§1.2 Separable Differential Equations

Office Hours

MWF 10:30-4:00 in Cowley 1026.

Homework Assignments

- Assignment 1: Due Monday, February 3, 2020 before class.
- Assignment 2: Due Wednesday, February 5, 2020 before class.
- Assignment 3: Due Friday, February 7, 2020 before class.

Notes

- Do the theory at the end of §1.2 on what is going on with separable equations.

Important Lessons

- A differential equation of **order** n is an equation that can be put in the form

$$F(t, x, x', x'', \dots, x^{(n)}) = 0,$$

where F is a function of $n + 2$ variables. A **solution** to the equation on an interval $I = (a, b)$ is a function $u = u(t)$ such that the first n derivatives of u are defined on I, and

$$F(t, u, u', u'', \dots, u^{(n)}) = 0.$$

- A **first-order differential equation** is an equation that can be written in the form

$$\frac{dx}{dt} = f(t, x).$$

- A differential equation is separable if it can be written in the form

$$\frac{dy}{dx} = M(x)N(y).$$

In this case we can rewrite the equation in the form

$$f(x) + g(y)\frac{dy}{dx} = 0$$

or

$$g(y)\,dy = f(x)\,dx$$

and solve by integrating both sides.

Investigations

1. Suppose we have a pond that will support 1000 fish, and the initial population is 100 fish. In order to determine the number of fish in the lake at any time t, we must find a solution to the initial value problem

$$\begin{aligned} \frac{dP}{dt} &= k\left(1 - \frac{P}{1000}\right)P \\ P(0) &= 100. \end{aligned}$$

It is easy to verify that $P(t) = 1000/(9e^{-kt}+1)$ is the solution to our initial value problem. Solve the IVP. If we know that the population is 200 fish after one year, then

$$200 = P(1) = \frac{1000}{9e^{-k}+1},$$

and we can determine that

$$k = \ln\left(\frac{9}{4}\right) \approx 0.8109.$$

Consequently, the solution to our intial-value problem is

$$P(t) = \frac{1000}{9e^{-0.8109t}+1}.$$

2. Solve the following initial value problems.

   (a) $x\,dx - y^2\,dy = 0$, $y(0) = 1$

   (b) $\dfrac{dy}{dx} = \dfrac{y}{x}$, $y(1) = -2$

3. Mr. Ratchett, an elderly American, was found murdered in his train compartment on the Orient Express at 7 A.M. When his body was discovered, the famous detective Hercule Poirot noted that Ratchett had a body temperature of 28 degrees. The body had cooled to a temperature of 27 degrees one hour later. If the normal temperature of a human being is 37 degrees and the air temperature in the train is 22 degrees, estimate the time of Ratchett's death using Newton's Law of Cooling.

4. Suppose that we have a large tank containing 1000 gallons of pure water and that water containing 0.5 pounds of salt per gallon flows into the tank at a rate of 10 gallons per minute. If the tank is also draining at a rate of 10 gallons per minute, the water level in the tank will remain constant. We will assume that the water in the tank is constantly stirred so that the mixture of salt and water is uniform in the tank.

   Solution: We can model the amount of salt in the tank using differential equations. If $x(t)$ is the amount of salt in the tank at time t, then the rate at which the salt is changing in the tank is the difference between the rate at which salt is flowing into the tank and the rate at which it is leaving the tank, or

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}.$$

   Of course, the salt flows into the tank at the rate of $10 \cdot 0.5 = 5$ pounds of salt per minute. However, the rate at which the salt leaves the tank depends on $x(t)$, the amount salt in the tank at time t. At time t, there is $x(t)/1000$ pounds of salt in one gallon. Therefore, salt flows out of the tank at a rate of $10x(t)/1000 = x(t)/100$ pounds per minute. Out differential equation now becomes

$$\begin{aligned} \frac{dx}{dt} &= 5 - \frac{x}{100} \\ x(0) &= 0. \end{aligned}$$

This equation is separable,

$$\frac{dx}{500 - x} = \frac{dt}{100}.$$

Integrating both sides of the equation, we have

$$-\ln|500 - x| = \frac{t}{100} + k$$

or

$$\ln|500 - x| = -\frac{t}{100} - k.$$

Consequently,

$$500 - x = Ce^{-0.01t},$$

where $C = e^{-0.01k}$. From our initial condition, we can quickly determine that $C = 500$ and

$$x(t) = 500 - 500e^{-0.01t}$$

models the amount of salt in the tank at time t. Notice that $x(t) \to 500$ as $t \to \infty$, as expected.