Q1

문제 1-1)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \qquad |A| = 1\times(2\times3-0\times3) + 0\times(0\times3-3\times0) + 1\times(0\times0-0\times2) = 6$$

문제 1-2)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A\cdot x = x\cdot\lambda$$

$$\Leftrightarrow x(A-I\cdot\lambda)=0$$

$$\therefore \det(A-I\cdot\lambda)=0$$

$$\Leftrightarrow A-I\lambda = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

$$\Rightarrow (1-\lambda)\left((2-\lambda)(3-\lambda)-3\times0\right) + 0\times(0\times(3-\lambda)-1\times0) + 1\times(0\times0-0\times(2-\lambda))$$

$$= -(\lambda-1)(\lambda-2)(\lambda-3)=0$$

$$\therefore \lambda = 1, 2, 3$$

case-1) $\lambda = 1$

$$(A-\lambda I)\cdot x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0+b+2c=0 \qquad \therefore b=-2c \rightarrow \vec{x}=(x, -2t, t)$$

case-2)

$$(A-\lambda I)\cdot x = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = -a+c=0 \qquad \therefore a=c \rightarrow \vec{x}=(t, y, t)$$

case-3)

$$(A-\lambda I)\cdot x = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = -2a-b=0 \qquad \therefore b=-2a \quad \therefore \vec{x}=(-2t, t, z)$$

Q2

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad B = P^{-1}DP$$

i) $$\det(B-\lambda I) = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix} = (-\lambda)(-\lambda)(1-\lambda)=0 \qquad \therefore \lambda = 0 \text{ or } 1$$

$$\therefore D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

① $\lambda=0$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right]$$

$x_2 = t_1$, $x_3 = s_1$

$x_1 = -\frac{1}{3}s_1$

$$X = t_1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s_1\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

② $\lambda_2 = 1$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{array}\right]$$

$x_1=0$, $x_2=0$, $x_3=u$

$$X = u\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow P^{-1} = \left[\begin{array}{ccc|ccc} 0 & -\frac{1}{3} & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right]$$

$$\Rightarrow B = PDP^{-1} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -3 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Q3

$p(\text{SPAM}) = 0.2 \qquad p(\text{당첨}|\text{SPAM}) = 0.5 \qquad p(\text{당첨}|\text{xSPAM}) = 0.01$

$p(\text{x SPAM}) = 0.8 \qquad p(\text{x당첨}|\text{SPAM}) = 0.5 \qquad p(\text{x당첨}|\text{xSPAM}) = 0.99$

$$p(\text{당첨}) = p(\text{당첨} \cap \text{SPAM}) + p(\text{당첨} \cap \text{xSPAM})$$

$$= p(\text{당첨}|\text{SPAM})\times p(\text{SPAM}) + p(\text{당첨}|\text{xSPAM})\times p(\text{xSPAM})$$

$$= 0.5 \times 0.2 + 0.01 \times 0.8$$

베이즈 정리를 이용하면,

$$p(\text{SPAM}|\text{당첨}) = \frac{p(\text{당첨}|\text{SPAM})\times p(\text{SPAM})}{p(\text{당첨})} = \frac{0.5\times0.2}{0.5\times0.2+0.01\times0.8}$$

$$\fallingdotseq 0.926$$

Q4

4-2) $$H(X) = -\sum_i p(x_i)\log p(x_i)$$

$$= -0.6\times\log_2 0.6 - 0.4\log_2 0.4$$

$$\fallingdotseq 0.970$$

4-3) 4-1)의 야구팀 hit 이산 분포를 P, 4-3에 주어진 야구팀의 hit 이산 분포를 Q라 할때,

$$D_{KL}(P\|Q) = \sum_i p(x_i)\log\left(\frac{p(x_i)}{Q(x_i)}\right)$$

$$= 0.6\times\log_2\left(\frac{0.6}{0.266}\right) + 0.4\times\log_2\left(\frac{0.4}{0.733}\right)$$

$$\fallingdotseq 0.3522$$

Q5)

<Logistic Regression>

$\Rightarrow$ Loss function : cross entropy

$$\therefore \min_w \left\{\sum_{i=1}^{m}(-y_i\log\hat{y}_i) - (1-y_i)\log(1-\hat{y}_i)\right\}, \quad \hat{y}_i = \frac{1}{1+e^{-w^T\cdot x_i}} = f(w)$$

$$J(w) = -y_i\log\frac{1}{1+e^{-w^T\cdot x_i}} - (1-y_i)\log\frac{1}{1+e^{w^T\cdot x_i}}$$

$$= -y_i\log\frac{1}{1+e^{-w^T\cdot x_i}} - (1-y_i)\left(\log\frac{1}{1+e^{-w^T\cdot x_i}} - w^Tx_i\right)$$

$$= (1-y_i)w^T\cdot x_i - (1-y_i)\log\frac{1}{1+e^{-w^T\cdot x_i}} - y_i\log\frac{1}{1+e^{-w^T\cdot x_i}}$$

$$= (1-y_i)w^T\cdot x_i - \log\frac{1}{1+e^{-w^T\cdot x_i}}$$

since $\nabla^2 f(w) \geq 0$, $J(w)$ is convex

$\therefore \min_w J(w)$, $w \in R$, $J(w)$: convex, R : convex set

$\therefore \nabla J(w) = 0$, $w = w^*$ $\left(w^* = \min_w J(w)\right)$