Probability and counting

S520

August 25, 2016

Read this in addition to (not in place of) Trosset ch. 2.2 and 3.1–3.3

Definitions

Sample space: A "universe" of possible *outcomes* for the experiment in question. Exactly one of the possible outcomes will happen.

Event: A subset of the sample space. It may be made up of one outcome, or many, or none.

Mathematical probability

Mathematical probability is essentially just a set of rules for putting numbers on events. For every event E, we want to assign a probability $P(E)$. In particular, we take the following three rules as axioms (Trosset p. 48):

1. If E is an event, then $0 \leq P(E) \leq 1$.
2. $P(S) = 1$.
3. If $\{E_1, E_2, E_3, \ldots\}$ is a countable collection of pairwise disjoint events, then

$$P\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

"Pairwise disjoint" means for any pair of events, the probability that they both happen is zero. Axiom 3 says that the probability that at least one of a collection of pairwise disjoint happens is the sum of the probabilities of the individual events.

Note that the math doesn't say anything about how you get your probabilities – it could be entirely subjective. So the following set of probabilities:

- P(I become Chief Justice of the Supreme Court) = 99%
- P(I do not become Chief Justice of the Supreme Court) = 1%

is perfectly fine as far as mathematical probability is concerned. As statisticians, however, we need to make sure our probabilities are connected with reality. There are two basic ways to do this:

- Sensible assumptions, e.g. equally likely outcomes.
- Data.

The goal for repeatable events is that events that have a probability of, say, 30%, should happen on 30% of repetitions in the long run. (How probability should interpreted for one-off events is controversial.)

We can use the rules above to make all kinds of crazy new rules. For example:

Complement rule. The probability that event A *doesn't* happen is $1 - P(A)$.

Proof. The events "A" and "Not A" are disjoint. So the probability that at least one of these two events happens is the sum of the two individual probabilities. But "at least of A and Not A" covers all possible outcomes, so it's certain to happen. That is, by Axiom 2:

$$P(A) + P(\text{Not } A) = 1$$

so

$$P(\text{Not } A) = 1 - P(A).$$

Equally likely outcomes

When we consider a set of *outcomes* for an experiment, we mean a set such that exactly one of the outcomes must occur. Suppose there are N equally likely outcomes. Since their probabilities must add to 1, the probability of any particular outcome is $1/N$.

This seems trivial, but we can use this to turn some quite complicated situations into counting problems.

Counting: The multiplication principle

(Trosset p. 29) Suppose there are two decisions to be made and that there are n_1 possible outcomes of the first decision. If, for each outcome of the first decision, there are n_2 possible outcomes of the second decision, then there are $n_1 n_2$ possible outcomes of the pair of decisions.

(If you like proving things, try proving this by induction.)

Urns and counting

The canonical model for random sampling is the urn (Trosset p. 18.)

1. Put a bunch of balls in an urn, one for each member of the population. Mix the balls around.
2. Draw one ball from the urn. By assumption, each ball has an equal chance of being drawn.
3. If *sampling with replacement*, put the ball back in the urn. If *sampling without replacement*, the ball remains outside the urn.
4. Repeat steps 2 and 3 until a sample of the desired size has been drawn.

The key idea: If we keep track of the order of the balls, then all possible samples are equally likely. We can use the multiplication principle to count the number of possible samples; the probability of a particular sample will be one divided by this count. Let's consider sampling with and without replacement separately.

Sampling with replacement

Suppose there are N distinct balls in the urn.

If you draw one ball, clearly there are N equally outcomes, so each ball has a probability of $1/N$.

Now suppose we draw **twice** from the urn, with replacement. How do we count the equally likely outcomes?

Let's start with the case where $N = 3$. Suppose the balls are labeled A, B, C. Then the equally likely outcomes are:

AA, AB, AC, BA, BB, BC, CA, CB, CC

Note that we count AB and BA as distinct. This means:

P(first ball A, second ball B) = 1/9

P(first ball B, second ball A) = 1/9

P(one A and one B, in some order) = 1/9 + 1/9 = 2/9

In general, when sampling with replacement, there are N equally likely ways to draw 1 ball, N^2 equally likely ways to draw 2 balls, and N^k equally likely ways to draw k balls (keeping track of order.)

Sampling without replacement

Now suppose there are three balls in the urn, and we draw all three, without replacement. How many equally likely ways are there to do this?

- There are three ways to choose the first ball.
- For each way to choose the first ball, there are two ways to choose the second ball.
- For each way to choose the first two balls, there's only one way to choose the third ball (since there's only one ball left.)

In other words, every time you take a ball out, there's one fewer ball to choose from next time. With three balls, there are $3 \times 2 \times 1 = 6$ ways to put them in order. With eight balls, there would be $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ ways of putting them in order. That was a lot to write down, so instead we write 8!, pronounced "eight factorial." "Factorial" means we multiply all the counting numbers from that number down to one. (NB: There's a special case: We define $0! = 1$. How many ways are there of putting zero objects in order? There's precisely one way: Don't do anything.)

Example. There are six balls in an urn, labeled A through F. What's the probability of drawing ABC if:

1. order matters?
2. order doesn't matter?
3. There are $6 \times 5 \times 4 = 120$ ways of drawing 3 out of 6 objects in order, so the probability is 1/120.
4. Out of the 120 equally likely ways of drawing 3 out of 6 objects in order, how many of them contain ABC in some order? We can list them:

ABC, ACB, BAC, BCA, CAB, CBA.

So 6 of the 120 equally likely ways are ABC. The probability is thus 6/120, or 1/20.

Note that we could have counted the orderings of ABC without listing them: all we're doing is putting A, B, C in an urn and sampling three letters without replacement. Hence there are $3 \times 2 \times 1 = 6$ ways.

We generalize these ideas using **permutations** and **combinations** (Trosset p. 29–32):

A **permutation** is a ordered choice of objects. The number of ways of choosing r out of n objects in order is:

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

A **combination** is an unordered choice of objects. There are $r!$ permutations for each combination of length r. So the number of ways of choosing r out of n objects if order doesn't matter is:

$$C(n,r) \equiv \left(\begin{array}{c} n \\ r \end{array} \right) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

Example

My top six Pokemon in Pokemon Go are Vaporeon, Hypno, Golbat, Flamethrower Growlithe, Persian, and Body Slam Growlithe.

1. I select two of these Pokemon without replacement. What's the probability they're both Growlithes?

There are $6 \times 5 = 30$ ways of choosing two out of six objects in order. But in this question, order doesn't matter: Flamethrower-Body Slam is the same as Body Slam-Flamethrower. So divide by 2: there are 15 ways of choosing two out of six objects if order doesn't matter. Each way has probability 1/15. That's the answer.

1. I select two of these Pokemon with replacement. What's the probability they're both Growlithes?

To use equally likely outcomes when sampling with replacement, we have to count them as if order matters. Then there are 6×6 equally likely outcomes. Of these, $2 \times 2 = 4$ of them are Growlithe-Growlithe (Flame-Flame, Flame-Body, Body-Flame, Body-Body.) So the probability is 4/36, or 1/9.

Next time: Conditional probability. Read Trosset ch. 3.4.