Trees

August 18, 2024

Connected-graphs

- Two vertices u and v are said to be connected (denoted by $u \sim v$ if there is a path between them.
- $\sim$ is an equivalence relation and $\sim$ partitions the vertices of a given graph into equivalence classes.
- The subgraph induced by each equivalence class is called a component of G.
- $\omega(G)$ denotes the number of componenets of G
- A graph G is said to be connected if every pair of distinct vertices are connected (i.e.) $\omega(G) = 1$.
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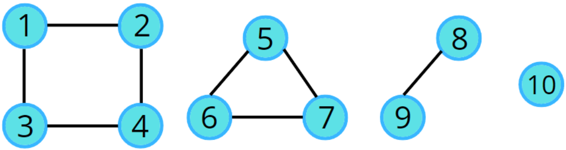
- Example : Disconnected Graph with 4 components.



- Equivalence Classes : $\{1, 2, 3, 4\}$, $\{5, 6, 7\}$, $\{8, 9\}$ and $\{10\}$.

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In a tree any two vertices are connected by a unique path.

Proof.

By contradiction. □

- Qn: Does the converse hold?

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Theorem

In a tree, $|E(G)| = |V(G)| - 1$.

Proof.

By induction on $|V(G)| = 1$.

1. $n = 1$: G is an isolated vertex which is a tree and $|E(G)| = 1$
2. Assume true for all trees with $n \leq k$ vertices.

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Proof Contd.

1. Let G be a tree with $n = k + 1$ vertices.
2. Choose an edge uv with end vertices u and v in G.
3. The graph $G \setminus uv$ has no $u - v$- path. (Why?)
4. uv is the unique $u - v$ path in G.
5. $G \setminus \{uv\}$ is disconnected with 2 components G_1 and G_2 with i and j vertices respectively where $i + j = k + 1$
6. By induction G_1 has $i - 1$ edges and G_2 has $j - 1$ edges.
7. Then G has $i - 1 + j - 1 + 1 = i + j - 1 = k$ edges.

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Corollary

Every non-trivial tree has two vertices of degree 1 (also called pendant vertices.)

Proof.

Hint: Use handshaking Lemma. (Degree sum formula). □

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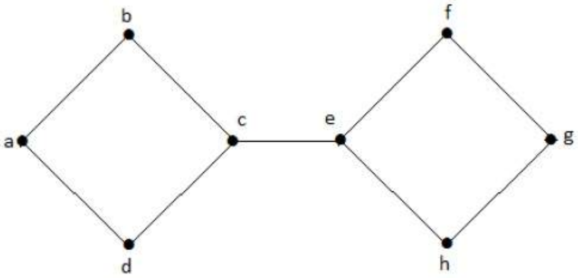
- Number of components of a graph G is denoted by $\omega(G)$.

Definition

A Cut-edge of G is an edge e such that $\omega(G - e) > \omega(G)$.

- Example :



Remarks

1. If $e = xy$ is a cut-edge of G, then x and y are in two different components of $G - e$.
2. Moreover, $\omega(G - e) = \omega(G) + 1$.

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Theorem

A edge e is a cut-edge of G iff e is not contained in any cycle of G.

Proof.

" $\Longrightarrow$ " Assume $e = xy$ is a cut-edge. To show e is not in any cycle of G.

1. Given e is a cut-edge.
2. By Definition $\omega(G \setminus e) > \omega(G)$.
3. There exists $P = u - v$ path in G but not in $G \setminus e$.
4. Let $P_1 = u - x$ path and $P_2 = y - v$ path in G as well as $G \setminus e$.
5. If C is a cycle containing e, then $C \setminus e$ is a path connecting xy.
6. Then $P_1 + P_2 + C \setminus e$ contains a $u - v$ path in $G \setminus e$, A contradiction.

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Corollary

A connected graph G is a tree iff every edge is a cut-edge.

Proof.

"$\Rightarrow$" Given G is a tree.

1. Let $e \in G$.
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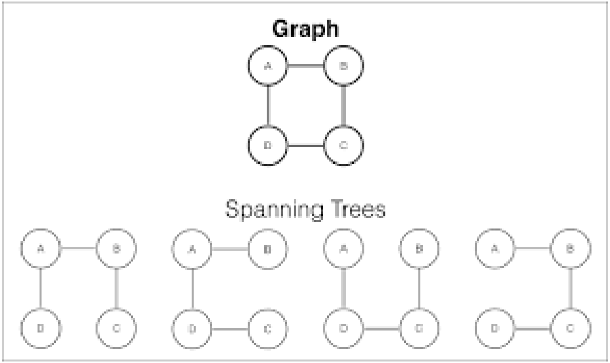
Spanning Tree

Definition

A spanning subgraph of a graph that is also a tree is a spanning tree of a given graph.



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Every connected graph contains a spanning tree.

Proof.

1. Let T be minimal connected spanning subgraph of G.
2. $\omega(T) = 1$ and $\omega(T \setminus e) > 1$ for all $e \in T$.
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Theorem

Let T be a Spanning tree of a given connected graph G. Let $e \notin T$. Then $T + e$ contains a unique cycle.

Proof.

1. T has no cycles.
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