# *Paths-Cycles-Trees*

August 11, 2024

# Walks

1. A walk in $G$ is a finite non-null sequence $W = v_0 e_1 v_1 e_2 v_2 \cdots e_k v_k$, whose terms are alternately vertices and edges, such that, for $1 \leq i \leq k$, the ends of $e_i$ are $v_{i-1}$ and $v_i$.
2. We say that $W$ is a walk from $v_0$ to $v_k$, or a $(v_0, v_k)$-walk.
3. The vertices $v_0$ and $v_k$ are called the origin and terminus of $W$, respectively, and $v_1, v_2, \cdots, v_{k-1}$ its internal vertices.
4. The integer $k$ is the length of $W$.

- Walk : Sequence of alternating vertices and incident edges with no restriction.
- Trail : A Walk in which no edge is repeated.
- Path : A Trail in which no vertex is repeated.
- Closed Walk : A Walk in which initial and final vertices are same.
- Circuit : A closed trail.
- Cycle : A closed path (Abuse of terminology).

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- Notation of Path, Cycle of length $n$ : $P_{n+1}$ and $C_n$ respectively.
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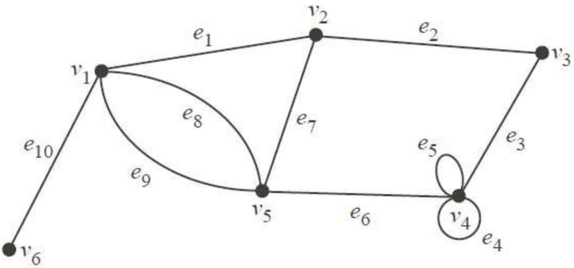



Graph G

**Open Walk:** $v_2, e_7, v_5, e_8, v_1, e_8, v_5, e_6, v_4, e_5, v_4, e_5, v_4$

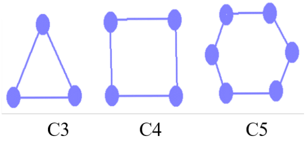
**Closed Walk:** $v_4, e_5, v_4, e_3, v_3, e_2, v_2, e_7, v_5, e_6, v_4$

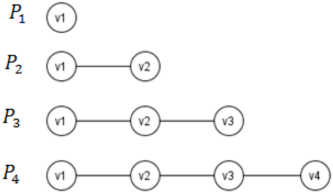
**Trail:** $v_1, e_8, v_5, e_9, v_1, e_1, v_2, e_7, v_5, e_6, v_4, e_5, v_4, e_4, v_4$

**Path:** $v_2, e_7, v_5, e_6, v_4, e_3, v_3$

**Circuit:** $v_2, e_7, v_5, e_6, v_4, e_3, v_3, e_2, v_2$

# Example



# Theorem

**Theorem**

*A graph is bipartite if and only if it contains no odd cycle.*

**Proof.**

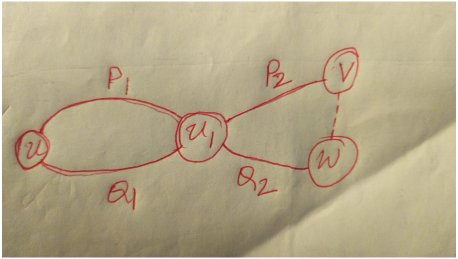
$" \Rightarrow "$ : Partition : $V = X \cup Y$. Then starting from a vertex in $X$, say $v_0$, to form a cycle with $v_0$, we need to visit even number of edges to come back to $v_0$. Hence, every cycle is even. $\square$

*Proof.*

" $\Leftarrow$ " :

1. Given that all cycles are even.
2. To show that $G$ is bipartitie.
3. $d(u, v)$ = length of the shortest path between vertices $u$ and $v$.
4. We find a bipartition of $V$. Choose a $u \in V$.
5. Define,
$$X = \{x \ : \ d(u, x) \text{ is even}\}$$
$$Y = \{x \ : \ d(u, x) \text{ is odd}\}$$
6. $V = X \cup Y$.

$\square$

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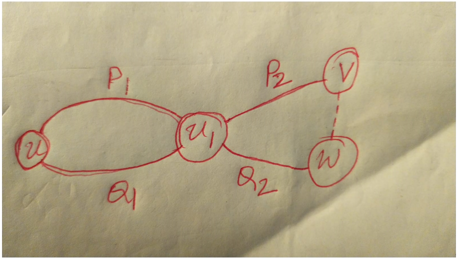
2. $u_1$ last common vertex in $P$ and $Q$.
3. $P_1, Q_1$ - shortest and $|Q_1| = |P_1|$.
4. $\Rightarrow P_2, Q_2$ have same parity.
5. $\Rightarrow |P_2| + |Q_2|$ is even.
6. $\Rightarrow |P_2| + |Q_2| +$ edge $vw$ = odd cycle. A contradiction. $\square$

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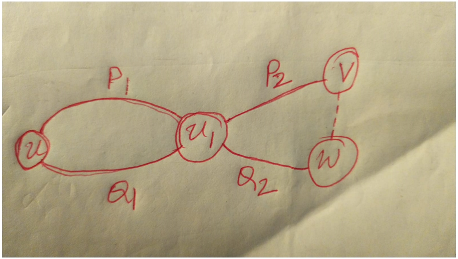


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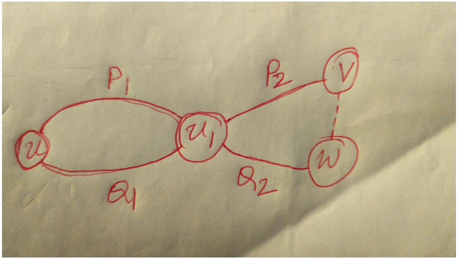


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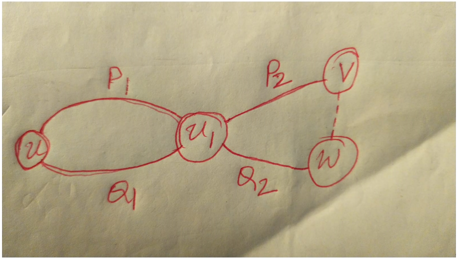


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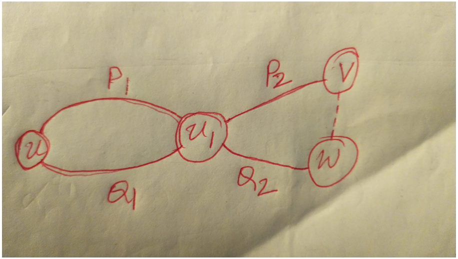


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# *Connected-graphs*

- Two vertices $u$ and $v$ are said to be connected (denoted by $u \sim v$ if there is a path between them.
- $\sim$ is an equivalence relation and $\sim$ partitions the vertices of a given graph into equivalence classes.
- The subgraph induced by each equivalence class is called a component of $G$.
- $\omega(G)$ denotes the number of componenets of $G$
- A graph $G$ is said to be connected if every pair of distinct vertices are connected (i.e.) $\omega(G) = 1$.
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