

Linear Algebra II: Determinants and Eigenvalues

Wigner Summer Camp

Data and Compute Intensive Sciences Research Group

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7-11 July 2025



What is the Determinant?

- ▶ The determinant is a scalar value associated with a square matrix A .
- ▶ It plays a key role in linear algebra:
 - ▶ A is invertible if and only if $\det(A) \neq 0$.
 - ▶ $|\det(A)|$ equals the volume scaling factor of the linear transformation defined by A .
- ▶ Determinants are widely used in solving linear systems, computing eigenvalues, and understanding geometric transformations.

Determinant of a 2×2 Matrix

- For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (1)$$

the determinant is given by

$$\det(A) = ad - bc. \quad (2)$$

- Example:

$$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = (1)(4) - (2)(3) = -2. \quad (3)$$

Exercise: 2×2 Determinant

Example 1 (Exercise)

Compute the determinant of the matrix:

$$A = \begin{pmatrix} 5 & 7 \\ 2 & 6 \end{pmatrix}. \quad (4)$$

Solution: 2×2 Determinant

Example 2 (Solution)

Using Equation (2):

$$\det(A) = (5)(6) - (7)(2) = 30 - 14 = 16. \quad (5)$$

Minor Matrix

Given an $n \times n$ matrix A , the *minor matrix* M_{ij} is the $(n - 1) \times (n - 1)$ matrix obtained by deleting row i and column j of A . Note that here and on the following slides the indexing starts from 1.

For example, if

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad (6)$$

then the minor M_{12} (removing row 1 and column 2) is:

$$M_{12} = \begin{pmatrix} d & f \\ g & i \end{pmatrix}. \quad (7)$$

Cofactor Sign Convention

The cofactor sign pattern for the Laplace expansion is given by the chessboard pattern:

+	-	+	-	...
-	+	-	+	...
+	-	+	-	...
-	+	-	+	...
⋮	⋮	⋮	⋮	⋮

The sign of the cofactor at position (i, j) is $(-1)^{i+j}$.

Determinant of an $n \times n$ Matrix

If we expand along the first row, the determinant of an $n \times n$ matrix A is defined recursively as:

$$\det(A) = \sum_{j=1}^n (-1)^{1+j} a_{1j} \det(M_{1j}), \quad (8)$$

where M_{1j} is the minor matrix obtained by removing the first row and j -th column of A and $a_{i,j}$ is the (i, j) -th element of A . This recursive formula is called the *Laplace expansion* and can be applied along any row or column.

Example: 3×3 Determinant

For a 3×3 matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad (9)$$

the determinant (expanding along the first row) is:

$$\det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}. \quad (10)$$

Each 2×2 minor is computed as:

$$\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = m_{11}m_{22} - m_{12}m_{21}. \quad (11)$$

Numerical Example

Compute:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}. \quad (12)$$

Expanding along the first row:

$$\det(A) = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \quad (13)$$

$$= 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) \quad (14)$$

$$= 1(-3) - 2(-6) + 3(-3) \quad (15)$$

$$= -3 + 12 - 9 = 0. \quad (16)$$

The determinant of this matrix is 0, so it is singular (non-invertible).

Exercise: Compute a 3×3 Determinant

Exercise

Compute the determinant of:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ 5 & 2 & 0 \end{pmatrix}. \quad (17)$$

Use expansion along the first row.

Solution: Exercise

Solution

$$\det(A) = 2 \begin{vmatrix} -1 & 4 \\ 2 & 0 \end{vmatrix} - 1 \begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3 \begin{vmatrix} 0 & -1 \\ 5 & 2 \end{vmatrix} \quad (18)$$

$$= 2((-1)(0) - 4(2)) - 1(0 - 4(5)) + 3((0)(2) - (-1)(5)) \quad (19)$$

$$= 2(0 - 8) - 1(0 - 20) + 3(0 + 5) \quad (20)$$

$$= -16 + 20 + 15 = 19. \quad (21)$$

Therefore, $\det(A) = 19$.

The Eigenvalue Problem — Step by Step

We want to find scalars λ and nonzero vectors \vec{v} such that:

$$A\vec{v} = \lambda\vec{v}. \quad (22)$$

Step 1: Rewrite as

$$(A - \lambda I)\vec{v} = 0. \quad (23)$$

Step 2: For a nontrivial solution ($\vec{v} \neq 0$), the matrix $A - \lambda I$ must be singular:

$$\det(A - \lambda I) = 0. \quad (24)$$

This is called the *characteristic equation*.

Solving the Eigenvalue Problem

Step 3: Expand $\det(A - \lambda I) = 0$ to get a polynomial in λ .

Step 4: The roots of this polynomial are the eigenvalues.

Step 5: For each eigenvalue λ , solve:

$$(A - \lambda I)\vec{v} = 0 \tag{25}$$

to find the corresponding eigenvector(s) by computing the nullspace.

Example: Eigenvalues and Eigenvectors

Given:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \quad (26)$$

Step 1: Compute $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{pmatrix}. \quad (27)$$

Step 2: Characteristic equation:

$$\det(A - \lambda I) = (2 - \lambda)(2 - \lambda) - 1 \quad (28)$$

$$= \lambda^2 - 4\lambda + 3 = 0. \quad (29)$$

Step 3: Roots:

$$\lambda_1 = 1, \quad \lambda_2 = 3. \quad (30)$$

Example: Eigenvectors

For $\lambda_2 = 3$:

$$A - 3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (31)$$

We solve:

$$(A - 3I)\vec{v} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \vec{v} = 0. \quad (32)$$

This gives $v_1 = v_2$, so an eigenvector is:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (33)$$

For $\lambda_1 = 1$:

$$A - I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (34)$$

This gives $v_1 = -v_2$, so an eigenvector is:

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (35)$$

Exercise: Find Eigenvalues and Eigenvectors

Exercise

Find the eigenvalues and one eigenvector for each eigenvalue of the matrix:

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}. \quad (36)$$

Solution: Exercise

Step 1: Compute characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 4 - \lambda & 2 \\ 1 & 3 - \lambda \end{vmatrix} = (4 - \lambda)(3 - \lambda) - 2 \quad (37)$$

$$= \lambda^2 - 7\lambda + 10 = 0. \quad (38)$$

Step 2: Roots:

$$\lambda_1 = 5, \quad \lambda_2 = 2. \quad (39)$$

Step 3: For $\lambda_1 = 5$:

$$A - 5I = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}. \quad (40)$$

Nullspace: $v_1 = 2v_2$, so:

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \quad (41)$$

Solution 2

For $\lambda_2 = 2$:

$$A - 2I = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}. \quad (42)$$

Nullspace: $v_1 = -v_2$, so:

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (43)$$