Problem 1

$Y = f(X) + U$
$E[U|X] = 0$
$E[U^2|X] = \sigma^2$

Show that

$$MSE(\hat{f}|X^*) := E[(\hat{f}(X^*) - Y^*)^2|X^*]$$

can be decomposed into.

$$MSE(\hat{f}|X^*) = E[(\hat{f}(X^*) - E[\hat{f}(X^*)|X^*])^2|X^*] + (E[\hat{f}(X^*)|X^*] - f(X^*))^2 + \sigma^2$$

First, we know that:

$$= E[(\hat{f}(X^*) - Y^*)^2|X^*] \implies E[((\hat{f}(X^*) - f(X^*)) + U^*)^2|X^*]$$

$$= E[(\hat{f}(X^*) - f(X^*))^2|X^*] + E[(U^*)^2|X^*] + 2E[(\hat{f}(X^*) - f(X^*))U^*|X^*]$$

$E[(\hat{f}(X^*) - f(X^*))U^*|X^*] = 0$ thus:

$$= E[(\hat{f}(X^*) - f(X^*))^2|X^*] + \sigma^2$$

when we $\pm E[\hat{f}(X^*)|X^*]$ to the first term, we get

$$= E[((\hat{f}(X^*) - E[\hat{f}(X^*)|X^*]) + (E[\hat{f}(X^*)|X^*] - f(X^*)))^2|X^*] + \sigma^2$$

$$\begin{aligned} &= E[(\hat{f}(X^*) - E[\hat{f}(X^*)|X^*])^2 + (E[\hat{f}(X^*)|X^*] - f(X^*))^2 \\ &+2(\hat{f}(X^*) - E[\hat{f}(X^*)|X^*])(E[\hat{f}(X^*)|X^*] - f(X^*))|X^*] + \sigma^2 \end{aligned}$$

$$\begin{aligned} &= E[(\hat{f}(X^*) - E[\hat{f}(X^*)|X^*])^2|X^*] + E[(E[\hat{f}(X^*)|X^*] - f(X^*))^2|X^*] \\ &+2E[(\hat{f}(X^*) - E[\hat{f}(X^*)|X^*])(E[\hat{f}(X^*)|X^*] - f(X^*))|X^*] + \sigma^2 \end{aligned}$$

$$= E[(\hat{f}(X^*) - E[\hat{f}(X^*)|X^*])^2|X^*] + (E[\hat{f}(X^*)|X^*] - f(X^*))^2$$

$$+2(E[\hat{f}(X^*)|X^*] - f(X^*))E[(\hat{f}(X^*) - E[\hat{f}(X^*)|X^*]|X^*] + \sigma^2$$

simplifying the third term:

$$2(E[\hat{f}(X^*)|X^*] - f(X^*))E[(\hat{f}(X^*) - E[\hat{f}(X^*)|X^*]|X^*]$$

$$\implies 2(E[\hat{f}(X^*)|X^*] - f(X^*))(E[\hat{f}(X^*)|X^*] - E[\hat{f}(X^*)|X^*])$$

since $E[(\hat{f}(X^*)|X^*] - E[\hat{f}(X^*)|X^*] = 0$, the term third is 0 leaving us with:

$$= E[(\hat{f}(X^*) - E[\hat{f}(X^*)|X^*])^2|X^*] + (E[\hat{f}(X^*)|X^*] - f(X^*))^2 + \sigma^2///$$

Problem 1: part 2

In [279…

```
import numpy as np #for matrixes
import math
import matplotlib.pyplot as plt #for plotting
from sklearn.linear_model import LinearRegression
#polynomial regression
from sklearn.preprocessing import PolynomialFeatures
```

In [280…

```
#a
np.random.seed(65211)

#create "true f(x)"
x_values = np.linspace(0, 2 * np.pi, 300)

# Calculate f(x) = sin(x)
f_x = np.sin(x_values)


X_vec = np.random.uniform(0, 2*np.pi, size=30)
Y_vec = np.random.normal(np.sin(X), 0.2, size=30)

print(X_vec, Y_vec)
```

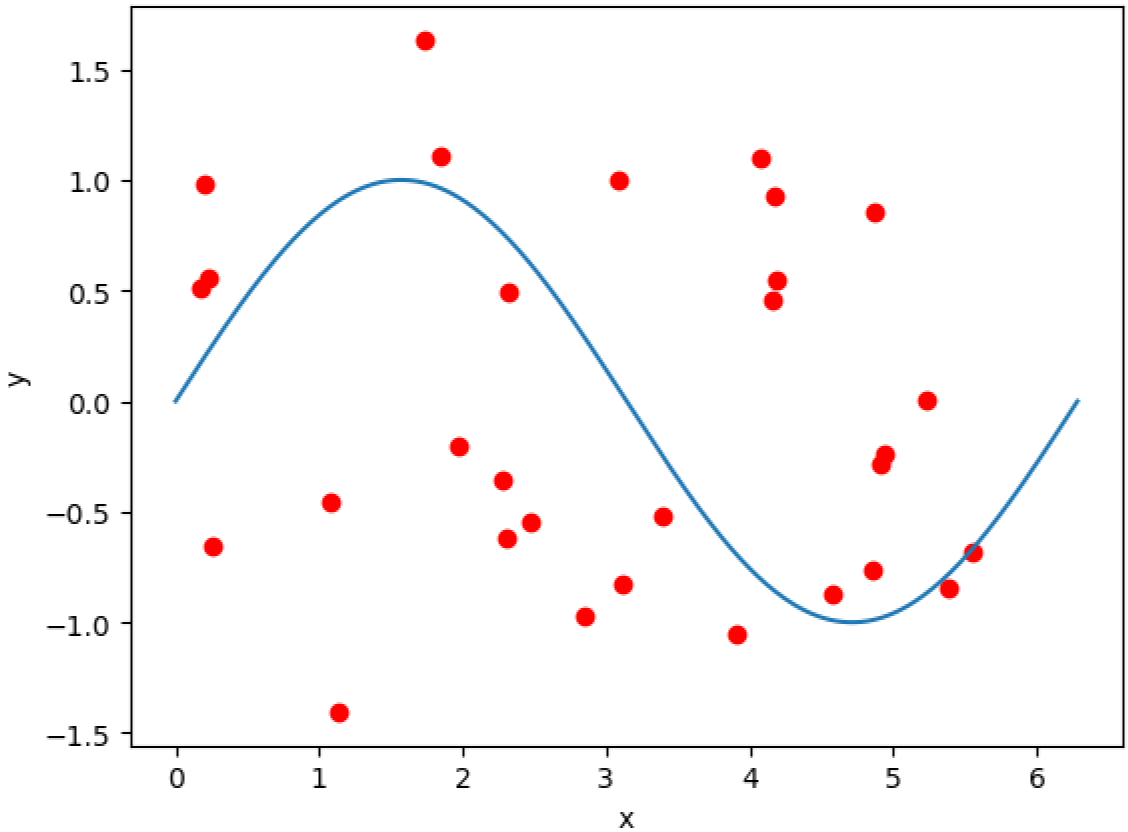
```
[4.93247475 4.86945915 2.47262067 0.26028025 3.10835807 5.23732458
 2.31204273 5.55431808 1.07381274 5.3788515  0.17191582 1.97494181
 4.08068639 2.27687508 2.32049142 0.22215632 1.14070405 2.8476873
 4.84945827 4.18113    3.90237511 1.72724692 3.08457644 4.17779842
 3.39683906 4.15490229 4.57111    4.9136611  0.20435054 1.84369289] [-0.240206
24  0.8557268  -0.5426165  -0.65081052 -0.8309233   0.00620705
 -0.61983897 -0.67851468 -0.45509092 -0.8481064   0.51298225 -0.19995107
  1.10183902 -0.35493945  0.49777488  0.56092525 -1.40462016 -0.97181998
 -0.76651806  0.54813158 -1.0561495   1.63330198  1.0022593   0.92769483
 -0.52096956  0.45305659 -0.87359675 -0.28425001  0.98015969  1.10476454]
```

In [281…

```
# PART B
fig = plt.figure()
plt.scatter(X_vec, Y_vec, c = 'red')
plt.plot(x_values, f_x) #sin(x), blue line
plt.xlabel('x')
plt.ylabel('y')
```

Out[281]:

```
Text(0, 0.5, 'y')
```

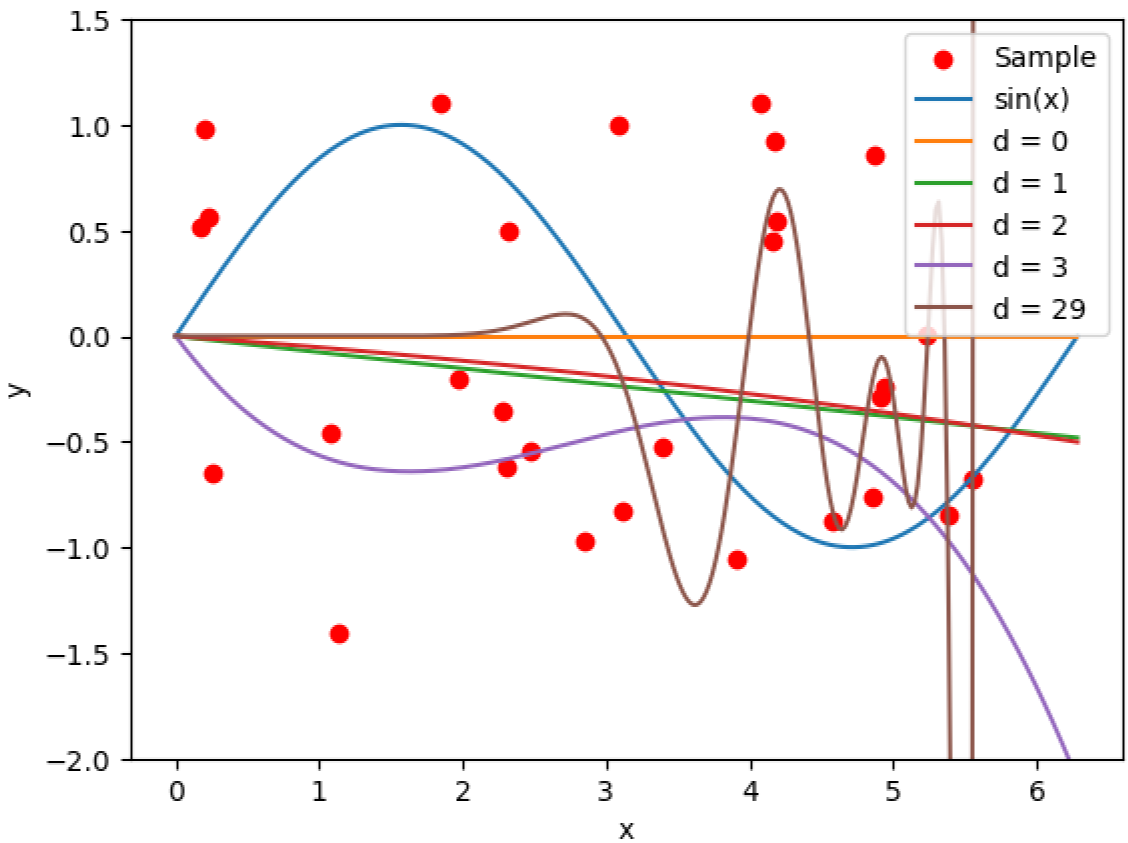


PART Ci

In [282…

```
# PART Ci
#function to make a + bx^2 + cX^3
def poly_fuc(x, coefficients):
    # Assuming coefficients is an array containing the coefficients for each te
    # The first element is the intercept, the rest are coefficients for x^1, x^
    degree = len(coefficients)

    # Create the polynomial features for the input x
    X_poly = np.array([x**i for i in range(degree)])

    # Calculate the predicted y using the coefficients
    y_pred = np.dot(coefficients, X_poly)

    return y_pred


def poly_reg(x_values, d):
    # Create polynomial features (transforms x to x^0, x^1 ,.., x^d)
    poly = PolynomialFeatures(degree = d)

    # .reshape(-1, 1)transforms to a 2D array
    # -1 means any number of rows, 1 means column, so this is a column vector
    # fit_transform does the transformtion on the column vector
    X_poly = poly.fit_transform(X_vec.reshape(-1, 1))

    # Fit linear regression model
    model = LinearRegression().fit(X_poly, Y_vec)

    # save the coefficients
    coefficients = model.coef_
```

```
    #make the estimated f D
    f_d = poly_fuc(x_values, coefficients)
    return f_d

fig = plt.figure()
plt.scatter(X_vec, Y_vec, c = 'red', label='Sample')
plt.plot(x_values, f_x, label='sin(x)')
plt.plot(x_values, poly_reg(x_values, 0), label='d = 0')
plt.plot(x_values, poly_reg(x_values, 1), label='d = 1')
plt.plot(x_values, poly_reg(x_values, 2), label='d = 2')
plt.plot(x_values, poly_reg(x_values, 3), label='d = 3')
plt.plot(x_values, poly_reg(x_values, 29),label='d = 29')
plt.xlabel('x')
plt.ylabel('y')
plt.ylim(-2, 1.5)
plt.legend()  # Add legends based on the provided labels
plt.show()


#fig = plt.figure()
```

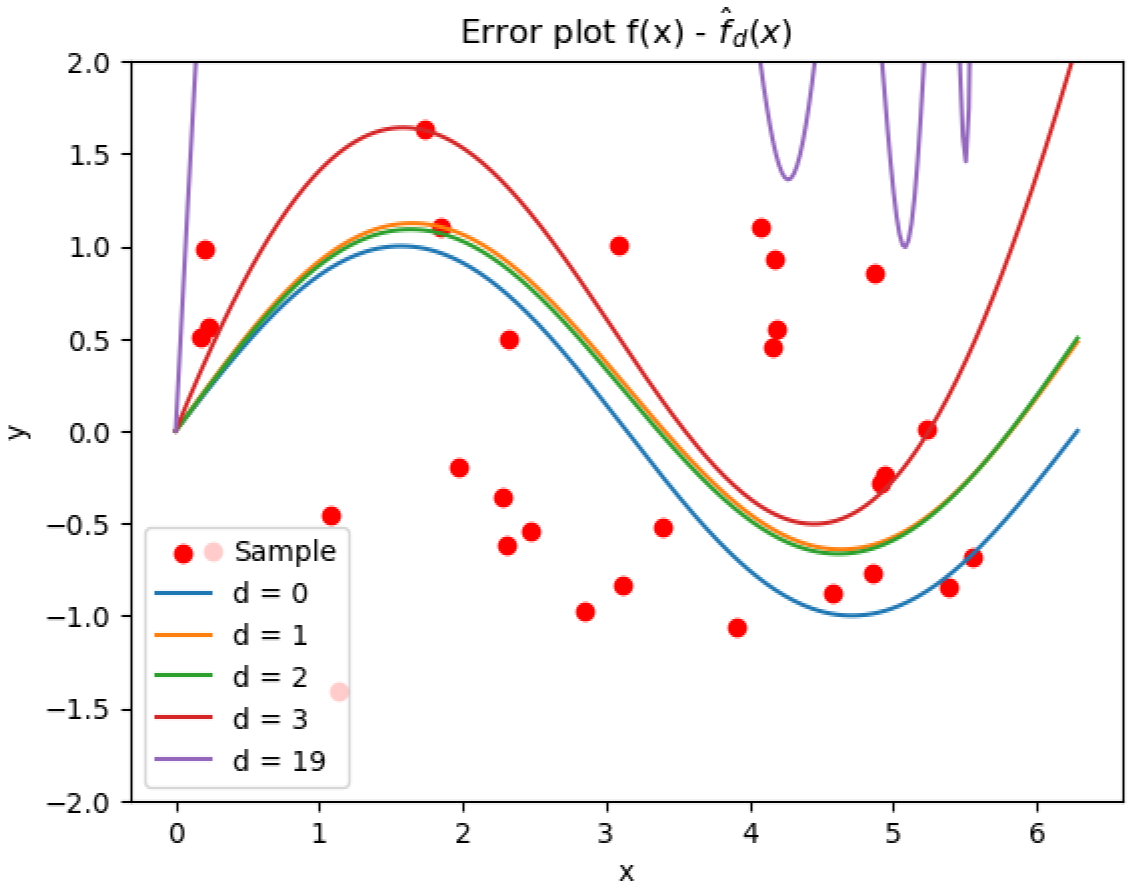


PART C II

In [284…

```
fig = plt.figure()
plt.scatter(X_vec, Y_vec, c = 'red', label = 'Sample')
#plt.plot(x_values, f_x, label='sin(x)')
plt.plot(x_values, f_x - poly_reg(x_values, 0), label='d = 0')
plt.plot(x_values, f_x - poly_reg(x_values, 1), label='d = 1')
plt.plot(x_values, f_x - poly_reg(x_values, 2), label='d = 2')
plt.plot(x_values, f_x - poly_reg(x_values, 3), label='d = 3')
plt.plot(x_values, f_x - poly_reg(x_values, 19),label='d = 19')
plt.xlabel('x')
```

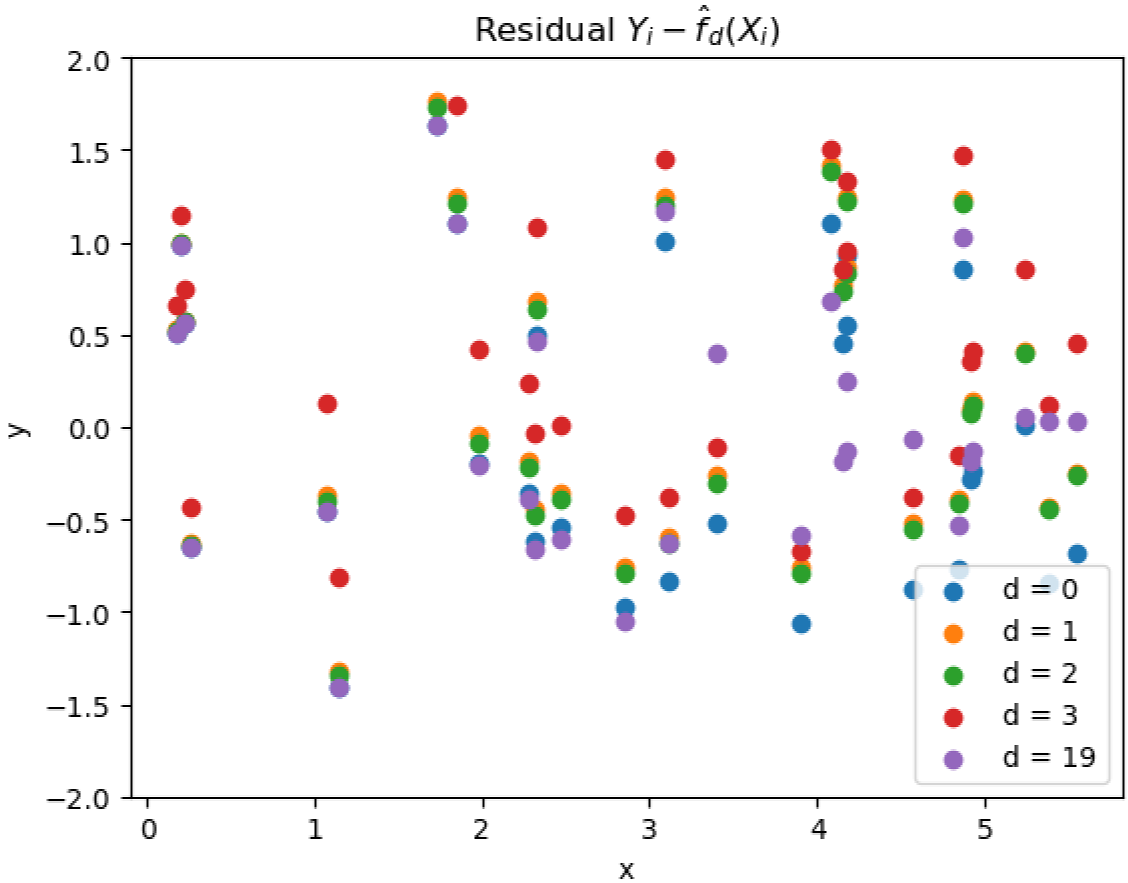
```
plt.ylabel('y')
plt.ylim(-2, 2)
plt.legend()  # Add legends based on the provided labels
plt.title('Error plot f(x) - $\hat f_d(x)$')
plt.show()
```



PART C III

In [285…

```
fig = plt.figure()

plt.scatter(X_vec, Y_vec - poly_reg(X_vec, 0), label='d = 0')

plt.scatter(X_vec, Y_vec - poly_reg(X_vec, 1), label='d = 1')

plt.scatter(X_vec, Y_vec - poly_reg(X_vec, 2), label='d = 2')

plt.scatter(X_vec, Y_vec - poly_reg(X_vec, 3), label='d = 3')

plt.scatter(X_vec, Y_vec - poly_reg(X_vec, 29),label='d = 19')

plt.xlabel('x')
plt.ylabel('y')
plt.ylim(-2, 2)
plt.legend()  # Add legends based on the provided labels
plt.title('Residual $Y_i - \hat f_d(X_i)$')
plt.show()
```

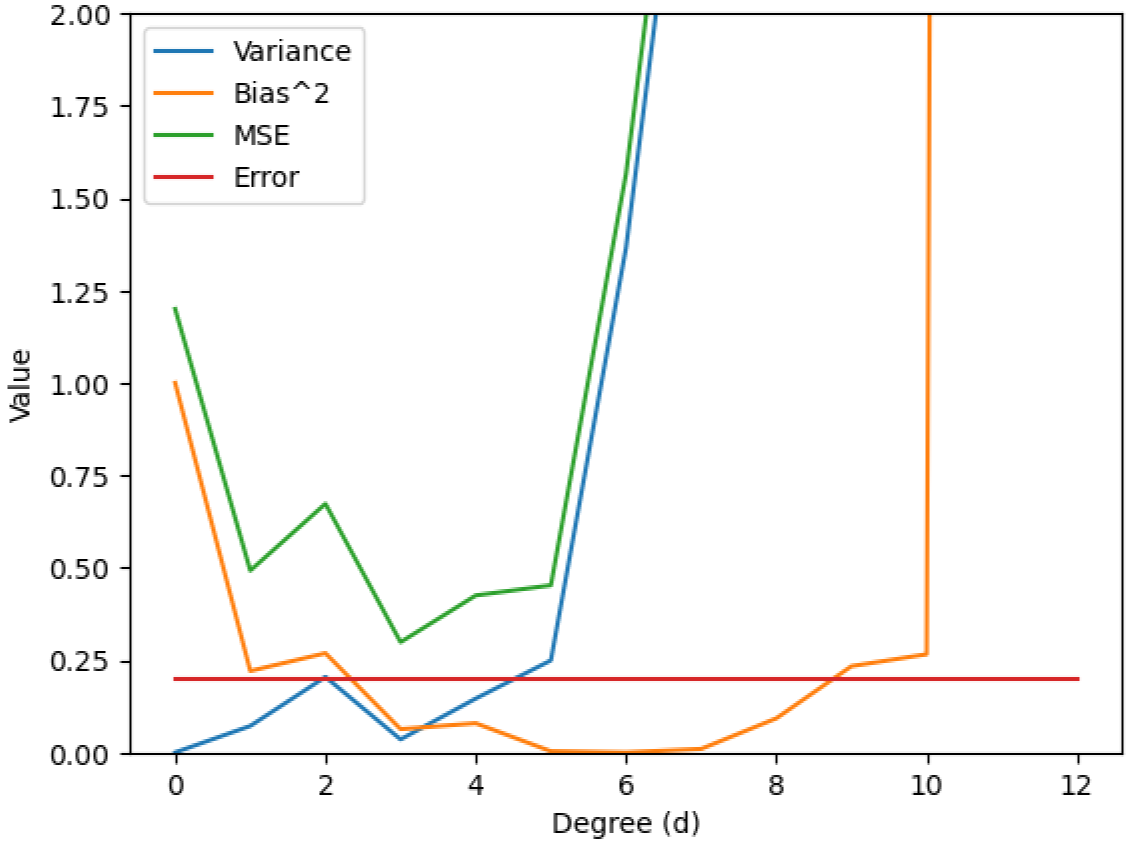


Part D

In [286…

```
def poly_fuc(x, coefficients):
    # Assuming coefficients is an array containing the coefficients for each te
    # The first element is the intercept, the rest are coefficients for x^1, x^
    degree = len(coefficients)

    # Create the polynomial features for the input x
    X_poly = np.array([x**i for i in range(degree)])

    # Calculate the predicted y using the coefficients
    y_pred = np.dot(coefficients, X_poly)

    return y_pred


def poly_reg(x_values, d):
    # Create polynomial features (transforms x to x^0, x^1 ,.., x^d)
    poly = PolynomialFeatures(degree = d)

    # .reshape(-1, 1)transforms to a 2D array
    # -1 means any number of rows, 1 means column, so this is a column vector
    # fit_transform does the transformtion on the column vector
    X_poly = poly.fit_transform(X.reshape(-1, 1))

    # Fit linear regression model
    model = LinearRegression().fit(X_poly, Y)

    # save the coefficients
    coefficients = model.coef_
```

```python
    #make the estimated f D
    f_d = poly_fuc(x_values, coefficients)
    return f_d


degree = range(0, 13)
n = 1000
x_test = 1.5*np.pi
f_test = np.sin(x_test)

d_var = []
d_bias = []
d_mse = []
error = 0.2
for d in degree:
    f_hat_test = []
    for i in range(n): #1000 times
        #get a random 30 points
        X = np.random.uniform(0, 2*np.pi, size=30)
        Y = np.random.normal(np.sin(X), 0.2, size=30)
        poly = PolynomialFeatures(degree = d)
        X_poly = poly.fit_transform(X.reshape(-1, 1))

        # Fit linear regression model
        model = LinearRegression().fit(X_poly, Y)

        # save coefficients
        coefficients = model.coef_

        #y hat
        y_test_pred = poly_fuc(x_test, coefficients)
        f_hat_test.append(y_test_pred)
    var = np.mean((f_hat_test - np.mean(f_hat_test))**2)
    bias = (np.mean(f_hat_test) - f_test)**2
    mse = var + bias + error
    d_var.append(var)
    d_bias.append(bias)
    d_mse.append(mse)

fig, ax = plt.subplots()
ax.plot(degree, d_var, label="Variance")
ax.plot(degree, d_bias, label="Bias^2")
ax.plot(degree, d_mse, label="MSE")
ax.plot(degree, [error] * len(degree), label="Error")
ax.set_xlabel('Degree (d)')
ax.set_ylabel('Value')
ax.legend()
ax.set_ylim(0, 2)
plt.show()
```



PART E

From about d < 4, we see that the model is underfitted due to high bias and low variance. Then from about 4 to 8 we see a switch with high variance and low bias, indicating overfitting. From about 8, both the bias and variance increase??

Problem 1 part 3

Breiman would think there are better ways to model this and would not do this exercise

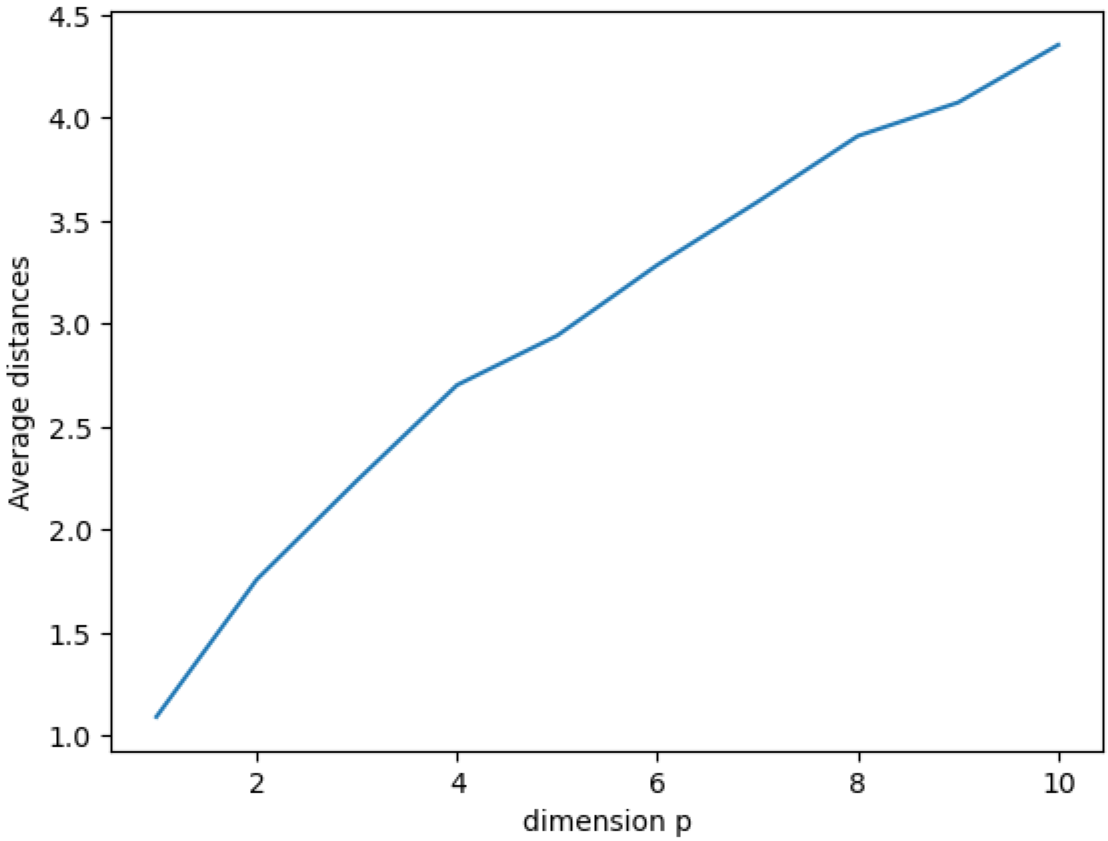
Problem 2

In [244…

```
def distance(vec1, vec2):
    store_data = []
    """The Euclidean distance between two arrays"""
    store_data.append((vec1 - vec2)**2)
    return math.sqrt(np.sum(store_data))

n = 1000
p = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
avg_dis = []
for i in p:
    store = []
    for k in range(n):
    #i.i.d and standard normal
        X1 = np.random.randn(i)
        X2 = np.random.randn(i)
        store.append(distance(X1, X2))
```

```
    mean_p = np.mean(store)
    avg_dis.append(mean_p)


fig = plt.figure()
plt.plot(p, avg_dis)
plt.xlabel('dimension p')
plt.ylabel('Average distances')
```

Out[244]: Text(0, 0.5, 'Average distances')



Problem 3

$$g^*(X) = \begin{cases} 1 & \text{if } P[Y=1|X=x] \geq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

want to prove that this minimizes the bayes error which is:

$$P[g(X) \neq Y|X=x] \tag{2}$$

It's enough to show that $P[g^*(X) \neq Y|X=x] \leq P[g(X) \neq Y|X=x]$

First:

$P[g(X) \neq Y|X=x] = 1 - P[g(X) = Y|X=x]$

since Y is binary and $Y = g(X)$:

$= 1 - (P[g(X) = 1, Y = 1|X=x] + P[g(X) = 0, Y = 0|X=x])$

$$= 1 - (1(g(X) = 1)P[Y = 1|X = x] + 1(g(X) = 0)P[Y = 0|X = x])$$

Let $P[Y = 1|X = x] = \pi$

$$= 1 - (1(g(X) = 1)\pi + 1(g(X) = 0)(1 - \pi))$$

For every x

$$P[g(X) \neq Y|X = x] - P[g^*(X) \neq Y|X = x]$$

$$1 - (1(g(X) = 1)\pi + 1(gX = 0)(1 - \pi)) - (1 - (1(g^*(X) = 1)\pi + 1(g^*(X) = 0)(1 - \pi)$$

For ease of notation, let
$1(gX = 0) = 1_0, 1(gX = 1) = 1_1, 1(g^*X = 0) = 1_0^*, 1(g^*X = 1) = 1_1^*$

$$= -(1_1\pi + 1_0(1 - \pi)) + (1_1^*\pi + 1_0^*(1 - \pi))$$

$$= \pi(1_1^* - 1_1) + (1 - \pi)(1_0^* - 1_0)$$

$$= \pi(1_1^* - 1_1) + (1 - \pi)((1 - 1_1^*) - (1 - 1_1))$$

$$= \pi(1_1^* - 1_1) + (1 - \pi)(1_1 - 1_1^*))$$

$$= \pi(1_1^* - 1_1) - (1 - \pi)(1_1^* - 1_1))$$

$$= (1_1^* - 1_1)(\pi - (1 - \pi))$$

$$= (1_1^* - 1_1)(2\pi - 1)$$

Case 1, when $g^*(X) = 1$ $(\text{and } \pi \geq \frac{1}{2})$:

$$= (\underbrace{1 - \underbrace{1_1}_{1 \text{ or } 0}}_{non-negative})(\underbrace{2\pi - 1}_{non-negative}) \implies \geq 0$$

Case 2, when $g^*(X) = 0$ $(\text{and } \pi = 0)$

$$= (\underbrace{0 - \underbrace{1_1}_{1 \text{ or } 0}}_{non-positive})(\underbrace{2\pi - 1}_{negative}) \implies \geq 0$$

$$\implies P[g(X) \neq Y|X = x] \geq P[g^*(X) \neq Y|X = x]$$ ///

In [ ]: