0.1 Group

A group is a set, $\mathbb{G}$, with an operation $*$ that combines two elements a and b to form $a * b$ satisfying:

- closure $a * b \in \mathbb{G}$.
- identity $e \in \mathbb{G}$ such that $e * a = a * e = a$.
- existence of an inverse $b \in \mathbb{G}$ such that $a * b = b * a = e$.
- associativity $(a * b) * c = a * (b * c)$.

Example: Exibit the group structure of a parabola $y = x^2$ with parametrization $\alpha(x) = [x, x^2]$.

proof: The identity is the origin $(0, 0)$. $a * b$ is the intersection of the line through $(0, 0)$ parallel to $\overline{ab}$ on the parabola in A^2. We want to find the point $a * b$ which is the intersection of the line $y = cx$ (with slope $c = (b_2 - a_2)/(b_1 - a_1))$ for $\overline{ab}$ and $y = x^2$. Hence, we obtain the intersection from equating $x^2 = cx$, which gives $x = c$ and $y = c^2$. This defines the point $a * b = [(b_2 - a_2/(b_1 - a_1), (b_2 - a_2)^2/(b_1 - a_1)^2]$.

0.2 Translation on a line A^1

A translation τ^1 (τ^{-1}) translates the points of A^1 by 1 to the right (left). The "orbit" generated by a point x: x under τ^1 and τ^{-1} and all iterates on A^1. The circle S^1 is the space of all orbits (i.e., the orbit of x and the orbit of $x + 1$ is the same orbit)

0.3 Rational Parametrization (for a unit circle)

$$e(m) = (\frac{1 - m^2}{1 + m^2}, \frac{2m}{1 + m^2}), \tag{1}$$

m is the slope of a line $y = m(x + 1)$ that passes through $(-1, 0)$ parameterizing the circle. $m = \infty, e(m) = (-1, 0)$. $m = 0, e(m) = (1, 0)$.

0.4 Transcendental Parametrization

$$\psi(\theta) = (\cos(\theta), \sin(\theta)) \tag{2}$$

0.5 Projective Space

Def: Sets of all lines through the origin of a vector space V.

$P^1(A^2)$ (projective lines) = space of all 1D subspaces of A^2

1. A circle S^1 is topologically $P^1(R^2) \longrightarrow P^1 \simeq S^1$.
2. A sphere S^2 is topologically $P^1(C^2) \longrightarrow P^1 \simeq S^2$.

Proof for 2. Suppose we have a lines through the origin of C^2, $t(w, k)$, where (w,k) is any vector in C^2 and $t \in R$. When $k \neq 0$, the line divided by k gives $t(w/k, 1)$, where w/k is any number in C. The line $t(w/k, 1)$ represents the entire set of lines through the origin except the line $t(w, 0)$, when $k = 0$, which is defined as the infinity on the complex plane. As we knew from stereographic projection, the complex plane (C) is homeomorphic to the sphere. Hence, the lines $t(w/k, 1)$ with $t(w, 0)$ forms the projective line of the sphere $\longrightarrow P^1(C^2) \simeq S^2$.

0.6 Homeomorphism

Def: Two topological spaces are homeomorphism if there exists a continuous bijective function with a continuous inverse function.

f is bijective, continuous and f^-1 is continuous.

0.7 Stereographic Projection P

Any point on a sphere can be represented with a point $[x, y, 0]$ on an equatorial plane. Starting with the south pole point $[0, 0, -1]$, we derive the projection for the sphere,

$$P(x, y) = (2x/(1 + x^2 + y^2), 2y/(1 + x^2 + y^2), (1 - x^2 - y^2)/(1 + x^2 + y^2)). \tag{3}$$

The south pole $[0, 0, -1]$ on the sphere is the infinity point (∞, ∞) on the equatorial plane. This completes the entire plane $\simeq$ sphere.

Remarks: 1) A line or circle on the equatorial plane is mapped to a circle on the sphere. 2) A conformal is preserved through P (i.e., the angle of two lines on the plane mapped to the sphere preserves this angle)

0.8 Conformal

Any two lines meeting at an angle on the equatorial plane preserves the angle when meeting at the projected circles on the sphere.

0.9 Inversive Geometry

In Euclidean geometry, a circle c has a inversion map sending points inside c to outside. $I_c : x \longrightarrow x'$,

$$d(o,x)d(o,x') = r^2, \tag{4}$$

where o is the origin, and r is the radius. c is fixed under I_c, I_c just maps a point to another point leaving c intact. The origin o will be sent infinitely far away, so I_c maps o to ∞.

Remarks :

1. A line sectioning through c mapped by I_c, is a circle through the origin.
2. A circle in c, is mapped by I_c to a bigger circle outside c.
3. Two circles inside c forming an angle, where the angle is preserved when mapped to the outside circles (i.e., conformal).

1 Algtop5, 2D objects: Torus and Genus

1.1 Translation group

$\langle\tau\rangle$: τ^n is the translation by an integer n, which is a group of transformation of A. τ^0 is the identity, $(\tau^n)^{-1} = \tau^{-n}$. O_x is the orbit of x, which consist of $\langle\tau\rangle$.

A Cylinder is topologically the space of all orbits in A^2.

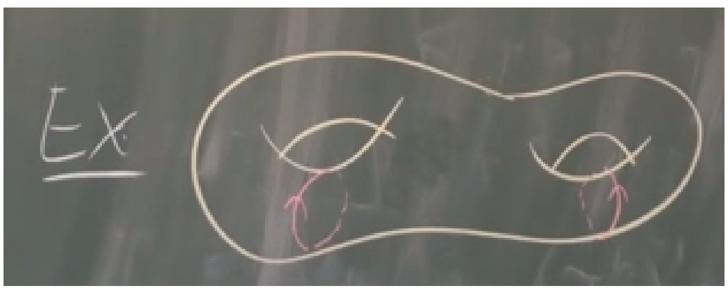
1.2 Torus

Doughnut shaped surface $T = S^1 \times S^1$.

The number of holes of a torus is defined as "Genus", g: maximum number of disjoint loops on the surface which do not disconnect the surface.

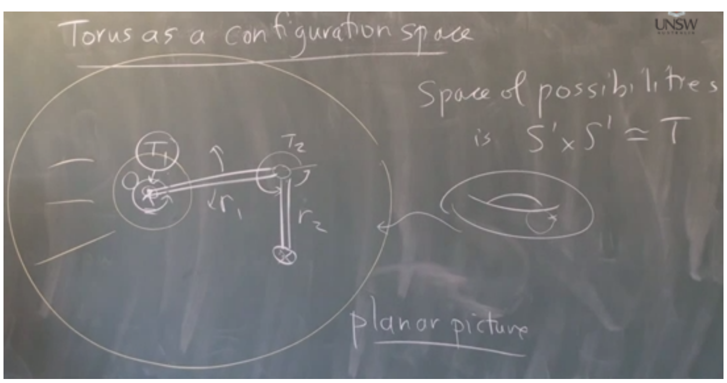
Jordan curve THM: prove that any loop on S^2 disconnects S^2.

The genus of a sphere $g(S^2) = 0$ (i.e., any number of loops will disconnet the sphere).
The genus of a torus $g(T) = 1$ (i.e., one loop cutting the torus into a cylinder does not disconnect the torus).
The genus of a 2-hole torus $g(\text{2-hole } T) = 2$ (i.e., also cutting the torus into a cylinder).



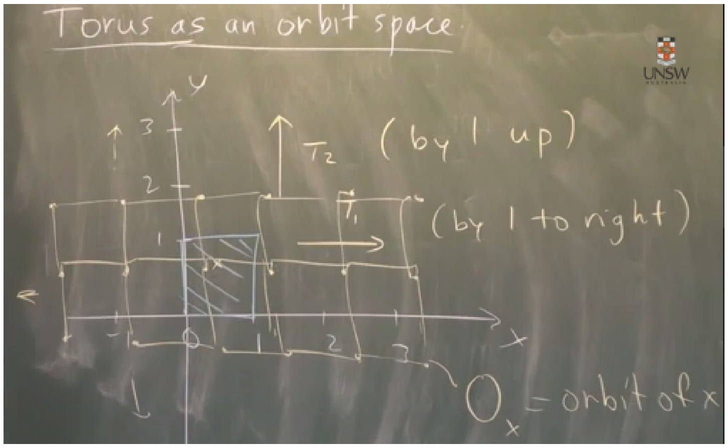
1.2.1 Torus as a configuration space

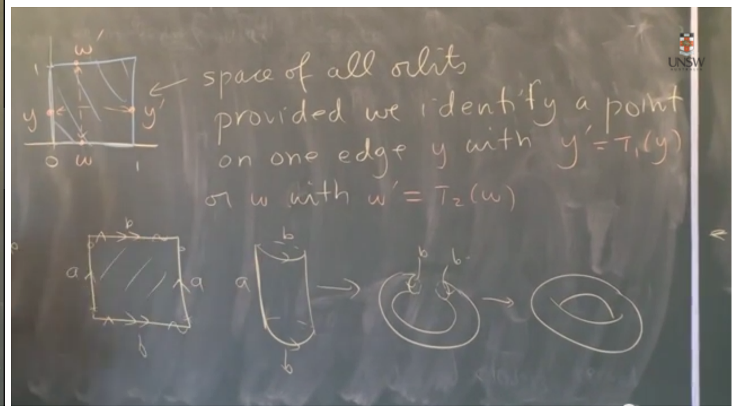
configuration space T_1, T_2, r_1, r_2.



1.2.2 Torus as an orbit space

Suppose τ_1 (τ_2) is a one translation to the right (up). $G = \langle \tau_1, \tau_2 \rangle$: group generated by all τ_1^n and τ_2^m, where $n, m \in Z$, and $G = Z \times Z$. Every orbit meets the unit square in 1, 2, or 4 points. Glue the edges together and get a torus.





1.3 Tessellation of a plane

Tessellation: A tiling of regular polygons (2-D), polyheda (3-D), polyhedron/polytope (n-D).

Prob Show that by gluing a hexagon we get a torus.