Linear Algebra II: Determinants and Eigenvalues

Wigner Summer Camp Data and Compute Intensive Sciences Research Group

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What is the Determinant?

- ► The determinant is a scalar value associated with a square matrix A.
- ▶ It plays a key role in linear algebra:
 - ightharpoonup A is invertible if and only if $det(A) \neq 0$.
 - $ightharpoonup |\det(A)|$ equals the volume scaling factor of the linear transformation defined by A.
- ▶ Determinants are widely used in solving linear systems, computing eigenvalues, and understanding geometric transformations.

Determinant of a 2×2 Matrix

 \triangleright For a 2 × 2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\tag{1}$$

the determinant is given by

$$\det(A) = ad - bc. \tag{2}$$

► Example:

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = (1)(4) - (2)(3) = -2. \tag{3}$$



Exercise: 2×2 Determinant

Example 1 (Exercise)

Compute the determinant of the matrix:

$$A = \begin{pmatrix} 5 & 7 \\ 2 & 6 \end{pmatrix}. \tag{4}$$

Solution: 2×2 Determinant

Example 2 (Solution)

Using Equation (2):

$$\det(A) = (5)(6) - (7)(2) = 30 - 14 = 16.$$
 (5)



Minor Matrix

Given an $n \times n$ matrix A, the minor matrix M_{ij} is the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j of A. Note that here and on the following slides the indexing starts from 1.

For example, if

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \tag{6}$$

then the minor M_{12} (removing row 1 and column 2) is:

$$M_{12} = \begin{pmatrix} d & f \\ g & i \end{pmatrix}. \tag{7}$$



Cofactor Sign Convention

The cofactor sign pattern for the Laplace expansion is given by the chessboard pattern:

+	_	+	_	
_	+	ı	+	
+	_	+	_	• • •
_	+	_	+	
:	:	•	:	٠

The sign of the cofactor at position (i, j) is $(-1)^{i+j}$.

Determinant of an $n \times n$ Matrix

If we expand along the first row, the determinant of an $n \times n$ matrix A is defined recursively as:

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(M_{1j}), \tag{8}$$

where M_{1j} is the minor matrix obtained by removing the first row and j-th column of A and $a_{i,j}$ is the (i,j)-th element of A. This recursive formula is called the *Laplace expansion* and can be applied along any row or column.

Example: 3×3 Determinant

For a 3×3 matrix

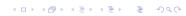
$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \tag{9}$$

the determinant (expanding along the first row) is:

$$\det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}. \tag{10}$$

Each 2×2 minor is computed as:

$$\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = m_{11}m_{22} - m_{12}m_{21}. \tag{11}$$



Numerical Example

Compute:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}. \tag{12}$$

Expanding along the first row:

$$\det(A) = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7)$$

$$= 1(-3) - 2(-6) + 3(-3)$$

$$(13)$$

$$= -3 + 12 - 9 = 0. (16)$$

The determinant of this matrix is 0, so it is singular (non-invertible).



Exercise: Compute a 3×3 Determinant

Exercise

Compute the determinant of:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ 5 & 2 & 0 \end{pmatrix}. \tag{17}$$

Use expansion along the first row.

Solution: Exercise

Solution

$$\det(A) = 2 \begin{vmatrix} -1 & 4 \\ 2 & 0 \end{vmatrix} - 1 \begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3 \begin{vmatrix} 0 & -1 \\ 5 & 2 \end{vmatrix}$$

$$= 2((-1)(0) - 4(2)) - 1(0 - 4(5)) + 3((0)(2) - (-1)(5))$$

$$= 2(0 - 8) - 1(0 - 20) + 3(0 + 5)$$

$$= -16 + 20 + 15 = 19.$$
(20)

Therefore, det(A) = 19.

The Eigenvalue Problem — Step by Step

We want to find scalars λ and nonzero vectors \vec{v} such that:

$$A\vec{v} = \lambda \vec{v}.\tag{22}$$

Step 1: Rewrite as

$$(A - \lambda I)\vec{v} = 0. (23)$$

Step 2: For a nontrivial solution $(\vec{v} \neq 0)$, the matrix $A - \lambda I$ must be singular:

$$\det(A - \lambda I) = 0. \tag{24}$$

This is called the *characteristic equation*.



Solving the Eigenvalue Problem

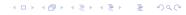
Step 3: Expand $det(A - \lambda I) = 0$ to get a polynomial in λ .

Step 4: The roots of this polynomial are the eigenvalues.

Step 5: For each eigenvalue λ , solve:

$$(A - \lambda I)\vec{v} = 0 \tag{25}$$

to find the corresponding eigenvector(s) by computing the nullspace.



Example: Eigenvalues and Eigenvectors

Given:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{26}$$

Step 1: Compute $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 2 - \lambda & 1\\ 1 & 2 - \lambda \end{pmatrix}. \tag{27}$$

Step 2: Characteristic equation:

$$\det(A - \lambda I) = (2 - \lambda)(2 - \lambda) - 1 \tag{28}$$

$$= \lambda^2 - 4\lambda + 3 = 0. (29)$$

Step 3: Roots:

$$\lambda_1 = 1, \quad \lambda_2 = 3. \tag{30}$$

Example: Eigenvectors

For $\lambda_2 = 3$:

$$A - 3I = \begin{pmatrix} -1 & 1\\ 1 & -1 \end{pmatrix}. \tag{31}$$

We solve:

$$(A - 3I)\vec{v} = \begin{pmatrix} -1 & 1\\ 1 & -1 \end{pmatrix} \vec{v} = 0.$$
 (32)

This gives $v_1 = v_2$, so an eigenvector is:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{33}$$

For $\lambda_1 = 1$:

$$A - I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{34}$$

This gives $v_1 = -v_2$, so an eigenvector is:

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{35}$$

Exercise: Find Eigenvalues and Eigenvectors

Exercise

Find the eigenvalues and one eigenvector for each eigenvalue of the matrix:

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}. \tag{36}$$

Solution: Exercise

Step 1: Compute characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 4 - \lambda & 2 \\ 1 & 3 - \lambda \end{vmatrix} = (4 - \lambda)(3 - \lambda) - 2$$

$$= \lambda^2 - 7\lambda + 10 = 0.$$
(38)

Step 2: Roots:

$$\lambda_1 = 5, \quad \lambda_2 = 2. \tag{39}$$

Step 3: For $\lambda_1 = 5$:

$$A - 5I = \begin{pmatrix} -1 & 2\\ 1 & -2 \end{pmatrix}. \tag{40}$$

Nullspace: $v_1 = 2v_2$, so:

$$\vec{v}_1 = \begin{pmatrix} 2\\1 \end{pmatrix}. \tag{41}$$

Solution 2

For $\lambda_2 = 2$:

$$A - 2I = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}. \tag{42}$$

Nullspace: $v_1 = -v_2$, so:

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{43}$$