0.1 Prove that a n point Gauss-quadrature is exact for a polynomial up to order $(2n-1)$

Quadrature Exactness for polynomial of order $n-1$:
The integration of the form

$$I(f) = \int_a^b \omega(x) f(x) dx \tag{1}$$

is exact when

$$I(f) \approx Q(f) = \sum_{i=1}^{n} w(x_i) f(x_i). \tag{2}$$

A function approximated by the n point Lagrange interpolant can be represented as a $(n-1)$th order polynomial

$$f(x) \approx \sum_{i=1}^{n} f(x_i) \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \tag{3}$$

Substituting into

$$\begin{aligned} I(f) &= \int \omega(x) f(x) dx \\ &\approx \int \omega(x) \sum_{i=1}^{n} f(x_i) \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} dx \\ &= \sum_{i=1}^{n} \Big(\int \omega(x) \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} dx \Big) f(x_i) \\ &= \sum_{i=1}^{n} w(x_i) f(x_i), \end{aligned} \tag{4}$$

and don't forget w is not a continuous function of the domain x, instead only has discrete values as a function of x_i. Therefore this shows that the ***quadrature is exact when the function f is a polynomial of order*** $(n-1)$.

Orthogonality:
Let p_i's be orthogonal polynomials of maximum order n over $[ab]$ such that

$$\int_a^b \omega(x) p_i(x) p_j(x) dx = c\delta_{ij}, \tag{5}$$

where c is a constant factor. Any polynomial $h_k(x)$ with order $k \leq (n-1)$ is a linear combination of p_k's such that

$$\int_a^b \omega(x) h_k(x) p_n(x) dx = 0 \text{ for all } k = 0, 1, \ldots, n-1, \tag{6}$$

shows the terms in h_k are all orthogonal to p_n.

Quadrature Exactness for Orthogonality:
The orthogonality integration (**??**) is "exact" if it is done over the roots of $p_n(x)$,

$$\sum_{i=1}^{n} \omega(x_i) h_k(x_i) p_n(x_i) = \int_a^b \omega(x) h_k(x) p_n(x) dx = 0. \tag{7}$$

Proof:
Using the two exactness conditions ((**??**) and (**??**)) and the orthogonal polynomials, we will prove that

$I(f) \approx Q(f)$ for f of order $2n-1$. The quotient form of polynomial $f(x) = p_n(x)q(x) + r(x)$, where q and r are both order $\leq n-1$,

$$\begin{aligned}
I(f) &= \int_a^b \omega f dx \\
&= \int_a^b \omega q p_n dx && + \int_a^b \omega r dx && \text{(quotient form)} \\
&= 0 && + \int_a^b \omega r dx && \text{(orthogonality)} \\
&\approx \sum w(x_i) q(x_i) p_n(x_i) && + \sum w(x_i) r(x_i) && \text{(exactness (??) + (??))} \\
&= \sum w(x_i) f(x_i) = Q(f).
\end{aligned} \tag{8}$$

Concluding Remarks:

- The quadrature exactness can go up to order $2n-1$ for f, and the weights are on the roots of orthogonal polynomials.

- If $\omega(x) = 1$ (associated with the Legendre orthogonal polynomials), then $I(f) = \int f dx$ is the regular integration for f.

Proof of Regression using orthonormal basis no better then arbitrary independent basis
Suppose f_1, f_2 are ON polynomial basis w.r.t. the operator $<>$ and $h_2 = c_1 f_1 + c_2 f_2$ is a polynomial of order 2. Notice that f_1 and h_2 are independent and could become a basis of $P_2(R)$.

We can approximate the true sample y by least squares using both basis

$$\begin{aligned}
y \approx & g = a_1 f_1 + a_2 f_2 \\
& h = b_1 f_1 + b_2 h_2.
\end{aligned} \tag{9}$$

Our goal is to see if h induces more error while approximating y by non-ON basis. From least squares $b = (A^T A)^{-1} A^T f$. To obtain the inverse of

$$A^T A = \begin{pmatrix} < f_1, f_1 > & < f_1, h_2 > \\ < h_2, f_1 > & < h_2, h_2 > \end{pmatrix}, \tag{10}$$

we apply Gauss-Jordan elimination

$$\begin{aligned}
&\begin{pmatrix} < f_1, f_1 > & < f_1, h_2 > & 1 & 0 \\ < h_2, f_1 > & < h_2, h_2 > & 0 & 1 \end{pmatrix} \\
\sim &\begin{pmatrix} < f_1^2 > & < f_1, h_2 > & 1 & 0 \\ 0 & \frac{<h_2^2><f_1^2> - <f_1,h_2>^2}{<f_1^2>} & \frac{-<h_2,f_1>}{<f_1^2>} & 1 \end{pmatrix} \\
\sim &\begin{pmatrix} < f_1^2 > & 0 & \frac{<h_2^2><f_1^2>}{<f_1^2><h_2^2> - <f_1,h_2>^2} & \frac{-<f_1^2><f_1,h_2>}{<f_1^2><h_2^2> - <f_1,h_2>^2} \\ 0 & \frac{<h_2^2><f_1^2> - <f_1,h_2>^2}{<f_1^2>} & \frac{-<h_2,f_1>}{<f_1^2>} & 1 \end{pmatrix} . \\
\sim &\begin{pmatrix} 1 & 0 & \frac{<h_2^2>}{<f_1^2><h_2^2> - <f_1,h_2>^2} & \frac{-<f_1,h_2>}{<f_1^2><h_2^2> - <f_1,h_2>^2} \\ 0 & 1 & \frac{-<h_2,f_1>}{<h_2^2><f_1^2> - <f_1,h_2>^2} & \frac{<f_1^2>}{<h_2^2><f_1^2> - <f_1,h_2>^2} \end{pmatrix}
\end{aligned} \tag{11}$$

Therefore

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{<h_2^2><f_1,f> - <f_1,h_2><h_2,f>}{<f_1^2><h_2^2> - <f_1,h_2>^2} \\ \frac{-<h_2,f_1><f_1,f> + <f_1^2><h_2,f>}{<f_1^2><h_2^2> - <f_1,h_2>^2} \end{pmatrix} = \begin{pmatrix} < f_1, f > - \frac{c_1}{c_2} < f_2, f > \\ \frac{<f_2,f>}{c_2} \end{pmatrix}. \tag{12}$$

This shows that b_1 has some projection coming from f_2. Substituting this into the equation gives $h = g$. $\blacksquare$