Introduction to Linear Algebra

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Introduction

Linear algebra deals with:

- Vectors.
- Matrices. (The linear transformations of vector spaces.)

It is used in many places:

- Physics (velocity, force, ...).
- Artificial intelligence - neural networks.
- **Adaptive Law-Based Transformation (ALT).**
- ...

What is a Vector?

- **Mathematician's view:** A set that satisfies the 8 axiom.
- **Physicist's view:** An arrow in space with direction and magnitude.
- **Computer Scientist's view:** A 1D array or list of numbers.

$$\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \langle 4, 0, -5 \rangle, \quad \vec{u} = (1, 2, 3). \tag{1}$$

- **Notation:**
  - Boldface: $\mathbf{v}$ (common in CS and mathematical books)
  - Arrow: $\vec{v}$ (common in physics)
  - Angled brackets or parentheses

Example 1 (Vectors)

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}, \quad \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2}$$

What is the Dot Product?

- The dot product of two vectors $\vec{a}$ and $\vec{b}$ is:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n. \tag{3}$$

- It can be shown that:

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cos\left(\angle\left(\vec{a}, \vec{b}\right)\right). \tag{4}$$

Example 2 (Dot product)

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix} = 2 \cdot 4 + 3 \cdot (-1) = 8 - 3 = 5. \tag{5}$$

Dot Product – Practice Exercises

- Calculate the dot product of the following vector pairs:
  1. $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}.$
  2. $\begin{bmatrix} 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

Dot Product – Solutions

- **Solution to (a):**

$$1 \cdot 4 + (-2) \cdot 0 + 3 \cdot (-1) = 4 + 0 - 3 = \boxed{1}.$$

- **Solution to (b):**

$$2 \cdot (-1) + 5 \cdot 1 = -2 + 5 = \boxed{3}.$$

- Interpretation:
  - (a) Zero $\Rightarrow$ the vectors are perpendicular.
  - (b) Positive $\Rightarrow$ angle between the vectors is acute.

What is a Matrix?

- A 2D array of numbers arranged in rows and columns:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \tag{6}$$

  where A is a 2×3 matrix.
- Types:
  - **Square:** same number of rows and columns (e.g., 2×2, 3×3). Mathematicians like them.
  - **Non-square:** different number of rows and columns (e.g., 2×3, 3×2).

Vector-Matrix Multiplication

- Each row of the matrix is dotted with the vector:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \tag{7}$$

- Multiplication:

$$A\vec{v} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot 1 \\ 3 \cdot 2 + 4 \cdot 1 \\ 5 \cdot 2 + 6 \cdot 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix}. \tag{8}$$

Vector-Matrix Multiplication – Exercises

- Multiply the vector $\vec{v}$ with matrix M in the following examples:
  1. General case:
  $$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad M = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix}. \tag{9}$$
  2. Interesting effect:
  $$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}. \tag{10}$$
  3. Even more interesting effect:
  $$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 5 & -5 \\ -3 & 3 \end{bmatrix}. \tag{11}$$
- Try to interpret the geometric meaning of each result.

Vector-Matrix Multiplication – Solutions

1. General case:

$$\vec{v} \cdot M = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 5 \end{bmatrix}. \tag{12}$$

2. Stretching cases:

$$\vec{v} \cdot M_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\vec{v}, \quad \vec{v} \cdot M_2 = \begin{bmatrix} -4 \\ -4 \end{bmatrix} = -4\vec{v}. \tag{13}$$

3. Nearly null output:

$$\vec{v} \cdot M_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\vec{v}, \quad \vec{v} \cdot M_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\vec{v}. \tag{14}$$

Matrices as linear transformation

- Each matrix denotes a transformation $\vec{v} \mapsto M\vec{v}$. It's always linear.
- It can be proven that all linear transformation can be described with a matrix.

Transformation examples

- Stretching: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.
- Rotation: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
- Reflection: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
- Projection (flattening onto a line or plane) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Matrix-Matrix Multiplication

Procedure is the same as the Vector-Matrix multiplication. If an $n \times m$ matrix is multiplied by an $m \times k$, the result is an $n \times k$ matrix.

- The number of columns in the first matrix must be equal to the number of rows in the second matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{15}$$

$$A \cdot B = \begin{bmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} \end{bmatrix} \tag{16}$$

Example 3

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \Rightarrow A \cdot B = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix} \tag{17}$$

Other examples

Example 4

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow A \cdot B = \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix} \tag{18}$$

Example 5

$$A = \begin{bmatrix} 8 & 4 & 11 & 5 \\ 4 & 4 & 21 & 6 \\ 6 & 9 & 11 & 8 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 4 & 6 \\ 1 & 4 \\ 7 & 2 \end{bmatrix} \Rightarrow A \cdot B = \begin{bmatrix} 86 & 94 \\ 91 & 128 \\ 121 & 126 \end{bmatrix} \tag{19}$$

Eigenvalues and Eigenvectors

- Notice that
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{20}$$
- Generally for a square matrix A if
$$A\vec{v} = \lambda\vec{v}, \tag{21}$$
then:
  - $\vec{v}$ is an eigenvector and
  - λ is the corresponding eigenvalue.
- These vectors don't change direction when multiplied by A.

Eigenvalues and Eigenvectors – Properties

- If $A\vec{v} = \lambda\vec{v}$, then $\vec{v}$ is an eigenvector of A with eigenvalue λ.
- Eigenvectors are defined up to a scalar: if $\vec{v}$ is an eigenvector, so is $c\vec{v}$ for any $c \neq 0$.
- The set of all eigenvectors corresponding to a single eigenvalue forms a vector subspace.
- A square matrix of size $n \times n$ has at most n eigenvalues (including complex ones and multiplicities).
- If all eigenvalues of a matrix are positive, the matrix is positive definite.
- Diagonal matrices (matrixes where for all $a_{ij} = 0$ where $i \neg j$) have their diagonal elements as eigenvalues, and the standard basis vectors as eigenvectors.
- The eigenvalues of a complex Hermitian or real symmetric matrix are real.

Finding eigenvalues

Using a computer:

```python
import torch
A = torch.tensor([[2., 0.], [0., 2.]])
eigenvalues, eigenvectors = torch.linalg.eigh(A)
print("Eigenvalues:", eigenvalues)
print("Eigenvectors:\n", eigenvectors)
```

Revision

In this presentation you learned about:

- Vectors.
- Matrices.
- Dot products.
- Vector-Matrix and Matrix-Matrix multiplication.
- Eigen-decomposition.

Extra: Matrix's determinant

Mathematical definition:

- The determinant is a scalar-valued funcion of the entries of a square matrix A.
  The determinant encodes:
  - A is invertibe if $\det(A) \neq 0$
  - Absolute value equals the volume change under the linear transformation of A.

Formula:

- for 2×2 case: $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$
- for 3×3 case:
  $$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - ceg - bdi - ahf.$$

Use in AI:

- Checking invertibility.
- Eigenvalue Problem.

Extra: Eigenvalue Problem (detailed)

- Step 1: Start with $A\vec{v} = \lambda\vec{v}$.
- Step 2: Rewrite as $(A - \lambda I)\vec{v} = 0$.
- Step 3: Solve $\det(A - \lambda I) = 0$ (this gives a polynomial).
- Step 4: Find the roots (eigenvalues), then solve the nullspace for each to get eigenvectors.

Example 6

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow \det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3. \tag{22}$$

- *Roots:* $\lambda = 1, 3$.
- *For* $\lambda = 3$*:* $(A - 3I)\vec{v} = 0 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- *For* $\lambda = 1$*:* $(A - I)\vec{v} = 0 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.