

Introduction to Linear Algebra

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Introduction

Linear algebra deals with:

- ▶ Vectors.
- ▶ Matrices. (The linear transformations of vector spaces.)

It is used in many places:

- ▶ Physics (velocity, force, ...).
- ▶ Artificial intelligence - neural networks.
- ▶ **Adaptive Law-Based Transformation (ALT).**
- ▶ ...

What is a Vector?

- ▶ **Mathematician's view:** A set that satisfies the 8 axiom.
- ▶ **Physicist's view:** An arrow in space with direction and magnitude.
- ▶ **Computer Scientist's view:** A 1D array or list of numbers.

$$\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \langle 4, 0, -5 \rangle, \quad \vec{u} = (1, 2, 3). \quad (1)$$

- ▶ **Notation:**
 - ▶ Boldface: \mathbf{v} (common in CS)
 - ▶ Arrow: \vec{v} (common in physics)
 - ▶ Angled brackets or parentheses

Example 1 (Vectors)

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}, \quad \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (2)$$

What is the Dot Product?

- The dot product of two vectors \vec{a} and \vec{b} is:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n. \quad (3)$$

- It can be shown that:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos\left(\angle\left(\vec{a}, \vec{b}\right)\right). \quad (4)$$

Example 2 (Dot product)

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix} = 2 \cdot 4 + 3 \cdot (-1) = 8 - 3 = 5. \quad (5)$$

Dot Product – Practice Exercises

- Calculate the dot product of the following vector pairs:

1. $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}.$

2. $\begin{bmatrix} 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

Dot Product – Solutions

► **Solution to (a):**

$$1 \cdot 4 + (-2) \cdot 0 + 3 \cdot (-1) = 4 + 0 - 3 = \boxed{1}.$$

► **Solution to (b):**

$$2 \cdot (-1) + 5 \cdot 1 = -2 + 5 = \boxed{3}.$$

► **Interpretation:**

- (a) Zero \Rightarrow the vectors are perpendicular.
- (b) Positive \Rightarrow angle between the vectors is acute.

What is a Matrix?

- ▶ A 2D array of numbers arranged in rows and columns:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad (6)$$

where A is a 2×3 matrix.

- ▶ Types:
 - ▶ **Square:** same number of rows and columns (e.g., 2×2 , 3×3). Mathematicians like them.
 - ▶ **Non-square:** different number of rows and columns (e.g., 2×3 , 3×2).

Vector-Matrix Multiplication

- ▶ Each row of the matrix is dotted with the vector:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \quad (7)$$

- ▶ Multiply:

$$A\vec{v} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot 1 \\ 3 \cdot 2 + 4 \cdot 1 \\ 5 \cdot 2 + 6 \cdot 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix}. \quad (8)$$

Vector-Matrix Multiplication – Exercises

- Multiply the vector \vec{v} with matrix M in the following examples:

1. General case:

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad M = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix}. \quad (9)$$

2. Interesting effect:

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}. \quad (10)$$

3. Even more interesting effect:

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 5 & -5 \\ -3 & 3 \end{bmatrix}. \quad (11)$$

- Try to interpret the geometric meaning of each result.

Vector-Matrix Multiplication – Solutions

1. General case:

$$\vec{v} \cdot M = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 5 \end{bmatrix}. \quad (12)$$

2. Stretching cases:

$$\vec{v} \cdot M_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\vec{v}, \quad \vec{v} \cdot M_2 = \begin{bmatrix} -4 \\ -4 \end{bmatrix} = -4\vec{v}. \quad (13)$$

3. Nearly null output:

$$\vec{v} \cdot M_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\vec{v}, \quad \vec{v} \cdot M_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\vec{v}. \quad (14)$$

Matrices as linear transformation

- ▶ Each matrix denotes a transformation $\vec{v} \mapsto M\vec{v}$. It's always linear.
- ▶ It can be proven that all linear transformation can be described with a matrix.

Transformation examples

- ▶ Stretching: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.
- ▶ Rotation: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
- ▶ Reflection: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
- ▶ Projection (flattening onto a line or plane) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Matrix-Matrix Multiplication

Procedure is the same as the Vector-Matrix multiplication. If an $n \times m$ matrix is multiplied by an $m \times k$, the result is an $n \times k$ matrix.

Example 3

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix} \quad (15)$$

Example 4

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix} \quad (16)$$

Eigenvalues and Eigenvectors

- ▶ Notice that

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \quad (17)$$

- ▶ Generally for a square matrix A if

$$A\vec{v} = \lambda\vec{v}, \quad (18)$$

then:

- ▶ \vec{v} is an eigenvector and
- ▶ λ is the corresponding eigenvalue.
- ▶ These vectors don't change direction when multiplied by A .

Eigenvalues and Eigenvectors – Properties

- ▶ If $A\vec{v} = \lambda\vec{v}$, then \vec{v} is an eigenvector of A with eigenvalue λ .
- ▶ Eigenvectors are defined up to a scalar: if \vec{v} is an eigenvector, so is $c\vec{v}$ for any $c \neq 0$.
- ▶ The set of all eigenvectors corresponding to a single eigenvalue forms a vector subspace.
- ▶ A square matrix of size $n \times n$ has at most n eigenvalues (including complex ones and multiplicities).
- ▶ If all eigenvalues of a matrix are positive, the matrix is positive definite.
- ▶ Diagonal matrices have their diagonal elements as eigenvalues, and the standard basis vectors as eigenvectors.
- ▶ The eigenvalues of a complex Hermitian or real symmetric matrix are real.

Finding eigenvalues

Using a computer:

```
1 import torch
2 A = torch.tensor([[2., 0.], [0., 2.]])
3 eigenvalues, eigenvectors = torch.linalg.eigh(A)
4 print("Eigenvalues:", eigenvalues)
5 print("Eigenvectors:\n", eigenvectors)
```


Revision

In this presentation you learned about:

- ▶ Vectors.
- ▶ Matrices.
- ▶ Dot products.
- ▶ Vector-Matrix and Matrix-Matrix multiplication.
- ▶ Eigen-decomposition.

Extra: Eigenvalue Problem: More Detail

- ▶ Step 1: Start with $A\vec{v} = \lambda\vec{v}$.
- ▶ Step 2: Rewrite as $(A - \lambda I)\vec{v} = 0$.
- ▶ Step 3: Solve $\det(A - \lambda I) = 0$ (this gives a polynomial).
- ▶ Step 4: Find the roots (eigenvalues), then solve the nullspace for each to get eigenvectors.

Example 5

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow \det \left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \right) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3. \quad (19)$$

- ▶ *Roots:* $\lambda = 1, 3$.
- ▶ *For $\lambda = 3$:* $(A - 3I)\vec{v} = 0 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- ▶ *For $\lambda = 1$:* $(A - I)\vec{v} = 0 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.