# Chapter 0 Review & Introduction

As a help to understand what we'll be doing in this (and next semester's) class, and to keep all the "basic" information in one place, this Section is a brief review of the important pre-AT731 information you should know.

## A. Interest Theory

$i$ is the Interest Rate

$v$ is $\dfrac{1}{1+i}$, the "discount" (but not the discount rate)

$d$ is the discount rate $= iv = \dfrac{i}{1+i}$

$\delta$ is the Force of Interest $= \ln(1+i)$

Also $e^{\delta} = (1+i)$; $e^{-\delta} = v$

$$(1+i) = \left(1 + {i^{(m)}}\!/\!{m}\right)^{m} = \left(1 - {d^{(p)}}\!/\!{p}\right)^{-p} = 1\!/\!(1-d)$$

$\ddot{a}_{\overline{n}|} = \dfrac{1-v^n}{d}$ Present value of annuity of 1 with first payment at beginning of year [today] ("due")

$a_{\overline{n}|} = \dfrac{1-v^n}{i}$ Present value of annuity of 1 with first payment at end of year ("immediate")

$\bar{a}_{\overline{n}|} = \dfrac{1-v^n}{\delta}$ Present value of annuity of 1 payable continuously throughout year

$\ddot{s}_{\overline{n}|} = \dfrac{(1+i)^n - 1}{d}$ Accumulated value of annuity of 1 with first payment at beginning of year

Increasing Annuities (due)

$$\left(I\ddot{a}\right)_{\overline{n}|} = 1 + 2v + 3v^2 + 4v^3 + \ldots + nv^{n-1} \rightarrow \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$

$$\left(I\ddot{a}\right)_{\overline{\infty}|} = 1 + 2v + 3v^2 + 4v^3 + \ldots \rightarrow \frac{1}{d^2}$$

Decreasing Annuities (Due)

$$\left(D\ddot{a}\right)_{\overline{n}|} = n + (n-1)v + (n-2)v^2 + \ldots + 2v^{n-1} + v^n \rightarrow \frac{n - \ddot{a}_{\overline{n}|}}{d}$$

Continuous Version

$$\left(\bar{I}\bar{a}\right)_{\overline{n}|} = \int_0^n t \cdot v^t \; dt = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}$$

## B. Probability & Statistics

### 1) Distribution Function and Probability Density Function

Distribution Function ("DF") is $F_X(x) = \Pr[X \le x]$

Probability Density Function ("pdf") is

- If ***F*** continuous $f_X(x_0) = F_X^{'}(x_0)$
- If ***F*** discontinuous $f_X(x_0) = Jump[F_X(x_0)] = F_X(x_0) - \lim_{x \to x_0^-} F_X(x_0)$

  (Don't worry; we won't use this)
- If ***F*** discrete $p(k) = \Pr[X = k]$

## 2) Expected Values

***T*** and ***K*** are Random Variables (we'll be using them in future weeks)

- <u>Discrete</u> Case $E[h(K)] = \sum_{k=0}^{\infty} h(k) f(k)$
- <u>Continuous</u> Case $E[h(T)] = \int_0^{\infty} h(t) f(t) dt$

Combining Expected Values

$$E[aX + b] = aE[X] + b$$

## 3) Variances

$$Var[T] = E[T^2] - E[T]^2$$

$$Var[aX + b] = a^2 Var[X]$$

# C. Calculus

## 1) Derivatives

$$\frac{d}{dx} x^n = nx^{x-1} \ (n \neq 0)$$

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}$$

$$\frac{d}{dx} \int_n^x f(t)dt = f(x)$$

## 2) Integrals

**General**

$$\int_a^b u^n du = \left. \frac{u^{n+1}}{n+1} \right|_a^b \ (n \neq 1)$$

$$\int_a^b \frac{du}{u} = \ln |u| \|_a^b = \ln\left(\frac{|u(b)|}{|u(a)|}\right)$$

**Important Integrals for AT731, AT732, Course MLC, Course 3…**

$$\int_0^n e^{-\alpha t}dt = \frac{1-e^{-\alpha n}}{\alpha} = \bar{a}_{\overline{n|}\alpha=\delta}$$

$$\int_0^n te^{-\alpha t}dt = \frac{\bar{a}_{\overline{n|}\delta=\alpha} - ne^{-\alpha n}}{\alpha} = \bar{I}\bar{a}_{\overline{n|}\delta=\alpha}$$

Note that "$\alpha$" can be ANYTHING (usually will be "$\delta$")

**When n=∞**

$$\int_0^\infty e^{-\alpha t}dt = \frac{1-e^{-\alpha\infty}}{\alpha} = 1/\alpha$$

$$\int_0^\infty te^{-\alpha t}dt = \frac{\bar{a}_{\overline{\infty|}} - \infty e^{-\alpha\infty}}{\alpha} = \frac{\frac{1-v^\infty}{\alpha} - 0}{\alpha} = 1/\alpha^2$$

**IMPORTANT: YOU SHOULD MEMORIZE ALL OF THESE FORMULAS, SINCE WE'LL BE SEEING THEM OVER AND OVER AND OVER AND OVER AND….**

### 3) Summation

Sometimes we need to sum a series.

The only formulas we need are simple ones:

Series $1, 0.500, 0.250, 0.125, \ldots$ can be written as $1 \cdot (0.50)^n$

Assigning $q$ as a constant and $r$ as the *ratio*,

$$\sum_{n=a}^{b} qr^n = \frac{\text{First term - Term AFTER last term}}{1-\text{ratio}}$$

$$= \frac{qr^a - qr^{b+1}}{1-r} \quad \text{(True for any real } q, r \text{ except } r = 1\text{)}$$

$$\text{If } |r| < 1, \text{ then } \sum_{n=a}^{\infty} qr^n = \frac{\text{First term - Term «after» last term } (\infty+1)}{1-\text{ratio}}$$

$$= \frac{qr^a}{1-r}$$

## D. Notation

There are **six** basic areas for any notation that we use.

$${}^{\{5\}}_{\{4\}}\overset{\{3\}}{\mathrm{A}}{}^{\{2\}}_{\{1\}}$$

### 1) Center (A)

Tells the type of actuarial function we're looking at.Examples you may have seen are $p, q, a, s$ .We'll look at other functions in this class:

| | |
|---|---|
| $e$ | Expected Lifetime |
| $A$ | Insurance |
| $a$ | Life Annuity |
| $P$ | Premium |

… and more

### 2) Lower Right (1)

Tells the age or the period (or both).Examples

| | |
|---|---|
| $a_{\overline{10}\rceil}$ | Annuity payable for 10 years |
| $A_{50}$ | Insurance for a 50-year old |
| $e_{50}$ | Expected lifetime of a 50-year old |
| $A^{\;1}_{50:\overline{10}\rceil}$ | 10 – year "Term" Life Insurance for a 50-year old |

### 3) Upper Right (2)

Tells us frequency (monthly = 12, semiannual = 2, annual = blank).Examples

| | |
|---|---|
| $a^{(12)}_{\overline{10}\rceil}$ | Annuity payable monthly (at a rate of 1/year) for 10 years |
| $A^{(2)}_{40}$ | Life Insurance payable at the end of half-year for a 40 year old |

### 4) Above (3)

Tells <u>when</u>. Examples

| | |
|---|---|
| $\ddot{a}_{\overline{10}\rceil}$ | Annuity payable at beginning of year for 10 years |
| $\ddot{a}_{30}$ | Life Annuity for 30 year old; first payment today (beginning of year) |
| $a_{30}$ | Life Annuity for 30 year old; first payment end of year |
| $\overline{a}_{30}$ | Life Annuity for 30 year old; payment continuously (rate of 1/year) |

### 5) Lower Left (4)

Tells period we're deferring or time since issue.Examples

| | |
|---|---|
| ${}_{10\rceil}\ddot{a}_{40}$ | Annuity for 40 year old, starts at (beginning of) age 50 |

### 6) Upper Left (5)

Reserves; tells how many Premiums payable

### 7) Big Finish

${}^{20}_{10}V^{(12)}_{40:\overline{10}\rceil}$ , this small notation (which we'll look at in AT732) means

- a Reserve (***V***)
- **10**-year term life Insurance purchased by a **40**-year old (Lower right)
- Annual premiums to age 60 (upper left **20** means 20 payments)
- Death benefits payable at the end of a month (upper right **(12)**)
- And we want to find the reserve at age 50 (lower left **10** means 10 years after purchase)

# E. Life Tables

"Life Tables": Summarizes survival model by specifying the proportion of lives that are expected to survive each age.

In 1600s, using London mortality, John Graunt: each newborn has 50% probability of surviving to age 16; probability of 1% surviving to age 70.

→ Recent US tables (94GAM) show that a 1-year old has probabilities of 99.7% to live to 16 and 82% to age 70.

## 1) Underwriting

- *Preferred*: low mortality risk: no recent record of smoking; no evidence of drug/alcohol abuse; no high-risk hobbies or occupations; no family history of disesase known to have a strong genetic component; no adverse medical indicators (high blood pressure; high cholesterol; high body mass index [BMI])
- *Normal life*: Higher-rated risk factors than preferred lives, but insurable at standard rates. Most applicants fall in this category
- *Rated lives*: One or more risk factors at raised levels; not insurable at standard levels. But can be insured at a higher premium (see select and ultimate discussion later in the semester)
- *Uninsurable*: Such high risk that insurance not available
- Further categorize by relative values of various risks (age, gender, smoking)

## 2) Common Actuarial Notation from life table

| | |
|---|---|
| $x$ | Current Age |
| $\boldsymbol{\alpha}$ | First age in table ("alpha") |
| $\boldsymbol{\omega}$ | Last age in table ("omega") |
| $\boldsymbol{q_x}$ | Probability of death between age $x$ and age $x$+1 |
| $\boldsymbol{p_x}$ | Probability someone who is age $x$ survives to age $x$+1 |
| $\boldsymbol{\ell_x}$ | Number of people alive at age $x$ |
| $\boldsymbol{d_x}$ | Number of people who die between $x$ and $x$+1 |
| $\boldsymbol{D_x}$ | $v^x \ell_x$ |
| $\boldsymbol{N_x}$ | $\sum_{t=x}^{\omega} D_x$ |

## 3) Minding *p*'s and *q*'s (get it?)

The basic probability of survival is called $p$, and the basic probability of not surviving (that is, dying) is called $q$.

As we discussed in Chapter 0, what goes $_{\textbf{here}}p$ and $\mathrm{p}_{\textbf{here}}$ are really important.

The general formula is $_tp_x$, where:

*x* is the age that we're looking at

*t* is the time that we're considering

So, when we see something like ${}_{40}p_{20}$, it means the probability that someone who is now 20 survives for 40 years (not lives to age 40).

Same can be said about $q$, but there's no need to repeat repeat repeat myself. We'll talk (in a bit) about what happens when we see something like ${}_{10|10}q_{10}$.

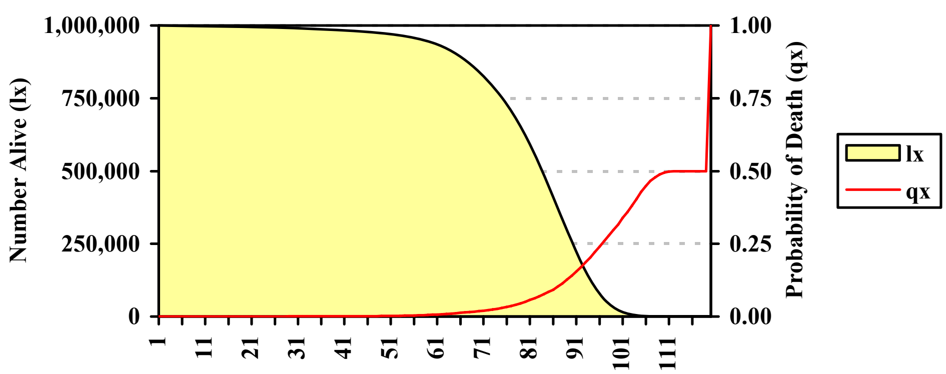
## 4) Example of Life Table

| $x$ | $q_x$ | $\ell_x$ | $d_x$ | $D_x$ | $N_x$ | Comment |
|---|---|---|---|---|---|---|
| **0** | 0.00051 | 1,000,000 | 514 | 1,000,000 | 15,137,709 | $\alpha=0$ |
| **10** | 0.00015 | 997,831 | 150 | 542,754 | 8,174,125 | |
| **20** | 0.00037 | 995,361 | 366 | 275,226 | 4,098,358 | |
| **30** | 0.00059 | 990,790 | 583 | 139,268 | 2,033,350 | |
| **31** | 0.00061 | 990,207 | 606 | 130,081 | 1,894,082 | |
| **40** | 0.00088 | 984,047 | 865 | 70,315 | 989,345 | |
| **50** | 0.00187 | 972,027 | 1,817 | 35,308 | 463,104 | |
| **60** | 0.00606 | 940,744 | 5,703 | 17,371 | 200,375 | |
| **90** | 0.14065 | 258,873 | 36,410 | 628 | 2,642 | |
| **100** | 0.31663 | 22,411 | 7,096 | 28 | 71 | |
| **110** | 0.49481 | 108 | 53 | 0 | 0 | |
| **120** | 1.00000 | 0 | 0 | 0 | 0 | $\omega=120$ |

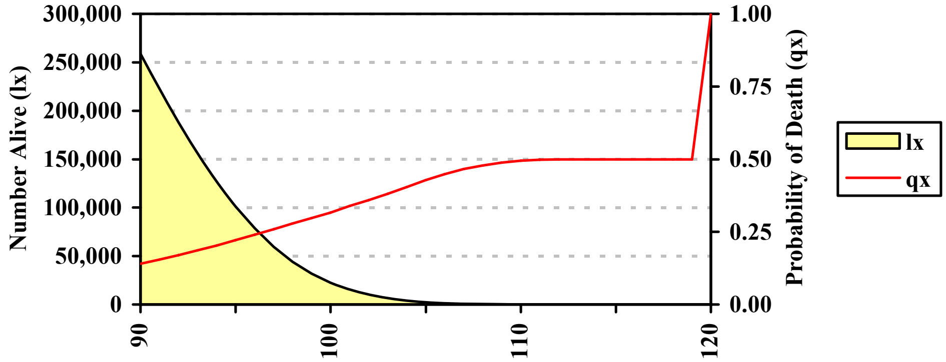
Example 0-1 – Basic Probabilities

From the table on the prior page, we can determine the probability that

1) a 30-year old survives for one year = $p_{30} = 1\text{-}q_{30} = 0.99941$
2) Or, we can say that $p_{30} = \ell_{31}/\ell_{30} = 990207/990790 = 0.9999$ (rounding)
3) The probability that a 30-year old lives to 80 is $\ell_{90}/\ell_{30} = 358873/990790 = 0.3624$; the probability the 30-year old dies before 90 is 1-0.3624=0.6376




Let's zoom in to 90 - 120




This table uses the "1983 Group Annuity Mortality" for $q_x$ .Interest is 7.5% (affects only $D_x$ and $N_x$ )

## F. What's the point of "Actuarial Math'?

### 1) The goal of pricing Insurance and Annuities:

## G. Comparison: "Interest Theory" and "Actuarial Mathematics"

| | **Interest Theory** | **Actuarial Math** |
|---|---|---|
| Payment of \$1 in the future; want value today | Know how many years in the future ($n$).Calculate as $v^n$ | Don't know when payment will be made.<br>**Example:** pay \$1 upon <u>death of</u> insured person.Can't use $v^n$ because we don't know value of $n$ |
| Payment of \$1 per year; want value today | Know how many years in the future ($n$).Calculate as $a_{\overline{n}\rvert}$ or $\ddot{a}_{\overline{n}\rvert}$ | Don't know how many years of payment.<br>**Example:** pay \$1/year while insured is alive; stop payment upon death.Can't use $a_{\overline{n}\rvert}$ or $\ddot{a}_{\overline{n}\rvert}$ because we don't know value of $n$ |

For each problem in Actuarial math, we can have several **Random Variables** (notation is to use bold, italic, capital letter: for example, $\boldsymbol{X}$)

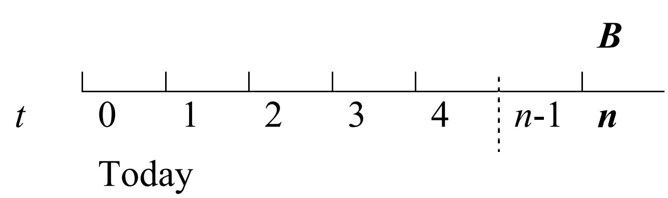
1) What is premium?
2) When will benefit be payable?
3) How much is benefit?

Usually, #3 is known. Which leaves:

$\boldsymbol{B}$=benefit payable in $n$ years

$\boldsymbol{E}$=event that causes payment of $B$

- Could be death (life insurance)
- Could be accident (car insurance)
- Could be failure of battery or expiration of warranty (casualty topics)
- Could be stopping being alive (annuity); this is opposite of life insurance, as we will see



Present Value is

- $B \cdot v^n$ with probability $\Pr(E)$
- $0 \cdot v^n$ with probability $1\text{-}\Pr(E)$

$$\text{Expected Present Value} = \text{How Much} \cdot \text{Interest Discount} \cdot \text{Probability of Occurence}$$

$$PV = B \cdot v^n \cdot \Pr(E)$$