Chapter 5 Reserves / Policy Values

A. Basics – What are "Reserves"?

Business model

Assets = Liabilities + Equity (Stock)

Insurance Company model

Assets = Amount held by company to pay future benefits + "Surplus"

A Benefit Reserve at any given time is the amount he insurance company needs to have on hand to be able to pay benefits that are owed (to policyholders) but not yet payable.

Benefit Reserves can be defined as

Present Value of Benefits – Present Value of Premiums.

Net Benefit Reserves assume no expenses are payable (we may change that assumption at a later time)

B. General Case

Assume that ℓ_x lives purchase $1 of insurance

Inflow	$P_x\ell_x$	$P_x\ell_{x+1}$	$P_x\ell_{x+2}$	$P_x\ell_{x+k}$	$P_x\ell_{x+k+1}$	$P_x\ell_{\omega-1}$	
Outflow		d_x	d_{x+1}	d_{x+k-1}	d_{x+k}	$d_{\omega-2}$	$d_{\omega-1}$
Age	x	$x+1$	$x+2$	$x+k$	$x+k+1$	ω-1	ω

1) Aggregate Liability (for Insurance Company)

If we look at what the Insurance Company needs prospectively, at time *t*, the Present Value of the future shortfall (i.e. how much more money the insurance company needs than it has) is equal to:

PV Future Shortfall$_t$ = Present Value Future Outflow$_t$ - Present Value Future Inflow$_t$

In actuarial math terms:

$$=\left[vd_x+v^2d_{x+1}+v^3d_{x+2}+v^4d_{x+3}+\cdots\right]-\left[P_x\ell_x+vP_x\ell_{x+1}+v^2P_x\ell_{x+2}+v^3P_x\ell_{x+3}+\cdots\right]$$

This is a **prospective** view of the amount the Insurance Company needs.

Put another way, the insurance company needs (or has too much) money at time *t* compared to how much it is expected to payout until the end of time. This is the **Aggregate Liability** for the insurance company.

2) Aggregate Assets (for Insurance Company)

If we ask the question looking retrospectively, the shortfall/excess for the Insurance Company is:

AV Past Excess$_t$ =
Accumulated Value Past Inflow$_t$ – Accumulated Value Past Outflow$_t$

In actuarial math terms:

$$=\left[P_x\ell_x(1+i)^t+P_x\ell_{x+1}(1+i)^{t-1}+\cdots+P_x\ell_{x+t-1}(1+i)\right]-\left[d_x(1+i)^{t-1}+d_{x+1}(1+i)^{t-2}+\cdots+d_{x+t-2}(1+i)+d_{x+t-1}\right]$$

This is a **retrospective** view of the amount the Insurance Company needs.

3) Creating the Formula

Since we want to have (at *t=0*), the Present Value of Liabilities to equal the Present Value of Assets (this is how we defined the Net Single Premium and Annual Premium), the present values will always be equal (at all future time *t*).

At each time *t*, therefore, the **Aggregate Liability = Aggregate Assets**, which we can combine as:

- (PRESENT VALUE Future Outflow – PRESENT VALUE Future Input) or
- (ACCUMULATED VALUE Past Inflow – ACCUMULATED VALUE Past Outflow)

C. Notation - Simple

Just as we used A_x for insurance, we have special notation for Reserves.

${}_tV_x$ is the general format of a Reserve at time *t* for an insurance purchased by *x*

But, we often (informally) use R_x to mean a general reserve (that is, instead of writing the specifics about *t*, *x*, continuous, etc.)

D. Definition of Reserve

1) Prospective

Look at the "Aggregate Liability" per person

$$\text{Excess/Shortfall} = \left[\frac{vd_x}{\ell_x}+\frac{v^2d_{x+1}}{\ell_x}+\frac{v^3d_{x+2}}{\ell_x}+\cdots\right]-\left[\frac{P_x\ell_x}{\ell_x}+\frac{vP_x\ell_{x+1}}{\ell_x}+\frac{v^2P_x\ell_{x+2}}{\ell_x}\cdots\right]$$
$$=\left[vq_x+v^2{}_{1|}q_x+v^3{}_{2|}q_x+\cdots\right]-P_x\left[1+vp_x+v^2{}_2p_x+\cdots\right]$$

You should recognize the quantities in parentheses as *insurance* and *annuity*.

Reserve at time *t* (i.e. age *x*+*t*) =
Insurance to someone (*x*+*t*) minus premium (from age *x*) times annuity to someone (*x*+*t*)

Using the actuarial notation that we learned in AT731 (or elsewhere), this is equivalent to

$$ {}_tV_x = A_{x+t} - P_x \ddot{a}_{x+t} $$

Note that we're looking at insurance at annuity at *x*+*t*, not *x*, but the Premium <u>is</u> computed at *x*.

2) Retrospective

Look at the "Aggregate Asset" per person. (I'm saving you the trouble of having to read a bunch more actuarial notation, just to get to the following point…)

$$ {}_tV_x = P_x \ddot{s}_{x:\overline{t|}} - \frac{A^{1}_{x:\overline{t|}}}{{}_tE_x} \frac{t \text{ year Term Insurance}}{t \text{ year Pure Endowment}} $$

In words: The reserve is equal to the <u>accumulated value</u> of the premium, recognizing survival and interest, minus a *t-year term insurance* divided by a *t-year pure endowment* (this reflects the <u>accumulated value</u> of that insurance, adjusted for survival and interest).

Example 5-1 – Simple Numerical Example

Given the following distribution for a group of 90-year olds (remember them?) who buy $1,000 of Whole Life Insurance

x	ℓ_x	d_x
90	100	28
91	72	33
92	39	39
93	0	--

i=6%; last Semester, we found out that $1000A_{90} = 885.30; \ddot{a}_x = 2.0263; 1000P_{90} = 436.90$

Using the "Aggregate Deterministic" approach to our calculation, we have (see development in class)

At t=1, what should Insurance Company do with **amount at end of year?** *Spend it?*

This is where we can look at what a "Reserve" really is:

	Year 1	Year 2
Outflow	33,000	39,000
Inflow	31,458	17,039

$$PV(Outgo): 33000v + 39000v^2 = 65{,}842$$
$$PV(Income): 31458 + 17039v = 47{,}531$$
$$PV(Shortfall): Outgo - Income = 18{,}311$$

So, the Insurance Company holds a "Reserve" of 18,311 at the end of year 1 (when the 90-year olds are now 91). This will be used to pay for future benefits. There are **72** people still alive, so the Reserve, which is computed per person, is

$$1000{}_1V_{90} = \frac{18311}{72} = 254.32$$
$$1000{}_2V_{90} = \frac{19754}{39} = 506.49$$

We've now calculated our first Net Premium Reserve!!

Let's use the formulas that we developed earlier.

3) <u>Prospective</u> Development

$${}_1V_{90} = A_{91} - P_{90}\ddot{a}_{91}$$
$$A_{91} = v\frac{d_{91}}{\ell_{91}} + v^2\frac{d_{92}}{\ell_{92}} = 0.91447$$
$$\ddot{a}_{91} = \left[1 + v\frac{d_{91}}{\ell_{91}}\right] \text{ or } \frac{1-A_{91}}{d} = 1.511$$
$$\therefore 1000 \cdot {}_1V_{90} = 1000\left[0.914471 - (.436895)(1.511)\right] = 1000 \cdot 0.25432 = 254.32$$

Example 5-2 – Another approach to numerical example

Let's use a 6% interest rate and the mortality table from *Bowers* to look at the purchase of a whole life insurance by 50-year olds.

$100000 q_{50} = 592; P_{50} = 0.0187722$, payable for the life of a 50-year old

If 100,000 policies are issued, the insurance company gets 1,877.22

In a year, the insurance company earns interest of 112.63

So, at the end of the year, the insurance company has 1,989.85

We expect to have 592 claims during the year ....(592.00)

Which means the insurance company has "left over" at year-end 1,397.85

Divide by the survivors (99,408) .... 0.01406 ← ${}_1V_{50}$

Compare this to a 1-year term insurance $A^{\,1}_{50:\overline{1}|} = vq_{50} = 0.0055848$.

If 100,000 policies are issued, the insurance company gets 558.48

In a year, the insurance company earns interest of 33.51

So, at the end of the year, the insurance company has 592.00

We expect to have 592 claims during the year ....(592.00)

Which means the insurance company has at year-end 0.00

In a similar way,

$$\begin{aligned} &A^{\,1}_{51:\overline{1}|} = 0.006058 \\ &\vdots \\ &A^{\,1}_{65:\overline{1}|} = 0.020113 > 0.0187722 \text{ (which is } P_{50}) \\ &\vdots \end{aligned}$$

So, there's a time at which the "one year term" is more expensive than the "whole life." One year term policies build no reserve

E. Notation – Complex

Remember what the location of certain of the notation means:

- Lower left (t in the ${}_tV_x$) means <u>how many years since issue</u> we're looking at the reserve
- Lower right (x in the ${}_tV_x$) means the <u>age at issue</u>
- If there's a bar above the middle $(\overline{VA})$, we have a continuous insurance
- If there's no bar, it's discrete.

- Just like Premiums, if we're dealing with a discrete insurance, we can leave the *A* out of the formula, as we did in this particular example

Continuous ${}_t\bar{V}(\bar{A}_x - something)$

Whole Life: ${}_t\bar{V}(\bar{A}_x) = \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t}$

n-year Term: ${}_t\bar{V}(\bar{A}^{1}_{x:\overline{n}|}) = \bar{A}^{1}_{x+t:\overline{n-t}|} - \bar{P}(\bar{A}^{1}_{x:\overline{n}|})\bar{a}_{x+t:\overline{n-t}|}$

n-year Pure Endowment: ${}_t\bar{V}(A_{x:\overline{n}|}^{\ \ 1}) = A_{x+t:\overline{n-t}|}^{\ \ \ \ 1} - P(A_{x:\overline{n}|}^{\ \ 1})\bar{a}_{x+t:\overline{n-t}|}$

n-year Endowment: ${}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x+t:\overline{n-t}|} - \bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{x+t:\overline{n-t}|}$

h Payment Whole Life: Premium is payable for a given number of years (*h*)

Premium	${}_hP_x$	${}_hP_x$	${}_hP_x$	${}_hP_x$			
Age	x	$x+1$	$x+2$	$x+h-1$	$x+h$	$x+k$	$x+k+1$

$${}^{h}_{t}\bar{V}(\bar{A}_x) = \bar{A}_{x+t} - {}_h\bar{P}(\bar{A}_x)\bar{a}_{x+t:\overline{h-t}|}\ , t < h$$

$${}^{h}_{t}\bar{V}(\bar{A}_x) = \bar{A}_{x+t}\ , t \geq h$$

n-year Deferred Annuity:

Annuity					\$1	\$1	
Premium	$P\left({}_{n\vert}\ddot{a}_x\right)$	$P\left({}_{n\vert}\ddot{a}_x\right)$	$P\left({}_{n\vert}\ddot{a}_x\right)$	$P\left({}_{n\vert}\ddot{a}_x\right)$			
Age	x	$x+1$	$x+2$	$x+n-1$	$x+n$	$x+k$	$x+k+1$

$${}_t\bar{V}({}_{n|}\bar{a}_x) = {}_{n-t|}\bar{a}_{x+t} - \bar{P}({}_{n|}\bar{a}_x)\bar{a}_{x+t:\overline{n-t}|}\ \ , t < n$$

$${}_t\bar{V}({}_{n|}\bar{a}_x) = \bar{a}_{x+t}\ \ , t \geq n$$

Discrete ${}_tV(A_x - something)$

It's important to remember that the Premium is payable at the *beginning of a year*, but Reserve is computed at the *end of the prior year*.

So, the Premium that is *about to be made* (i.e. on January 1) *is included* in the Reserve computed the day before (December 31).

Terminal Reserve in year $x+k$ ${}_kV_x = A_{x+k} - P_x\ddot{a}_{x+k}$

Initial Reserve in year $x+k+1$ ${}_kV_x + P_x$

> Problem 5-1
>
> $A_{x:\overline{n}|} = 0.20; d = 0.08$
>
> Find ${}_{n-1}V_{x:\overline{n}|}$

Problem 5-2

$$\mu_{35}(t) = 0.03, t \geq 0$$

$$\delta = 0.05$$

$$1000\bar{A}^{\,1}_{35:\overline{25|}} = 324.25$$

$$\bar{a}_{35:\overline{15|}} = 8.7351$$

$${}^{15}_{5}\bar{V}\left(\bar{A}_{35:\overline{25|}}\right) = ???$$

Problem 5-3

Special Fully Discrete Whole Life insurance of \$1 purchased by a 45-year old

Premiums P_1 *and* P_2 *are calculated under Equivalence Principle*

Premium P_1 *is paid for first 20 years, then* P_2 *is paid*

$$A_{45} = 0.25; A_{65} = 0.50; \ddot{a}_{45} = 15.71; \ddot{a}_{65} = 10.60; \ddot{a}_{45:\overline{20|}} = 12.43$$

${}_{20}V$ *is the Terminal Reserve under this premium payment pattern*

If the premium at/after 20 years is changed to P_1 *and the benefit is reduced to 75¢,* ${}_{20}V$ *doesn't change.*

Find ${}_{20}V$

Problem 5-4

A 30-year old purchases a 20-payment 35-year endowment.

n	$\ddot{a}_{30:\overline{n\|}}$	$\ddot{a}_{40:\overline{n\|}}$
10	7.75	7.60
20	11.96	11.76
25	13.25	12.95
35	14.84	14.27

d=.06

What is the ***Net Benefit Reserve*** *at the end of the* ***10th*** *year?*

F. Loss Random Variable

1) x purchases A_x with premium P_x
2) x survives to $x+t$
3) The "Loss" to the Insurance Company (just as it was in the review of Premiums) is a good thing, since no payment is made: $PVFBen_t - PVFP_t$
4) Using J as the curtate future lifetime of $x+t$

$$_tL = 1 \cdot v^{J+1} - P_x \cdot \ddot{a}_{\overline{J+1}|}$$

Prospective Future Loss Function = PV \$1 payable in J+1 years minus PVFP at J+1

Note that this looks like the "Loss Random Variable at Issue" that we studies when we looked at Premiums $(L \text{ or } {}_0L)$**. That's why we used the "0" before the Loss Random Variable.**

5) $_tL$ is a conditional distribution
   a. x is alive at $x+t$
   b. Policy is still "in force"

If $s > t$, $\Pr[T(x) > s \mid T(x) > t] = {}_{s-t}p_{x+t} = \Pr[T(x+t) > s]$

$$\text{Example}: \Pr[T(50) > 15 \mid T(50) > 10] = \frac{{}_{15}p_{50}}{{}_{10}p_{50}} = \frac{{}_{10}p_{50} \cdot {}_5p_{60}}{{}_{10}p_{50}} = {}_5p_{60}$$

1) Benefit Reserve at *t*

$$\begin{aligned} {}_tV &= E\left[{}_tL \mid T(x) > t\right] \\ &= PVFB_{x+t} - PVFP_{x+t} \\ &= E\left[v^{J+1}\right] - P_x E\left[\ddot{a}_{\overline{J+1}|}\right] \\ &= A_{x+t} - P_x \ddot{a}_{x+t} \end{aligned}$$

Note that this is the Prospective Form that we've already looked at. But, this time, it's all about the Expected Value of the Loss Random Variable.

> Problem 5-5
>
> *3 Year fully discrete term insurance for* x. *Death benefit = \$1,000. Interest = 10%*
>
> $q_x = 0.1; q_{x+1} = 0.2; q_{x+2} = 0.3$
>
> *Find the formula for each of* ${}_0L,\ {}_1L,\ {}_2L$ *and the Expected Value of each.*

G. Calculation of Risk

Look at the Variance of the Loss Random Variable

$$ {}_{T}L = v^{T} - \pi \bar{a}_{\overline{T|}} \text{ (Note: Interest Only!)} $$
$$ = Z - \pi Y $$

Remember (from Premiums, last semester) that we can only do the following if we have

- a **whole life insurance** with premiums payable for life or
- an ***n*-year endowment insurance** with premiums payable for *n* years

so that we can use the fact that $Y = \dfrac{1-Z}{d \text{ or } \delta}$.

$$
\begin{aligned}
Var\left[{}_{t}L\right] &= Var\left[v^{t} - \pi \bar{a}_{\overline{t|}}\right] \\
&= Var\left[v^{t} - \pi \frac{1-v^{t}}{\delta}\right] \\
&= Var\left[v^{t}\left(1+\frac{\pi}{\delta}\right) - \frac{\pi}{\delta}\right] \\
&= \left(1+\frac{\pi}{\delta}\right)^{2} Var\left[v^{t}\right] \\
&= \left(1+\frac{\pi}{\delta}\right)^{2} \left({}^{2}\bar{A}_{x+t} - \bar{A}_{x+t}^{2}\right)
\end{aligned}
$$

Problem 5-6

A 40-year old purchases a fully continuous whole life insurance with a face value of $10,000. Premium is calculated under Equivalence Principle.

UDD applies.

$$
\begin{aligned}
A_{40} &= 0.16132 \\
A_{50} &= 0.24905 \\
{}^{2}A_{50} &= 0.09476 \\
\bar{a}_{40} &= 14.3166 \\
i &= 6\%
\end{aligned}
$$

$Var\left[{}_{10}L \mid T(40) > 10\right] = ???$

Problem 5-7

Premiums are computed under the Equivalence Principle for a fully continuous 20-year endowment payable to a 45-year old that pays $4,000. Mortality follows DeMoivre's Law with $\omega = 95$. $\delta = 0.075$. *Find* $Var\left[{}_{5}L \mid T(45) > 5\right]$.

Problem 5-8

$_1L$ is the Loss Random Variable one year after issue for a Fully Discrete n-year endowment insurance of $1. $i = 5.4\%$. Premiums calculated using Equivalence Principle.

	k=0	*k=1*
$A_{x+k:\overline{n-k}\rvert}$	*0.7439*	*0.7703*
$^2A_{x+k:\overline{n-k}\rvert}$	*0.5622*	*0.6007*

Find $Var[{}_1L \mid L > 1]$

H. Using the Definition of $\left[{}_tL \mid T(x) > t\right]$ to find $Var[{}_tL]$

- Use this approach when we have nonstandard benefit and/or premium
- Need a formula for the function that defines $\left[{}_tL \mid T(x) > t\right]$ in terms of $T(x) - t = T(x+t)$
- Know that ${}_tL = \underbrace{Z_t}_{\substack{LRV\text{ for}\\ \text{Insurance}}} - \underbrace{\pi Y_t}_{\substack{LRV\text{ for}\\ \text{Annuity}}}$
- $$E\left[{}_tL\right] = \text{Reserve at time } t$$
  $$= PVFB_t - PVFP_t$$

Approach:

1) State function defining LRV in terms of $T(x)$ or $K(x)$
2) Find Variance

Problem 5-9

A 4-year Fully Discrete Increasing Term Insurance is purchased by (x) with a death benefit of $1, $2, $3, $4. Premiums are payable in a decreasing pattern: 4π, 3π, 2π, π, and are calculated using the Equivalence Principle.

$v = 0.9$.

k	$_{k\vert}q_x$	$_kp_x$
0	*0.10*	*1.00*
1	*0.15*	*0.90*
2	*0.20*	*0.75*
3	*0.25*	*0.55*

Find $_1L$*, given death at 3.3*

Find $_2L$*, given death at 2.3*

Problem 5-10

In problem Problem 5-10, Find $Var\left({}_1L\right)$ *given x survives to* 1

Problem 5-11

For a special continuous whole life insurance for (x), *you are given:*

$\delta = 0.05;\ b_t = 900e^{0.009t};\ \mu(t) = 0.02,\ t \geq 10;\ \pi_t = 10e^{0.02t},\ t \geq 10$

Calculate the Benefit Reserve at the end of year 10

I. Other Reserve Formulas

Remember (works for both continuous and discrete):

- Prospective: ${}_tV_x = A_{x+t} - P_x\ddot{a}_{x+t}$
- Retrospective: ${}_tV_x = P_x\ddot{s}_{x:\overline{t|}} - \dfrac{A^{\ 1}_{x:\overline{t|}}}{{}_tE_x}$

1) "Premium Difference" Formula

Start with Prospective…

$${}_tV_x = \left(P_{x+t} - P_x\right)\ddot{a}_{x+t}$$

2) "Ratio of Annuities" Formula

Need relationship between insurance and annuity to get "$P_x + d$"

Start with Prospective…

$${}_tV_x = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$$

3) "Paid up Insurance" Formula

Start with Prospective…

$${}_tV_x = A_{x+t}\left(1 - {}^{P_x}\!/\!_{P_{x+t}}\right)$$

4) Reserve in terms of Life Insurance APV

Start with Prospective…

$${}_tV_x = \frac{A_{x+t} - A_x}{1 - A_x}$$

5) "Recursive" Formula

Start with "Aggregate Deterministic" relationship

Divide and regroup

$$\underbrace{\left({}_tV_x + P_x\right)}_{\substack{\text{Amount at Start}\\ \text{of Year}}} = \underbrace{1 \cdot v \cdot q_{x+t}}_{\substack{\text{PV}\\ \text{Death Benefit}}} + \underbrace{{}_{t+1}V_x \cdot v \cdot p_{x+t}}_{\text{PV to Survivors}}$$

6) Another Look at the "Retrospective" Formula

${}_nk_x = \dfrac{A^{\,1}_{x:\overline{n}|}}{{}_nE_x}$ is equal to the accumulated value of a n-year term insurance at age $x+k$

$${}_nk_x = \frac{d_x(1+i)^{n-1} + d_{x+1}(1+i)^{n-2} + \cdots + d_{x+n-1}}{\ell_{x+n}}$$

$${}_1k_x = \frac{d_x}{\ell_{x+1}}, \text{ Interest doesn't matter (no } i\text{)}$$

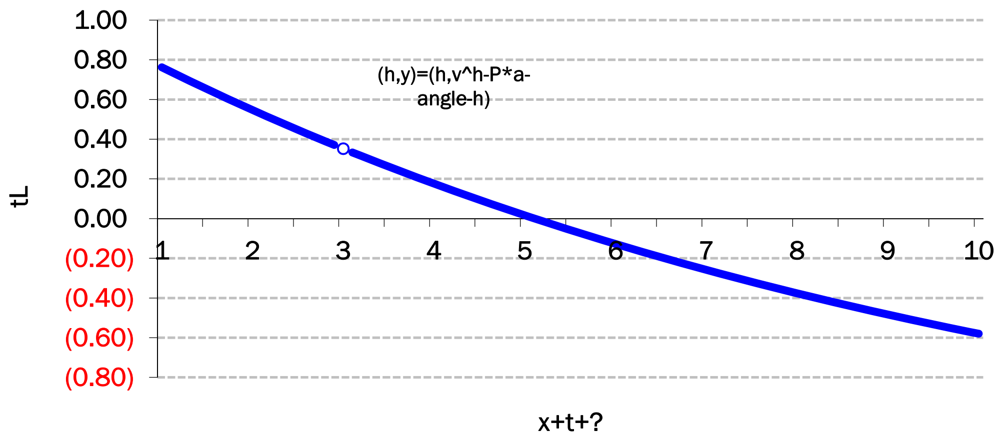
For ${}_nk_x, n > 1,$ Interest does matter

> Problem 5-12
>
> *The reserve at the end of year 1 for a semi-continuous* n-*year term insurance of $1,000 is $15. Benefits are payable at the moment of death, premiums payable at the beginning of the year.*
>
> *δ= 7.0%; μ=0.04*
>
> *What is the premium?*

> Problem 5-13
>
> ${}_{10}V_{25} = 0.1;\ {}_{10}V_{35} = 0.2;\ {}_{20}V_{25} = ???$

> Problem 5-14
>
> ${}_5V_{20:\overline{15}|} = 2/7;\ A_{20:\overline{15}|} = 0.30;\ A_{30:\overline{5}|} = 0.70$
>
> ${}_5V_{25:\overline{10}|} = ???$

J. Density and Distribution of $\left[{}_tL \mid L > t\right]$



> Problem 5-15
>
> *A fully continuous whole life insurance of $1 is purchased by a 20-year old. DeMoivre's Law applies, with* $\omega = 100$. $\delta = 0.06$
>
> *Given: the 20-year old survives to the expected age at death*
>
> *Find: the probability that the Loss Random Variable is greater than the Reserve at that time*

K. Recursive ("Fackler") Formula

We know that $\left({}_kV_x + \pi_x\right) = b_{k+1} \cdot v \cdot q_{x+k} + {}_{k+1}V_x \cdot v \cdot p_{x+k}$ (see page 104), using b_{k+1} instead of "1" and generalizing the premium. We're also making it clear that we're using a discrete insurance (k instead of t). Let's rearrange:

$$ {}_{k+1}V_x = \frac{\left({}_kV_x + \pi_k\right)(1+i) - b_{k+1} \cdot q_{x+k}}{p_{x+k}} \quad \text{and} $$

$$ {}_{k+1}V_x = \left({}_kV_x + \pi_k\right)(1+i) - \left(b_{k+1} - {}_{k+1}V_x\right) \cdot q_{x+k} $$

So, the reserve at the end of year $k+1$ can be described in terms of the reserve at the end of year k, the benefit payable at the end of year $k+1$, and the reserve itself!

1) Net Amount at Risk

Looking at the equation, we can see that the $\left({}_kV_x + \pi_k\right)$, the beginning of year reserve plus the beginning of year premium, is not subject to any probability. But the term $\left(b_{k+1} - {}_{k+1}V_x\right)$ depends on the probability of death between age $x+k$ and $x+k+1$.

Looking at the expression, it says (in words) that the benefit payable at the end of the year minus the reserve at the end of the year is multiplied by the chance of not living from the beginning to the end of the year. **Another way to look at this is that** if the amount the insurance company may have to pay out an amount that's equal to the benefit it must pay (b_{k+1}) if that's more than the reserve that the insurance company has set up (${}_{k+1}V_x$).

The Net Amount at Risk is equal to $\left(b_{k+1} - {}_{k+1}V_x\right)$ and represents how much the insurance company may have to pay out in excess of what it has reserved, so is the amount the insurance company is risking for the period.

Problem 5-16

$1000P^{1}_{35:\overline{8}|} = 117.30$

$1000\,{}_6V^{1}_{35:\overline{8}|} = 1.59$

$q_{41} = 0.12$

$q_{42} = 0.13$

Find i

Problem 5-17

Given

$u(x) = vq_{64+x} + vp_{64+x} \cdot u(x+1) - P_{65},\ x = 0,1,\ldots 36$

$u(36) = 1$

$\omega = 100$

Find an expression for $u(1)$

Problem 5-18

A 30-year old buys a Special discrete 3-year term insurance. Death benefit is 2^{k} *at the end of years* k*=1,2,3. Premiums are payable at the beginning of the year and are 3***π** *at the beginning of year 1, and* **π** *at the beginning of year 2.*

$p_x = 0.9,\ x \geq 0$

$v = 0.9$

${}_1V = 0.6954$

${}_2V = ???$

After the policyholder has paid the premium for a year (π) [we will assume that premiums are constant for all years], some of it goes to pay death benefits. From the previous section, the value of amount that we need to take from the premium (computed at the beginning of the year) is $vq_{x+k}\left(b_{k+1} - {}_{k+1}V_x\right)$. This means that the remainder is available to build up reserves. The available amount for a year, then, is $\pi - vq_{x+k}\left(b_{k+1} - {}_{k+1}V_x\right)$.

The amount that remains has to grow to ${}_nV$ by the time we get to age $x+n$.

$$
\begin{aligned}
{}_nV &= \sum_{k=0}^{n-1}\left[\pi - vq_{x+k}\left(b_{k+1} - {}_{k+1}V\right)\right](1+i)^{n-k} \\
&= \pi\ddot{s}_{\overline{n}|} - \sum_{k=0}^{n-1}\left[vq_{x+k}\left(b_{k+1} - {}_{k+1}V\right)\right](1+i)^{n-k}
\end{aligned}
$$

2) Constant Net Amount at Risk

When we know that the **benefit payable at the end of the year is a constant amount more than the end of the year Reserve** (this includes the case where that constant equals zero), we have the following (using ***B*** as the amount in excess of Reserve).

We know that we want to accumulate an amount such that the total of this accumulation will – when accumulated from today (time 0) to the end of the year we're looking at (time *n*-1) be sufficient to pay the reserve (following the discussion above).

Algebrically:

General Recursion Formula

$$\underbrace{\left({}_tV+\pi\right)}_{\substack{\text{Amount at Start}\\ \text{of Year}}}=\underbrace{1\cdot v\cdot q_{x+t}}_{\substack{\text{PV}\\ \text{Death Benefit}}}+\underbrace{{}_{t+1}V\cdot v\cdot p_{x+t}}_{\text{PV to Survivors}}$$

Convert to "Fackler"

$${}_{k+1}V=\left({}_kV+\pi\right)(1+i)-\left(b_{k+1}-{}_{k+1}V\right)q_{x+k} \text{ [via Algebra]}$$

$${}_kV=\left({}_{k-1}V+\pi\right)(1+i)-\left(b_k-{}_kV\right)q_{x+k-1} \text{ [substitute } k+1 \text{ for } k\text{]}$$

$$v\,{}_kV-{}_{k-1}V=\pi-\underbrace{\left(b_k-{}_kV\right)}_{\substack{\text{Constant NAR}\\ \text{Substitute with } B}}vq_{x+k-1} \text{ [Divide by } 1+i \text{ and rearrange]}$$

$$v\,{}_kV-{}_{k-1}V=\pi-B\cdot vq_{x+k-1}$$

$$v^k\,{}_kV-v^{k-1}\,{}_{k-1}V=v^{k-1}\pi-B\cdot v^kq_{x+k-1} \text{ [Multiply by } v^{k-1}\text{]}$$

Remember that what we wanted to do was to sum from 0 to *n*-1, but since we're using "*k*-1", from 1 to *n*:

$$\sum_{k=1}^{n}\left[v^k\,{}_kV-v^{k-1}\,{}_{k-1}V\right]=\sum_{k=1}^{n}\left[v^{k-1}\pi-B\cdot v^kq_{x+k-1}\right]$$

First, the left side

$$\sum_{k=1}^{n}\left(v^k\,{}_kV-v^{k-1}\,{}_{k-1}V\right)=\left.\begin{array}{l}\left[v^n\,{}_nV-v^{n-1}\,{}_{n-1}V\right]\\ +\left[v^{n-1}\,{}_{n-1}V-v^{n-2}\,{}_{n-2}V\right]\\ +\left[v^{n-2}\,{}_{n-2}V-v^{n-3}\,{}_{n-3}V\right]\\ +\cdots\\ +\left[v^2\,{}_2V-v\,{}_1V\right]\\ +\left[v\,{}_1V-0\right]\end{array}\right\}v^n\,{}_nV$$

Now the right side

$$\sum_{k=1}^{n}\left(v^{k-1}\pi-Bv^kq_{x+k-1}\right)=\pi\ddot{a}_{\overline{n}|}-B\sum_{k=1}^{n}v^kq_{x+k+1}$$

Bring together left side and right side

$$v^n\,{}_nV=\pi\ddot{a}_{\overline{n}|}-B\sum_{k=1}^{n}v^kq_{x+k+1}$$

$$\pi=\frac{v^n\,{}_nV+B\sum_{k=0}^{n-1}v^{k+1}q_{x+k}}{\ddot{a}_{\overline{n}|}} \text{ [Switch back so } k\rightarrow k+1\text{]}$$

Solving for ${}_nV$:

$${}_nV=(1+i)^n\left[\pi\ddot{a}_{\overline{n}|}-B\sum_{k=0}^{n-1}v^{k+1}q_{x+k}\right]$$

$$=\left[\pi\ddot{s}_{\overline{n}|}-(1+i)^nB\sum_{k=0}^{n-1}v^{k+1}q_{x+k}\right]$$

3) Net Amount at Risk = 0

When we know told that the **benefit equals the Reserve at the end of the year** (instead of a specific dollar amount), we can write that as $b_{k+1} = {}_{k+1}V$. The equation becomes:

$$
\begin{aligned}
{}_nV &= \pi \ddot{s}_{\overline{n}|} - (1+i)^n \sum_{k=0}^{n-1} \left[v^{k+1} q_{x+k} (0) \right] \\
&= \pi \ddot{s}_{\overline{n}|} = \frac{\pi \ddot{a}_{\overline{n}|}}{v^n} = \\
\rightarrow \pi &= \frac{v^n \cdot {}_nV}{\ddot{a}_{\overline{n}|}}
\end{aligned}
$$

Mortality is not used when the "Net Amount at Risk" is equal to zero!

Problem 5-19

For a Fully Discrete 10-year endowment purchased by a 30-year old:

1) *Premiums are level for 10 years*
2) *Death Benefit at end of year is $1 more than the Net Benefit Reserve at that date*
3) *Payment if survival to 40 is $1 (that is, the "endowment" portion is $1)*
4) ${}_k p_x = 0.98^k, k = 0,1,\ldots,10$
5) *Interest is 6.0%*

Find the Net Benefit Reserve at the end of year 3

Problem 5-20

A 45-year old purchases a Fully Discrete 10-year endowment.

1) *Death Benefit at end of year is $5000 more than the Net Benefit Reserve at that date*
2) $q_{x+k} = 0.015, \text{ for all } k$
3) *Interest is 10.0%*

Find the level premium for this insurance.

Problem 5-21

A 45-year old buys a Fully Discrete 10-year Deferred Whole Life Insurance with a face value of $1. The insurance is paid for with 10 level annual premiums.

1) During the deferral period, the death benefit is equal to the Net Benefit Reserve at the end of the year.

2) $A_{55} = 0.350$

3) $q_{45+k} = 0.01 + .001k, k = 0,1,\ldots,9$

4) Interest is 8.0%

Reserves are based on Net Benefit Premiums.

What is the terminal Reserve for the 7th Policy Year?

L. m^{th}ly Reserves

(for 2012 (if still using these notes) move this to just before the reserves between…" later in the notes)

Premiums paid every m^{th} instead of at beginning of year

Problem 5-22

Whole life insurance to (x) with a Death Benefit $1,000 at the moment of death. Premiums are payable at the beginning of each month. Deaths are Uniformly Distributed throughout the year.

$A_x = 0.375; A_{x+k} = 0.500; \delta = 0.06$

${}_kV_x^{(12)} = ???$

M. Reserves between Premium Dates: m^{th}ly Policies

Example: We have a calendar year policy that has annual premiums, but want the Net Benefit Reserve at April 30 (maybe that's the end of the Insurance Company's Fiscal Year).

We don't have to worry about this when we're looking at a continuous insurance, because the Reserve is always calculable.

1) Policy with Premium of $P^{(m)}$, Reserve at Nonintegral t

Define terms (for this analysis only)

$t =$ Time we're looking for Reserve

t is not a multiple of m

$k =$ Integral Part of t

$h =$ Largest Integer such that $k + \frac{h}{m} < t$

$r =$ Remainder of t in years; Always true that $r < \frac{1}{m}$

Example 5-3 – Reserve between "end points"

For example, let's look at the reserve at 2 years, 5 months for a Premium payable quarterly

$$t = 2\tfrac{5}{12}$$

$$k = 2$$

$$m = 4$$

$$h = 1$$

$$r = \underbrace{2\tfrac{5}{12}}_{\substack{t\\ \text{Time we're}\\ \text{looking for}\\ \text{Reserve}}} - \underbrace{2\tfrac{3}{12}}_{\substack{\text{Last time}\\ \text{Premium}\\ \text{was paid}}} = \tfrac{2}{12}$$

h=1 (since when h=1, k+h/m=2.25 and when h=2, k+h/m=2.50, the value we want is h=1)

h=1			r=1/6					
P			P		**V=?**	P		
2	2y 1m	2y 2m	2y 3m	2y 4m	2y 5m	2y 6m	2y 7m	2y 8m

Gray area is portion of premium paid (on April 1) but not yet earned. Need to add this into otherwise-calculated formula for reserve. Use linear interpolation from $t+\frac{h}{m}$ to $t+\frac{h+1}{m}$

$$\underbrace{{}_{k}V^{(m)}}_{\substack{\text{Reserve we're}\\ \text{looking for}}} = \underbrace{\left(1-\frac{h}{m}-r\right){}_{k}V^{(m)} + \left(\frac{h}{m}+r\right){}_{k+1}V^{(m)}}_{\text{Interpolated Terminal Reserve over whole year}} + \underbrace{\left(\frac{1}{m}-r\right)P^{(m)}}_{\text{Unearned Premium}}$$

In our example:

$$\underbrace{{}_{2\frac{5}{12}}V^{(4)}}_{\substack{\text{Reserve we're}\\ \text{looking for}}} = \underbrace{\left(1-\underbrace{\frac{1}{4}}_{\substack{\text{Fraction}\\ \text{through}\\ \text{last Prem}}}-\underbrace{\frac{1}{6}}_{\substack{\text{Fraction}\\ \text{between last}\\ \text{Prem and}\\ {}_{t}V_{x}^{(4)}\ \text{date}}}\right){}_{2}V^{(4)} + \left(\frac{1}{4}+\frac{1}{6}\right){}_{3}V^{(4)}}_{\text{Interpolated Terminal Reserve over whole year}} + \underbrace{\left(\frac{1}{4}-\frac{1}{6}\right)P^{(4)}}_{\text{Unearned Premium}}$$

$${}_{2\frac{5}{12}}V^{(4)} = \frac{7}{12}\,{}_{2}V^{(4)} + \frac{5}{12}\,{}_{3}V^{(4)} + \frac{1}{12}P^{(4)}$$

Problem 5-23

A Whole Life Insurance policy is issued on 9/15/2007 to a 35-year old. Death benefits are payable at the moment of death. Premiums are level and payable at the beginning of each calendar quarter; the Equivalence Principle is used to determine Premiums.

$q_{35} = 0.003;\ i = 8.0\%;\ 1000\mathrm{P}^{(4)}\left(\bar{A}_{35}\right) = 7.00$

a) *Assuming the deaths follow the Uniform Distribution of Death, what is the Terminal Reserve at time 1?*

b) *What is the Net Benefit Reserve at 12/31/2007, if we assume each month has 30 days?*