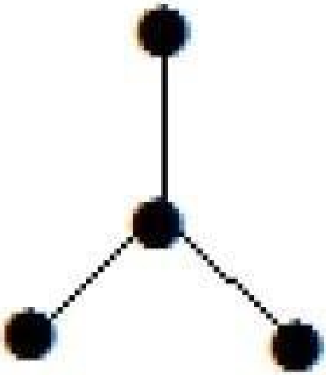
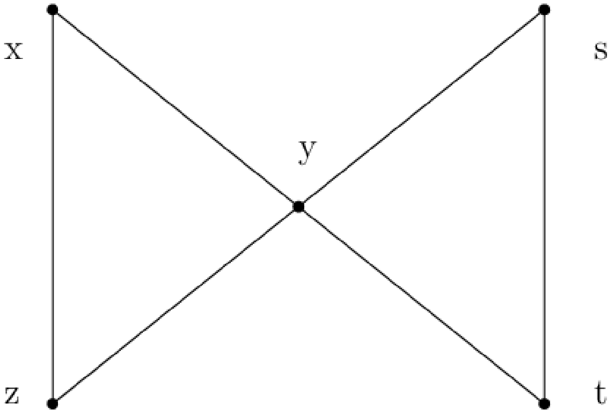
Connectivity

September 2, 2024

Examples



- Deletion of any edge disconnects the graph.



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- Deletion of vertex disconnects the graph.

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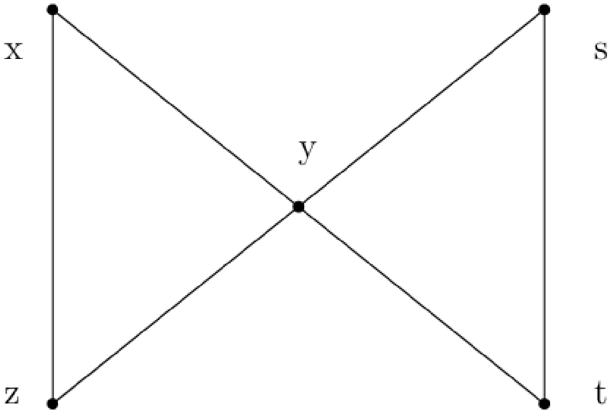

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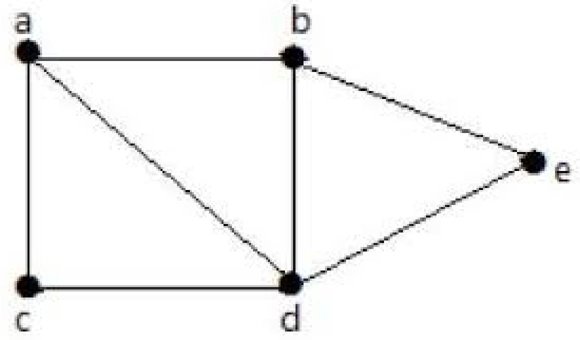

- No cut edges and no cut vertices.

- Each successive graph is more strongly connected than the previous one.
- We define two concepts that measures the extent to which a given graph is connected.
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- (Vertex)-Connectivity and Edge-Connectivity.

Definition

A vertex subset $W \subseteq V(G)$ is called a vertex-cut if $G \setminus W$ is disconnected or $G \setminus W$ is trivial.

1. $\kappa(G) = min\{|W| \ : \ W \subseteq V(G)$ and W is a vertex-cut$\}$
2. $\kappa(G)$ is called the vertex-connectivity of G.
3. That is, $\kappa(G)$ is the minimum number of vertices whose deletion disconnects G or results in a graph with a single vertex.
4. Any vertex-cut $K \subseteq V(G)$ such that $|K| = \kappa(G)$ is called a minimum vertex-cut of G.
5. A graph G is said to be k-vertex-connected, if $\kappa(G) \geq k$.
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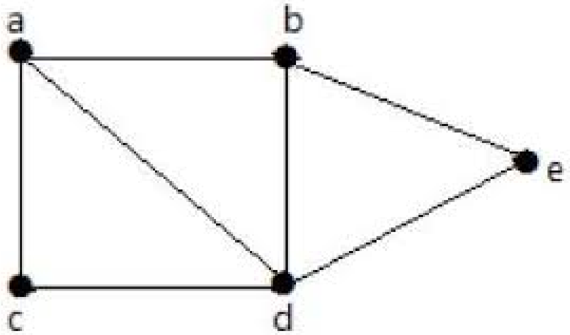
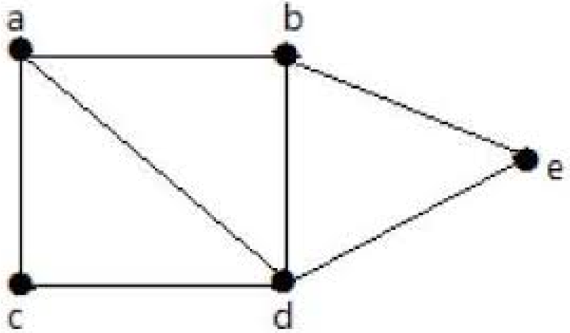


- Vertex cut : $\{a, d\}$ or $\{b, d\}$ or $\{a, b, d\}$



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An edge cut is a subset $F \subseteq E(G)$ of the form $[S, S']$ where V is partitioned into $S \neq \emptyset$ and $S' \neq \emptyset$ such that each edge in F has one end in S and other in S'.

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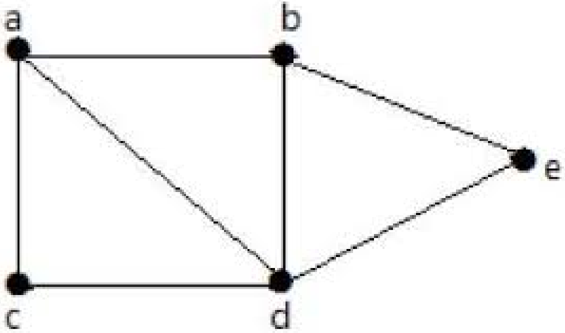

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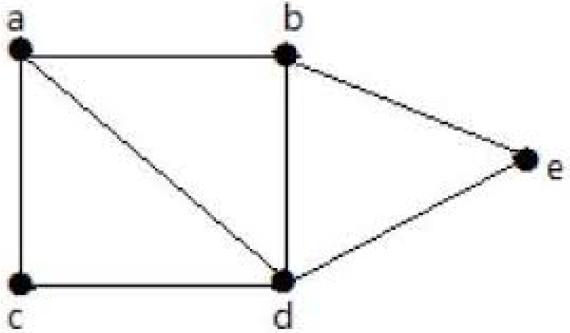

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Properties-Edge cut

1. $\kappa'(G) = 0$ if and only if G is disconnected or G is trivial.
2. $\kappa'(G) = 1$ if and only if G is connected and G has a cut-edge.
3. $\kappa'(G) \leq \delta(G)$; this follows since by deleting all the edges incident with a vertex of minimum degree, we disconnect the graph.
4. A graph G may have many minimum edge-cuts but $\kappa'(G)$ is unique.
5. If G is k-edge-connected, then it is t-edge-connected, for every t, $1 \leq t \leq k$.
6. If F is a minimum edge-cut, then $G \setminus F$ contains exactly two components A and B such that $[V(A), V(B)] = F$.

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Theorem

For any graph G with at least two vertices,

$$\kappa(G) \leq \kappa'(G) \leq \delta(G)$$

Proof.

1. We have observed : $\kappa'(G) \leq \delta(G)$.
2. Only need to prove $\kappa(G) \leq \kappa'(G)$.
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Proof Contd

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Now we assume that G is connected.

1. If $|V(G)| = 2$, then $G = K_2$.
2. Then,
$$\kappa(K_2) = 1$$
(makes th graph trivial).
3. $\kappa'(G) = 1$
(Makes the graphs disconnected with two vertices).
4. $\delta(G) = 1$.
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Let G be connected and $|V(G)| \geq 3$. We now discuss two possible cases:

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Case 1 :G is simple, connected and $|V(G)| \geq 3$. Need to show $\kappa(G) \leq \kappa'(G)$.

1. Let $k = \kappa'(G)$.
2. Let $F = \{e_1, e_2, \cdots, e_k\}$ be an edge cut of G where $e_i = u_i v_i$, $u_i, v_j \in V$. (Note : u_i's and v_j's need not distinct.)
3. Need to construct a vertex cut from F.
4. Let $H = G \setminus \{\{u_i : 1 \leq i \leq k-1\} \setminus \{u_k, v_k\}\}$
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1. If H is already disconnected, then $\kappa(G) \leq k - 1 < \kappa'(G)$.
2. If H is connected, then e_k is a cut-edge of H and hence, removal of one of u_k or v_K from H makes the graph disconnected or trivial (H can be K_2).
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Case 2 : G is any graph.

1. Let H be an underlying simple spanning subgraph of G.
2. Then, $\kappa(H) = \kappa(G)$ and $\kappa'(H) \leq \kappa'(G)$.
3. From Case 1 we know $\kappa(H) \leq \kappa'(H)$.
4. $\implies \kappa(G) = \kappa(H) \leq \kappa'(H) \leq \kappa'(G)$. □

Hence, in all cases $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

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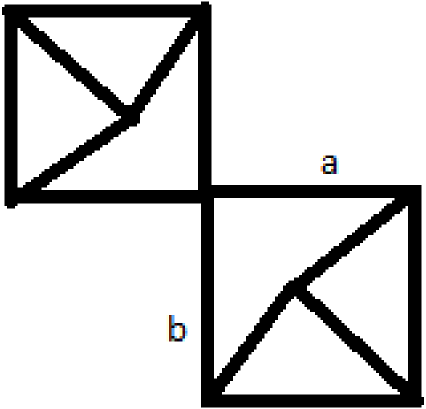
Example

An Example where the above inequality is strict.

$$\kappa(G) = 1 < \kappa'(G) = 2 < \delta(G) = 3$$



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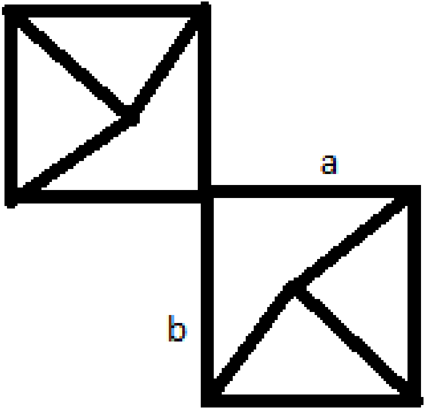

Definition

A maximal connected subgraph B of a graph G that has a no cut- vertex of its own is called a block of G.

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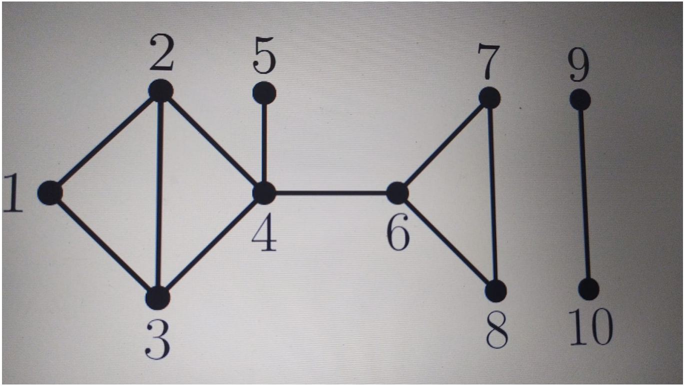

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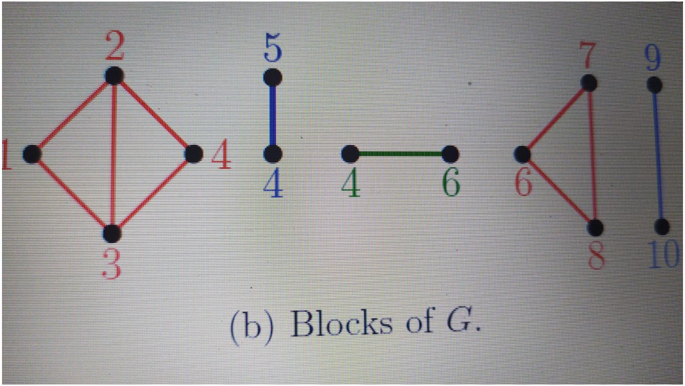

Observations

1. A block of a graph does not have a cut-vertex of its own. However, it may contain cut-vertices of the whole graph.
2. By definition, G itself is a block if it is connected and it has no cut-vertex.
3. Two blocks in a graph share at most one vertex; else, the two blocks together form a block, thus contradicting the maximality of blocks.
4. If two distinct blocks of G share a vertex v, then v is a cut-vertex of G.
5. Any two distinct blocks are edge disjoint; so the edge sets of blocks partition the edge set of G.
6. To establish a property P of a graph G, often it is enough to establish P for each of its blocks, and thereby simplify the proofs.

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Definition

Two paths $P : x - y$ and $Q : x - y$ are said to be internally disjoint if they have no internal vertex common.

Theorem

A graph G is k-vertex connected $(1 \leq k \leq n-1)$ if and only if given any two distinct vertices u and v, there exist k internally disjoint $u - v$ paths (that is, no two of the paths have an internal vertex common).

The following result is a special case of a theorem proved by Menger (1932).

Theorem

Let G be a graph with $|V(G)| \geq 3$. Then G is a block if and only if given any two vertices x and y of G, there exist at least two internally disjoint $x-y$-paths in G.

Proof

" $\Rightarrow$ " Given any two vertices x, y of G, there exist at least two internally disjoint $x - y$ paths. To show that G is a block.

1. G is connected.
2. So, we have to only show that G has no cut-vertices.
3. On the contrary, suppose v is a cut-vertex of G.
4. Then by a previous Theorem there exist vertices x, y such that every $x - y$ path passes through v.
5. This implies that there do not exist two internally disjoint $x - y$ paths, which is a contradiction to the hypothesis.
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Proof-Converse

" $\Leftarrow$ " Given G is a block. To show that given any two vertices x, y of G, there exist at least two internally disjoint $x - y$ paths.

Proof by induction on $d(x, y)$.

Base Case :

1. $d(x, y) = 1 \Rightarrow$ x and y are adjacent.
2. Let $e = xy$ the edge between x and y.
3. $G - e$ is connected. If not e is a cut edge and one of x or y is a cut-vertex of G.
4. A contradiction as G is a block.
5. Hence, there exists $P = x - y$ path in $G - e$.
6. $P = x - y$ in $G - e$ and $e = xy$ are two internally disjoint $x - y$ paths in G.

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Induction Hypothesis : Assume true for all vertices with $d(x, y) \leq k$.

Induction Step : Let $d(x, y) = k + 1$.

1. Let $P : x - y$ be a shortest $x - y$ path.
2. so length of $P = d(x, y)$.
3. Let w be a vertex that precedes y in P.
4. So, $d(x, w) = d(x, y) - 1$.
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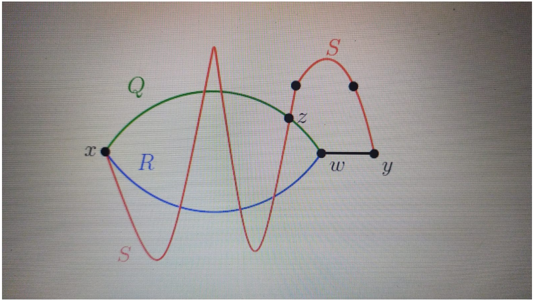
4. For definiteness, let $z \in V(Q)$. Then $Q : x - z$, $S : z - y$ and $R : x - w{+}wy$ are two internally disjoint $x - y$ paths. $\square$



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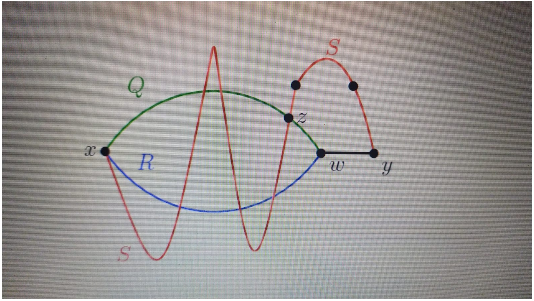

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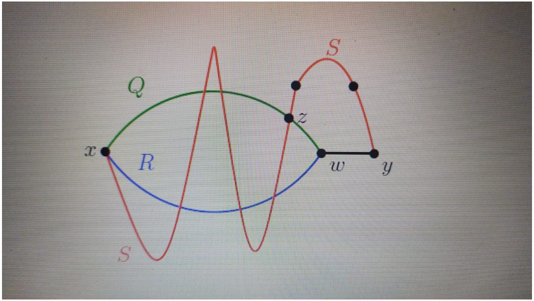

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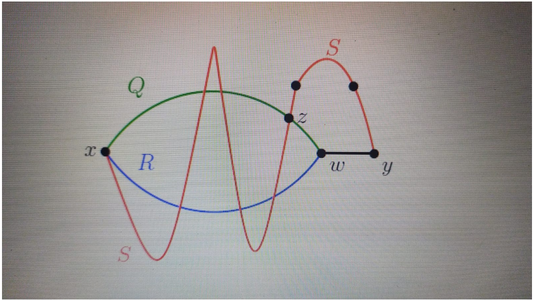
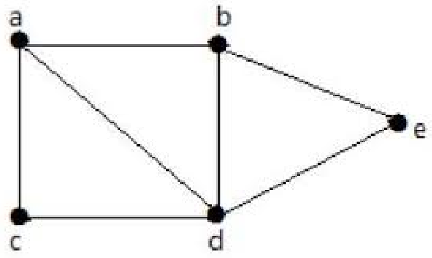


Theorem

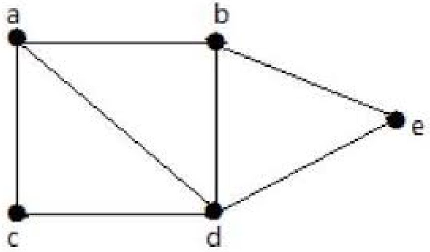
A graph G is k-edge connected if and only if given any two distinct vertices u and v, there exists k edge-disjoint $u - v$ paths (that is, no two paths have an edge common).

There are many sufficient conditions for a graph to be k-vertex-connected or k-edge-connected. (In advance course on GT).



1. $\kappa(G) = 2$ and $\kappa'(G) = 2$
2. The given graph is 2-connected and 2-edge connected.
3. Between any two vertices there exists two internally disjoint paths and two edge disjoint paths.

Example



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