§1.1 Modeling with Differential Equations

Important Lessons

- A **differential equation** is an equation relating a function to one or more of its derivatives.
- An **initial value problem** is a differential equation

$$\frac{dx}{dt} = f(t, x),$$

where the **initial condition**, $x(t_0) = x_0$, is specified.

- A population that is not affected by overcrowding can be modeled by the differential equation $P' = kP$ and is said to grow **exponentially**.
- A population that must compete for limited resources can be modeled by the **logistic equation**,

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P,$$

where N is the carrying capacity of the population.

- Some phenomenon, such as the relationship between a population of predators and a population of prey, are best modeled by systems of differential equations.
- The three principle steps in modeling any phenomenon with differential equations are:
  - Discovering the differential equation or equations that best describe a specified physical situation.
  - Finding—either exactly or approximately—the appropriate solution of the equation or equations.
  - Interpreting the solution in terms of the phenomenon.

Investigations

1. The black rhinoceros, once the most numerous of all rhinoceros species, is now critically endangered. The black rhino, native to eastern and southern Africa, was estimated to have a population of about 100,000 around 1900. Because of hunting, habitat changes, competing species, and most of all illegal poaching, the number of black rhinos today is estimated to be below 3000. If the wild population becomes too low, the animals may not be able to find suitable mates and the black rhino will become extinct. There must be a minimum population for the species to continue. Suppose the this minimum or threshold population for the black rhino is 1000 animals and that remaining habitant in Africa will support no more that 20,000 rhinos. How might we model the current population, $P(t)$ of black rhinos?

   (a) For what values of P is the rhino population increasing? What can be said about the value of dP/dt for these values of P?

   (b) For what values of P is the rhino population decreasing? What can be said about the value of dP/dt for these values of P?

(c) For what values of P is the rhino population in equilibrium? What can be said about the value of dP/dt for these values of P?

(d) Find a differential equation that models the population of rhinos at time t.

2. Verify that $y(t) = -7e^{t^2} - \frac{1}{2}$ is a solution of $y' = 2ty + t$.

3. Find all values of a such that $y(t) = e^{at}$ is a solution to $y'' - 3y' + 2y = 0$.

4. Verify that $y(t) = c_1 e^{-2t} \cos 3t + c_2 e^{-3t} \sin 3t$ is a solution of $y'' + 4y' + 13y = 0$. Then find values of c_1 and c_2 such that $y(0) = 1$ $y'(0) = 0$.

5. Consider the following predator-prey systems of differential equations

$$\frac{dx}{dt} = -\frac{x}{2} + \frac{xy}{2+y},$$
$$\frac{dy}{dt} = y(1-y) - \frac{xy}{2+y}.$$

(a) Which equation models the prey population and which equation models the predator population?

(b) How does the prey population grow if there are no predators present?

(c) What happens if there are a lot of prey present?