0.1 Holder's Inequality

Proof by Jensen's inequality (convexity) on convex function $g(h) = |h|^p$

$$g(\mathbf{E}h) \leq \mathbf{E}(g) \;\Rightarrow\; |\int hd\mu|^p \leq \int |h|^p d\mu \;\Rightarrow\; \int hd\mu \leq \Big(\int |h|^p d\mu\Big)^{1/p} \tag{1}$$

0.2 Minkowski Inequality

Given the measure μ on the L^p space, $1 \leq p \leq \infty$, where $f(x)$ and $g(x)$ are in $L^p(S)$, $x \in S$, we have the triangle inequality

$$||f+g||_p \leq ||f||_p + ||g||_p, \tag{2}$$

where the p-norm of f is $||f||_p = (\int_S |f|^p d\mu)^{1/p}$.

Proof:

First prove that p-norm of $f+g$ is bounded if f and g are both bounded, i.e., proving that $|f+g|^p$ is bounded by $|f|^p$ and $|g|^p$.

$|0.5f + 0.5g|^p \leq |0.5|f| + 0.5|g||^p \leq 0.5|f|^p + 0.5|g|^p$ (convexity of $h(x) = |x|^p$ for $p \geq 1$ gives $h(0.5|f| + 0.5|g|) \leq 0.5h(|f|) + 0.5h(|g|) \Rightarrow |0.5|f| + 0.5|g||^p \leq 0.5|f|^p + 0.5|g|^p$). Therefore the inequality becomes

$$|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p), \tag{3}$$

finishing the boundedness proof.

The Minkowski inequality is proven by using triangular inequality and Holder's inequality as, $||f+g||_p^p = \int |f+g|^p d\mu = \int |f+g||f+g|^{p-1} d\mu \leq \int (|f|+|g|)|f+g|^{p-1} d\mu = \int |f||f+g|^{p-1} d\mu + \int |g||f+g|^{p-1} d\mu \leq$