Linear algebra II: Determinants and Eigenvalues

Wigner Summer Camp
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What is the Determinant?

- The determinant is a scalar value associated with a square matrix A.
- It plays a key role in linear algebra:
  - A is invertible if and only if $\det(A) \neq 0$.
  - $|\det(A)|$ equals the volume scaling factor of the linear transformation defined by A.
- Determinants are widely used in solving linear systems, computing eigenvalues, and understanding geometric transformations.

Determinant of a 2×2 Matrix

- For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1}$$

the determinant is given by

$$\det(A) = ad - bc. \tag{2}$$

- Example:

$$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = (1)(4) - (2)(3) = -2. \tag{3}$$

Exercise: 2×2 Determinant

Example 1 (Exercise)

Compute the determinant of the matrix:

$$A = \begin{pmatrix} 5 & 7 \\ 2 & 6 \end{pmatrix}. \tag{4}$$

Solution: 2×2 Determinant

Example 2 (Solution)

Using Equation (2):

$$\det(A) = (5)(6) - (7)(2) = 30 - 14 = 16. \tag{5}$$

Minor Matrix

Given an $n \times n$ matrix A, the *minor matrix* M_{ij} is the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j of A. Note that here and on the following slides the indexing starts from 1.

For example, if

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \tag{6}$$

then the minor M_{12} (removing row 1 and column 2) is:

$$M_{12} = \begin{pmatrix} d & f \\ g & i \end{pmatrix}. \tag{7}$$

Cofactor Sign Convention

The cofactor sign pattern for the Laplace expansion is given by the chessboard pattern:

$+$	$-$	$+$	$-$	$\dots$
$-$	$+$	$-$	$+$	$\dots$
$+$	$-$	$+$	$-$	$\dots$
$-$	$+$	$-$	$+$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The sign of the cofactor at position (i, j) is $(-1)^{i+j}$.

Determinant of an $n \times n$ Matrix

If we expand along the first row, the determinant of an $n \times n$ matrix A is defined recursively as:

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(M_{1j}), \tag{8}$$

where M_{1j} is the minor matrix obtained by removing the first row and j-th column of A and $a_{i,j}$ is the (i,j)-th element of A. This recursive formula is called the *Laplace expansion* and can be applied along any row or column.

Example: 3×3 Determinant

For a 3×3 matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \tag{9}$$

the determinant (expanding along the first row) is:

$$\det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}. \tag{10}$$

Each 2×2 minor is computed as:

$$\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = m_{11}m_{22} - m_{12}m_{21}. \tag{11}$$

Numerical Example

Compute:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}. \tag{12}$$

Expanding along the first row:

$$\begin{aligned}
\det(A) &= 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} && (13) \\
&= 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) && (14) \\
&= 1(-3) - 2(-6) + 3(-3) && (15) \\
&= -3 + 12 - 9 = 0. && (16)
\end{aligned}$$

The determinant of this matrix is 0, so it is singular (non-invertible).

Exercise: Compute a 3×3 Determinant

Exercise

Compute the determinant of:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ 5 & 2 & 0 \end{pmatrix}. \tag{17}$$

Use expansion along the first row.

Solution: Exercise

Solution

$$\det(A) = 2\begin{vmatrix} -1 & 4 \\ 2 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & -1 \\ 5 & 2 \end{vmatrix} \tag{18}$$

$$= 2((-1)(0) - 4(2)) - 1(0 - 4(5)) + 3((0)(2) - (-1)(5)) \tag{19}$$

$$= 2(0 - 8) - 1(0 - 20) + 3(0 + 5) \tag{20}$$

$$= -16 + 20 + 15 = 19. \tag{21}$$

Therefore, $\det(A) = 19$.

The Eigenvalue Problem — Step by Step

We want to find scalars λ and nonzero vectors $\vec{v}$ such that:

$$A\vec{v} = \lambda\vec{v}. \tag{22}$$

Step 1: Rewrite as

$$(A - \lambda I)\vec{v} = 0. \tag{23}$$

Step 2: For a nontrivial solution ($\vec{v} \neq 0$), the matrix $A - \lambda I$ must be singular:

$$\det(A - \lambda I) = 0. \tag{24}$$

This is called the *characteristic equation*.

Solving the Eigenvalue Problem

Step 3: Expand $\det(A - \lambda I) = 0$ to get a polynomial in λ.
Step 4: The roots of this polynomial are the eigenvalues.
Step 5: For each eigenvalue λ, solve:

$$(A - \lambda I)\vec{v} = 0 \tag{25}$$

to find the corresponding eigenvector(s) by computing the nullspace.

Example: Eigenvalues and Eigenvectors

Given:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{26}$$

Step 1: Compute $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}. \tag{27}$$

Step 2: Characteristic equation:

$$\det(A - \lambda I) = (2-\lambda)(2-\lambda) - 1 \tag{28}$$

$$= \lambda^2 - 4\lambda + 3 = 0. \tag{29}$$

Step 3: Roots:

$$\lambda_1 = 1, \quad \lambda_2 = 3. \tag{30}$$

Example: Eigenvectors

For $\lambda_2 = 3$:

$$A - 3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{31}$$

We solve:

$$(A - 3I)\vec{v} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \vec{v} = 0. \tag{32}$$

This gives $v_1 = v_2$, so an eigenvector is:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{33}$$

For $\lambda_1 = 1$:

$$A - I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{34}$$

This gives $v_1 = -v_2$, so an eigenvector is:

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{35}$$

Exercise: Find Eigenvalues and Eigenvectors

Exercise

Find the eigenvalues and one eigenvector for each eigenvalue of the matrix:

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}. \tag{36}$$

Solution: Exercise

Step 1: Compute characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2 \tag{37}$$

$$= \lambda^2 - 7\lambda + 10 = 0. \tag{38}$$

Step 2: Roots:

$$\lambda_1 = 5, \quad \lambda_2 = 2. \tag{39}$$

Step 3: For $\lambda_1 = 5$:

$$A - 5I = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}. \tag{40}$$

Nullspace: $v_1 = 2v_2$, so:

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{41}$$

Solution 2

For $\lambda_2 = 2$:

$$A - 2I = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}. \tag{42}$$

Nullspace: $v_1 = -v_2$, so:

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{43}$$