

Introduction to Linear Algebra

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Introduction

Linear algebra deals with:

- ▶ Vectors.
- ▶ Matrices. (The linear transformations of vector spaces.)

It is used in many places:

- ▶ Physics (velocity, force, ...).
- ▶ Artificial intelligence - neural networks.
- ▶ **Adaptive Law-Based Transformation (ALT).**
- ▶ ...

What is a Vector?

- ▶ **Mathematician's view:** A set that satisfies the 8 axioms.
- ▶ **Physicist's view:** An arrow in space with direction and magnitude.
- ▶ **Computer Scientist's view:** A 1D array or list of numbers.

$$\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \langle 4, 0, -5 \rangle, \quad \vec{u} = (1, 2, 3). \quad (1)$$

- ▶ **Notation:**
 - ▶ Boldface: \mathbf{v} (common in CS and mathematical books)
 - ▶ Arrow: \vec{v} (common in physics)
 - ▶ Angled brackets or parentheses

Example vectors

Example 1 (Vectors)

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}. \quad (2)$$

Example 2 (Common vectors)

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (3)$$

What is the Dot Product?

- ▶ The dot product of two vectors \vec{a} and \vec{b} is:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n. \quad (4)$$

- ▶ It can be shown that:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos\left(\angle\left(\vec{a}, \vec{b}\right)\right). \quad (5)$$

Example 3 (Dot product)

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \end{bmatrix} = 2 \cdot 4 + 3 \cdot (-1) = 8 - 3 = 5. \quad (6)$$

Dot Product – Practice Exercises

- Calculate the dot product of the following vector pairs:

1. $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}.$

2. $\begin{bmatrix} 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

Dot Product – Solutions

► **Solution to (a):**

$$1 \cdot 4 + (-2) \cdot 0 + 3 \cdot (-1) = 4 + 0 - 3 = \boxed{1}.$$

► **Solution to (b):**

$$2 \cdot (-1) + 5 \cdot 1 = -2 + 5 = \boxed{3}.$$

► **Interpretation:**

- (a) Zero \Rightarrow the vectors are perpendicular.
- (b) Positive \Rightarrow angle between the vectors is acute.

What is a Matrix?

- ▶ A 2D array of numbers arranged in rows and columns:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad (7)$$

where A is a 2×3 matrix.

- ▶ Types:
 - ▶ **Square:** same number of rows and columns (e.g., 2×2 , 3×3). Mathematicians like them.
 - ▶ **Non-square:** different number of rows and columns (e.g., 2×3 , 3×2).

Vector-Matrix Multiplication

- Each row of the matrix is dotted with the vector:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \quad (8)$$

- Multiplication:

$$A\vec{v} = \begin{bmatrix} 1 \cdot 2 + 2 \cdot 1 \\ 3 \cdot 2 + 4 \cdot 1 \\ 5 \cdot 2 + 6 \cdot 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix}. \quad (9)$$

Vector-Matrix Multiplication – Exercises

- Multiply the vector \vec{v} with matrix M in the following examples:

1. General case:

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad M = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix}. \quad (10)$$

2. Interesting effect:

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}. \quad (11)$$

3. Even more interesting effect:

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 5 & -5 \\ -3 & 3 \end{bmatrix}. \quad (12)$$

- Try to interpret the geometric meaning of each result.

Vector-Matrix Multiplication – Solutions

1. General case:

$$\vec{v} \cdot M = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 5 \end{bmatrix}. \quad (13)$$

2. Stretching cases:

$$\vec{v} \cdot M_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\vec{v}, \quad \vec{v} \cdot M_2 = \begin{bmatrix} -4 \\ -4 \end{bmatrix} = -4\vec{v}. \quad (14)$$

3. Nearly null output:

$$\vec{v} \cdot M_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\vec{v}, \quad \vec{v} \cdot M_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\vec{v}. \quad (15)$$

Matrices as linear transformations

- ▶ Each matrix denotes a transformation $\vec{v} \mapsto M\vec{v}$. It's always linear.
- ▶ It can be proven that all linear transformations can be described with a matrix.

Transformation examples

- ▶ Stretching: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.
- ▶ Rotation: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
- ▶ Reflection: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
- ▶ Projection (flattening onto a line or plane) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Matrix-Matrix Multiplication

Procedure is the same as the Vector-Matrix multiplication. If an $n \times m$ matrix is multiplied by an $m \times k$, the result is an $n \times k$ matrix.

- ▶ The number of columns in the first matrix must be equal to the number of rows in the second matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad (16)$$

$$A \cdot B = \begin{bmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} \end{bmatrix} \quad (17)$$

Example 4

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \Rightarrow A \cdot B = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix} \quad (18)$$

Other examples

Example 5

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow A \cdot B = \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix} \quad (19)$$

Example 6

$$A = \begin{bmatrix} 8 & 4 & 11 & 5 \\ 4 & 4 & 21 & 6 \\ 6 & 9 & 11 & 8 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 4 & 6 \\ 1 & 4 \\ 7 & 2 \end{bmatrix} \Rightarrow A \cdot B = \begin{bmatrix} 86 & 94 \\ 91 & 128 \\ 121 & 126 \end{bmatrix} \quad (20)$$

Eigenvalues and Eigenvectors

- ▶ Notice that

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \quad (21)$$

- ▶ Generally for a square matrix A if

$$A\vec{v} = \lambda\vec{v}, \quad (22)$$

then:

- ▶ \vec{v} is an eigenvector and
- ▶ scalar λ is the corresponding eigenvalue.
- ▶ These vectors don't change direction when multiplied by A .

Eigenvalues and Eigenvectors – Properties

- ▶ If $A\vec{v} = \lambda\vec{v}$, then \vec{v} is an eigenvector of A with eigenvalue λ .
- ▶ Eigenvectors are defined up to a scalar: if \vec{v} is an eigenvector, so is $c\vec{v}$ for any $c \neq 0$.
- ▶ The set of all eigenvectors corresponding to a single eigenvalue forms a vector subspace.
- ▶ A square matrix of size $n \times n$ has at most n eigenvalues (including complex ones and multiplicities).
- ▶ If all eigenvalues of a matrix are positive, the matrix is positive definite.
- ▶ Diagonal matrices (matrices where for all $a_{ij} = 0$ where $i \neq j$) have their diagonal elements as eigenvalues, and the standard basis vectors as eigenvectors.
- ▶ The eigenvalues of a complex Hermitian or real symmetric matrix are real.

Finding eigenvalues

Using a computer:

```
1 import torch
2 A = torch.tensor([[2., 0.], [0., 2.]])
3 eigenvalues, eigenvectors = torch.linalg.eigh(A)
4 print("Eigenvalues:", eigenvalues)
5 print("Eigenvectors:\n", eigenvectors)
```

Revision

In this presentation you learned about:

- ▶ Vectors.
- ▶ Matrices.
- ▶ Dot products.
- ▶ Vector-Matrix and Matrix-Matrix multiplication.
- ▶ Eigendecomposition.

What is the Determinant?

- ▶ The determinant is a scalar value associated with a square matrix A .
- ▶ It plays a key role in linear algebra:
 - ▶ A is invertible if and only if $\det(A) \neq 0$.
 - ▶ $|\det(A)|$ equals the volume scaling factor of the linear transformation defined by A .
- ▶ Determinants are widely used in solving linear systems, computing eigenvalues, and understanding geometric transformations.

Determinant of a 2×2 Matrix

- For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (23)$$

the determinant is given by

$$\det(A) = ad - bc. \quad (24)$$

- Example:

$$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = (1)(4) - (2)(3) = -2. \quad (25)$$

Exercise: 2×2 Determinant

Example 7 (Exercise)

Compute the determinant of the matrix:

$$A = \begin{pmatrix} 5 & 7 \\ 2 & 6 \end{pmatrix}. \quad (26)$$

Solution: 2×2 Determinant

Example 8 (Solution)

Using Equation (24):

$$\det(A) = (5)(6) - (7)(2) = 30 - 14 = 16. \quad (27)$$

Determinant of a 3×3 Matrix

We compute the determinant of a 3×3 matrix by expanding along the first row:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}. \quad (28)$$

The formula is:

$$\det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}. \quad (29)$$

Each 2×2 minor is the determinant of the smaller matrix obtained by deleting the corresponding row and column.

Why does this work?

The determinant of A is defined recursively by expanding into 2×2 minors:

$$\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = m_{11}m_{22} - m_{12}m_{21}. \quad (30)$$

So:

$$\det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}. \quad (31)$$

This is called the *Laplace expansion*.

Signs come from the cofactor sign pattern:

+	-	+
-	+	-
+	-	+

The determinant of an $n \times n$ matrix is computed similarly: expanding along any row or column into $(n - 1) \times (n - 1)$ minors.

Numerical Example

Compute:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}. \quad (32)$$

Expanding along the first row:

$$\det(A) = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \quad (33)$$

$$= 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) \quad (34)$$

$$= 1(-3) - 2(-6) + 3(-3) \quad (35)$$

$$= -3 + 12 - 9 = 0. \quad (36)$$

The determinant of this matrix is 0, so it is singular (non-invertible).

Exercise: Compute a 3×3 Determinant

Exercise

Compute the determinant of:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ 5 & 2 & 0 \end{pmatrix}. \quad (37)$$

Use expansion along the first row.

Solution: Exercise

Solution

$$\det(A) = 2 \begin{vmatrix} -1 & 4 \\ 2 & 0 \end{vmatrix} - 1 \begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3 \begin{vmatrix} 0 & -1 \\ 5 & 2 \end{vmatrix} \quad (38)$$

$$= 2((-1)(0) - 4(2)) - 1(0 - 4(5)) + 3((0)(2) - (-1)(5)) \quad (39)$$

$$= 2(0 - 8) - 1(0 - 20) + 3(0 + 5) \quad (40)$$

$$= -16 + 20 + 15 = 19. \quad (41)$$

Therefore, $\det(A) = 19$.

The Eigenvalue Problem — Step by Step

We want to find scalars λ and nonzero vectors \vec{v} such that:

$$A\vec{v} = \lambda\vec{v}. \quad (42)$$

Step 1: Rewrite as

$$(A - \lambda I)\vec{v} = 0. \quad (43)$$

Step 2: For a nontrivial solution ($\vec{v} \neq 0$), the matrix $A - \lambda I$ must be singular:

$$\det(A - \lambda I) = 0. \quad (44)$$

This is called the *characteristic equation*.

Solving the Eigenvalue Problem

Step 3: Expand $\det(A - \lambda I) = 0$ to get a polynomial in λ .

Step 4: The roots of this polynomial are the eigenvalues.

Step 5: For each eigenvalue λ , solve:

$$(A - \lambda I)\vec{v} = 0 \tag{45}$$

to find the corresponding eigenvector(s) by computing the nullspace.

Example: Eigenvalues and Eigenvectors

Given:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \quad (46)$$

Step 1: Compute $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{pmatrix}. \quad (47)$$

Step 2: Characteristic equation:

$$\det(A - \lambda I) = (2 - \lambda)(2 - \lambda) - 1 \quad (48)$$

$$= \lambda^2 - 4\lambda + 3 = 0. \quad (49)$$

Step 3: Roots:

$$\lambda_1 = 1, \quad \lambda_2 = 3. \quad (50)$$

Example: Eigenvectors

For $\lambda_2 = 3$:

$$A - 3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (51)$$

We solve:

$$(A - 3I)\vec{v} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \vec{v} = 0. \quad (52)$$

This gives $v_1 = v_2$, so an eigenvector is:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (53)$$

For $\lambda_1 = 1$:

$$A - I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (54)$$

This gives $v_1 = -v_2$, so an eigenvector is:

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (55)$$

Exercise: Find Eigenvalues and Eigenvectors

Exercise

Find the eigenvalues and one eigenvector for each eigenvalue of the matrix:

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}. \quad (56)$$

Solution: Exercise

Step 1: Compute characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 4 - \lambda & 2 \\ 1 & 3 - \lambda \end{vmatrix} = (4 - \lambda)(3 - \lambda) - 2 \quad (57)$$

$$= \lambda^2 - 7\lambda + 10 = 0. \quad (58)$$

Step 2: Roots:

$$\lambda_1 = 5, \quad \lambda_2 = 2. \quad (59)$$

Step 3: For $\lambda_1 = 5$:

$$A - 5I = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}. \quad (60)$$

Nullspace: $v_1 = 2v_2$, so:

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \quad (61)$$

Solution 2

For $\lambda_2 = 2$:

$$A - 2I = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}. \quad (62)$$

Nullspace: $v_1 = -v_2$, so:

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (63)$$