DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH1020 Physics II **Problem Set 4** **March 2024**

Simply Dielectrics

1. Prove the following identity for any vector field $\mathbf{P}(\mathbf{r})$:

$$\int_V dV\,\mathbf{P}(\mathbf{r}) = \int_S \mathbf{r}\,[\mathbf{P}(\mathbf{r})\cdot\mathbf{dS}] - \int_V dV\;\mathbf{r}\;[\nabla\cdot\mathbf{P}(\mathbf{r})]\;,$$

where V is the volume enclosed by the surface S.
Hint: Take the dot product of the above identity with a constant vector $\mathbf{a}$ to obtain a new identity. Prove this new identity and then argue that it implies the required identity.

2. A point dipole of moment $\mathbf{p} = p_0(\hat{e}_x + 2\hat{e}_y + 3\hat{e}_z)$ is placed in an electrostatic potential ϕ given by

$$\phi(x,y,z) = \phi_0\left[1 + \frac{(x^2+y^2+z^2)}{a^2} + \frac{(x^4+y^4+z^4)}{a^4}\right]$$

where ϕ_0 and a are appropriate constants. Find the force and the couple acting on the dipole when it is located at the point (a,a,a). Find also the torque of the force about the origin.

3. Starting from the equations for the polarization charge density and the polarization surface charge density, show that the net polarization charge in the volume and on the surface of a dielectric body is always zero.

4. A dielectric medium carries a *uniform* polarization $\mathbf{P}_0$. A spherical cavity is scooped out inside the medium. Find (i) the bound surface charge density on the surface of the cavity, and (ii) the electric field at the centre of the cavity due to this surface charge.

5. A cylinder of length $2L$ and radius a is centred at the origin, with the z-axis as its symmetry axis. The cylinder is uniformly polarized with polarization $\mathbf{P} = P_0\,\hat{e}_z$, where P_0 is a constant.

   (a) Find the bound charge densities ρ_b and σ_b.
   (b) Find the electric field at all points on the positive z-axis, and verify that it satisfies the appropriate boundary condition at $z = L$.
   (c) Find the electric field at the origin, and sketch its magnitude as a function of the ratio a/L.

6. Consider a uniform spherical free charge distribution of radius a and charge density ρ_0. This region is filled with a medium of dielectric constant κ_1, and surrounded by a medium of dielectric constant κ_2. Find (i) the bound volume charge density everywhere in space, and (ii) the bound surface charge density on the surface of the sphere of radius a.

7. A sphere of radius R and dielectric constant κ, centred at the origin of coordinates, is placed in a constant electric field $\mathbf{E}_0$ directed along the z axis. The corresponding electrostatic potential is given by $\phi\,(r,\theta,\varphi) = (-E_0\,r + b_1\,r^{-2})\,\cos\theta$ outside the sphere, and $\phi\,(r,\theta,\varphi) = b_2\,r\,\cos\theta$ inside the sphere. Find

   (a) the constants b_1 and b_2, in terms of κ, E_0 and R;

   (b) the electric field at all points in space;

   (c) the polarization $\mathbf{P}$ of the sphere, and the dipole moment of the sphere about the origin;

   (d) the volume and surface densities of the bound charge in the sphere.

8. A cylindrical coaxial cable, lying along the z axis has conducting surfaces at $\rho = a$ and $\rho = 4a$, which carry uniform surface charge densities σ_0 and $-\sigma_0/4$, respectively. The region between the two conductors is filled with linear dielectric media with dielectric constants K_1 and K_2 in the regions $a < \rho \leq 2a$ and $2a < \rho < 4a$, respectively.

   - Find the energy density between $a < \rho \leq 2a$.
   - Determine the ratio of the magnitude of the polarization just inside and just outside the boundary at $\rho = 2a$.
   - Sketch $|E|$ as a function of ρ in the interval $0 < \rho \leq 5a$ when $K_1 = 1.5$ and $K_2 = 2$.