Continuous random variables

S520

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Motivation: Random number generator

(Trosset chapter 5.1.)

Discrete random variables are intuitive: you list the outcomes and then you put probabilities on them. But they're not the only kind of random variable.

For example, suppose we create a random number generator that picks a real number X between 0 and 1. We could do it in the following way:

1. Roll a ten-sided die labeled $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. This is the first digit after the decimal point.
2. Roll the die again. This is the second digit.
3. Roll the die again. This is the third digit.
4. Keep going for an infinite number of die rolls.

Now what's the probability that X is exactly 0.5? To get 0.5, you'd need to roll a 5, then a 0, then another 0, then another 0, and so on ad infinitum. But the chance of rolling an infinite number of zeroes in a row is zero.

However, we can easily answer the question: What's the probability that $X \leq 0.5$? To be less than or equal to 0.5, either X is less than 0.5 or X equals 0.5. But we already said $P(X = 0.5) = 0$.

$$\begin{aligned} P(X \leq 0.5) &= P(X < 0.5) + P(X \leq 0.5) \\ &= P(X < 0.5) + 0 \\ &= P(X < 0.5) \\ &= P(\text{first die is 0, 1, 2, 3, or 4}) \\ &= 0.5 \end{aligned}$$

In fact, for any y between 0 and 1, $P(X \leq y) = y$. We can write this down formally as a CDF:

$$F(y) = \begin{cases} 0 & y < 0 \\ y & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$$

This is a special case of a **uniform distribution**.

The CDF and the PDF

(Trosset ch. 5.2 p. 120-123.)

Continuous random variables have a continuous CDF. They have an uncountable number of possible outcomes.

The PMF fails for continuous random variables because the probability of any single number is zero. However, probability still works because the chance of being in an *interval* is not zero.

It turns out we can define something analagous to the PMF for the continuous case. A **probability density function**, or **PDF**, is a function $f(x)$ such that:

1. $f(x) \geq 0$ for all real numbers x.
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

To verify that a function is a PDF, check these two conditions hold.

A continuous random variable X has a probability density function that allows you to find the probability of being in any interval by *integration*, i.e. by finding the area under the curve:

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

(Note that since the probability of any single number is zero, we could have also used "less than" in place of "less than or equal to" above.)

The above holds even when $a = -\infty$ or $b = \infty$. The CDF is just the probability of being in the interval $(-\infty, y]$:

$$F(y) = \int_{-\infty}^{y} f(x)\,dx.$$

To get the PDF from the CDF, differentiate. Finally, note that:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

The probability something happens still has to be 1.

Expected value and variance

(Trosset p. 123-124.) We define expected value and variance by analogy to the discrete case, only instead of taking sums, we take integrals.

Expected value:

$$\mu = EX = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

Variance:

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = 1$$

Note: It's often easier to find the variance using the alternative formula

$$\mathrm{Var}(X) = EX^2 - (EX)^2$$

where

$$EX^2 = \int_{-\infty}^{\infty} x^2\, f(x)\,dx$$

Note: The formula $EX^2 - (EX)^2$ also works in the discrete case.

Finally, the standard deviation σ is once again just the square root of the variance.

Example: Uniform random variables

In a **uniform random variable**, the probability of being in an interval is proportional to the length of that interval, as long as the interval is between a minimum a and a maximum b. We can write down this idea formally as a PDF:

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

This is just a rectangle, scaled so that the total area under the rectangle is equal to 1.

To find the CDF, we integrate. Again, this is just equivalent to finding the area of a rectangle.

$$F(y) = \begin{cases} 0 & y < a \\ \frac{y-a}{b-a} & a \le y < b \\ 1 & y \ge b \end{cases}$$

The expression to find the expected value is

$$EX = \int_a^b \frac{x}{b-a}\, dx$$

If you don't like calculus, we can find the expected value by thinking of it as the long-run average. Since the PDF is symmetric, the expected value must be at the point of symmetry, i.e. halfway between a and b. That is, $EX = (a+b)/2$.

To find the variance, there's no real alternative to doing the integration. We find that for the uniform, $\sigma^2 = (b-a)^2/12$.

Example. *The minute hand of my watch moves continuously until it stops at a random time. Let X be the position of the minute hand (in minutes after the hours) – it doesn't have to be a whole number. What are the expected value and variance of X?*

X is Uniform$(0, 60)$, so $EX = 30$ and $\text{Var}(X) = (60-0)^2/12 = 300$.

Example. *Let X be Uniform$(0, 1)$. Generate Y by first rolling a fair die:*

- *If the die shows 1 or 2, then $Y = X$.*
- *If the die shows 3, 4, 5, or 6, then $Y = X + 10$.*

What's the PDF of Y? What's the CDF? What's the expected value?

One-third of the time, Y is between 0 and 1, while 2/3rds of the time, it's between 10 and 11. The PDF is thus

$$f(x) = \begin{cases} \frac{1}{3} & 0 < x < 1 \\ \frac{2}{3} & 10 < x < 11 \\ 0 & \text{otherwise.} \end{cases}$$

The CDF is

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{3}y & 0 \le y < 1 \\ \frac{1}{3} & 1 \le y < 10 \\ \frac{1}{3} + \frac{2}{3}(y-10) & y \le 10 < 11 \\ 1 & y \ge 11 \end{cases}$$

The expected value is $(1/3 \times 0.5) + (2/3 \times 10.5) = 43/6$.

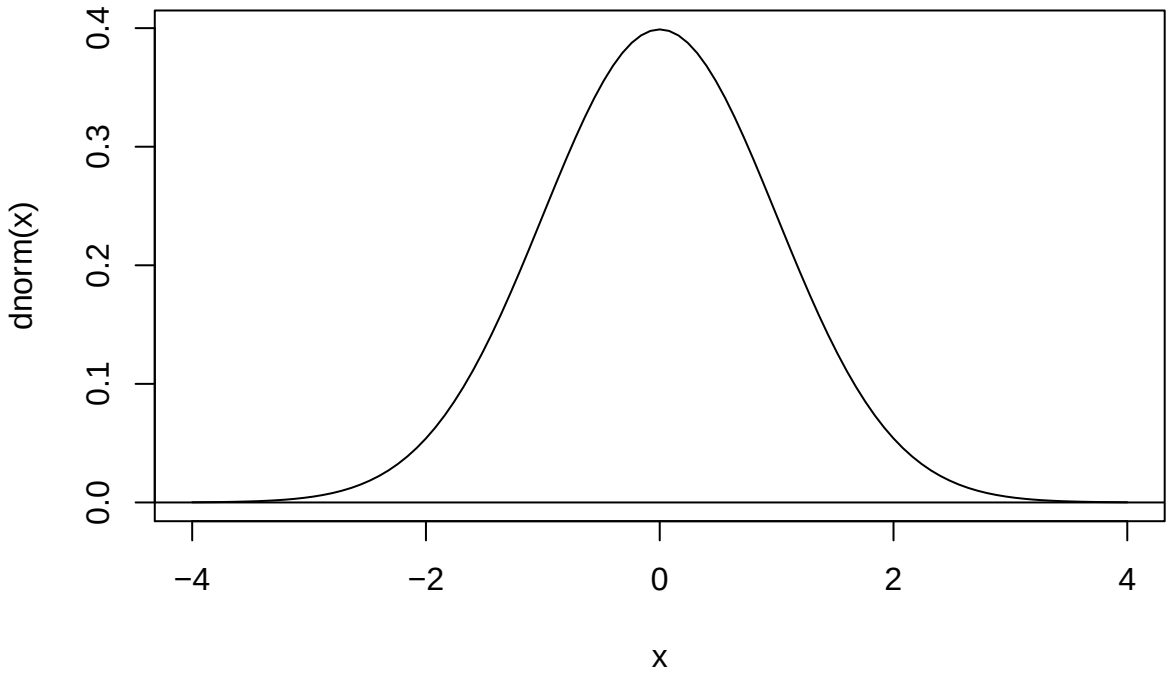
The Normal distribution

Warning: Notes here are incomplete.

The standard normal curve

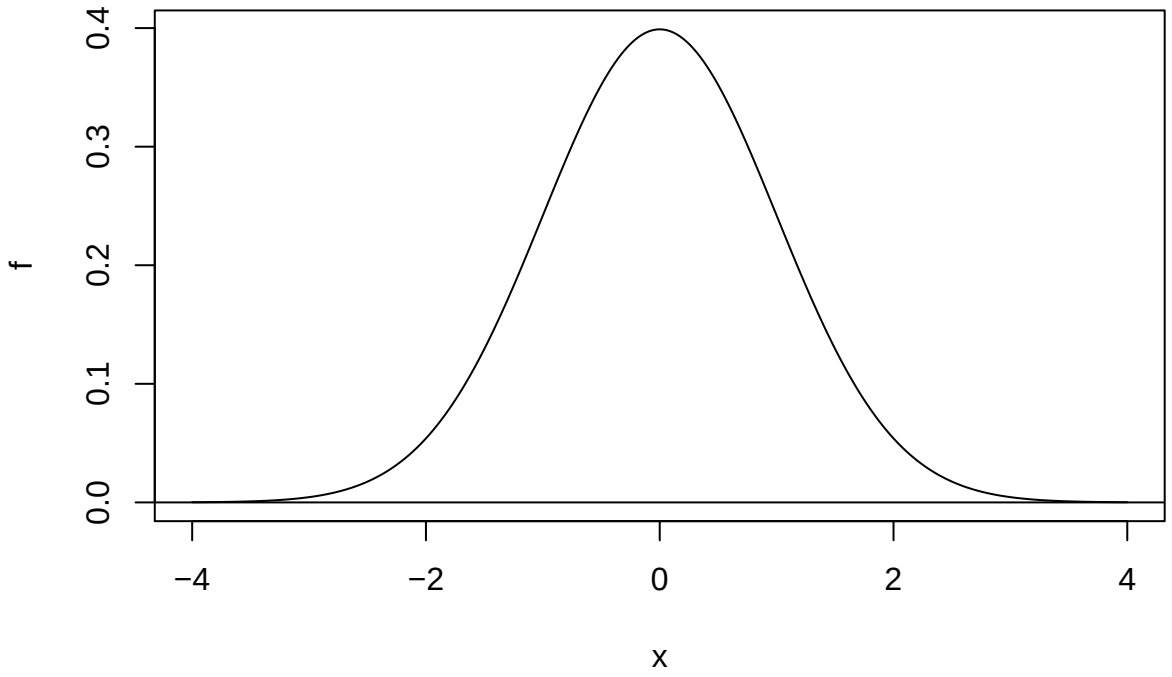
Let's draw a picture with the help of the `curve()` function in R:

```
curve(dnorm, -4, 4)
abline(h = 0)
```



Alternatively, we could specify the x-values:

```
x = seq(-4, 4, 0.01)
f = dnorm(x)
plot(x, f, type="l")
abline(h=0)
```

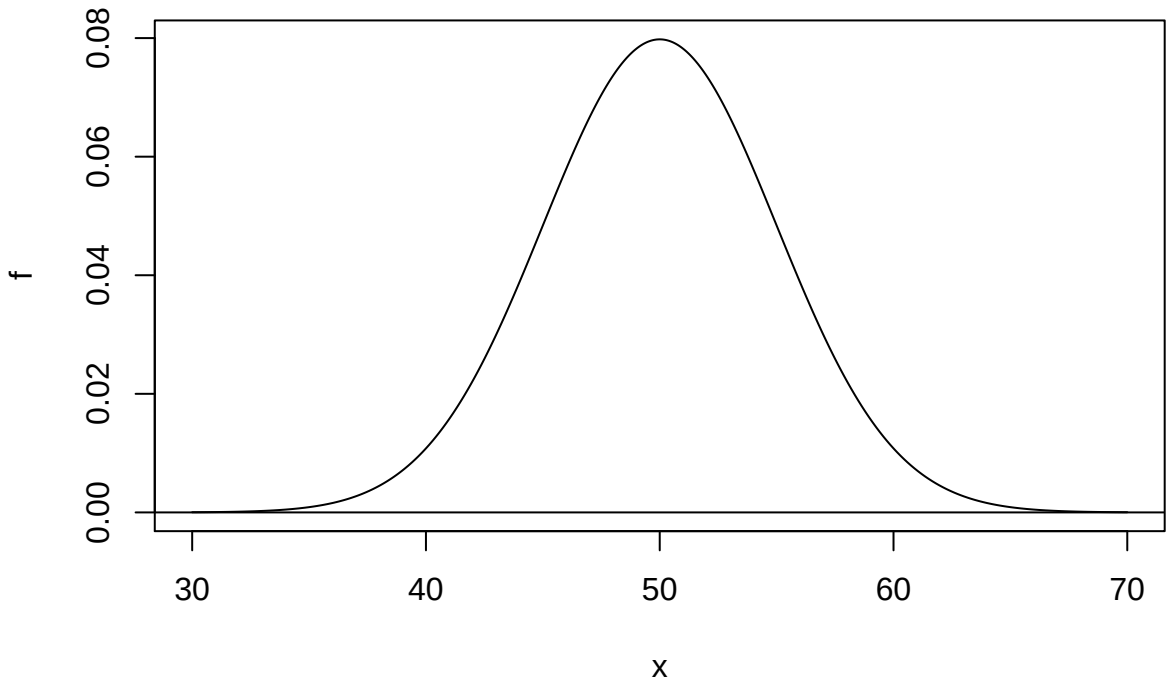


The curve has a "bell" shape.

Other normal random variables

Here's the PDF of a Normal$(50, 5^2)$ random variable.

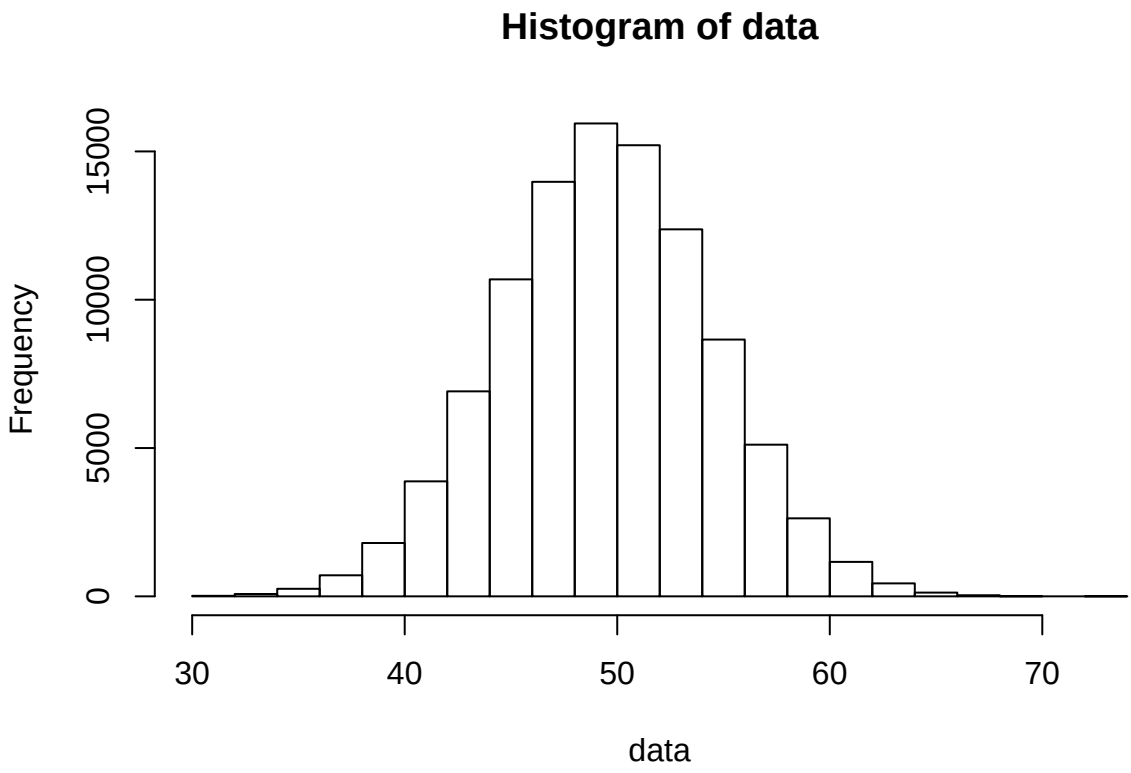
```
x = seq(30, 70, 0.01)
f = dnorm(x, mean=50, sd=5)
plot(x, f, type="l")
abline(h = 0)
```



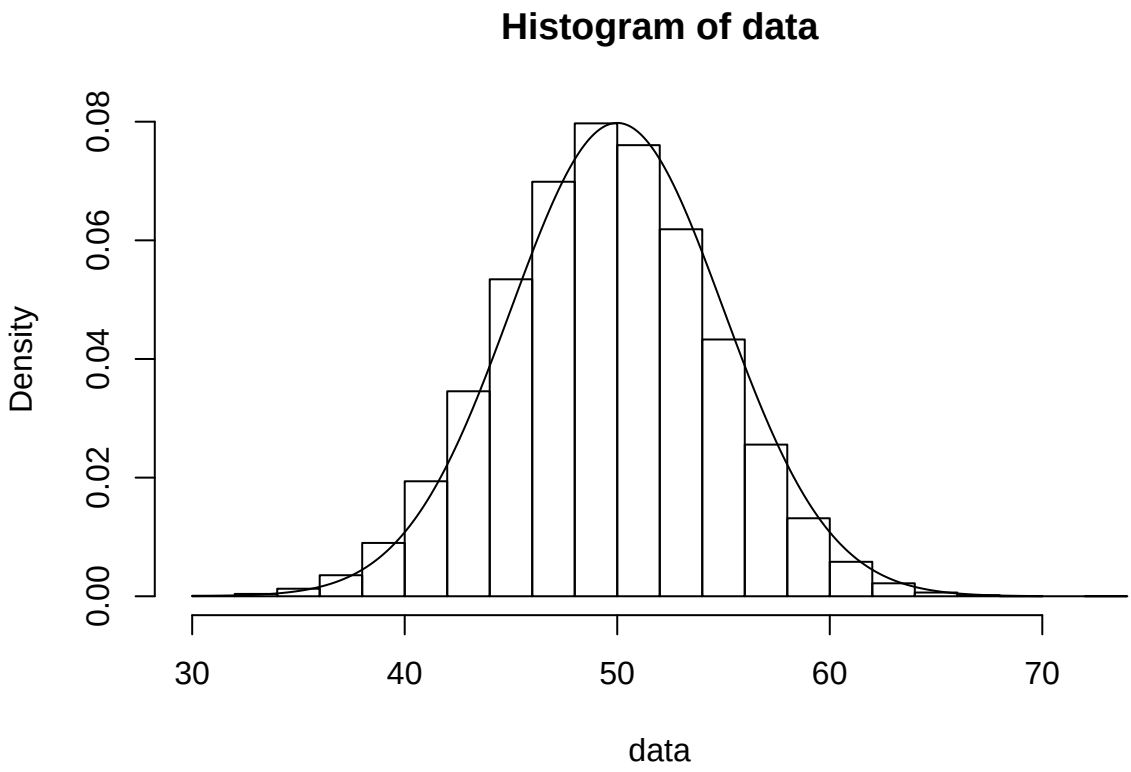
Do these look normal?

Simulate the number of heads in 100 fair coin flips:

```
data = rbinom(100000, 100, 0.5)
hist(data)
```

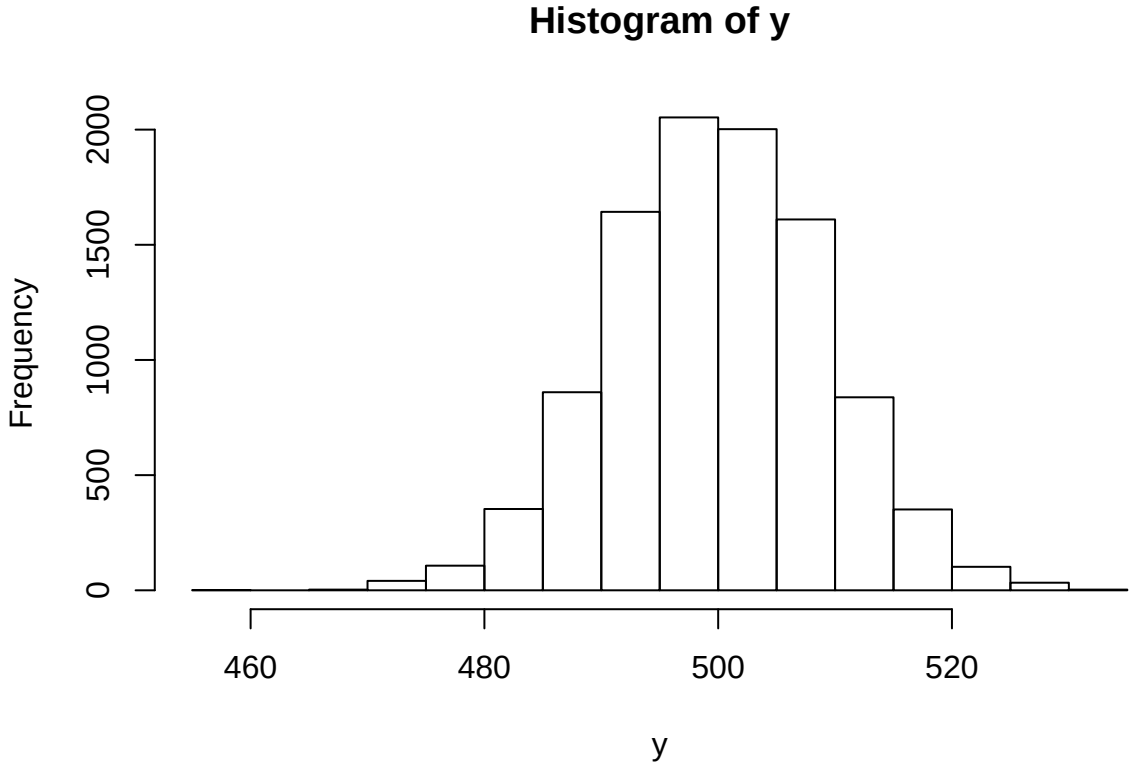



```
hist(data, prob=TRUE)
lines(x, f)
```



Simulate the sum of 1000 independent Uniform$(0, 1)$ random variables:

```
y = replicate(10000, sum(runif(1000)))
hist(y)
```



Normal probabilities

Let Z have a standard normal distribution. Let's find some probabilities!

```
# P(Z <= 1) = P(Z < 1)
pnorm(1)
```

```
## [1] 0.8413447
```

```
# P(Z > 1) = P(Z >= 1)
1 - pnorm(1)
```

```
## [1] 0.1586553
```

```
# P(-1 < Z < 1) = P(|Z| < 1)
pnorm(1) - pnorm(-1)
```

```
## [1] 0.6826895
```

```
# P(-2 < Z < 2)
pnorm(2) - pnorm(-2)
```

```
## [1] 0.9544997
```

```
# P(-3 < Z < 3)
pnorm(3) - pnorm(-3)
```

```
## [1] 0.9973002
```

```
# P(|Z| > 1) = P(Z < -1) + P(Z > 1)
pnorm(-1) + (1 - pnorm(1))
```

```
## [1] 0.3173105
```

```
2 * pnorm(-1)
```

```
## [1] 0.3173105
```

```
2 * (1 - pnorm(1))
```

```
## [1] 0.3173105
```

```
# P(|Z| > 6)
2 * (1 - pnorm(6))
```

```
## [1] 1.973175e-09
```

Now let X have a normal distribution with mean 100 and variance 10^2. Here are some more probabilities:

```r
# P(90 < X < 110)
pnorm(110, mean=100, sd=10) -
  pnorm(90, mean=100, sd=10)
```

```
## [1] 0.6826895
```

```r
pnorm(110, 100, 10) - pnorm(90, 100, 10)
```

```
## [1] 0.6826895
```

```r
# P(80 < X < 120)
pnorm(120, 100, 10) - pnorm(80, 100, 10)
```

```
## [1] 0.9544997
```

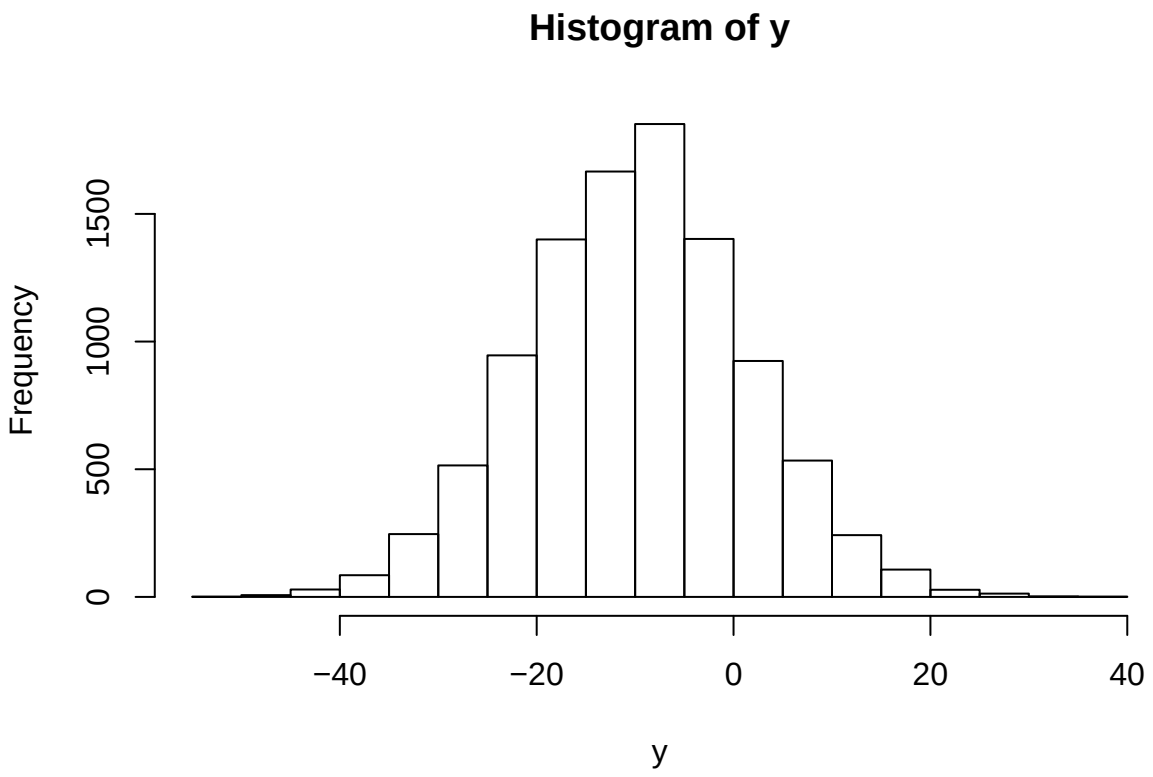
```r
# P(70 < X < 130)
pnorm(130, 100, 10) - pnorm(70, 100, 10)
```

```
## [1] 0.9973002
```

Combining independent normals

Adding/subtracting independent normals gives you a new normal random variable:

```r
x1 = rnorm(10000, mean=10, sd=5)
x2 = rnorm(10000, mean=-20, sd=10)
y = x1 + x2
hist(y)
```



The same is NOT true for multiplying, squaring, dividing, etc. For example:

```
z = rnorm(10000)
hist(z^2) # NOT normal
```