Introduction to Linear Algebra

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Introduction

Linear algebra deals with:

- Vectors.
- Matrices. (The linear transformations of vector spaces.)

It is used in many places:

- Physics (velocity, force, ...).
- Artificial intelligence - neural networks.
- **Adaptive Law-Based Transformation (ALT).**
- ...

What is a Vector?

- **Mathematician's view:** A set that satisfies the 8 axioms.
- **Physicist's view:** An arrow in space with direction and magnitude.
- **Computer Scientist's view:** A 1D array or list of numbers.

$$\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \langle 4, 0, -5 \rangle, \quad \vec{u} = (1, 2, 3). \tag{1}$$

- **Notation:**
  - Boldface: $\mathbf{v}$ (common in CS and mathematical books)
  - Arrow: $\vec{v}$ (common in physics)
  - Angled brackets or parentheses

Example vectors

Example 1 (Vectors)

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}. \tag{2}$$

Example 2 (Common vectors)

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3}$$

What is the Dot Product?

- The dot product of two vectors $\vec{a}$ and $\vec{b}$ is:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n. \tag{4}$$

- It can be shown that:

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cos\left(\angle\left(\vec{a}, \vec{b}\right)\right). \tag{5}$$

Example 3 (Dot product)

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix} = 2 \cdot 4 + 3 \cdot (-1) = 8 - 3 = 5. \tag{6}$$

Dot Product – Practice Exercises

- Calculate the dot product of the following vector pairs:
  1. $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$.
  2. $\begin{bmatrix} 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Dot Product – Solutions

- **Solution to (a):**

$$1 \cdot 4 + (-2) \cdot 0 + 3 \cdot (-1) = 4 + 0 - 3 = \boxed{1}.$$

- **Solution to (b):**

$$2 \cdot (-1) + 5 \cdot 1 = -2 + 5 = \boxed{3}.$$

- Interpretation:
  - (a) Zero $\Rightarrow$ the vectors are perpendicular.
  - (b) Positive $\Rightarrow$ angle between the vectors is acute.

What is a Matrix?

- A 2D array of numbers arranged in rows and columns:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \tag{7}$$

  where A is a 2×3 matrix.
- Types:
  - **Square:** same number of rows and columns (e.g., 2×2, 3×3). Mathematicians like them.
  - **Non-square:** different number of rows and columns (e.g., 2×3, 3×2).

Vector-Matrix Multiplication

- Each row of the matrix is dotted with the vector:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \tag{8}$$

- Multiplication:

$$A\vec{v} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot 1 \\ 3 \cdot 2 + 4 \cdot 1 \\ 5 \cdot 2 + 6 \cdot 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix}. \tag{9}$$

Vector-Matrix Multiplication – Exercises

- Multiply the vector $\vec{v}$ with matrix M in the following examples:
  1. General case:
$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad M = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix}. \tag{10}$$
  2. Interesting effect:
$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}. \tag{11}$$
  3. Even more interesting effect:
$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 5 & -5 \\ -3 & 3 \end{bmatrix}. \tag{12}$$
- Try to interpret the geometric meaning of each result.

Vector-Matrix Multiplication – Solutions

1. General case:

$$\vec{v} \cdot M = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 5 \end{bmatrix}. \tag{13}$$

2. Stretching cases:

$$\vec{v} \cdot M_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\vec{v}, \quad \vec{v} \cdot M_2 = \begin{bmatrix} -4 \\ -4 \end{bmatrix} = -4\vec{v}. \tag{14}$$

3. Nearly null output:

$$\vec{v} \cdot M_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\vec{v}, \quad \vec{v} \cdot M_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\vec{v}. \tag{15}$$

Matrices as linear transformations

- Each matrix denotes a transformation $\vec{v} \mapsto M\vec{v}$. It's always linear.
- It can be proven that all linear transformations can be described with a matrix.

Transformation examples

- Stretching: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.
- Rotation: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
- Reflection: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
- Projection (flattening onto a line or plane) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Matrix-Matrix Multiplication

Procedure is the same as the Vector-Matrix multiplication. If an $n \times m$ matrix is multiplied by an $m \times k$, the result is an $n \times k$ matrix.

- The number of columns in the first matrix must be equal to the number of rows in the second matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{16}$$

$$A \cdot B = \begin{bmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} \end{bmatrix} \tag{17}$$

Example 4

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \Rightarrow A \cdot B = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix} \tag{18}$$

Other examples

Example 5

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow A \cdot B = \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix} \tag{19}$$

Example 6

$$A = \begin{bmatrix} 8 & 4 & 11 & 5 \\ 4 & 4 & 21 & 6 \\ 6 & 9 & 11 & 8 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 4 & 6 \\ 1 & 4 \\ 7 & 2 \end{bmatrix} \Rightarrow A \cdot B = \begin{bmatrix} 86 & 94 \\ 91 & 128 \\ 121 & 126 \end{bmatrix} \tag{20}$$

Eigenvalues and Eigenvectors

- Notice that
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{21}$$
- Generally for a square matrix A if
$$A\vec{v} = \lambda\vec{v}, \tag{22}$$
then:
    - $\vec{v}$ is an eigenvector and
    - scalar λ is the corresponding eigenvalue.
- These vectors don't change direction when multiplied by A.

Eigenvalues and Eigenvectors – Properties

- If $A\vec{v} = \lambda\vec{v}$, then $\vec{v}$ is an eigenvector of A with eigenvalue λ.
- Eigenvectors are defined up to a scalar: if $\vec{v}$ is an eigenvector, so is $c\vec{v}$ for any $c \neq 0$.
- The set of all eigenvectors corresponding to a single eigenvalue forms a vector subspace.
- A square matrix of size $n \times n$ has at most n eigenvalues (including complex ones and multiplicities).
- If all eigenvalues of a matrix are positive, the matrix is positive definite.
- Diagonal matrices (matrices where for all $a_{ij} = 0$ where $i \neq j$) have their diagonal elements as eigenvalues, and the standard basis vectors as eigenvectors.
- The eigenvalues of a complex Hermitian or real symmetric matrix are real.

Finding eigenvalues

Using a computer:

```
import torch
A = torch.tensor([[2., 0.], [0., 2.]])
eigenvalues, eigenvectors = torch.linalg.eigh(A)
print("Eigenvalues:", eigenvalues)
print("Eigenvectors:\n", eigenvectors)
```

Revision

In this presentation you learned about:

- Vectors.
- Matrices.
- Dot products.
- Vector-Matrix and Matrix-Matrix multiplication.
- Eigendecomposition.

What is the Determinant?

- The determinant is a scalar value associated with a square matrix A.
- It plays a key role in linear algebra:
  - A is invertible if and only if $\det(A) \neq 0$.
  - $|\det(A)|$ equals the volume scaling factor of the linear transformation defined by A.
- Determinants are widely used in solving linear systems, computing eigenvalues, and understanding geometric transformations.

Determinant of a 2×2 Matrix

- For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{23}$$

the determinant is given by

$$\det(A) = ad - bc. \tag{24}$$

- Example:

$$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = (1)(4) - (2)(3) = -2. \tag{25}$$

Exercise: 2×2 Determinant

Example 7 (Exercise)

Compute the determinant of the matrix:

$$A = \begin{pmatrix} 5 & 7 \\ 2 & 6 \end{pmatrix}. \tag{26}$$

Solution: 2×2 Determinant

Example 8 (Solution)

Using Equation (24)*:*

$$\det(A) = (5)(6) - (7)(2) = 30 - 14 = 16. \tag{27}$$

Determinant of a 3×3 Matrix

We compute the determinant of a 3×3 matrix by expanding along the first row:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}. \tag{28}$$

The formula is:

$$\det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}. \tag{29}$$

Each 2×2 minor is the determinant of the smaller matrix obtained by deleting the corresponding row and column.

Why does this work?

The determinant of A is defined recursively by expanding into 2×2 minors:

$$\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = m_{11}m_{22} - m_{12}m_{21}. \tag{30}$$

So:

$$\det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}. \tag{31}$$

This is called the *Laplace expansion*.
Signs come from the cofactor sign pattern:

$+$	$-$	$+$
$-$	$+$	$-$
$+$	$-$	$+$

The determinant of an $n \times n$ matrix is computed similarly: expanding along any row or column into $(n-1) \times (n-1)$ minors.

Numerical Example

Compute:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}. \tag{32}$$

Expanding along the first row:

$$\begin{aligned}
\det(A) &= 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} && (33) \\
&= 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) && (34) \\
&= 1(-3) - 2(-6) + 3(-3) && (35) \\
&= -3 + 12 - 9 = 0. && (36)
\end{aligned}$$

The determinant of this matrix is 0, so it is singular (non-invertible).

Exercise: Compute a 3×3 Determinant

Exercise

Compute the determinant of:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ 5 & 2 & 0 \end{pmatrix}. \tag{37}$$

Use expansion along the first row.

Solution: Exercise

Solution

$$\det(A) = 2\begin{vmatrix} -1 & 4 \\ 2 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & -1 \\ 5 & 2 \end{vmatrix} \tag{38}$$

$$= 2((-1)(0) - 4(2)) - 1(0 - 4(5)) + 3((0)(2) - (-1)(5)) \tag{39}$$

$$= 2(0 - 8) - 1(0 - 20) + 3(0 + 5) \tag{40}$$

$$= -16 + 20 + 15 = 19. \tag{41}$$

Therefore, $\det(A) = 19$.

The Eigenvalue Problem — Step by Step

We want to find scalars λ and nonzero vectors $\vec{v}$ such that:

$$A\vec{v} = \lambda\vec{v}. \tag{42}$$

Step 1: Rewrite as

$$(A - \lambda I)\vec{v} = 0. \tag{43}$$

Step 2: For a nontrivial solution ($\vec{v} \neq 0$), the matrix $A - \lambda I$ must be singular:

$$\det(A - \lambda I) = 0. \tag{44}$$

This is called the *characteristic equation*.

Solving the Eigenvalue Problem

Step 3: Expand $\det(A - \lambda I) = 0$ to get a polynomial in λ.
Step 4: The roots of this polynomial are the eigenvalues.
Step 5: For each eigenvalue λ, solve:

$$(A - \lambda I)\vec{v} = 0 \tag{45}$$

to find the corresponding eigenvector(s) by computing the nullspace.

Example: Eigenvalues and Eigenvectors

Given:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{46}$$

Step 1: Compute $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}. \tag{47}$$

Step 2: Characteristic equation:

$$\det(A - \lambda I) = (2-\lambda)(2-\lambda) - 1 \tag{48}$$
$$= \lambda^2 - 4\lambda + 3 = 0. \tag{49}$$

Step 3: Roots:

$$\lambda_1 = 1, \quad \lambda_2 = 3. \tag{50}$$

Example: Eigenvectors

For $\lambda_2 = 3$:

$$A - 3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{51}$$

We solve:

$$(A - 3I)\vec{v} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \vec{v} = 0. \tag{52}$$

This gives $v_1 = v_2$, so an eigenvector is:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{53}$$

For $\lambda_1 = 1$:

$$A - I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{54}$$

This gives $v_1 = -v_2$, so an eigenvector is:

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{55}$$

Exercise: Find Eigenvalues and Eigenvectors

Exercise

Find the eigenvalues and one eigenvector for each eigenvalue of the matrix:

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}. \tag{56}$$

Solution: Exercise

Step 1: Compute characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2 \tag{57}$$

$$= \lambda^2 - 7\lambda + 10 = 0. \tag{58}$$

Step 2: Roots:

$$\lambda_1 = 5, \quad \lambda_2 = 2. \tag{59}$$

Step 3: For $\lambda_1 = 5$:

$$A - 5I = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}. \tag{60}$$

Nullspace: $v_1 = 2v_2$, so:

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{61}$$

Solution 2

For $\lambda_2 = 2$:

$$A - 2I = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}. \tag{62}$$

Nullspace: $v_1 = -v_2$, so:

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{63}$$