarrow r_2}}{\sim}\left[\begin{array}{ccc|ccc}1&0&0&0&1&0\\0&1&0&-3&0&0\\0&0&1&3&0&1\end{array}\right],$$

Diagonalization: $P^{-1}BP=D$, where D is a diagonal matrix

$$P^{-1}BP=\begin{bmatrix}0&1&0\\-3&0&0\\3&0&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\3&0&1\end{bmatrix}\begin{bmatrix}0&-\frac{1}{3}&0\\1&0&0\\0&1&1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\3&0&1\end{bmatrix}\begin{bmatrix}0&-\frac{1}{3}&0\\1&0&0\\0&1&1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

Q 3. $P(\text{스팸}) = 0.2$, $P(\text{스팸}^c) = 0.8$

$P(\text{당첨} | \text{스팸}) = 0.5 = \frac{P(\text{당첨} \cap \text{스팸})}{P(\text{스팸})} = \frac{P(\text{당첨} \cap \text{스팸})}{0.2}$, $P(\text{당첨} \cap \text{스팸}) = 0.2 \cdot 0.5 = 0.1$

$P(\text{당첨} | \text{스팸}^c) = 0.01 = \frac{P(\text{당첨} \cap \text{스팸}^c)}{P(\text{스팸}^c)} = \frac{P(\text{당첨} \cap \text{스팸}^c)}{0.8}$, $P(\text{당첨} \cap \text{스팸}^c) = 0.8 \cdot 0.01 = 0.008$

$P(\text{당첨}) = P(\text{당첨} \cap \text{스팸}) + P(\text{당첨} \cap \text{스팸}^c) = 0.1 + 0.008 = 0.108$

$P(\text{스팸} | \text{당첨}) = \frac{P(\text{당첨} \cap \text{스팸})}{P(\text{당첨})} = \frac{0.1}{0.108} = 0.9259$

Q4.

(2.) $n = 15$, # of Hits $= 9$, # of Outs $= 6$,

$P(\text{Hit}) = \frac{9}{15} = \frac{3}{5} = 0.6$, $P(\text{Out}) = 0.4$

$H(X) = -P(\text{Hit}) \cdot \log_2(P(\text{Hit})) - P(\text{Out}) \cdot \log_2(P(\text{Out}))$

$= -0.6 \cdot \log_2(0.6) - 0.4 \cdot \log_2(0.4) = 0.971$

(3.) $Q(\text{Hit}) = \frac{4}{15}$, $Q(\text{Out}) = \frac{11}{15}$

$KL(P \| Q) = H(P, Q) - H(P) = P(\text{Hit}) \cdot \log_2\left(\frac{P(\text{Hit})}{Q(\text{Hit})}\right) + P(\text{Out}) \cdot \log_2\left(\frac{P(\text{Out})}{Q(\text{Out})}\right)$

$= 0.6 \cdot \log_2\left(\frac{0.6}{\frac{4}{15}}\right) + 0.4 \cdot \log_2\left(\frac{0.4}{\frac{11}{15}}\right) = 0.3522$

Q5. Loss Function of Logistic Regression:

$$\min_{w} \underbrace{\sum_{i=1}^{m} -y^{(i)} \cdot \log\left(\hat{y}^{(i)}\right) - \left(1-y^{(i)}\right) \cdot \log\left(1-\hat{y}^{(i)}\right)}_{\text{Cross Entropy}}$$

$$= \min_{w} \sum_{i=1}^{m} \underbrace{\boxed{-y^{(i)} \cdot \log\left(\frac{1}{1+e^{-w^T x^{(i)}}}\right)} - \boxed{\left(1-y^{(i)}\right) \cdot \log\left(\frac{e^{-w^T x^{(i)}}}{1+e^{-w^T x^{(i)}}}\right)}}_{J(w)}$$

$J(w)$의 2개의 Equation 모두 Convex in w 를 만족해야 함!
하지만 둘 다 Convex 를 만족함으로 $J(w)$ 도 Convex 하며
we can then find $w = w^*$ s.t. $\nabla J(w) = 0$