1-1)

1. $P_i = \frac{1}{6}$

$$H(X) = -\sum_{i=1}^{6} \frac{1}{6}\log_2\left(\frac{1}{6}\right) = -6\times\left(\frac{1}{6}\log_2\left(\frac{1}{6}\right)\right)$$

$$= -\log_2\left(\frac{1}{6}\right) \fallingdotseq 2.585$$

2. $P(6)=1$

$$H(X) = -\left[P(6)\log_2 P(6)\right) + \sum_{i=1}^{5} P(i)\log_2 P(i))\right]$$

$$= -1\log_2(1) = 0$$

3. 홀수 : p 짝수 : $2p \Rightarrow 3(p+2p)=1 \therefore p=\frac{1}{9}$

$$H(X) = -\left[3\times\left(\frac{1}{9}\log_2\left(\frac{1}{9}\right)\right) + 3\times\left(\frac{2}{9}\log_2\left(\frac{2}{9}\right)\right)\right]$$

$$= -\left[\frac{1}{3}\log_2\left(\frac{1}{9}\right) + \frac{2}{3}\log_2\left(\frac{2}{9}\right)\right] \fallingdotseq 2.528$$

4. $P(1)=0.5$

$$H(X) = -[0.5\log_2(0.5) + 5\times(0.1\log_2(0.1))]$$

$$= -[0.5(-1) + 0.5(-\log_2(0.1))]$$

$$\approx 2.161$$

$H(X)$는 균등분포일 때 max

$\therefore 1 > 3 > 4 > 2$

1-2)

- $H(X) \geq 0$

모든 결과 x에 대하여

$0 \leq P(x) \leq 1$

$$\Rightarrow \frac{1}{P(x)} \geq 1$$

$$\Rightarrow \log_2\left(\frac{1}{p(x)}\right) \geq 0$$

$$\Rightarrow \therefore H(X) = \sum p(x) \log_2\left(\frac{1}{p(x)}\right) \geq 0$$

- $H(X) \leq \log_2 |X|$

$$H(X) = \sum -p(x) \log_2(p(x))$$

$$= \sum p(x) \log_2\left(\frac{1}{p(x)}\right)$$

$$= \sum p(x) f\left(\frac{1}{p(x)}\right)$$

$$\sum p(x) \log_2\left(\frac{1}{p(x)}\right) \leq \log_2\left(\sum p(x) \cdot \frac{1}{p(x)}\right)$$

$$\sum p(x) \log_2\left(\frac{1}{p(x)}\right) \leq \log_2(|X|)$$

$$H(X) = \sum p(x) \log_2\left(\frac{1}{p(x)}\right) \leq \log_2(|X|)$$

(1-3)

$$P(X=0)=0.45+0.15=0.6$$

$$P(X=1)=0.1+0.3=0.4$$

$$P(Y=0)=0.45+0.1=0.55$$

$$P(Y=1)=0.15+0.3=0.45$$

1. $H(X), H(Y)$

$$H(X)=-[0.6\log_2(0.6)+0.4\log_2(0.4)]$$

$$\approx 0.971$$

$$H(Y)=-[0.55\log_2(0.55)+0.45\log_2(0.45)]$$

$$\approx 0.993$$

2. $H(X,Y)$

$$H(X,Y) = -\sum P(x,y)\log_2(P(x,y))$$

$$= -[0.45\log_2(0.45) + 0.15\log_2(0.15) + 0.1\log_2(0.1) + 0.3\log_2(0.3)]$$

$$\approx 1.782$$

3. $H(X|Y)$

$$H(X|Y) = H(X,Y) - H(Y)$$

$$\approx 1.782 - 0.993$$

$$= 0.789$$

4. $I(X;Y)$

$$I(X;Y) = H(X) - H(X|Y)$$

$$0.991 - 0.789$$

$= 0.182$

5. $I(X;Y)$

0.182 비트는 한사람의 흡연여부(Y)를 아는것이 그사람의 폐암 발병(X) 여부에 대해 얼마나 정보를 제공하는 것, 즉 두변수간의 통계적의존도를 표시한 값이다.

0이 아니므로 두 변수는 독립이 아니다.

- $I(X;Y)=0 \Rightarrow$ X와 Y가 독립

  $\Rightarrow$ 흡연여부 아는것과 폐암 발병 가능성과 아무런 관련X

- $I(X;Y)=H(X) \Rightarrow$ Y를 알면 X의

모든 불확실성 해소

=> 흡연해보았던
폐암발병 확실히
알수있음

2-1.

1.

1) 고유값 찾기

$$\det\begin{pmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{pmatrix} = (2-\lambda)((2-\lambda)(1-\lambda)-0)$$

$$=(2-\lambda)^2(1-\lambda)=0$$

$\therefore\ \lambda_1=1\ ,\ \lambda_2=2$

2) 고유벡터 찾기

$\cdot\ \lambda=1 : (A-I)v=0$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x=0, \\ y+z=0 \end{matrix}$$

$$P_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

- $\lambda = 2$: $(A-2I)V = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x+z=0$$

$y=1, z=0 \Rightarrow x=0$, $P_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$y=0, z=1 \Rightarrow x=-1$, $P_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

3. P, D

$$P = \begin{array}{ccc} P_1 & P_2 & P_3 \\ \downarrow & \downarrow & \downarrow \end{array}$$

$$P = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

2-3.

1) B^TB & 고유값/벡터 찾기

$$B^TB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

고유 : $\lambda_1 = 2, \lambda_2 = 1$

특이 : $\sigma_1 = \sqrt{2}, \sigma_2 = 1$

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \cdots & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\therefore u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\circ\, u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow U = [u_1 \; u_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & \cdots \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

3-1 목적함수

1. $J(w)$

$$J(w) = -y^{(i)} \log \hat{y}^{(i)} - (1-y^{(i)}) \log(1-\hat{y}^{(i)})$$

$$\hat{y}^{(i)} = \sigma(w^T x^{(i)}) = \frac{1}{1+e^{-w^T x^{(i)}}}$$

2. Gradient

$$d\sigma(z) = \sigma(z)(1-\sigma(z))$$

$$\overline{dz}$$

$$\nabla_w J(w) = \frac{\partial J}{\partial \hat{y}} \cdot \frac{\partial \hat{y}}{\partial z} \cdot \frac{\partial z}{\partial w}$$

1) $$\frac{\partial J}{\partial \hat{y}} = \frac{\partial}{\partial \hat{y}} \left[-y\log(\hat{y}) - (1-y)\log(1-\hat{y}) \right]$$

$$= -\frac{y}{\hat{y}} + \frac{1-y}{1-\hat{y}}$$

2) $$\frac{\partial \hat{y}}{\partial z} = \frac{d}{dz}(1+e^{-z})^{-1}$$

$$= -1(1+e^{-z})^{-2}(-e^{-z})$$

$$= \frac{e^{-z}}{(1+e^{-z})^2}$$

$$= \frac{1}{1+e^{-z}} \cdot \frac{e^{-z}}{1+e^{-z}}$$

$$= \sigma(z)\,(1-\sigma(z))$$

3) $$\frac{\partial z}{\partial w} = \frac{d}{dw}(w^T x)$$

$$= x$$

$$\nabla_w J(w) = \left(-\frac{y}{\hat{y}} + \frac{1-y}{1-\hat{y}}\right) \cdot \hat{y}(1-\hat{y})\,x$$

$$= \left(\frac{-y(1-\hat{y}) + (1-y)\hat{y}}{\hat{y}(1-\hat{y})}\right) \hat{y}(1-\hat{y})\,x$$

$$= (\hat{y} - y)\,x$$

$$(\sigma(w^T x) - y)\,x$$

$$= (\Delta(w^T x) - y)$$

2. Hessian

$$H = \nabla_w^2 J(w) = \nabla_w\left[(\Delta(w^T x) - y)x\right]$$

$$H = \nabla_w\left[\Delta(w^T x)\right] \cdot x^T$$

$$\nabla_w\left[\Delta(w^T x)\right] = \frac{\partial \Delta}{\partial z} \cdot \frac{\partial z}{\partial w}$$

$$= \Delta(z)(1-\Delta(z)) \cdot x$$

$$H = \left[\Delta(z)(1-\Delta(z))x\right] \cdot x^T$$

$$= \Delta(z)(1-\Delta(z))x \cdot x^T$$

$$= \Delta(w^T x)(1-\Delta(w^T x))xx^T$$

$$0 \le \underbrace{\Delta(Z)}_{0 \le \ \le 1}\underbrace{(1-\Delta(Z))}_{0 \le \ \le 1}$$

$$\times$$

$x \cdot x^T \Rightarrow$ 양의 준정부호 행렬

$$\Downarrow$$

$$H = \nabla_w^2 J(w) \ge 0$$

$$\Downarrow$$

$\therefore J(w)$: convex function

3-2

1. $f(x)$

$f(x) = 4x^3 - 6x^2 - 6x + 1$

$f'(x) = \ldots$

2. $t=0$

$\cdot\ x_0 = 1$

$f'(1) = 4 - 6 - 6 + 1 = -7$

$\Rightarrow x_1 = x_0 - r f'(x_0)$ (update)

$= 1 - (0.2)(-7)$

$= 2.4$

3. $t=1$

$\cdot\ x_1 = 2.4$

$f'(2.4) = 4\cdot(2.4)^3 - 6\cdot(2.4)^2 - 6(2.4) + 1$

$= 7.336$

$\ldots f(x_1)$ (update)

$\Rightarrow x_2 = x_1 - \gamma f'(x_1)$

$= 2.4 - 0.2(7.336)$

$= 2.4 - 1.4672$

$= 0.9328$

$\therefore x_2 = 0.9328$

3-3.

1) lagranglian 설정

$g(x, y) = x + y - 2 = 0$

$L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$

$= x^2 + y^2 + \lambda(x + y - 2)$

stationary point

2). stationary

$$\frac{\partial L}{\partial x} = 2x + \lambda = 0$$

$$\frac{\partial L}{\partial y} = 2y + \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x + y - 2 = 0 \quad \cdots \quad ③$$

3. x, y, λ 계산

$$x = -\frac{\lambda}{2}, \quad y = -\frac{\lambda}{2}$$

$$\Rightarrow x = y$$

③ $x + x - 2 = 0 \Rightarrow x = 1$, $y = 1$

$2(1)+\lambda=0 \Rightarrow \lambda=-2$

$\therefore$ stationary point $(x,y)=(1,1)$

4. 최솟값 계산

$f(1,1)=1^2+1^2=2$