0.1 Material derivative

The material derivative of a scalar or vector quantity ψ following a particle is

$$\frac{\partial \psi}{\partial t} + \boldsymbol{u} \cdot \nabla \psi. \tag{1}$$

Imagine a flow with cold water upstream and warm downstream. The fluid temperature is the scalar quantity. Therefore, the particle at any instant is locally heated by the sun which changes the first term. The changes due to motion, particle flows from cold to warm with velocity $\boldsymbol{u}$, then it recieves the temperature gradient advection from the second term.

0.2 Tangential Vector of a functional surface

The tangential vector at a point x relative to the point at y is $(y - x, f(y) - f(x))$, where the first order Taylor expansion of $f(y) = f(x) + \nabla f(x)^T (y - x)$, which gives $(y - x, \nabla f(x)^T (y - x))$.

0.3 Tensor

0.3.1 Properties

Tensor, T, can be seen as a "**multilinear function**" that eats in vectors and spits out a scalar number, or a "**linear operator**" that transforms vectors into vectors. Multilinear $f : V_1 \times \cdots \times V_n \to W$, means linear in each vector space (closed under additivity and scalar multiplication).

Example: Moment of Inertia tensor I

$$KE = \frac{1}{2}(\omega_x,\ \omega_y,\ \omega_z) \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{2}$$

where

$$L = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z . \end{pmatrix} \tag{3}$$

The first equation, I is seen as a multilinear function that eats in two vectors of ω and spits out a number $2KE$, $I(\omega, \omega) = 2KE$. The second equation, I is seen as a linear operator that eats in ω vector and spits *out L vector,* $I(\omega) = L$.

0.3.2 Components

A component of T is just a value of the function T on the given basis vectors, e.g. $T_{xx} = T(\hat{x}, \hat{x})$.

0.3.3 Multilinearity

Closed under addtivity and scalar multiplication on the separate vector spaces.

Example: rank 2 tensor

$T : V \times W \Rightarrow \mathcal{R}$. *$T$ is a rank 2 multilinear function on v and w with basis vectors $\hat{x}_1, \hat{x}_2$ if*

$$\begin{aligned}
T(v, w) &= T(v_1\hat{x}_1 + v_2\hat{x}_2, w_1\hat{x}_1 + w_2\hat{x}_2) \\
&= v_1w_1T(\hat{x}_1, \hat{x}_1) + v_1w_2T(\hat{x}_1, \hat{x}_2) + v_2w_1T(\hat{x}_2, \hat{x}_1) + v_2w_2T(\hat{x}_2, \hat{x}_2) \\
&= v_iw_jT_{ij}.
\end{aligned}$$

0.3.4 Covariant and Contravariant components of a vectors

In Cartesian coordinates, the Length of vector A, $L_A = A_1^2 + A_2^2$ and the dot-product of A and B, $A \cdot B = A_1B_1 + A_2B_2$. In non-Cartesian coordinates, the Length of the vector A, L_A and $A \cdot B$ doesn't work, which is why we need covariant and contravariant components.

Covariant components for A is (A_1, A_2) and Contravariant is (A^1, A^2). $L_A = A_1A^1 + A_2A^2$. The Covariant and Contravariant is defined so that L_A and $A \cdot B$ is unchanged.

Original definition of convariant and contravariant:

Given two basis, e and f, for any Euclidean vector $v = Ae = Bf$, $A = (a_1, \ldots, a_n)^T, B = (b_1, \ldots, b_n)^T$. Suppose the transformation matrix, T, from e to f satisfies $f = Te$. Therefore, $v = BTe$, thus $BT = A$, which shows that $B = AT^{-1}$. In conclusion, the basis vector e that is transformed covaries with f, hence e is transformed covariantly to a new basis f, and the components A is transformed contravariantly to the new basis f, with respect to the transformation T

Covariant transformation: $Ae = BTe = Bf$.

Contravariant transformation: $Ae = AT^{-1}f = Bf$.

0.4 Upscale KE backscatter

The 2D Vorticity equation by streamfunction ψ

$$\frac{D\nabla^2\psi}{Dt} = F. \tag{4}$$

Suppose there is no F, then by multiplying $-\psi$ and integrating over a domain A

$$\begin{aligned}
&\int_A \left(-\psi\frac{D\nabla^2\psi}{Dt}\right) dA = 0 \\
&= \int_A \left(-\psi\nabla\frac{D\nabla\psi}{Dt}\right) dA \\
&= \int_A \left(-\nabla\left(\cancel{\psi\frac{D\nabla\psi}{Dt}}\right) + \nabla\psi\frac{D\nabla\psi}{Dt}\right) dA \\
&= \int_A \left(\frac{1}{2}\frac{D(\nabla\psi)^2}{Dt}\right) dA \\
&= \frac{D}{Dt}\int_A \frac{1}{2}(\nabla\psi)^2 dA,
\end{aligned} \tag{5}$$

which gives the domain integrated kinetic energy $E = \int_A \frac{1}{2}(\nabla\psi)^2 dA$ by using periodic boundary condition or $v \cdot n = 0$, and $\frac{DE}{Dt} = 0$. Any function of the vorticity is conserved after integrating over the domain A, $\frac{D}{Dt}\int_A g(\xi)dA = 0$. Therefore

$$\begin{aligned}
\int_A -\frac{D}{Dt}(\psi\nabla^2\psi)dA = 0 = \int_A \left(-\psi\frac{D\nabla^2\psi}{Dt}\right) dA \\
\frac{D}{Dt}\int_A -(\psi\nabla^2\psi)dA = \frac{D}{Dt}\int_A \frac{1}{2}(\nabla\psi)^2 dA,
\end{aligned} \tag{6}$$

where $E = -\int_A(\psi\nabla^2\psi)dA$.

0.5 Pressure Gradient Force

Given surface depth $Z_1(x, y)$, and the second layer surface $Z_2(x, y)$. Pressure at a constant depth z in layer 1 equals $p_1 = \rho g(Z_1 - z)$. Pressure at a constant depth z in layer 2 equals $p_2 = \rho g Z_1 + \rho g'(Z_2 - z)$. Therefore, the horizontal pressure gradient at layer 1 is $\nabla p_1 = \rho g Z_1$ where the constant depth z is eliminated. Likewise, layer 2 is $\nabla p_2 = \nabla p_1 + \rho g' Z_2$.

Remarks: This suggests the pressure gradient in each layer is just a function of the varying surface height.

0.6 Flux

Definition: rate of flow of a property (with some unit U) per unit area ($Um^{-2}s^{-1}$).