Euler Tours

October 2, 2024

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2. A closed Eulerian trial is called an Eulerian Tour.
3. A graph is called an Eulerian graph if it contains a Eulerian tour.

1. An Eulerian graph is necessarily connected.
2. If G is Eulerian then one can choose an Eulerian trail starting and ending from any given vertex.

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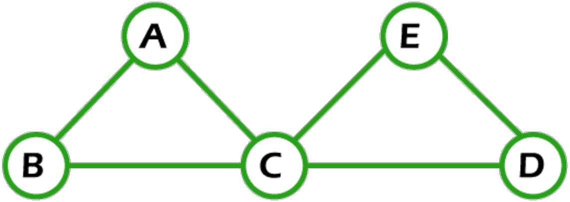
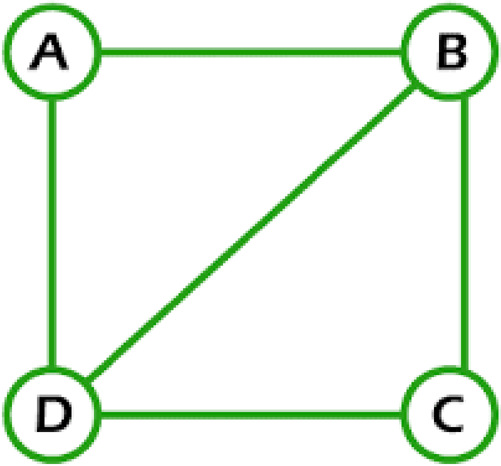
Example



Example of Euler Graph

Non-Eulerian Graph



Theorem

A connected graph G is Eulerian iff every vertex has even degree.

Proof.

$" \Rightarrow "$ Assume G is Eulerian. We show that every vertex has even degree.

1. Let $W = (v_0, e_1, v_1, e_2, v_2, \cdots, v_{t-1}, e_t, v_t(= v_0))$ be an Euler tour in G.
2. If v_i is an internal vertex ($\neq v_0$) of W appearing k times, then $deg_G(v_i) = 2k$.
3. If v_0 appears r times internally, then $deg_G(v_i) = 2r + 2$.

$\square$

Converse

G is a graph such that every vertex is of even degree. To prove that G is Eulerian.

1. By induction on number of edges m.
2. Base Case : $m \leq 2$: Easy verification.
3. Assume true for all $m \leq k - 1$.
4. Consider a graph G with k edges.
5. x - any vertex of G.
6. y, z- vertices (need not be distinct) adjacent to x.
7. Delete edges xy and xz and add a new edge yz.
8. The new graph H has $\leq m - 1$ edges and its every vertex has even degree.
9. H may be connected or disconnected.

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We consider two cases.

Case 1 : H is connected.

1. By induction hypothesis, H contains an Eulerian tour.
$$W = (v_0, e_1, v_1, e_2, v_2, \cdots, y, f = yz, z, \cdots, v_0)$$
2. Then, by replacing edge f with yx, x, xz in W we get,
$$W^* = (v_0, e_1, v_1, e_2, v_2, \cdots y, yx, x, xz, z, \cdots, v_0)$$
3. W^* an Euler tour in G.

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Case 2 : H is disconnected.

1. Let C and D be two components of H such that $x \in C$ and edge $f = yz \in D$.
2. $|E(C)| \leq m - 1$ and $|E(D)| \leq m - 1$.
3. By induction, C and D contain Euler tours.
4. $W_1 = (x, \cdots, x)$ in C.
5. $W_2 = (v_0, e_1, v_1, e_2, v_2, \cdots y, f, z, \cdots, v_0)$ in D.
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Motivation : TSP

Travelling salesman Problem

Given n cities and distance between any two cities, a traveling salesman wishes to start from one of these cities, visits each city exactly once and comes back to the starting point. Design an algorithm to find a shortest route.

Motivation : TSP

Graph theoretical model of TSP

Given a weighted graph G on n vertices, design an algorithm to find a cycle with minimum weight containing all the vertices of G.

1. A path (cycle) in a graph containing all its vertices is called a Hamilton path (respectively Hamilton cycle).
2. A graph is called a Hamilton graph or Hamiltonian if it contains a Hamilton cycle.
3. A graph is Hamiltonian iff its underlying simple graph is Hamilton. Hence, the study of Hamilton graphs is limited to simple graphs.

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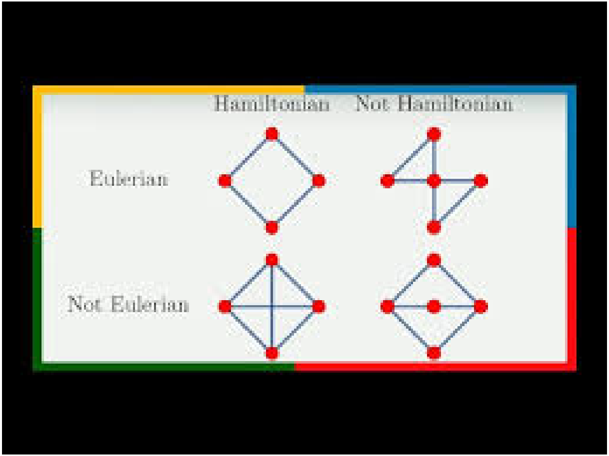

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Necessary conditions and sufficient conditions

Theorem

(Necessary condition). If G is Hamiltonian, then

$$\omega(G - S) \leq |S| \quad \textit{for every } S \subseteq V(G)$$

Proof.

Two way counting technique □

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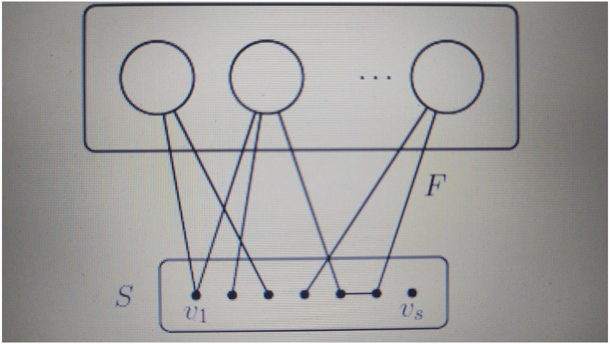
Two way counting technique $\square$

1. Let C be a Hamilton cycle in G and let $S \subseteq V(G)$ and $|S| = s$.
2. Consider the set of edges $F \subseteq E(C)$ with one end in S and another end in $G - S$.
3. Every vertex in G is incident with two edges of C.



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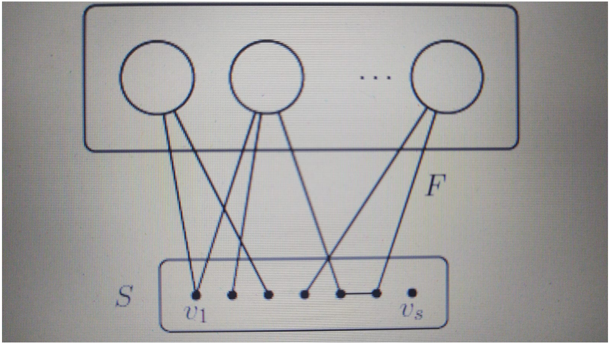

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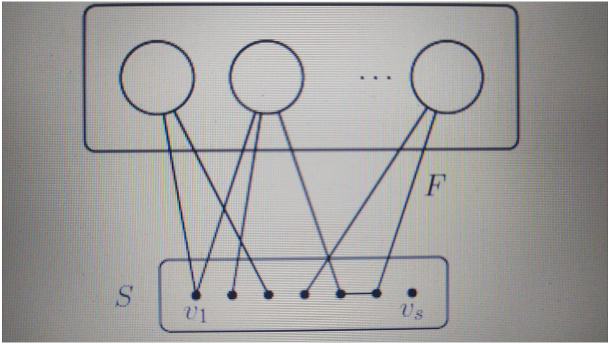

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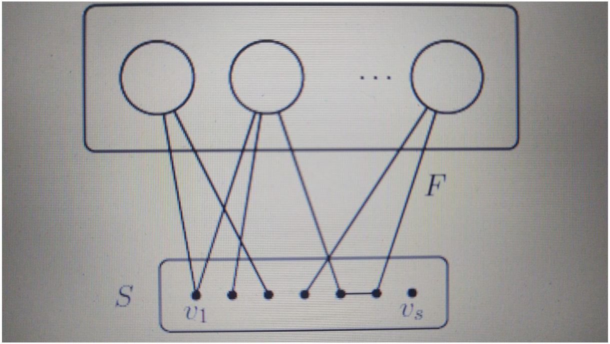

1. Therefore, every vertex in S is incident with at most two edges of F.
2. $\implies |F| \leq 2s = 2|S|$.
3. Also, every component of $G - S$ is incident with at least two edges of F.
4. Hence, $|F| \geq 2\omega(G - S)$.
5. $\omega(G - S) \leq \frac{|F|}{2} \leq |S|$.
6. $\implies \quad \omega(G - S) \leq |S|$ for all $S \subseteq V(G)$ $\square$

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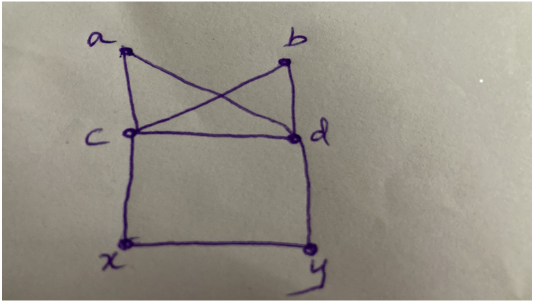

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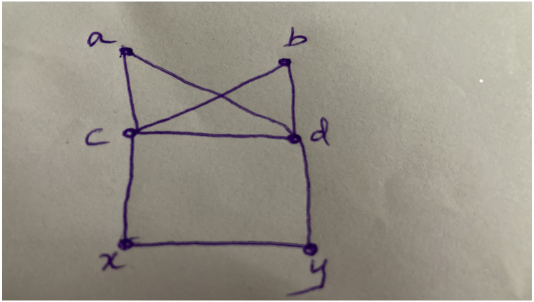

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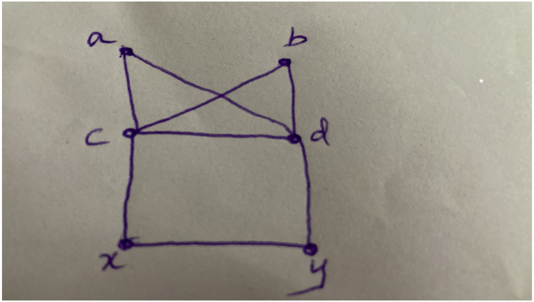

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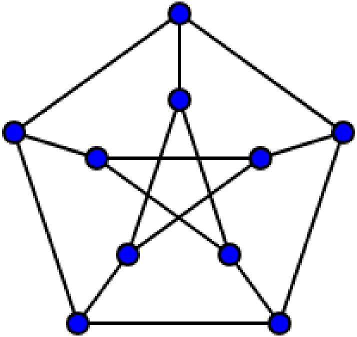

Figure: Petersen Graph

For any $S \subseteq V(G)$, $\omega(G - S) \leq |S|$.
But G is not Hamiltonian.

Example-Petersen Graph

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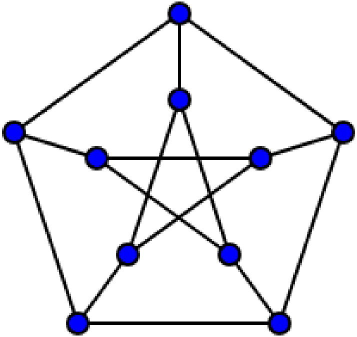

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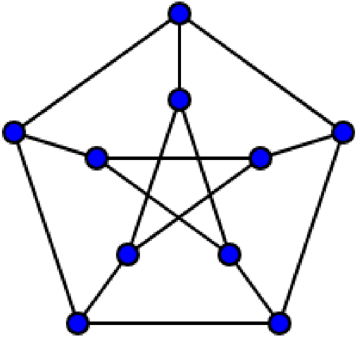

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A graph G is called maximal non-Hamiltonian if it satisfies the following.

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Dirac's Theorem

Theorem

If G is a simple graph with n (≥ 3) vertices and $\delta(G) \geq \frac{n}{2}$ then G is Hamiltonian.

Proof.

1. Proof by Contradiction.
2. Let G be such that, G is maximal non-Hamiltonian and $\delta(G) \geq \frac{n}{2}$.
3. G is not complete as $n \geq 3$.
4. Choose $u, v \in V(G)$ and u and v are non-adjacent.
5. $G + uv$ is Hamiltonian.
6. Every Hamilton cycle of $G + uv$ contains the edge uv.
7. Let $P \; : \; u = v_1 - v_2 - \cdots - v_n = v$ be a Hamilton Path in G.

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3. G is not complete as $n \geq 3$.
4. Choose $u, v \in V(G)$ and u and v are non-adjacent.
5. $G + uv$ is Hamiltonian.
6. Every Hamilton cycle of $G + uv$ contains the edge uv.
7. Let $P \ : \ u = v_1 - v_2 - \cdots - v_n = v$ be a Hamilton Path in G.

Dirac's Theorem

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If G is a simple graph with n (≥ 3) vertices and $\delta(G) \geq \frac{n}{2}$ then G is Hamiltonian.

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Define

$$S = \{v_i \ : \ uv_{i+1} \in E(G)\}$$
$$T = \{v_i \ : \ v_i v \in E(G)\}$$

1. $v_n \notin S \cup T$.
2. $|S \cup T| < n$
3. $S \cap T = \emptyset$.
4. If not, let $v_j \in S \cap T$, then

$$v_1 v_2 \cdots v_j v_n v_{n-1} \cdots v_{j+1} v_1$$

is a Hamilton cycle of G, contrary to our assumption.
5. $deg(u) + deg(v) = |S| + |T| = |S \cup T| - |S \cap T| < n$.
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□

Theorem

If a graph G on n (≥ 3) *is such that*

$$deg(u) + deg(v) \geq n$$

for every pair of non-adjacent vertices u and v then G is Hamiltonian.

Note : We did not prove this in class. We only proved the consequence of it. (only the consequence in your text book :)).

Consequence of Ore's theorem

Theorem

If a graph G on n (≥ 3) is such that

$$deg(u) + deg(v) \geq n$$

for a pair of non-adjacent vertices u and v then G is Hamiltonian iff $G + uv$ is Hamiltonian.

Proof.

1. Let $u, v \in G$ be such that $deg_G(u) + deg_G(v) \geq n$.
2. If $uv \in G$, then there is nothing to prove.
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Proof : $\Rightarrow$

This is trivial.

1. Given G is Hamiltonian.
2. G has a Hamilton cycle C (say)
3. C is also a Hamilton cycle of $G + uv$.
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Suppose given that $G + uv$ is Hamiltonian.

We show that G is Hamiltonian.

1. Denote $e = uv$.
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Claim : There exists an i with $1 < i < n$ such that $uv_i \in G$ and $v_{i-1}v \in G$.

1. u and v not adjacent in G.
2. $deg_G(u) + deg_G(v) \geq n$.
3. $\implies (n-1) - deg_G(v) \leq deg_G(u) - 1$.
4. Atmost $deg_G(u) - 1$ vertices other than v are not adjacent to v.
5. Let $S = \{v_{i-1} \ : \ uv_i \in E(G)\}$ in the Hamilton path P.
6. $v \notin S$ and $|S| = deg_G(u)$.
7. Therefore, at least one vertex say v_k of S is adjacent to v_n.

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Closure of a graph

Definition

For a graph G, define inductively a sequence $G_0, G_1, \cdots, G_k$ of graphs such that

$$G_0 = G \text{ and } G_{i+1} = G_i + uv$$

where u and v are any vertices such that

1. $uv \notin G_i$ and
2. $deg_{G_i}(u) + deg_{G_i}(v) \geq n$.

1. This procedure stops when no new edges can be added to G_k for some k, that is, in G_k, for all $u, v \in G$ either $uv \in G_k$ or $deg_{G_k}(u) + deg_{G_k}(v) < n$.
2. The result of this procedure is the closure of G, and it is denoted by $cl(G)(= G_k)$.
3. In each step of the construction of $cl(G)$ there are usually alternatives which edge uv is to be added to the graph, and therefore the above procedure is not deterministic.
4. However, the final result $cl(G)$ is independent of the choices.
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2. The result of this procedure is the closure of G, and it is denoted by $cl(G)(= G_k)$.
3. In each step of the construction of $cl(G)$ there are usually alternatives which edge uv is to be added to the graph, and therefore the above procedure is not deterministic.
4. However, the final result $cl(G)$ is independent of the choices.
5. Example

Well Defined

Lemma

The closure $cl(G)$ is uniquely defined for all graphs G of order $n \geq 3$.

Proof.

Suppose there are two ways to close G, say

$$H = G + \{e_1, \cdots, e_r\} \text{ and } H' = G + \{f_1, \cdots, f_s\}$$

where the edges are added in the given orders.

We show that

$$\{e_1, \cdots, e_r\} = \{f_1, \cdots, f_s\}$$

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1. Let $H_i = G + \{e_1, \cdots, e_i\}$ and $H'_i = G + \{f_1, \cdots, f_i\}$.
2. For the initial values, we have $G = H_0 = H'_0$.
3. Let $e_k = uv \in H_k$ be the first edge such that $e_k \notin H'$
4. Now, $e_k \notin H_{k-1}$ we have

$$deg_{H_{k-1}}(u) + deg_{H_{k-1}}(v) \geq n,$$

5. $H_{k-1} \subseteq H'$ (Subgraph).
6. $deg_{H'}(u) + deg_{H'}(v) \geq n$, which means that $e = uv$ must be in H' ; a contradiction.
7. Therefore $H \subseteq H'$.
8. Similarly we can show that $H \subseteq H'$.
9. Hence, $H = H'$. □

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Theorem

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Let G be a graph such that $|V(G)| = n \geq 3$.

1. *G is Hamiltonian if and only if its closure $cl(G)$ is Hamiltonian.*
2. *If $cl(G)$ is a complete graph, then G is Hamiltonian.*

Proof.

Exercise. □