Velocities: u longitudinal, v transverse, w spanwise.

1 Linear-Instability Theory

1.1 Orr-Sommerfeld equation

Linearized Navier-Stokes equation with the condition (uniform ρ and mean flow field $(\bar{u}(y), 0, 0)$), and use *normal mode analysis* (waveform for the perturbation) to obtain the Orr-Sommerfeld equation. Special condition of Orr-Sommerfeld gives the Rayleigh equation (2D inviscid) and Kuo equation (2D rotation).

1.1.1 Normal mode analysis

The analysis is done by inserting wave form solution of the perturbation (e.g., velocity, streamfunction, etc) to obtain instability criterions for the equations that result in a positive complex growth rate $\omega_i = \alpha c_i > 0$. The variables for the criterions include the mean background flow, perturbed wavenumber, Reynolds number, Coriolis parameter etc.

1.1.2 Rayleigh equation (2D, $\nu = 0$)

- Rayleigh instability criterion: An inflection point must exist in the flow ($\bar{u}''$ change sign).
- Fjortoft instability criterion (sufficient condition): The absolute vorticity $|du/dy|$ must reach maximum at the inflection point (see figure (d)).

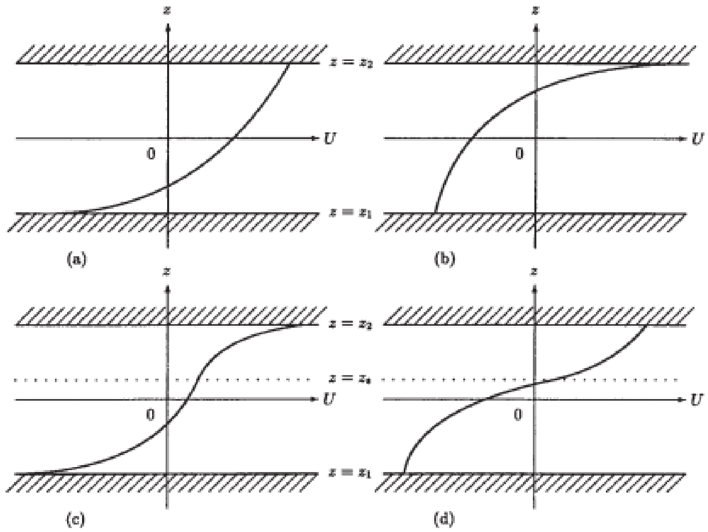


Figure 8.3 Some examples of flows governed by the Rayleigh–Fjørtoft necessary conditions for instability. (a) Stable because $U'' < 0$ everywhere. (b) Stable because $U'' > 0$ everywhere. (c) Stable because $U''(z_s) = 0$ but $U''(U - U_s) \geq 0$. (d) Possibly unstable because $U''(z_s) = 0$ and $U''(U - U_s) \leq 0$.

Figure 5.3.2: Fjortoft Theorm , From Drazin 2002.

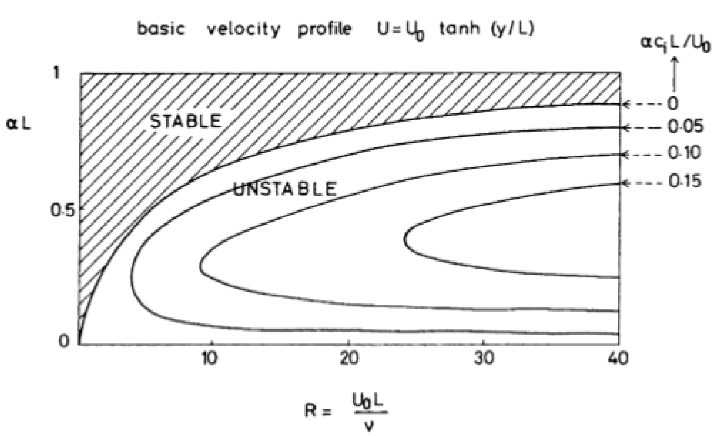
notice figure (c) has inflection, but vorticity goes to zero at inflection point, which kills the instability.

1.1.3 Kuo equation (with rotational β-plane)

Instability criterion: $\bar{u}'' - \frac{df}{dy}$ must change sign.

1.1.4 3D Orr-Sommerfeld equation

Dependence of velocity perturbation growth rate ω_i (imaginery part of angular frequency) on Reynolds number ($Re = \frac{UL}{\nu}$) and wavenumber.



2 Moments, Homogeneity, and Stationarity

- Moments: "n"th order moment is the ensemble average of any tensorial product of n-components of the velocity field
- Homogeneity: Invariant of any mean quantity after translation over space.
- Stationarity: Invariant of any mean quantity after translation over time.
- Isotropy: Invariant of any mean quantity after simultaneous rotation of the set of n points.

2.1 Ensemble average

An experiment gives a single realization of a set of $(n \times m)$-point (x_i, t_j) solution $u(x_i, t_j)$. When given an ensemble of N realizations one could define the enbsemble average operator $\langle \cdot \rangle$ applied over the set of random variables, e.g.,

$$\langle u(x_1,t_1)u(x_2,t_2)\dots u(x_n,t_n)\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} u^i(x_1,t_1)u^i(x_2,t_2)\dots u^i(x_n,t_n) \tag{1}$$

Above is the nth moment of the velocity.

2.2 Homogeneity

2.2.1 Definition

Invariant of the mean quantity after tranlation over space (i.e., uniform in space),

$$\langle u_{\alpha_1}(x_1,t_1)u_{\alpha_2}(x_2,t_2)\dots u_{\alpha_n}(x_n,t_n)\rangle = \langle u_{\alpha_1}(x_1+r,t_1)u_{\alpha_2}(x_2+r,t_2)\dots u_{\alpha_n}(x_n+r,t_n)\rangle \tag{2}$$

where α_i is the index of the coordinate (e.g., $\vec{u} = (u, v, w)$). For instance, the velocity correlation between two spatial points x_1 and $x_1 + r$ at t_1 and t_2, respectively, denoted as a tensorised correlation matrix

$$[(u(x_1,t_1))^i, (v(x_1,t_1))^i][(u(x_1{+}r,t_2))^i, (v(x_1{+}r,t_2))^i]^T = \begin{pmatrix} \langle u(x_1,t_1)u(x_1+r,t_2)\rangle & \langle u(x_1,t_1)v(x_1+r,t_2)\rangle \\ \langle v(x_1,t_1)u(x_1+r,t_2)\rangle & \langle v(x_1,t_1)v(x_1+r,t_2)\rangle \end{pmatrix}, \tag{3}$$

shorthanded as

$$U_{ij}(r, t_1, t_2) = \langle u_i(x_1, t_1) u_j(x_1 + r, t_2) \rangle . \tag{4}$$

2.2.2 Remarks

1. For homogeneous turbulence, any mean quantity is invariant with a translation of the spatial points (i.e., mean quantity is homogeneous in space). e.g., the mean velocity $\langle u(x,t) \rangle$ is independent of x (i.e., mean velocity is homogeneous in space).

2. An ergodic hypothesis allows one to calculate enesemble average $\langle \cdot \rangle$ as a spatial average

$$U_{ij}(r, t_1, t_2) = \lim_{V \to \infty} \frac{1}{V} \int_V u_i(x_1, t_1) u_j(x_1 + r, t_2) dx_1, \tag{5}$$

essentially using one realization assuming spatial indices as the ensemble indices. Therefore, the correlation between the two points, x_1 and $x_1 + r$, averaged over all realizations is the same as spatially averaging between all pairs of points x_i and $x_i + r$.

2.3 Stationarity

2.3.1 Definition

Invariant of the mean quantity after tranlation over time,

$$\langle u_{\alpha_1}(x_1, t_1) u_{\alpha_2}(x_2, t_2) \ldots u_{\alpha_n}(x_n, t_n) \rangle = \langle u_{\alpha_1}(x_1, t_1 + \tau) u_{\alpha_2}(x_2, t_2 + \tau) \ldots u_{\alpha_n}(x_n, t_n + \tau) \rangle \tag{6}$$

2.3.2 Remarks

An ergodic hypothesis allows one to calculate enesemble average as a time average

$$U_{ij}(x_1, x_2, \tau) = \lim_{T \to \infty} \frac{1}{T} \int_T u_i(x_1, t_1) u_j(x_2, t_1 + \tau) dt_1, \tag{7}$$

2.4 Isotropy

2.4.1 Definition

Invariant of the mean quantity after simultaneous rotation of the set of n points (i.e., no directional preference $\langle u(x,t) \rangle$).

2.4.2 Example

- non-isotropy: suppose an ensemble of velocity pointing in the +x direction with mean quantity equal 3, then a π rotation of orientation with coordinate -x will give -3, which indicates a orientation preference of the flow.

- isotropy: mean velocity is zero $\langle u(x,t) \rangle = 0$ since a π rotation of orientation yields $\langle u(x,t) \rangle = -\langle u(x,t) \rangle = 0$.