Optional Reading:
Derivations for comparing two paired means using Bayes factors

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Draft

Paired Data

In the example in the video, we have $n = 10$ paired observations Y_{iB} and Y_{iS} for $i = 1, \ldots, n$ representing the concentrations of zinc at the bottom and surface, respectively.

Rather than working with the two groups of observations, we will work with the differences $D_i \equiv Y_{iB} - Y_{iS}$ to make inference about the difference in the means $\mu_1 - \mu_2 \equiv \mu$ converting this problem to a one group Normal problem.

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> zinc
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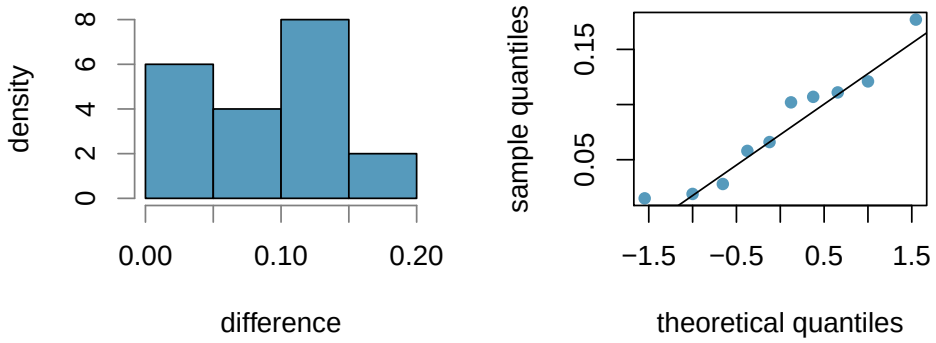
	bottom	surface	difference
1	0.430	0.415	0.015
2	0.266	0.238	0.028
3	0.567	0.390	0.177
4	0.531	0.410	0.121
5	0.707	0.605	0.102
6	0.716	0.609	0.107
7	0.651	0.632	0.019
8	0.589	0.523	0.066
9	0.469	0.411	0.058
10	0.723	0.612	0.111

We will make the same assumptions about the distributions of the differences as in the case of the frequentist paired t-test. That is conditional on the parameters μ and σ^2 the observed differences are independently and identically distributed from a normal distribution expressed as

$$D_i \mid \mu, \sigma^2 \overset{\text{iid}}{\sim} \mathsf{N}(\mu, \sigma^2)$$

. To check the assumption of normality we can look at a histogram or normal quantile plot of the sampled differences.



Likelihood

The normal sampling model leads to a likelihood function

$$\mathcal{L}(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{\sigma^2 2\pi}} \exp\left(-\frac{1}{2}\frac{(D_i - \mu)^2}{\sigma^2}\right) \tag{1}$$

where the likelihood function is proportional to the sampling distribution of the data. To simplify our calculations we can reduce the data down to two "sufficient" statistics, where

$$\bar{D} \mid \mu, \sigma^2 \sim \mathsf{N}(\mu, \sigma^2/n)$$

and is independent of

$$s^2 \mid \sigma^2 \sim \mathsf{Ga}\left(\frac{n-1}{2}, \frac{n-1}{2\sigma^2}\right)$$

where s^2 is the sample variance, $s^2 = \sum(D_i - \bar{D})^2/(n-1)$, and Ga is a gamma distribution. Note, we will use the rate parameterization of the gamma, so if $Y \sim \mathsf{Ga}(a, b)$ then Y has a probability density function

$$p(y) = \frac{1}{\Gamma(a)} b^a y^{a-1} e^{-yb}$$

with expected value a/b. From this we can find that $\mathsf{E}[s^2] = \sigma^2$ so that the sample variance is an unbiased estimator of the population variance. *Note that the rate parameterization that we are using here is different from the scale parameterization that is used in Week 2 for the Conjugate Poisson-Gamma.* The rate parameterization leads to easier updating rules as we will see.

For ease of derivation, we are going to create a new parameter $\phi \equiv 1/\sigma^2$ to help with specifying a conjugate prior distribution. The parameter ϕ is known as the precision; if the variance is small we have high precision, while if the variance is small we have more uncertainty and low precision. In the new parameterization our two statistics have sampling distributions

$$\bar{D} \mid \mu, \phi \sim \mathsf{N}\left(\mu, 1/(\phi n)\right) \tag{2}$$

$$s_d^2 \mid \phi \sim \mathsf{Ga}(\nu/2, \nu\phi/2) \tag{3}$$

where $\nu = n - 1$ is the usual degrees of freedom leading to a likelihood function based on taking the product of the independent distributions

$$\mathcal{L}(\mu, \phi) \propto (n\phi)^{1/2} \frac{1}{\sqrt{(2\pi)}} \exp\left\{-\frac{1}{2} n\phi(\bar{D} - \mu)^2\right\} \frac{1}{\Gamma(\nu/2)} \left(\frac{\nu\phi}{2}\right)^{\nu/2} s_d^{2\nu/2-1} \exp -\frac{\phi\nu s_d^2}{2}.$$

Note: you could just start with the independent normal samples and through some algebra rearrange to get to this.

Conjugate Normal-Gamma Prior Distribution

For Bayesian inference we need to assign prior distributions to all of the unknown parameters under all hypotheses. As a first attempt, conjugate prior distributions are a convenient choice or as we will encounter later provide building blocks for more complex distributions. Recall a conjugate prior distribution is one where the posterior distribution and the prior distribution are in the same family.

Conjugate Prior and Posterior for μ given ϕ

In Week 2 we studied the conjugate prior for a normal mean assuming that σ^2 or ϕ was known. While in this case the variance is unknown, conditional on σ^2 (or ϕ now), the conjugate prior for μ given ϕ is a normal distribution,

$$\mu \mid \phi \sim \mathsf{N}\left(m_0, \frac{1}{n_0\phi}\right)$$

where m_0 is the prior mean and n_0 is a hyper-parameter that is used to represent how concentrated or less concentrated the distribution is about m_0 relative to the precision ϕ, and may be thought of as a prior imaginary sample size upon which the prior distribution is based if there are no historical observations. Taking $n_0 = 1$ implies that our prior distribution is worth the equivalent of one observation.

Bayes theorem in proportional form leads to

$$\begin{aligned} p(\mu \mid \phi, \text{data}) \propto & \mathcal{L}(\mu, \phi) p(\mu \mid \phi) && (4)\\ = & (n\phi)^{1/2} \frac{1}{\sqrt{(2\pi)}} \exp\left\{-\frac{1}{2} n\phi(\bar{D} - \mu)^2\right\} p(s^2 \mid \phi) && (5)\\ & \cdot (n_0\phi)^{1/2} \frac{1}{\sqrt{(2\pi)}} \exp\left\{-\frac{1}{2} n_0\phi(\mu - m_0)^2\right\} && (6) \end{aligned}$$

where we have left the sampling distribution for s^2 as a density as it does not involve μ. Ignoring constants that do not involve ϕ or μ we may simplify further

$$\begin{aligned} & && (7)\\ p(\mu \mid \phi, \text{data}) \propto & \phi^{1/2} \exp\left\{-\frac{1}{2} n\phi(\bar{D} - \mu)^2 - \frac{1}{2} n_0\phi(\mu - m_0)^2\right\} \left(\phi^{1/2} p(s^2 \mid \phi)\right) && (8) \end{aligned}$$

where the above expression includes the sum of two quadratic expressions in the exponential. This almost looks like a normal. Can these be combined to form one quadratic expression that looks like a normal density? Yes! This is known as "completing the square". Taking the a normal distribution for a parameter μ with mean m and precision p, the quadratic term in the exponential may be expanded as

$$p(\mu - m)^2 = p\mu^2 - 2p\mu m + pm^2$$

where we can read off that the precision is the term that multiplies the quadratic in μ and term that multiplies the linear term in μ is the product of two times the mean and precision; if we know the precision, we can identify the mean. The last term is the precision times the mean squared. For our posterior, we need to expand the quadratics and recombine terms to identify the new precision (the coefficient multiplying the quadratic in μ) and the new mean and completing the square or quadratic so that it may be factored. Any left over terms will be independent of μ but may depend on ϕ. Applying to our case we have

$$-\frac{1}{2}n\phi(\bar{D} - \mu)^2 - \frac{1}{2}n_0\phi(\mu - m_0)^2 = -\frac{1}{2}\left(\phi(n + n_0)\mu^2 - 2\phi\mu(n\bar{D} + n_0m_0) + \phi(n\bar{D}^2 + n_0m_0^2)\right)$$

where we can read off that the posterior precision is $\phi(n + n_0)$. The linear term is not yet of the form of the posterior precision times the posterior mean (times 2), but if we multiply and divide by $n + n_0$ it does satisfy that

$$-\frac{1}{2}\left(\phi(n + n_0)\mu^2 - 2\phi(n + n_0)\mu\frac{(n\bar{D} + n_0m_0)}{n + n_0} + \phi(n\bar{D}^2 + n_0m_0^2)\right) \tag{9}$$

so that we may identify that the posterior mean is $(n\bar{D} + n_0m_0)/(n + n_0)$ which combined with the precision (or inverse variance) is enough to identity the conditional posterior distribution for μ. This leads to the result

$$\mu \mid \phi, \text{data} \sim \mathsf{N}(m_n, (\phi n_n)^{-1})$$

where $m_m = (n\bar{D} + n_0m_0)/(n + n_0)$ a weighted average of the sample mean and the prior mean, and $n_n = n + n_0$ the sample and prior combined sample size. This is exactly the result from earlier, but written in terms of relative prior precision and sampling precision n_0 and n respectively to obtain the relative (to ϕ) posterior precision $n_n = n + n_0$.

Conjugate prior for ϕ

Since σ^2 and ϕ can only take on values greater than zero and are continuous rather than discrete, any reasonable prior distribution needs to incorporate those constraints. Out of the distributions that we have encountered so far, the gamma distribution fits the bill and is in fact the conjugate prior distribution for ϕ. We will use the following parameterization

$$\phi \sim \mathsf{Ga}(\nu_0/2, \nu_0 s_0^2/2)$$

with hyperparameters ν_0 (the prior degrees of freedom) and a rate parameter $\nu_0 s_0^2$ where s_0^2 is the best prior estimate of σ^2 (based on real or imaginary data) with prior degrees of freedom ν_0 with a density

$$p(\phi) = \frac{1}{\Gamma(\nu_0/2)}(\nu_0 s_0^2)^{\nu_0/2}\phi^{\nu_0/2-1}e^{-\phi\frac{\nu_0 s_0^2}{2}}$$

Joint Posterior

Together these form what is called a **Normal-Gamma**(m_0, n_0, ν_0, s_0^2) family of distributions for μ, ϕ:

$$p(\mu, \phi) = \frac{(n_0\phi)^{1/2}}{\sqrt{2\pi}} e^{-\frac{\phi n_0}{2}(\mu - m_0)^2} \frac{1}{\Gamma\nu_0/2} (\nu_0 s_0^2)^{\nu_0/2-1} e^{-\phi\frac{\nu_0 s_0^2}{2}} \tag{10}$$

based on taking the product of the conditional normal distribution for μ given ϕ and the marginal Gamma distribution for ϕ.

Using the Normal-Gamma prior, the joint posterior is proportional to

$$p(\mu, \phi \mid \text{data}) \propto \mathcal{L}(\mu, \phi) p(\mu \mid \phi) p(\phi))$$

the likelihood in (1) times the prior in (10). is proportional to the product of the likelihood and priors. If we substitute all of the above expressions for the likelihood and priors and simplify we can show that the posterior is in the Normal-Gamma family. When we found the conditional normal posterior for μ given ϕ, we stopped after identifying the mean and variance. The extra terms that are needed to complete the square involve ϕ so for the joint posterior distribution, we need to be a bit more careful about keeping track of terms.

Conjugate Posterior Distribution

Given the data $\bar{D}$, n, ν and s^2 the Normal-Gamma prior is updated to obtain posterior distribution which is Normal-Gamma(m_n, n_n, ν_n, s_n^2) where the posterior hyperparameters are obtained using the following updating rules

- m_n: posterior mean of μ

  $$m_n = \frac{n\bar{D} + n_0 m_0}{n + n_0}$$

  which is a weighted combination of the sample mean and the prior mean with weights proportional to the relative precisions.

- n_n: relative posterior precision of the estimate $n_n = n + n_0$ based on combined observed sample size and prior sample size.

- ν_n: posterior degrees of freedom $\nu_n = \nu + \nu_0 + 1$ where the extra 1 comes from the distribution on μ

- s_n^2: posterior scale (squared)

  $$s_n^2 = \frac{s^2\nu + s_0^2\nu_0 + \frac{nn_0}{n+n_0}(\bar{D} - m_0)^2}{\nu_n}$$

  which combines the observed sum of squared deviations of the data, from the sample mean (νs^2), the prior sum of squares ($\nu_0 s_0^2$), and the last term which is deviation of the observed sample mean from the prior mean. If our prior mean is v<ery far from the sample mean, this may in fact increase our posterior uncertainty, although this effect goes away as the sample size increases.

Marginal Distribution for μ

The conditional distribution for μ given ϕ is normal with mean m_m and variance $1/(n_n\phi)$, however, this does not directly help for obtaining credible intervals or inference as ϕ is unknown. For posterior inference about μ we need to obtain the marginal distribution by "averaging" over the posterior uncertainty of ϕ. This requires integration (which we will show in the derivations), so for now we simply state the result

$$\mu \mid \text{data} \sim \mathsf{t}_{\nu_n}(m_n, s_n^2/n_n) \text{ or } \frac{\mu - m_n}{\sqrt{(s_n^2/n_n)}} \sim \mathsf{t}_{\nu_n}(0,1)$$

that μ given the data has a Student t distribution centered at m_n (the posterior mean) and with a scale parameter that is s_n^2/n_n. As with normals, subtracting the mean and dividing by the square root of the scale parameter leads to a scale-free distribution centered at 0. Credible intervals or highest posterior density intervals with coverage $(1-\alpha)100\%$ may be obtained by taking

$$m_n \pm t_{1-\alpha/2,\nu_n} s_n$$

Reference Prior

If you wish to use the Bayesian interpretation of probability, but want to try to be as objective as possible, you might think that a reasonable approach would be to construct your imaginary prior data letting your prior sample size and degrees of freedom go to zero. A limiting case of the conjugate Normal-Gamma prior is what is refered to as a reference prior for μ, ϕ and corresponds to taking $m_0 = n_0 = s_0^2 = 0$ but letting $\nu_0 = -1$. The negative prior degrees of freedom do not make any sense, but this slight difference is important as the resulting prior distribution does not depend on the units of measurenent, a form of "invariance". While this is not a proper prior distribution, it does lead to a proper posterior distribution for μ

$$\mu \mid \text{data} \sim \mathsf{t}_{\nu}(\bar{D}, s^2/n) \text{ or } \frac{\mu - \bar{D}}{\sqrt{(s^2/n}} \mid \text{data} \sim \mathsf{t}_{\nu}(0,1)$$

where $\nu = n-1$. (Technically it is proper if we have at least 2 observations). The right hand distribution has the same form as the sampling distribution for $\bar{D}$ (when conditioning on μ), providing a duality between the frequentist and Bayesian paradigms for estimation, e.g. 95% credible intervals ar of the form

$$\bar{D} \pm t_{1-\alpha/2,n-1} s/\sqrt{n}$$

where $t_{1-\alpha/2,n-1}$ is the usual Student t This allows the objective Bayesian to calculate a classical confidence interval, while providing the Bayesian probabilistic interpretation of the interval.

Bayes Factors and Hypothesis Testing

The following were the hypotheses of interest in terms of the original parameters and the mean of the differences:

no differences $H_1 : \mu_B = \mu_S \Leftrightarrow \mu = 0$

means are different $H_2 : \mu_B \neq \mu_S \Leftrightarrow \mu \neq 0$

sub-hypotheses $H_3 : \mu_B > \mu_S \Leftrightarrow \mu > 0$

$H_4 : \mu_B < \mu_S \Leftrightarrow \mu < 0$

It should be clear that H_3 and H_4 are included in H_2, so that we first need to find the probability of H_1 and H_2. To find the posterior probabilities, we start with the Bayes factor for comparing H_1 to $H2$,

$$BF[H_1 : H_2] = \frac{p(\text{data} \mid H_1)}{p(\text{data} \mid H_2)}$$

which depends on the prior predictive distribution of the data or sufficient statistics $\bar{D}$ and s^2 under the two hypotheses.

To start, lets look at the distribution of $\bar{D}$. One way to think of $\bar{D}$ is that it is a noisy version of the population mean

$$\bar{D} = \mu + \epsilon$$

where ϵ has a normal distribution with mean 0 and variance $1/(\phi n)$. Under H_1, Since our prior distribution for μ was also normal and independent of the added noise (given ϕ), we can add these two sources of variation to get the prior predictive distribution of $\bar{D}$ (given ϕ) by adding the two means, $m_0 + 0$, and adding the two variances, $\frac{1}{\phi n} + \frac{1}{\phi n_0}$,

$$\bar{D} \mid \phi \sim \mathsf{N}\left(m_0, \frac{1}{\phi}\left(\frac{1}{n} + \frac{1}{n_0}\right)\right)$$

where the variance combines the uncertainty due to sampling variation and our prior uncertainty. Of course, we do not know ϕ so if we average over our prior uncertainty for ϕ, the resulting predictive distribution would be a Student t distribution

$$\bar{D} \sim \mathsf{t}_{\nu_0}\left(m_0, s_0^2\left(\frac{1}{n} + \frac{1}{n_0}\right)\right)$$

with degrees of freedom ν_0, location m_0 and squared scale parameter $s_0^2\left(\frac{1}{n} + \frac{1}{n_0}\right)$, or the standardized version where

$$\frac{\bar{D} - m_0}{s_0\sqrt{\left(\frac{1}{n} + \frac{1}{n_0}\right)}} \sim \mathsf{t}_{\nu_0}(0, 1)$$

a standard t distribution with ν_0 degrees of freedom. The extra uncertainty due to the unknown variance leads to a distribution with heavier tails and wider intervals.

The above derivation does not include the data from s^2. Let's see how we can incorporate that. From Bayes theorem we have that conditional on H_i (for i equal 1 or 2) that

$$p(\mu, \phi \mid \text{data}, H_i) = \frac{p(\mu, \phi \mid H_i)p(\text{data} \mid \mu, \phi, H_i)}{p(\text{data} \mid H_i)}$$

. If we happen to know the conjugate updating rules and the forms of the densities then we can solve for $p(\text{data} \mid H_i)$ as

$$p(\text{data} \mid H_i) = \frac{p(\mu, \phi \mid H_i)p(\text{data} \mid \mu, \phi, H_i)}{p(\mu, \phi \mid \text{data}, H_i)}$$

For those that are comfortable with integration,

$$p(\text{data} \mid H_i) = \int_0^\infty \int_{-\infty}^\infty p(\mu, \phi \mid H_i) p(\text{data} \mid \mu, \phi, H_i) d\mu \, d\phi.$$

With some algebra we can simplify the expression of the ratio of the predictive distributions of the data to find the Bayes factor.

Under a limiting case with $\nu_0 = s_0^2 = 0$ the Bayes factor is

$$BF[H_1 : H_2] = \left(\frac{n + n_0}{n_0} \right)^{1/2} \left(\frac{t^2 \frac{n_0}{n+n_0} + \nu}{t^2 + \nu} \right)^{\frac{\nu+1}{2}}$$

which is a function of the

- t-statistic

$$t = \frac{|\bar{D}|}{s/\sqrt{n}}$$

- sample standard deviation s
- degrees of freedom $\nu = n - 1$

This provides a way to provide a posterior probability of the hypothesis through the Bayes factor that depends on the usual t statistic.

Add R-code from github