- 1. $P_2(\mathcal{R})$ is a inner-product space with $< p, q >= \int_0^1 p(x)q(x)dx$. Define $T \in L(P_2(\mathcal{R}))$ by $T(a_0 + a_1x + a_2x^2) = a1x$ on the basis $(1, x, x^2)$. a) Show that T isn't self-adjoint.

  Proof:

  a) The inner product $< T(a_0 + a_1x + a_2x^2), b_0 + b_1x + b_2x^2 >=< a_1x, b_0 + b_1x + b_2x^2 >= \int_0^1 a_1(b_0 + b_1x + b_2x^2)dx = a_1b_0 + 1/2a_1b_1 + 1/3a_1b_2$. If $T^* = T$, then $< Tv, w >=< v, Tw >$, but $< a_0 + a_1x + a_2x^2, T(b_0+b_1x+b_2x^2) >=< a_0+a_1x+a_2x^2, b_1x >= \int_0^1 b_1(a_0+a_1x+a_2x^2)dx = b_1a_0+1/2b_1a_1+1/3b_1a_2$ implies $< Tv, w >\neq< v, Tw >$, hence $T \neq T^*$.

  b) The reason to the matrix of T wrt the basis $(1, x, x^2)$ equals its conjugate transpose, but still not self-adjoint is due to the basis not being orthonormal over the inner product $\int_0^1 dx$. $(M(T, (e_1, \ldots, e_n), (f_1, \ldots$ $M(T^*, (f_1, \ldots, f_n), (e_1, \ldots, e_n))^T$ if the two basis vectors are both orthonormal)

- 2. Product of two self-adjoint operators are not self-adjoint.

  Proof:

$$\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} \tag{1}$$