[ML/DL을 위한 수학 1,2]

9기 조세린

Q1.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

(1) determinant 구하기

$$1\begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 3 \\ 0 & 3 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 6 - 0 + 0 = 6$$

→ 세 벡터 $(1,0,1)$, $(0,2,3)$, $(0,0,3)$으로 만들어지는 평행육면체의 부피

(2) eigenvalue, eigen vector 구하기

$$Ae = \lambda I e$$

$$\det(A - \lambda I) = 0$$

$$\det\begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 3-\lambda \end{pmatrix} = 0 \qquad (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$\lambda = 1, 2, 3 \Rightarrow$ eigenvalue

i) $\lambda = 1$ $\quad \therefore \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \qquad \begin{aligned} z &= 0 \\ y &= 0 \\ x &= 1 \end{aligned}$$

· 고유벡터 : 선형 변환(A)시, 방향은 바뀌지 않고 크기만 변하는 벡터

· 고유값 : 그 크기가 얼마만큼 바뀌었는가.

ii) $\lambda = 2$ $\quad \therefore \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \qquad \begin{aligned} z &= 0 \\ y &= 1 \\ x &= 0 \end{aligned}$$

iii) $\lambda = 3$ $\quad \therefore \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} -2 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \qquad \begin{aligned} y &= 3z \\ z &= 2x \end{aligned} \qquad \begin{pmatrix} \frac{1}{2} \\ 3 \\ 1 \end{pmatrix}$$

Q2.

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

(1) eigen value 구하기

$$\det \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 3 & 0 & 1-\lambda \end{pmatrix} = 0 \qquad -\lambda(-\lambda)(1-\lambda) = 0 \qquad \lambda = 0, 1$$

(2) eigen vector 구하기

i) $\lambda = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \qquad 3x = -z \qquad \therefore \begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

ii) $\lambda = 1$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \qquad \therefore \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$P^{-1} = \left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

(3) D

$$D = P^{-1}BP = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Q3.

$P(\text{spam}) = 0.2$

$P(\text{당첨} \mid \text{spam}) = 0.5$

$P(\text{당첨} \mid \text{not spam}) = 0.01$

$$P(\text{spam} \mid \text{당첨}) = \frac{P(\text{당첨} \mid \text{spam})P(\text{spam})}{P(\text{당첨})} = \frac{0.5 \times 0.2}{P(\text{당첨})} = \frac{0.1}{0.108} = \boxed{0.9259}$$

$$P(\text{당첨}) = P(\text{당첨} \mid \text{spam})P(\text{spam}) + P(\text{당첨} \mid \text{not spam})P(\text{not spam})$$

$$= 0.5 \times 0.2 + 0.01 \times 0.8 = 0.108$$

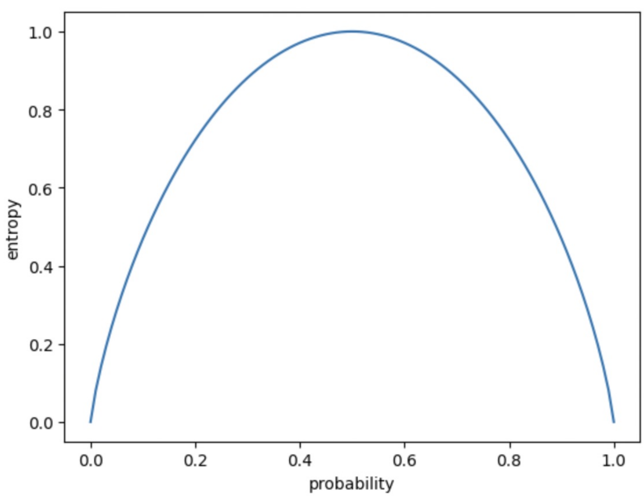
Q4.

(1)

스윙 결과의 분포: 이항분포 → 안타 or 아웃이 연속적으로 반복됨.

이항분포의 엔트로피 　 p = 안타를 칠 확률

$$H(p) = -p\log_2(p) - (1-p)\log_2(1-p)$$



(2)

$$p(hit) = \frac{9}{15} = \frac{3}{5} = 0.6$$

$$H = -0.6\log_2(0.6) - 0.4\log_2(0.4) \approx 0.971$$

(3)

$$p(hit\_other) = \frac{4}{15} = q$$

$$KL(p\|q) = H(p,q) - H(p)$$

$$H(p,q) = -0.6\log_2\left(\frac{4}{15}\right) - 0.4\log_2\left(\frac{11}{15}\right) = 1.666$$

$$KL(p\|q) = H(p,q) - H(p)$$

$$= 1.666 - 0.971 = 0.695$$

(4) logistic regression의 convex optimization

· cost function of logistic regression → cross entropy

목적 : $$\min_w \sum_{i=1}^{m} \underbrace{-y^{(i)}\log(\hat{y}^{(i)}) - (1-y^{(i)})\log(1-\hat{y}^{(i)})}_{J(w)}$$

→ cost function을 최소화하는 가중치 w 찾기

i) J(w)가 convex function 인지 확인

$$-y^{(i)}\log(\hat{y}^{(i)}) - (1-y^{(i)})\log(1-\hat{y}^{(i)}) = -y^{(i)}\log\frac{1}{1+e^{-w^Tx^{(i)}}} - (1-y^{(i)})\log\frac{e^{-w^Tx^{(i)}}}{1+e^{-w^Tx^{(i)}}}$$

$$-\log\frac{e^{-w^Tx^{(i)}}}{1+e^{-w^Tx^{(i)}}} = w^Tx^{(i)} - \log\frac{1}{1+e^{-w^Tx^{(i)}}}$$

Hessian이 positive semi definite, 즉 미분을 2번 하더라도 0 이상의 양수 범위에 존재

affine function (constant + linear)
→ affine function은 concave 이면서 convex

$$= \text{convex} + \text{convex} \Rightarrow \text{convex}$$

$t \in (0,1)$ $(x,y) \in \mathbb{R}$ $\Rightarrow tx + (1-t)y \in \mathbb{R}$ ⇒ 모든 실수 집합 $\mathbb{R}$에 대해, J(w)는 convexity 만족

ii) convex optimization

convex 함수의 local minimum은 항상 global minimum 이므로, J(w)가 convex function 일때 GD를 이용하여 $\nabla J(w) = 0$ 을 찾으면, 이때의 $w = w^*$은 global minimum 이다.