

0.1 Wavenumber, angular frequency, complex plane

We know that for a wave, $1(\text{cycle}) = 2\pi(\text{rad}) = 1(\lambda = x(m))$.

- wavenumber spatial frequency of a wave:

$$k = \frac{2\pi(\text{rad})}{\lambda(\text{m})} = \frac{1(\text{cycle})}{\lambda(\text{m})} \quad (1)$$

shows how many cycles per unit distance or radians per unit distance a single wavelength (λ) has. Its a measure of spatial frequency (number of rads or cycles "per" meter), so the longer (shorter) wavelength occurs less (more) frequent in a unit distance.

- angular frequency of a wave:

$$\omega = \frac{2\pi(\text{rad})}{T(\text{m})} = \frac{1(\text{cycle})}{T(\text{m})}, \quad (2)$$

shows how many cycles or radians a wave has per second.

- relationship to complex plane:

The unit circle on the complex plane with $|z| = 1$ is $z = e^{i(kx - \omega t)}$. The units for kx is "cycle", thus k is a measure of rate of rotation in a circle as a wave propagates.

Example 1. For a wave of $k = 0.1$, it needs to travel $2\pi/0.1 = 62.8(m)$ (single wavelength) for a full circle/cycle. Whereas $k = 10$ only requires $6.28(m)$ to travel a full circle. Therefore a wave rotates faster (shorter wavelength) in a complex circle as the k increases.

- Visualizing a wave interacting with a complex plane:

Draw a three dimensional cartesian coordinate with x on one axis and the other two are the complex plane axes. Draw a wave $f(x) = \cos(kx)$ running with a fixed k to an arbitrary x . The complex number can be seen as a vector-valued function $z = g(x) = (\cos(kx), \sin(kx))$ mapping x to the complex plane coordinate. Thus as the wave propagates along x , the z will be spinning in a circle on the complex plane. The "graph" $(x, g(x))$ is seen as a spring swirling along/around the x axis. The numbers of wave in a unit distance $x = 1$ can be seen on the graph $f(0 \leq x \leq 1)$.

- Visualizing the angular velocity vector (in a rotating reference plane):

A vector that is perpendicular to the rotational circle on a physical plane. Lets focus on three orthogonal vectors, v (cross radial velocity vector), ω (angular velocity vector), r (radial vector). The three coordinates are related by $\omega = \frac{d\theta}{dt} = \frac{1}{r} \frac{dx}{dt} = \frac{v}{r}$, . Thus we see that with a constant angular velocity, the smaller r gives higher v . For an arbitrary velocity vector $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.

0.2 Inertial and non-inertial reference frame

A rotating reference frame (frame that accelerates w.r.t. an inertial frame) is a non-inertial reference frame. The Navier-Stokes equation of the absolute velocity (inertial frame) is composed of the rotational velocity (rotating frame) plus the velocity contributed by the angular velocity

$$\left(\frac{du_{\text{inert}}}{dt}\right)_{\text{inert}} = \left(\frac{du_{\text{rot}}}{dt}\right)_{\text{rot}} + 2\boldsymbol{\Omega} \times \mathbf{u}_{\text{rot}}. \quad (3)$$

Example 2. Suppose a ball going straight from north to south along a longitude when viewed from a fixed point in the outer space. This ball will seem to curve right, with westward velocity, when viewed on earth. Thus adding back the eastward rotational velocity will cancel the westward velocity and transform the velocity back to the inertial frame, which is a straight southward velocity.

0.3 Field, Eulerian, Lagrangian

- Eulerian framework (reference frame fixed in space):
Different particles with the QoI measured at the same local point with time. This gives the "field" of QoI if all spatial points are measured/modeled.
- Lagrangian framework (reference frame moving in space):
A particle with the QoI is measured by following the particle parametrized by the initial position. The instantaneous position \mathbf{x} becomes a QoI instead of a fixed value as in the Eulerian view.
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0.4 streamline

Derivation. A streamline can be mapped out by solving for the instantaneous spatial points that matches

$$dt = \frac{dx}{u} = \frac{dy}{v}. \quad (4)$$

This condition must be satisfied when following a streamline. When given $u(t_1)$, $v(t_1)$ and $\Delta t = t_2 - t_1$, find (x_2, y_2) for the entire field, connect the dots of (x_2, y_2) with the closest (x_1, y_1) in the field gives the streamline. The streamlines show the instantaneous directional path of the flow field.

0.5 Material derivative/Lagrangian derivative

The acceleration of a particle in the Eulerian field could be derived from the first order Taylor expansion of u by following a particle from (x, t) to $(x + \delta x, t + \delta t)$,

$$\begin{aligned} u(x + \delta x, t + \delta t) &\approx u(x, t) + \delta x \cdot \nabla u + \delta t \frac{\partial u}{\partial t} \\ \lim_{\delta t \rightarrow 0} \frac{u(x + \delta x, t + \delta t) - u(x, t)}{\delta t} &= \frac{\partial u}{\partial t} + \mathbf{u} \cdot \nabla u. \end{aligned} \quad (5)$$

Imagine a flow with cold water upstream and warm downstream. The particle temperature is the scalar QoI. Therefore, the particle at any instant is locally heated by the sun which changes the first term. The changes due to motion (spatial derivative), particle flows from cold to warm with velocity \mathbf{u} , then it receives the temperature gradient advection from the second term.

0.6 Volume material derivative

A 2D squared volume at the initial time

$$\delta V = \delta x \delta y = (x_2 - x_1)(y_2 - y_1). \quad (6)$$

At the next time step

$$\begin{aligned} \delta V' &= \delta x' \delta y' \\ &= (x_2 + u_2 \delta t - (x_1 + u_1 \delta t))(y_2 + v_2 \delta t - (y_1 + v_1 \delta t)) \\ &= (\delta x + \delta u \delta t)(\delta y + \delta v \delta t) \\ &= \delta V + \delta x \delta v \delta t + \delta y \delta u \delta t + \delta t^2 \delta u \delta v. \end{aligned} \quad (7)$$

After some algebra

$$\frac{\Delta \delta V}{\Delta t} = \delta x \delta y \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \delta t \delta u \delta v. \quad (8)$$

The last term goes to zero as $\delta t \rightarrow 0$,

$$\frac{D \delta V}{Dt} = \delta V \nabla \cdot \vec{u}. \quad (9)$$

0.7 Gauss' Theorem (divergence theorem)

Definition.

$$\int \nabla \cdot Q dV = \int dA \cdot Q, \quad (10)$$

which relates volume integral to a surface integral of a scalar, vector or tensor field Q (see [1], p155 for proof). Intuitively, the sum of all sources minus the sinks in the volume equals the net flow going outward.

0.8 linear strain rate

Definition. The rate of change of volume per unit volume (expansion or contraction)

Derivation. The rate of change per unit distance in direction x is (see Fig 1)

$$\begin{aligned} \frac{1}{\delta x} \frac{D\delta x}{Dt} &= \frac{1}{\delta x} \frac{D}{Dt} \left(\left(u + \frac{\partial u}{\partial x} \delta x \right) \delta t \right) - u \delta t \\ &= \frac{\partial u}{\partial x}. \end{aligned} \quad (11)$$

Therefore the rate of change in volume

$$\begin{aligned} \frac{1}{\delta V} \frac{D\delta V}{Dt} &= \frac{1}{\delta V} \frac{D\delta x \delta y \delta z}{Dt} \\ &= \frac{1}{\delta x} \frac{D\delta x}{Dt} + \frac{1}{\delta y} \frac{D\delta y}{Dt} + \frac{1}{\delta z} \frac{D\delta z}{Dt} \\ &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\ &= \frac{\partial u_i}{\partial x_i} \end{aligned} \quad (12)$$

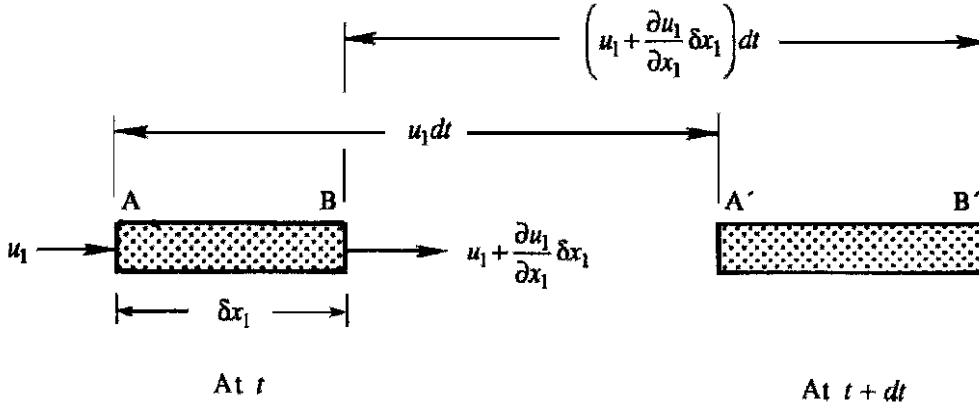


Figure 1: From [2].

0.9 shear strain rate

Definition. The rate of decrease in angles of “two” perpendicular lines (change in shape caused by squeezing, and no change in volume) (see Fig 2). For a single angle α ,

$$\begin{aligned}
 \frac{d\alpha}{dt} &= \frac{1}{\delta x_2} \frac{dx_2}{dt} \\
 &= \frac{1}{\delta x_2} \frac{((u_1 + \frac{\partial u_1}{\partial x_2} \delta x_2) \delta t - u_1 \delta t)}{\delta t} \\
 &= \frac{1}{\delta x_2} \frac{d}{dt} \left(\frac{\partial u_1}{\partial x_2} \delta x_2 \delta t \right) \\
 &= \frac{\partial u_1}{\partial x_2}
 \end{aligned} \tag{13}$$

(notice the small angle $\delta\theta = \frac{\delta x}{r}$, where r is the radius and δx is the arc length from C to A). Thus the total squeezing

$$\frac{d\alpha + d\beta}{dt} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \tag{14}$$

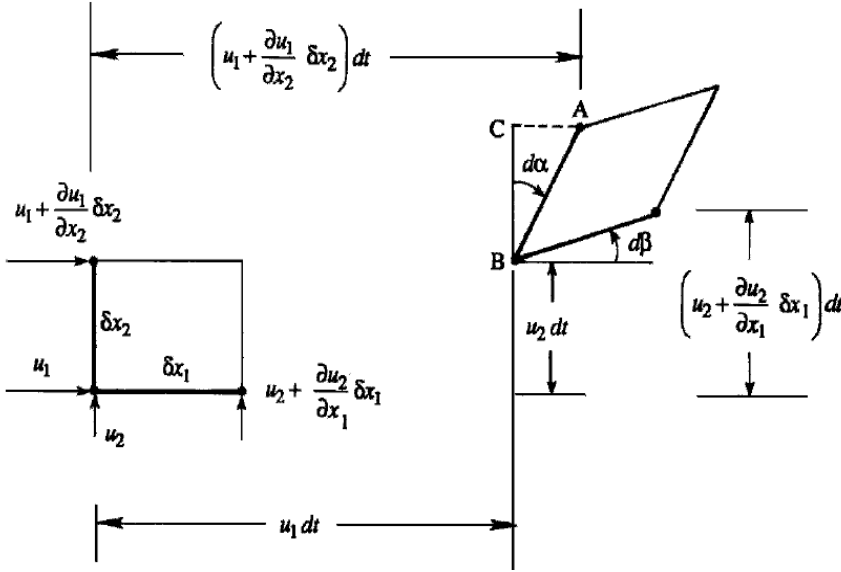


Figure 2: From [2].

0.10 strain rate tensor

Definition. The strain rate tensor (combining the linear and shear strain rate)

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{15}$$

where the off-diagonal terms of e_{ij} is half the shear strain rate. This tensor describes how a fluid element deforms.

0.11 Mass continuity via material derivative

By definition, mass is conserved following a particle,

$$\begin{aligned}
\frac{D}{Dt}(\rho \delta V) &= 0 \\
\left(\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v}\right) \delta V &= 0 \\
\int \left(\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v}\right) dV &= 0 \\
\int \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right) dV &= 0
\end{aligned} \tag{16}$$

Applied (9) to the second line. Applying divergence theorem to the last line gives

$$\int_V \left(\frac{\partial \rho}{\partial t}\right) dV = - \int_V (\nabla \cdot (\rho \mathbf{v})) dV = - \int_S \mathbf{v} \cdot d\mathbf{S} \tag{17}$$

indicates the increase rate of density inside a particle equals the inflow rate from the surface of the particle.

0.12 Tangential Vector of a functional surface

The tangential vector at a point x relative to the point at y is $(y - x, f(y) - f(x))$, where the first order Taylor expansion of $f(y) = f(x) + \nabla f(x)^T(y - x)$, which gives $(y - x, \nabla f(x)^T(y - x))$.

0.13 Tensor

0.13.1 Properties

Tensor, T , can be seen as a “**multilinear function**” that eats in vectors and spits out a scalar number, or a “**linear operator**” that transforms vectors into vectors. Multilinear $f : V_1 \times \cdots \times V_n \rightarrow W$, means linear in each vector space (closed under additivity and scalar multiplication).

Example. Moment of Inertia tensor \mathbf{I}

By definition, moment of inertia is the angular momentum per unit angle per unit time.

$$KE = \frac{1}{2} (\omega_x, \omega_y, \omega_z) \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{18}$$

where

$$L = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{19}$$

The first equation, I is seen as a multilinear function that eats in two vectors of ω and spits out a number $2KE$, $I(\omega, \omega) = 2KE$. The second equation, I is seen as a linear operator that eats in ω vector and spits out L vector, $I(\omega) = L$.

0.13.2 Components

A component of T is just a value of the function T on the given basis vectors, e.g. $T_{xx} = T(\hat{x}, \hat{x})$.

0.13.3 Multilinearity

Closed under additivity and scalar multiplication on the separate vector spaces.

Example. rank 2 tensor $T : V \times W \Rightarrow \mathcal{R}$. T is a rank 2 multilinear function on v and w with basis vectors \hat{x}_1, \hat{x}_2 if

$$\begin{aligned} T(v, w) &= T(v_1\hat{x}_1 + v_2\hat{x}_2, w_1\hat{x}_1 + w_2\hat{x}_2) \\ &= v_1w_1T(\hat{x}_1, \hat{x}_1) + v_1w_2T(\hat{x}_1, \hat{x}_2) + v_2w_1T(\hat{x}_2, \hat{x}_1) + v_2w_2T(\hat{x}_2, \hat{x}_2) \\ &= v_iw_jT_{ij}. \end{aligned}$$

0.13.4 Covariant and Contravariant components of a vectors

In Cartesian coordinates, the Length of vector A , $L_A = A_1^2 + A_2^2$ and the dot-product of A and B , $A \cdot B = A_1B_1 + A_2B_2$. In non-Cartesian coordinates, the Length of the vector A , L_A and $A \cdot B$ doesn't work, which is why we need covariant and contravariant components.

Covariant components for A is (A_1, A_2) and Contravariant is (A^1, A^2) . $L_A = A_1A^1 + A_2A^2$. The Covariant and Contravariant is defined so that L_A and $A \cdot B$ is unchanged.

Original definition of covariant and contravariant:

Given two basis, e and f , for any Euclidean vector $v = Ae = Bf$, $A = (a_1, \dots, a_n)^T$, $B = (b_1, \dots, b_n)^T$. Suppose the transformation matrix, T , from e to f satisfies $f = Te$. Therefore, $v = BTe$, thus $BT = A$, which shows that $B = AT^{-1}$. In conclusion, the basis vector e that is transformed covaries with f , hence e is transformed covariantly to a new basis f , and the components A is transformed contravariantly to the new basis f , with respect to the transformation T

Covariant transformation: $Ae = BTe = Bf$.

Contravariant transformation: $Ae = AT^{-1}f = Bf$.

0.14 Upscale KE backscatter

The 2D Vorticity equation by streamfunction ψ

$$\frac{D\nabla^2\psi}{Dt} = F. \quad (20)$$

Suppose there is no F , then by multiplying $-\psi$ and integrating over a domain A

$$\begin{aligned} \int_A \left(-\psi \frac{D\nabla^2\psi}{Dt} \right) dA &= 0 \\ &= \int_A \left(-\psi \nabla \frac{D\nabla\psi}{Dt} \right) dA \\ &= \int_A \left(-\cancel{\nabla \left(\psi \frac{D\nabla\psi}{Dt} \right)} + \nabla\psi \frac{D\nabla\psi}{Dt} \right) dA \\ &= \int_A \left(\frac{1}{2} \frac{D(\nabla\psi)^2}{Dt} \right) dA \\ &= \frac{D}{Dt} \int_A \frac{1}{2} (\nabla\psi)^2 dA, \end{aligned} \quad (21)$$

which gives the domain integrated kinetic energy $E = \int_A \frac{1}{2} (\nabla\psi)^2 dA$ by using periodic boundary condition or $v \cdot n = 0$, and $\frac{DE}{Dt} = 0$. Any function of the vorticity is conserved after integrating over the domain A , $\frac{D}{Dt} \int_A g(\xi) dA = 0$. Therefore

$$\begin{aligned} \int_A -\frac{D}{Dt}(\psi \nabla^2 \psi) dA &= 0 = \int_A \left(-\psi \frac{D \nabla^2 \psi}{Dt} \right) dA \\ \frac{D}{Dt} \int_A -(\psi \nabla^2 \psi) dA &= \frac{D}{Dt} \int_A \frac{1}{2} (\nabla \psi)^2 dA, \end{aligned} \tag{22}$$

where $E = -\int_A (\psi \nabla^2 \psi) dA$.

0.15 static pressure gradient force

Given surface depth $Z_1(x, y)$, and the second layer surface $Z_2(x, y)$. Pressure at a constant depth z in layer 1 equals $p_1 = \rho g(Z_1 - z)$. Pressure at a constant depth z in layer 2 equals $p_2 = \rho g Z_1 + \rho g'(Z_2 - z)$. Therefore, the horizontal pressure gradient at layer 1 is $\nabla p_1 = \rho g \nabla Z_1$ where the constant depth z is eliminated. Likewise, layer 2 is $\nabla p_2 = \nabla p_1 + \rho g' \nabla Z_2$.

Remarks: This suggests the pressure gradient in each layer is just a function of the varying surface height.

0.16 Flux

Definition. rate of flow of a property (with some unit U) per unit area ($Um^{-2}s^{-1}$).

0.17 anelastic approximation

Definition. Suppose $\rho = \rho_0(z) + \rho^*(x, y, z, t)$ and $\rho \approx \rho_0$. From hydrostasy ($\frac{1}{\rho_0} \frac{\partial \bar{p}}{\partial z} - g = 0$), the vertical equation of motion becomes

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0} \frac{\partial p^*}{\partial z} - g \frac{\rho^*}{\rho_0} + F. \tag{23}$$

Assume density doesn't fluctuate too fast with time ($\frac{D\rho}{Dt} = 0$), then the mass continuity equation becomes

$$\nabla \cdot \rho_0 v = 0. \tag{24}$$

0.18 Boussinesq approximation

Definition. Same as anelastic approximation but assuming shallow layer ($\rho_0 = \text{constant}$), thus

$$\nabla \cdot v = 0. \tag{25}$$

This allows a flux form substitution of the advection term for any variable.

0.19 dynamic viscosity

Definition. $\mu [Nsm^{-2}]$, the “required” tangential force per unit area to move a parallel plane at unit speed relative to a fixed plane at unit distance apart ($\mu = \tau / (\frac{\partial u}{\partial z}) = \tau$ per unit speed per unit distance = standardized τ). A bigger dynamic viscosity means more force is required to move the two planes, thus more viscous.

0.20 kinematic viscosity

Definition. $\nu = \frac{\mu}{\rho} [m^2s^{-1}]$, dynamic viscosity per unit density (per unit mass per unit volume). Kinematic is the study over a fixed body or point, so the kinematic viscosity is the viscosity for a fixed body.

0.21 conservation of momentum

Definition. The force (stress per unit area multiplied by the area) per unit volume on an infinitesimal box (i.e. point) in the x_1 direction (see Fig 3)

$$\frac{1}{\delta V} \left(\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} \right) dx_1 dx_2 dx_3 = \frac{\partial \tau_{j1}}{\partial x_j}. \quad (26)$$

Therefore, the equation of motion in a non-rotating system becomes (from symmetry of stress tensor proven in [2] p91, $\tau_{ij} = \tau_{ji}$)

$$\rho \frac{Du_i}{Dt} = \rho g_k + \frac{\partial \tau_{ij}}{\partial x_j}. \quad (27)$$

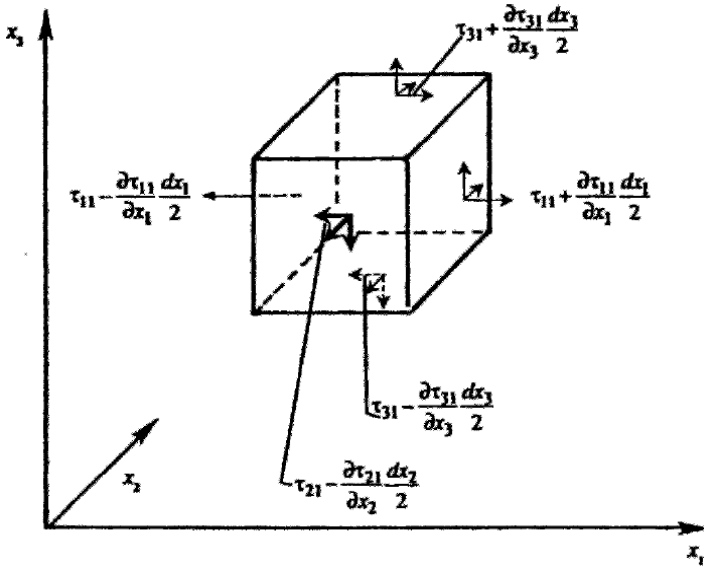


Figure 3: From [2].

0.22 down gradient turbulent transport

Definition.

$$\overline{w's'} = -K \frac{\partial \bar{s}}{\partial z} \quad (28)$$

Notice that when the vertical gradient of the mean quantity s is positive (negative), then the mean turbulent vertical transport is downward (upward), and negatively (positively) correlated.

0.23 Websites

Great lecture notes of numerical methods of hydrodynamics

References

- [1] C. S, “An introduction to vector analysis,” 2010.
- [2] P. Kundu and I. Cohen, “Fluid mechanics. 2004,” *Elsevier Academic Press, San Diego*). *Two-and three-dimensional self-sustained flow oscillations*, vol. 307, pp. 471–476, 2008.