

# Introduction to Linear Algebra

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# Introduction

Linear algebra deals with:

- ▶ Vectors.
- ▶ Matrices. (The linear transformations of vector spaces.)

It is used in many places:

- ▶ Physics (velocity, force, ...).
- ▶ Artificial intelligence - neural networks.
- ▶ **Adaptive Law-Based Transformation (ALT).**
- ▶ ...

# What is a Vector?

- ▶ **Mathematician's view:** A set that satisfies the 8 axiom.
- ▶ **Physicist's view:** An arrow in space with direction and magnitude.
- ▶ **Computer Scientist's view:** A 1D array or list of numbers.

$$\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \langle 4, 0, -5 \rangle, \quad \vec{u} = (1, 2, 3). \quad (1)$$

- ▶ **Notation:**
  - ▶ Boldface:  $\mathbf{v}$  (common in CS)
  - ▶ Arrow:  $\vec{v}$  (common in physics)
  - ▶ Angled brackets or parentheses

## Example 1 (Vectors)

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}, \quad \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (2)$$

# What is the Dot Product?

- The dot product of two vectors  $\vec{a}$  and  $\vec{b}$  is:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n. \quad (3)$$

- It can be shown that:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos\left(\angle\left(\vec{a}, \vec{b}\right)\right). \quad (4)$$

## Example 2 (Dot product)

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix} = 2 \cdot 4 + 3 \cdot (-1) = 8 - 3 = 5. \quad (5)$$

# Dot Product – Practice Exercises

- Calculate the dot product of the following vector pairs:

1.  $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}.$

2.  $\begin{bmatrix} 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

# Dot Product – Solutions

► **Solution to (a):**

$$1 \cdot 4 + (-2) \cdot 0 + 3 \cdot (-1) = 4 + 0 - 3 = \boxed{1}.$$

► **Solution to (b):**

$$2 \cdot (-1) + 5 \cdot 1 = -2 + 5 = \boxed{3}.$$

► **Interpretation:**

- (a) Zero  $\Rightarrow$  the vectors are perpendicular.
- (b) Positive  $\Rightarrow$  angle between the vectors is acute.

# What is a Matrix?

- ▶ A 2D array of numbers arranged in rows and columns:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad (6)$$

where  $A$  is a  $2 \times 3$  matrix.

- ▶ Types:
  - ▶ **Square:** same number of rows and columns (e.g.,  $2 \times 2$ ,  $3 \times 3$ ). Mathematicians like them.
  - ▶ **Non-square:** different number of rows and columns (e.g.,  $2 \times 3$ ,  $3 \times 2$ ).

# Vector-Matrix Multiplication

- ▶ Each row of the matrix is dotted with the vector:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \quad (7)$$

- ▶ Multiply:

$$A\vec{v} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot 1 \\ 3 \cdot 2 + 4 \cdot 1 \\ 5 \cdot 2 + 6 \cdot 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix}. \quad (8)$$



# Vector-Matrix Multiplication – Exercises

- Multiply the vector  $\vec{v}$  with matrix  $M$  in the following examples:

1. General case:

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad M = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix}. \quad (9)$$

2. Interesting effect:

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}. \quad (10)$$

3. Even more interesting effect:

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 5 & -5 \\ -3 & 3 \end{bmatrix}. \quad (11)$$

- Try to interpret the geometric meaning of each result.

# Vector-Matrix Multiplication – Solutions

1. General case:

$$\vec{v} \cdot M = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 5 \end{bmatrix}. \quad (12)$$

2. Stretching cases:

$$\vec{v} \cdot M_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\vec{v}, \quad \vec{v} \cdot M_2 = \begin{bmatrix} -4 \\ -4 \end{bmatrix} = -4\vec{v}. \quad (13)$$

3. Nearly null output:

$$\vec{v} \cdot M_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\vec{v}, \quad \vec{v} \cdot M_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\vec{v}. \quad (14)$$

# Matrices as linear transformation

- ▶ Each matrix denotes a transformation  $\vec{v} \mapsto M\vec{v}$ . It's always linear.
- ▶ It can be proven that all linear transformation can be described with a matrix.

# Transformation examples

- ▶ Stretching:  $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ .
- ▶ Rotation:  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ .
- ▶ Reflection:  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ .
- ▶ Projection (flattening onto a line or plane)  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ .

# Matrix-Matrix Multiplication

Procedure is the same as the Vector-Matrix multiplication. If an  $n \times m$  matrix is multiplied by an  $m \times k$ , the result is an  $n \times k$  matrix.

## Example 3

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix} \quad (15)$$

## Example 4

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix} \quad (16)$$

# Eigenvalues and Eigenvectors

- ▶ Notice that

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \quad (17)$$

- ▶ Generally for a square matrix  $A$  if

$$A\vec{v} = \lambda\vec{v}, \quad (18)$$

then:

- ▶  $\vec{v}$  is an eigenvector and
- ▶  $\lambda$  is the corresponding eigenvalue.
- ▶ These vectors don't change direction when multiplied by  $A$ .

# Eigenvalues and Eigenvectors – Properties

- ▶ If  $A\vec{v} = \lambda\vec{v}$ , then  $\vec{v}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ .
- ▶ Eigenvectors are defined up to a scalar: if  $\vec{v}$  is an eigenvector, so is  $c\vec{v}$  for any  $c \neq 0$ .
- ▶ The set of all eigenvectors corresponding to a single eigenvalue forms a vector subspace.
- ▶ A square matrix of size  $n \times n$  has at most  $n$  eigenvalues (including complex ones and multiplicities).
- ▶ If all eigenvalues of a matrix are positive, the matrix is positive definite.
- ▶ Diagonal matrices have their diagonal elements as eigenvalues, and the standard basis vectors as eigenvectors.
- ▶ The eigenvalues of a complex Hermitian or real symmetric matrix are real.

# Finding eigenvalues

Using a computer:

```
1 import torch
2 A = torch.tensor([[2., 0.], [0., 2.]])
3 eigenvalues, eigenvectors = torch.linalg.eigh(A)
4 print("Eigenvalues:", eigenvalues)
5 print("Eigenvectors:\n", eigenvectors)
```



# Revision

In this presentation you learned about:

- ▶ Vectors.
- ▶ Matrices.
- ▶ Dot products.
- ▶ Vector-Matrix and Matrix-Matrix multiplication.
- ▶ Eigen-decomposition.

## Extra: Eigenvalue Problem: More Detail

- ▶ Step 1: Start with  $A\vec{v} = \lambda\vec{v}$ .
- ▶ Step 2: Rewrite as  $(A - \lambda I)\vec{v} = 0$ .
- ▶ Step 3: Solve  $\det(A - \lambda I) = 0$  (this gives a polynomial).
- ▶ Step 4: Find the roots (eigenvalues), then solve the nullspace for each to get eigenvectors.

### Example 5

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow \det \left( \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \right) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3. \quad (19)$$

- ▶ *Roots:*  $\lambda = 1, 3$ .
- ▶ *For  $\lambda = 3$ :*  $(A - 3I)\vec{v} = 0 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .
- ▶ *For  $\lambda = 1$ :*  $(A - I)\vec{v} = 0 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .