1-1

1) $-\left(\frac{1}{6}\log_2\frac{1}{6}\times 6\right) = -\left(\log_2\frac{1}{6}\right) = \log_2 6$

2) $-\left(1\log_2 1\right) = 0$

3) $\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}$ $\quad -\left(\frac{1}{9}\log_2\frac{1}{9}\times 3 + \frac{2}{9}\log_2\frac{2}{9}\times 3\right)$

$= \frac{1}{3}\log_2 9 - \frac{2}{3}\left(1-\log_2 9\right)$

4) $-\left(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{10}\log_2\frac{1}{10}\times 5\right) = -\left(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{10}\right)$

$= -\left(\frac{1}{2}\log_2\frac{1}{20}\right)$

$= \frac{1}{2}\log_2 20$

1-2

1) $H(X) \geq 0$ 증명

$H(X) = \sum_{x\in X} -p(x)\log_2 p(x)$ (1-1 엔트로피정의)

$0 \leq p(x) \leq 1$ 이므로 $\log_2 p(x) \leq 0$

$\therefore p(x)\log_2 p(x) \leq 0$

그러므로, $H(X) = \sum_{x\in X} p(x)\log_2 p(x) \times (-1) \geq 0$

2) $H(x) = E[-\log_2 p(x)]$

$-\log_2 p(x)$는 convex 함수므로 Jensen's Inequality 적용

$E[-\log_2 p(x)] \le \log_2 |x|$ 증명

⇒ convex 함수 기댓값은 균등할때 최대다.

$p(x) = \frac{1}{x}$ 일때 최대가정

$-\sum \frac{1}{|x|} \log_2 \frac{1}{|x|} = -\log_2 \frac{1}{|x|} = \log_2 |x|$

$\therefore 0 \le H(x) \le \log_2 |x|$

1-3

1) ① $H(X)$ 계산

$p(X=0) = 0.6$, $p(X=1) = 0.4$

$H(X) = -(0.6 \log_2 0.6 + 0.4 \log_2 0.4) \approx 0.971$

② $H(Y)$ 계산

$p(Y=0) = 0.55$ $p(Y=1) = 0.45$

$H(Y) = -(0.55 \log_2 0.55 + 0.45 \log_2 0.45)$
≈ 0.993

2) $H(X,Y)$

$$= -\left(0.45\log_2 0.45 + 0.15\log_2 0.15 + 0.1\log_2 0.1 + 0.3\log_2 0.3\right)$$

$$= 1.781$$

3) $H(X|Y) = H(X,Y) - H(Y)$

$$= 1.781 - 0.993 = 0.788$$

4) $I(X;Y) = H(X) + H(Y) - H(X,Y)$

$$= 0.971 + 0.993 - 1.781 = 0.183$$

18.3%만 예측?

5) ① 18.3% ⇒ Y로 X를 완벽히 예측 X but 완벽히 독립도 X

② $I(X;Y)=0$ 이라면 X와 Y는 완전히 독립

③ $I(X;Y)=H(X)$라면 Y를 알면 X 완벽히 예측가능

2-1

$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

고유값 λ는

$\det(A - \lambda I) = 0$ 을 만족

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 2-\lambda \left[(2-\lambda)(1-\lambda) \right]$$

$$- 0 + 0 = 0$$

$$\lambda = 2 \text{ or } \lambda = 1$$

λ= 2일때

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ x_1+2x_2+x_3 \\ -x_1+x_3 \end{bmatrix} = \begin{matrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{matrix}$$

$x_1 =$ 다가능

$x_2 =$ 다가능 $\Rightarrow \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$x_3 =$ 다가능

$\lambda = 1$일 때

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ x_1+2x_2+x_3 \\ -x_1+x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1 = 0$

$x_2 + x_3 = 0$

x_3은 다 가능 → 1

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \\ 1 & -1 & \\ -1 & 1 & = \end{bmatrix}$$

λ가 3개가 나와야하는게 아닌가요? ㅠㅠ

2-3

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1) $$BB^T = \underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$B^TB = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2) $$\begin{bmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} \Rightarrow \text{det 구하기}$$

$$(2-\lambda)(1-\lambda) = 0$$

$$\lambda = 2 \text{ or } 1$$

① $\lambda = 2$ 일때

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$x_1 =$ 다 가능

$x_2 = 0$ $\Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

② $\lambda = 1$ 일때

$$\begin{bmatrix} 2x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$x_1 = 0$

$x_2 =$ 다 가능

$\Rightarrow 1$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\therefore U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{1} & 0 \end{bmatrix}$

3) B^TB 의 고유값은 2, 1, 0 일것이므로

$[\underline{2, 1}, 0]$ — 3x3 행렬이므로 0 하나추가

↑ BB^T에서 구한 고유값

→ 왜 그렇지?

$\lambda = 2$ 일때

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$\underline{x_1 = x_2} \Rightarrow$ 1

$x_3 = 0$

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 1$ 일때

$$\begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_2 = 0$

$x_1 = 0$

$x_3 =$ 다가능 $\Rightarrow 1$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 0$ 일때

$$\begin{bmatrix} x_1 + x_2 \\ x_2 + x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 = 0$

$x_1 = -x_2 \Rightarrow 1, -1$

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \therefore V = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$$

(4) $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

3-1

1) $y = \dfrac{1}{1+e^{-w^Tx}} = \dfrac{1}{1+e^{-z}}$ (시그모이드 함수) $= \sigma(z)$

$$L(z) = -y\log(\sigma(z)) - (1-y)\log(1-\sigma(z))$$

$$= -y\log\left(\frac{1}{1+e^{-z}}\right) - (1-y)\log\left(\frac{e^{-z}}{1+e^{-z}}\right)$$

$$= y\log(1+e^{-z}) + (1-y)\left(z+\log(1+e^{-z})\right)$$

$$= \log(1+e^{-z}) + (1-y)z$$

2) 목적함수 $L(z)$를 나눠서

각각 $f''(x) \geq 0$, $g''(x) \geq 0$ 이라면

$f''(x) + g''(x) \geq$ 이므로 최종목적함수도 convex

①

$$\frac{dL}{dz} = \frac{d}{dz}\left(\log(1+e^{-z})\right)$$

i)

$$= \boxed{\frac{-e^{-z}}{1+e^{-z}}} = -(1-\sigma(z))$$

$$= \sigma(z)-1$$

ii)

$$\frac{d^2L}{dz^2} = \frac{d}{dz}\left(\sigma(z)-1\right) = \frac{e^{-z}}{(1+e^{-z})^2}$$

$$\frac{d}{dz}\left(\frac{1}{1+e^{-z}}\right) = \frac{-(-e^{-z})}{(1+e^{-z})^2}$$

② i)

$$\frac{dL}{dz} = \frac{d(1-y)z}{z} = (1-y)$$

ii)

$$\frac{d^2L}{dz^2} = \frac{d}{dz}(1-y) = 0$$

$\therefore$ $0 \le y \le 1$ (상수취급)

$$\frac{e^{-z}}{(1+e^{-z})^2}$$ 에서 $(1+e^{-z})^2 \geq 0$ 이고

$e^{-z} \geq 0$ 이므로

$L(z)$ 를 두번 미분한 값이 ≥ 0 이다

$\Rightarrow$ convex 하다

3-2

$$f(x) = x^4 - 2x^3 - 3x^2 + x$$

$x_0 = 1 \quad \gamma = 0.2$

$$x_{t+1} = x_t - \gamma f'(x_t)$$

$$f'(x) = 4x^3 - 6x^2 - 6x + 1$$

1) x_1 계산

$f'(1) = 4 - 6 - 6 + 1 = -7$

$x_1 = x_0 - 0.2(-7) = 1 + 1.4 = 2.4$

2) x_2 계산

$f'(2.4) = 4(2.4)^3 - 6(2.4)^2 - 6(2.4) + 1$

$= 4(13.824) - 6(5.76) - 6(2.4) + 1$

$= 7.336$

$x_2 = 2.4 - 0.2(7.336)$

$= 0.9328$

3-3

$\min f(x,y) = x^2+y^2$ s.t $x+y \geq 2$

⇒ 제약조건이 있으므로 라그랑주 승수를 이용하여 새로운 함수를 설정

1) inequality ⇒ $\lambda \geq 0$

$\min (x^2+y^2)$, $2-x-y \leq 0$

$L(x,y,\lambda) = x^2+y^2 + \lambda(2-x-y)$

2) $\frac{dL}{dx} = 2x - \lambda \overset{set}{=} 0$

$\frac{dL}{dy} = 2y - \lambda \overset{set}{=} 0$

$\frac{dL}{d\lambda} = 2-x-y \overset{set}{=} 0$

3) $\lambda = 2x$ $\quad 2 = x + y$

$\lambda = 2y$ $\quad 2 = \frac{\lambda}{2} + \frac{\lambda}{2}$

$\lambda = 2$

$x = 1$

$y = 1$

4) $\min f(x, y) = f(1, 1)$

$= 2$