Coloring

October 17, 2024

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2. It also arises in many circuit board problems where the wires connecting a device have to be of different color.

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Vertex Coloring

Definition

1. A *k-vertex colouring* of G is an assignment of k colours, $1, 2, \cdots, k$, to the vertices of G.
2. The colouring is proper if no two distinct adjacent vertices have the same colour.
3. A proper k-vertex colouring of a loopless graph G is a partition $(V_1, V_2, \cdots, V_t)$ of V into k (possibly empty) independent sets.
4. G is k-vertex-colourable if G has a proper k-vertex colouring.

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1. A simple graph is 1-colourable if and only if it is empty (No vertices or totally disconnected).
2. A simple graph 2-colourable if and only if it is bipartite.
3. The chromatic number, $\chi(G)$, of G is the minimum k for which G is k-colourable.
4. If $\chi(G) = k$, G is said to be *k-chromatic.*

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Example

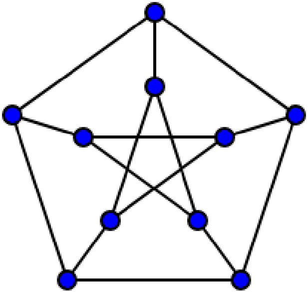

Figure: Petersen

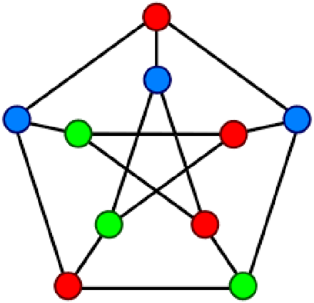

Figure: Proper coloring

We study the properties of a special class of graphs called critical graphs. (Introduced by Dirac in 1952).

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2. Which n? If $n > 1$, then on the removal of any vertex, we still have an empty graph with chromatic number 1, and so the graph is not 1-critical.
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Exercise : Classify all 3-critical graphs.

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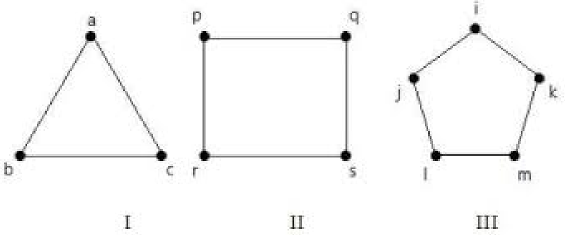

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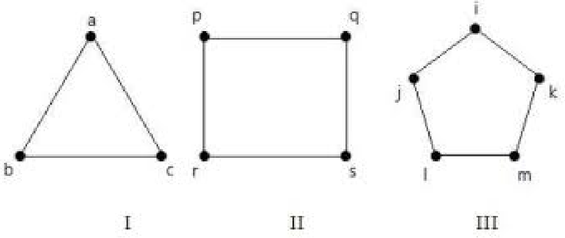

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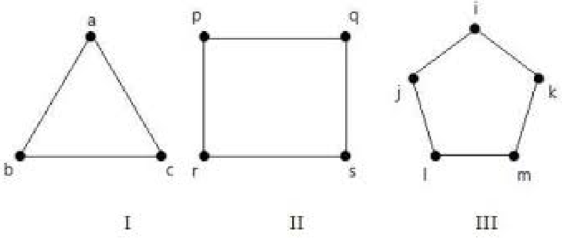

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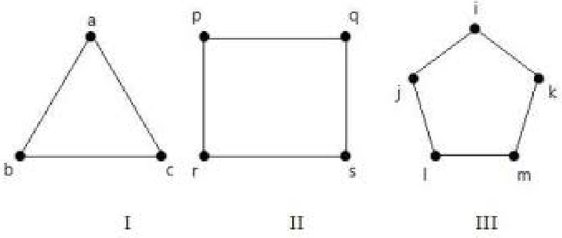

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Every k-critical graph is connected.

Proof.

Easy by definition. □

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Lemma

If G is k-critical, then $\delta \geq k - 1$.

Proof.

1. By contradiction.
2. Suppose $\delta(G) \leq k - 2$.
3. Let v be a vertex of degree δ in G.
4. Since G is k-critical, $G - v$ is $(k-1)$-colourable.
5. Let $(V_1, V_2, \cdots, V_{k-1})$ be a $(k-1)$- colouring of $G - v$.
6. By definition, v is adjacent in G to $\delta < k - 1$ vertices.

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Corollary

Every k-chromatic graph has at least k vertices of degree at least $k-1$.

Proof.

1. Let G be a k-chromatic graph.
2. Let H be a k-critical subgraph of G.
3. By previous Lemma, each vertex of H has degree at least $k-1$ in H, and hence also in G.
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Corollary

For any graph G,

$$\chi \leq \Delta + 1$$

S-Components

Definition

1. Let S be a vertex cut of a connected graph G.
2. Let the components of $G - S$ have vertex sets $V_1, V_2, \cdots, V_n$.
3. Then the subgraphs $G_i = G[V_i \cup S]$ are called the S-components of G.
4. We say that colourings of $G_1, G_2, \cdots, G_n$ agree on S if, .for every $v \in S$, vertex v is assigned the same colour in each of the colourings.

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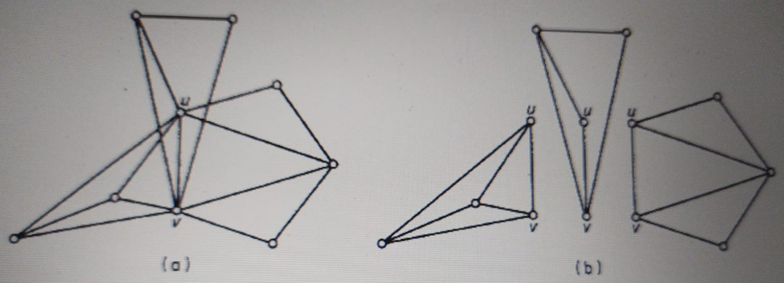

Figure 8.3. (a) G; (b) the $\{u, v\}$-components of G

Theorem

In a critical graph, no vertex cut is a clique.

Proof.

1. By contradiction.
2. Let G be a k-critical graph.
3. Suppose that G has a vertex cut S that is a clique.
4. Denote the S-components of G by $G_1, G_2, \cdots, G_n$.
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Proof Contd.

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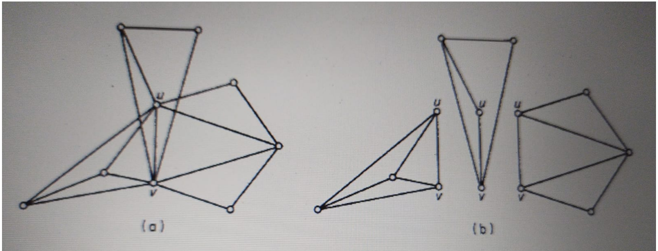

Figure 8.3. (a) G; (b) the $\{u, v\}$-components of G

Is G critical? Why?

Corollary

Every critical graph G is a block.

Proof.

1. If v is a eut vertex, then $\{v\}$ is a vertex cut.
2. $\{v\}$ (trivially,) is à clique.
3. Given that G is k-critical for some k.
4. By previous theorem, G does not have a Clique.
5. A single cut-vertex is a clique.
6. So, G does not have a cut-vertex.
7. $\Rightarrow$ G is a block.

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4. By previous theorem, G does not have a Clique.
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Brook's Theorem

Theorem

If G is a connected simple graph and is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta$.

1. Proof...in Advanced course.
2. Use critical graphs/existence of a critical subgraph.
3. Not a strong result.
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2. The colouring $\mathcal{C}$ is proper if no two adjacent edges have the same colour.
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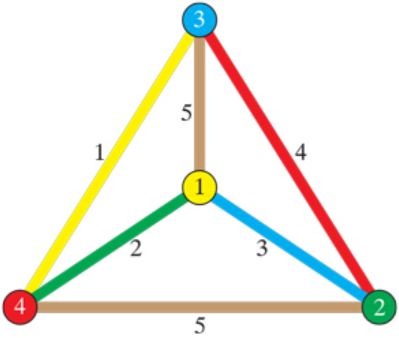
Example



1. Yellow = $E_1 = \{34\}$
2. Brown = $E_2 = \{13, 42\}$
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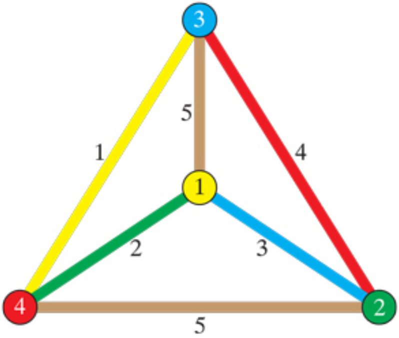
Note : Even though we have a proper coloring with 5 colors, the coloring is not optimal.



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1. G is k-edge colourable if G has a proper k-edge-colouring.
2. Trivially, every loopless graph G is $|E(G)|$-edge-colourable.
3. If G is k-edge-colourable, then G is also l-edge-colourable for every $l > k$.
4. The edge chromatic number $\chi'(G)$, of a loopless graph G, is the minimum k for which G is k-edge colourable.
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Theorem

If G is simple, then either $\chi' = \Delta$ *or* $\chi' = \Delta + 1$.

No Proof...(Advanced Course)

Vizing's theoem divides the class of simple graphs into two classes.

Definition

A simple graph G is said to be of Class-1 if $\chi'(G) = \Delta(G)$ and it is said to be of Class-2 if $\chi'(G) = \Delta(G) + 1$.

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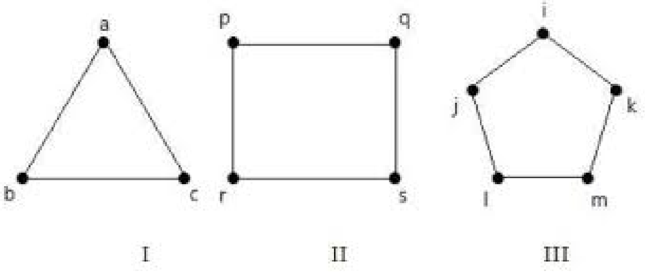

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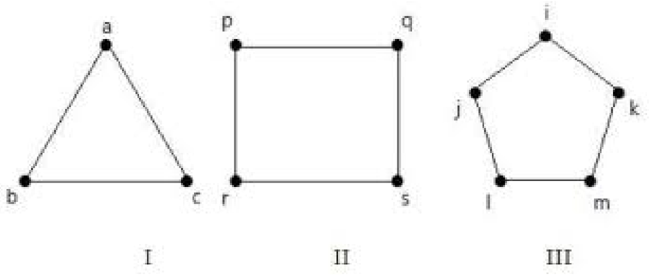

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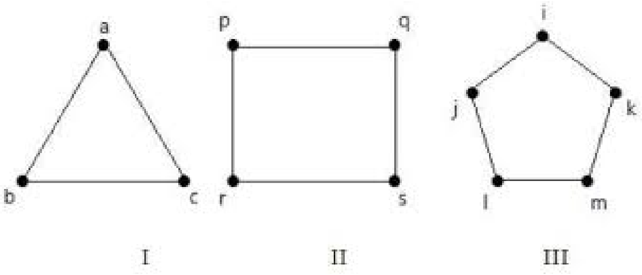

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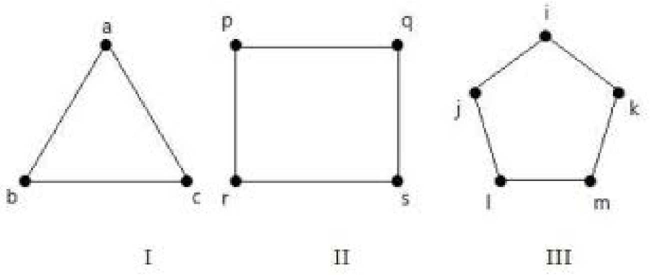

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For any bipartite graph G, then $\chi'(G) = \Delta$.

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Before proving we need the following Lemma.

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Let G be a bipartite graphs and let $d = \Delta(G)$. Then G is a subgraph of a d-regular bipartite graph H.

Proof.

1. Let $G[X, Y]$ be the bipartition of $V(G)$, where $X = \{x_1, x_2, \cdots x_s\}$ and $Y = \{y_1, y_2, \cdots y_t\}$.
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3. Construct the bipartite graph $H_1[X \cup Y_1, X_1 \cup Y]$ by adding the following edges:
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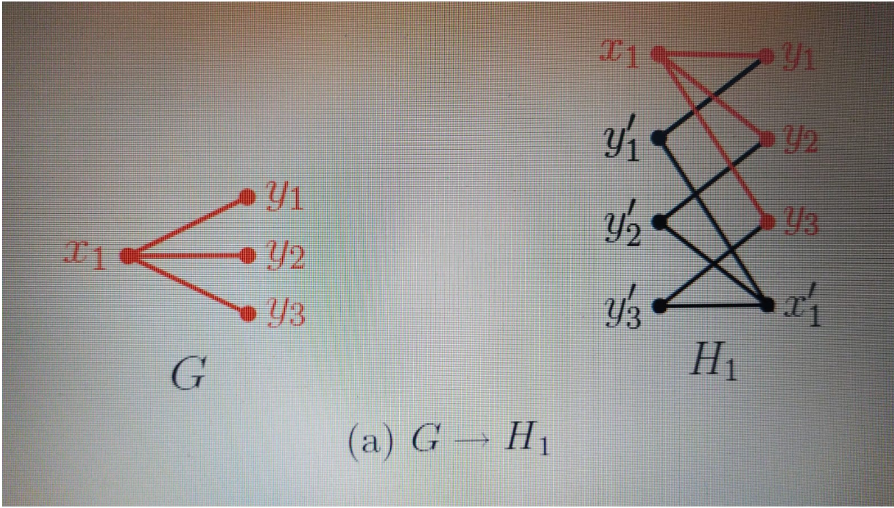

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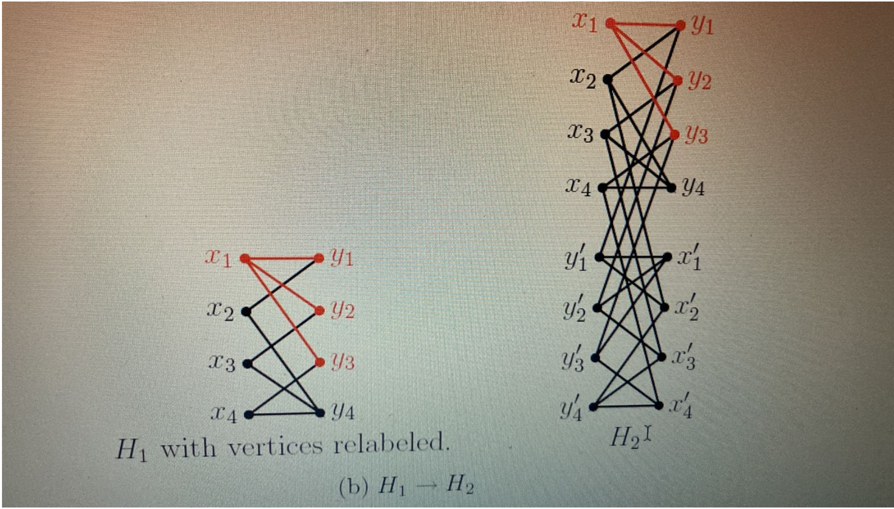

x_1
y_1
x_2
y_2
x_3
y_3
x_4
y_4
H_1 with vertices relabeled.
x_1
y_1
x_2
y_2
x_3
y_3
x_4
y_4
y'_1
x'_1
y'_2
x'_2
y'_3
x'_3
y'_4
x'_4
H_2
(b) $H_1 \to H_2$

Proof of Konig's Theorem

Theorem

For any bipartite graph G, then $\chi'(G) = \Delta$.

Proof.

1. We know that (By observation) $\chi'(G) \geq \Delta(G)$.
2. We only need to show $\chi'(G) \leq \Delta(G)$.
3. We prove by induction on $\Delta(G)$.
4. Base case : $\Delta = 0$.. trivially true.
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4. Base case : $\Delta = 0$.. trivially true.
5. $\Delta(G) = 1$. $G = K_2$, if connected or all components of G are K_2's and isolated vertices.
6. Need only one color to color the edges of G.
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8. Induction Hypothesis : Assume true for $\Delta(G) = l$.

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For any bipartite graph G, then $\chi'(G) = \Delta$.

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1. Let G be a bipartite graph with $\Delta(G) = l + 1$.
2. By Previous lemma, there exists a H_k, which is $\Delta(G)$-regular, bipartite and G is an induced subgraph of H_k.
3. By Hall's matching theorem. H_k contains a Perfect Matching M.
4. $H_k - M$ has a maximum degree $\Delta(G) - 1 = l$.
5. By induction, $H_k - M$ has a proper $\Delta - 1$ edge coloring.
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We derive a few sufficient conditions for a graph to be of Class-2.

Theorem

For any graph G,

$$\chi'(G) \geq \left\lceil \frac{|E(G)|}{\alpha'(G)} \right\rceil$$

Proof.

1. Let $C = (M_1, \cdots, M_{\chi'})$ be a χ'-edge-coloring of G.
2. Then, $|E(G)| = |M_1| + |M_2| + \cdots + |M_{\chi'}| \leq |\alpha'(G)| + |\alpha'(G)| + \cdots + |\alpha'(G)| = \alpha'(G) \times \chi'(G)$.
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Corollary

If G is a simple graph with $|E(G)| > \Delta(G) \times \chi'(G)$, then G is a Class-2 graph.

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