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PH1020 Physics II — Problem Set 4 (Solutions) — March 2024

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1. Let $\boldsymbol{a}$ be a constant vector. Consider
$$\boldsymbol{a}\cdot\int_V dV\ \boldsymbol{r}(\boldsymbol{\nabla}\cdot\boldsymbol{P})\ .$$
This is equal to
$$\begin{aligned}\boldsymbol{a}\cdot\int_V dV\ \boldsymbol{r}(\boldsymbol{\nabla}\cdot\boldsymbol{P}) &= \int_V dV\ (\boldsymbol{a}\cdot\boldsymbol{r})(\boldsymbol{\nabla}\cdot\boldsymbol{P}) \quad \text{(as } \boldsymbol{a} \text{ is a constant)}\\ &= \int_V dV\ \boldsymbol{\nabla}\cdot[(\boldsymbol{a}\cdot\boldsymbol{r})\boldsymbol{P}] - \int dV\ [\boldsymbol{\nabla}(\boldsymbol{a}\cdot\boldsymbol{r})]\cdot\boldsymbol{P}\\ &= \int_S (\boldsymbol{a}\cdot\boldsymbol{r})(\boldsymbol{P}\cdot d\boldsymbol{S}) - \boldsymbol{a}\cdot\int_V dV\ \boldsymbol{P}\\ &= \boldsymbol{a}\cdot\left(\int_S \boldsymbol{r}(\boldsymbol{P}\cdot d\boldsymbol{S}) - \int_V dV\ \boldsymbol{P}\right)\end{aligned}$$
This is true for all $\boldsymbol{a}$. So
$$\int_V dV\boldsymbol{r}(\boldsymbol{\nabla}\cdot P) = \int_S \boldsymbol{r}(P\cdot d\boldsymbol{S}) - \int_V dV\ \boldsymbol{P}$$

2. We are given the following electrostatic potential
$$\phi(x,y,z) = \phi_0 + \frac{\phi_0}{a^2}\left(x^2+y^2+z^2\right) + \frac{\phi_0}{a^4}\left(x^4+y^4+z^4\right)\ ,$$
whose electric field is
$$\begin{aligned}\boldsymbol{E} &= -\boldsymbol{\nabla}\phi\\ &= -\left(\frac{\phi_0}{a^2}\right)\left[\left(2x+\frac{4x^3}{a^2}\right)\hat{e}_x + \left(2y+\frac{4y^3}{a^2}\right)\hat{e}_y + \left(2z+\frac{4z^3}{a^2}\right)\hat{e}_z\right]\ .\end{aligned}$$
Using the standard formula for the force on a point dipole in a spatially inhomogenous electrostatic field, i.e., $\boldsymbol{F} = (\boldsymbol{p}\cdot\boldsymbol{\nabla})\ \boldsymbol{E}$. [1]
$$\begin{aligned}\boldsymbol{F} &= (\boldsymbol{p}\cdot\boldsymbol{\nabla})\boldsymbol{E}\Big|_{(a,a,a)}\\ &= p_0\left(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}\right)\boldsymbol{E}\Big|_{(a,a,a)}\\ &= -\frac{p_0\phi_0}{a^2}\left[\left(2+\frac{12x^2}{a^2}\right)\hat{e}_x + 2\left(2+\frac{12y^2}{a^2}\right)\hat{e}_y + 3\left(2+\frac{12z^2}{a^2}\right)\hat{e}_z\right]\Bigg|_{(a,a,a)}\end{aligned}$$
$$\boxed{\boldsymbol{F} = -\frac{14p_0\phi_0}{a^2}\left(\hat{e}_x + 2\hat{e}_y + 3\hat{e}_z\right)\ .}$$
The torque, $\boldsymbol{\tau}$, on a point dipole in an electric field is given by $\boldsymbol{\tau} = (\boldsymbol{p}\times\boldsymbol{E})$, where $\boldsymbol{E}$ is the electric field at the location of the point dipole. Computing
$$\boldsymbol{\tau} = (\boldsymbol{p}\times\boldsymbol{E})|_{(a,a,a)}$$
$$\boxed{\boldsymbol{\tau} = \frac{6p_0\phi_0}{a}\left(\hat{e}_x - 2\hat{e}_y + \hat{e}_z\right)\ .}$$

[1] This is easily derived using the construction of a point dipole as a limit of two charges with equal in magnitude but with opposite signs.

The torque, $\boldsymbol{\tau}$, about the origin is

$$\boldsymbol{\tau}|_{(a,a,a)} + a(\hat{e}_x + \hat{e}_y + \hat{e}_z) \times \boldsymbol{F} = -\frac{8p_0\phi_0}{a}(\hat{e}_x - 2\hat{e}_y + \hat{e}_z) \ .$$

3. Consider a surface S that bounds a volume V. Due to the Polarization, $\mathbf{P}$, there is a surface charge density $\sigma_p = \mathbf{P}.\hat{n}$, and a volume charge density given by $\rho_p = -\nabla \cdot \mathbf{P}$. The net polarization charge is

$$\begin{aligned} &\int_{\mathrm{S}} \sigma_p dS + \int_{\mathrm{V}} \rho_p dV \\ = &\int_{\mathrm{S}} \mathbf{P}.\hat{n} dS + \int_{\mathrm{V}} -\nabla \cdot \mathbf{P} dv = 0, \end{aligned}$$

wherein in the last line we use Gauss's divergence theorem.

4. (i) Choose coordinate axis such that $\hat{e}_z$ points along $\mathbf{P}_0$. Now, the bound surface charge is given by

$$\boxed{\sigma_b = \mathbf{P}_0 \cdot \hat{n} = -P\cos\theta \ .}$$

Notice that $\hat{n}$ points radially *inwards*.

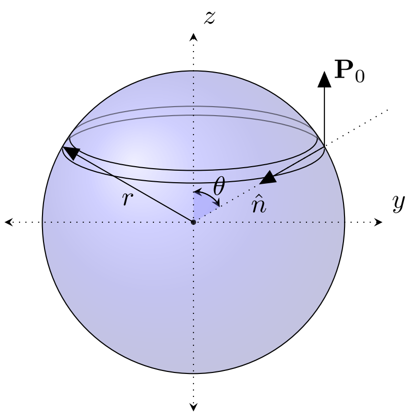

Figure 1: A spherical cavity inside a dielectric medium

(ii) We need to find the electric field due to a hollow sphere with surface charge density $-P_0\cos(\theta)$. The most direct way to do this is to divide the sphere into rings at constant θ and integrate their electric fields.

$$\begin{aligned} dE_z &= -\frac{1}{4\pi\epsilon_0}\frac{(2\pi r\sin\theta \ r \ d\theta)\sigma_b}{r^2}\cos\theta \\ E_z &= -\frac{-P_0}{2\epsilon_0}\int_0^\pi \cos^2\theta \sin\theta \ d\theta \\ &= \frac{P_0}{\epsilon_0}\int_0^1 \cos^2\theta \ d(\sin\theta) \\ &= \frac{P_0}{3\epsilon_0} \end{aligned}$$

The other components of the electric field vanish due to the cylindrical symmetry about the z-axis, and therefore

$$\boxed{\mathbf{E} = \frac{P_0}{3\epsilon_0}\hat{e}_z}$$

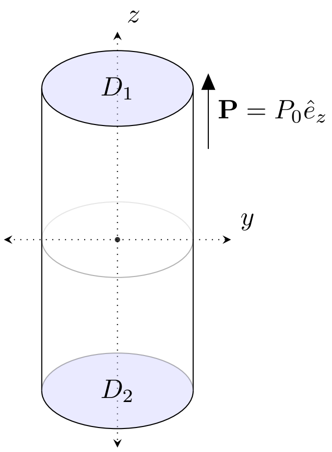

Figure 2: A uniformly polarized cylinder

5. (a) Since the cylinder is uniformly polarized with $\mathbf{P} = P_0\ \hat{e}_z$,

$$\begin{aligned}
\rho_b &= -\boldsymbol{\nabla}\cdot\mathbf{P} = 0\\
\sigma_b|_{\text{curved part}} &= P_0\hat{e}_z\cdot\hat{\varrho} = 0\\
\sigma_b|_{D_1} &= P_0\hat{e}_z\cdot\hat{e}_z = P_0\\
\sigma_b|_{D_2} &= P_0\hat{e}_z\cdot(-\hat{e}_z) = -P_0
\end{aligned}$$

(b) The electric field on the z axis is the vector sum of electric fields due to discs D_1 and D_2. Electric field along the z axis due to a disc with center at the origin and charge density σ

$$\frac{\sigma}{2\epsilon_0}\left(1-\frac{z}{\sqrt{z^2+a^2}}\right)\hat{e}_z \quad z>0$$

So the net electric field on the z-axis, for $-L<z<L$ is

$$\begin{aligned}
\mathbf{E} &= -\frac{P_0}{2\epsilon_0}\left(1-\frac{L-z}{\sqrt{(L-z)^2+a^2}}\right)\hat{e}_z - \frac{P_0}{2\epsilon_0}\left(1-\frac{L+z}{\sqrt{(L+z)^2+a^2}}\right)\hat{e}_z\\
&= \frac{P_0}{2\epsilon_0}\left(\frac{L-z}{\sqrt{(L-z)^2+a^2}}+\frac{L+z}{\sqrt{(L+z)^2+a^2}}-2\right)\hat{e}_z
\end{aligned}$$

For $z>L$,

$$\begin{aligned}
\mathbf{E} &= \frac{P_0}{2\epsilon_0}\left(1-\frac{z-L}{\sqrt{(z-L)^2+a^2}}\right)\hat{e}_z - \frac{P_0}{2\epsilon_0}\left(1-\frac{L+z}{\sqrt{(L+z)^2+a^2}}\right)\hat{e}_z\\
&= \frac{P_0}{2\epsilon_0}\left(\frac{z+L}{\sqrt{(z+L)^2+a^2}}-\frac{z-L}{\sqrt{(z-L)^2+a^2}}\right)\hat{e}_z
\end{aligned}$$

Similarly, for $z<-L$,

$$\begin{aligned}
\mathbf{E} &= -\frac{P_0}{2\epsilon_0}\left(1-\frac{L-z}{\sqrt{(L-z)^2+a^2}}\right)\hat{e}_z + \frac{P_0}{2\epsilon_0}\left(1+\frac{L+z}{\sqrt{(L+z)^2+a^2}}\right)\hat{e}_z\\
&= \frac{P_0}{2\epsilon_0}\left(\frac{L+z}{\sqrt{(L+z)^2+a^2}}+\frac{L-z}{\sqrt{(L-z)^2+a^2}}\right)\hat{e}_z
\end{aligned}$$

It is now a simple exercise to see that the discontinuity in the electric field at $z=L$, i.e,. $(\mathbf{E}(z=L^+)-\mathbf{E}(z=L^-))\cdot\hat{e}_z$ equals $\frac{\sigma_b}{\epsilon_0}=\frac{P_0}{\epsilon_0}$.

(c) The electric field at the origin is given by setting $z = 0$ in the relevant expression for the electric field part (b)

$$\mathbf{E}|_{z=0} = -\frac{P_0}{\epsilon_0}\left(1 - \frac{L}{\sqrt{L^2 + a^2}}\right)\hat{e}_z$$
$$= -\frac{P_0}{\epsilon_0}\left(1 - \frac{1}{\sqrt{1 + (a/L)^2}}\right)\hat{e}_z$$

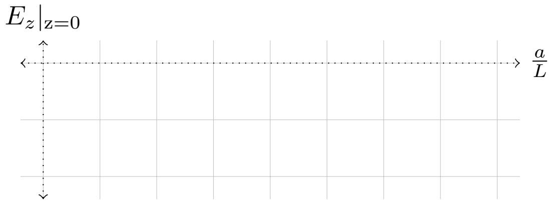

Figure 3: Electric field at the origin

6. (i) To find the bound charge density, we need to find the polarization of the medium. Once we have $\mathbf{P}$, the bound charge density ρ_b can be found using $\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P}$
Gauss' Law in the presence of dielectrics reads

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f$$

where $\mathbf{D}$ is the electric displacement and ρ_f is the free charge density. Using this, we get

$$\mathbf{D} = \begin{cases} \frac{\rho_0 a^3}{3r^2}\hat{e}_r & \text{for } r \geq a \\ \frac{\rho_0 r}{3}\hat{e}_r & \text{for } r < a \end{cases}$$

Given that the dielectric constants are κ_1 inside the spherical region and κ_2 outside it, we can find the polarization using $\mathbf{D} = \epsilon_0 \kappa \mathbf{E}$ and $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} = \epsilon_0(\kappa - 1)\mathbf{E}$.

$$\mathbf{P} = \frac{(\kappa - 1)}{\kappa}\mathbf{D}$$
$$\implies \mathbf{P} = \begin{cases} \left(1 - \frac{1}{\kappa_2}\right)\frac{\rho_0 a^3}{3r^2}\hat{e}_r & \text{for } r > a \\ \left(1 - \frac{1}{\kappa_1}\right)\frac{\rho_0 r}{3}\hat{e}_r & \text{for } r < a \end{cases}$$

Therefore, the bound volume charge density is given by

$$\boxed{\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P} = \begin{cases} 0 & \text{for } r > a \\ -\left(1 - \frac{1}{\kappa_1}\right)\rho_0 & \text{for } r < a \end{cases}}$$

(ii) The bound surface charge density is given by $\sigma_b = -(\mathbf{P}_{\text{outer}} - \mathbf{P}_{\text{inner}}) \cdot \hat{e}_r$. So,

$$\mathbf{P} \cdot \hat{e}_r = \begin{cases} \left(1 - \frac{1}{\kappa_2}\right)\frac{\rho_0 a}{3} & \text{on the outer surface} \\ \left(1 - \frac{1}{\kappa_1}\right)\frac{\rho_0 a}{3} & \text{on the inner surface} \end{cases}$$

Therefore, the total charge density on the surface of the sphere is

$$\boxed{\sigma_b = \left(\frac{1}{\kappa_2} - \frac{1}{\kappa_1}\right)\frac{\rho_0 a}{3}}$$

7. The electrostatic potential is given to be

$$\phi(r,\theta,\varphi) = \begin{cases} (-E_0\, r + b_1\, r^{-2}) \cos\theta \ , & \text{for } r > R \\ b_2\, r \cos\theta \ , & \text{for } r < R \ . \end{cases}$$

The corresponding electric field is given by

$$\mathbf{E} = \begin{cases} E_0\ \hat{e}_z + \frac{b_1}{r^3}(2\cos\theta\ \hat{e}_r + \sin\theta\hat{e}_\theta) \ , & \text{for } r > R \\ -b_2\ \hat{e}_z \ , & \text{for } r < R \ . \end{cases} \tag{1}$$

The displacement $\mathbf{D}$ can be computed using $\mathbf{D} = \epsilon_0\kappa\mathbf{E}$ for $r < R$ and $\mathbf{D} = \epsilon_0\mathbf{E}$ for $r > R$. We obtain

$$\frac{\mathbf{D}}{\epsilon_0} = \begin{cases} E_0\ \hat{e}_z + \frac{b_1}{r^3}(2\cos\theta\ \hat{e}_r + \sin\theta\hat{e}_\theta) \ , & \text{for } r > R \\ -b_2\kappa\ \hat{e}_z \ , & \text{for } r < R \ . \end{cases}$$

(a) In order to fix the constants b_1 and b_2 we use the following matching conditions at $r = R$ and arbitrary (θ,φ). First, we expect the normal component of $\mathbf{D}$, i.e., $\mathbf{D}\cdot\hat{e}_r$ to be continuous as there is no **free** charge on the interface. Second, the tangential component of $\mathbf{E}$, i.e., $\mathbf{D}\times\hat{e}_r$ should be continuous. We obtain (using $\hat{e}_r \times \hat{e}_z = -\sin\theta\hat{e}_\varphi$)

$$\left(E_0 + 2\frac{b_1}{R^3}\right)\ \cos\theta = -b_2\kappa\ \cos\theta$$

$$\left(-E_0 + \frac{b_1}{R^3}\right)\ \sin\theta\hat{e}_\varphi = b_2\sin\theta\hat{e}_\varphi$$

which can be solved to obtain

$$\boxed{b_1 = \left(\frac{\kappa-1}{\kappa+2}\right)\ E_0R^3 \quad , \quad b_2 = -\left(\frac{3}{\kappa+2}\right)\ E_0 \ .}$$

(b) Inserting the values of b_1 and b_2 in Eq. (1), we get

$$\boxed{\mathbf{E} = \begin{cases} E_0\ \hat{e}_z + E_0\left(\frac{\kappa-1}{\kappa+2}\frac{R^3}{r^3}\right)(2\cos\theta\ \hat{e}_r + \sin\theta\hat{e}_\theta) \ , & \text{for } r > R \\ \frac{3}{\kappa+2}\ E_0\ \hat{e}_z \ , & \text{for } r < R \ . \end{cases}}$$

(c) The polarization is given by $\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E}$ and we find a non-zero answer for $r < R$. We obtain

$$\boxed{\mathbf{P} = \frac{3(\kappa-1)}{\kappa+2}\ \epsilon_0 E_0\ \hat{e}_z \ ,}$$

which is (somewhat surprisingly) a constant vector. The total dipole moment of the sphere is $4\pi R^3/3$ times the constant value of $\mathbf{P}$ given above.

(d) The volume and bound charge densities can be computed easily using the standard formulae:

$$\rho_b(\theta,\varphi) = -\boldsymbol{\nabla}\mathbf{P} = 0\ ,\sigma_b(\theta,\varphi) = \mathbf{P}\cdot\hat{n} = \mathbf{P}\cdot\hat{e}_r = \frac{3(\kappa-1)}{\kappa+2}\ \epsilon_0 E_0\cos\theta \ .$$

8. We use the Gauss's law for dielectric to evaluate D in the various regions of interest: For $0 < \rho < a$, we see that Q_f the free charge enclosed is zero. Thus, $D = 0$ and hence $E = 0$. In similar fashion, $\rho > 4a$, the enclosed charge is once again zero thus once again $D = 0$, and $E = 0$ in this region. For $a < \rho < 4a$, $Q_f^{\text{enc}} = 2\pi\sigma_0 aL$. Thus by using Gauss's law for dielectrics we get

$$\int_S \mathbf{D}\cdot d\mathbf{S} = Q_f^{\text{enc}}.$$

Using a cylindrical Gaussian surface, one sees that

$$D_\rho = \frac{\sigma_0 a}{\rho}.$$

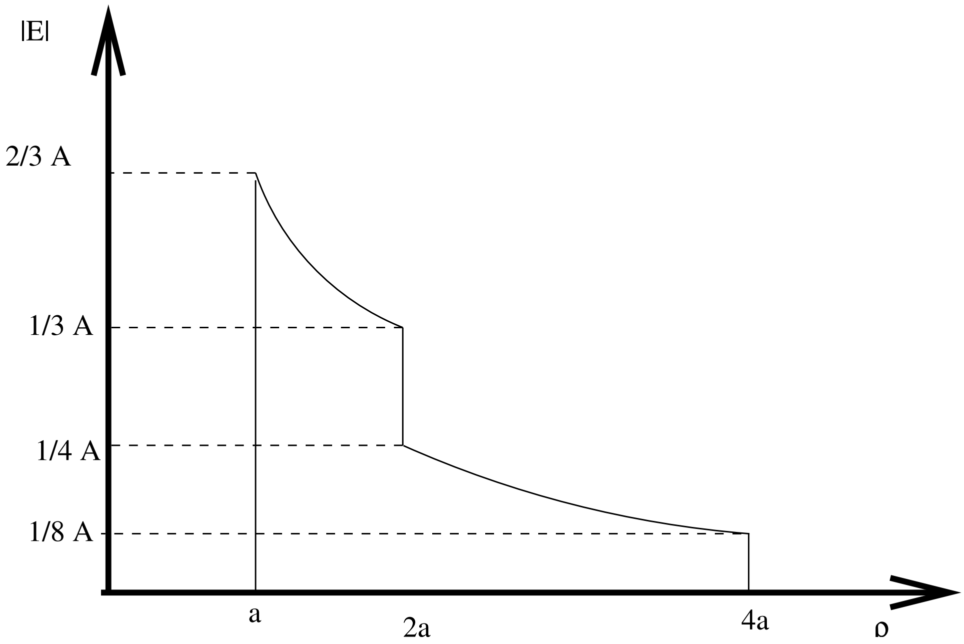

Figure 4: :
A plot of $E(\rho)$ vs ρ: The discontinuities of the Electric field across the various boundaries are shown. The constant $A = \sigma_0/\epsilon_0$.

- To evaluate the energy density in the region $a < \rho < 2a$, we use Energy density $= \frac{1}{2}\mathbf{D}\cdot\mathbf{E}$, with $\mathbf{E}(a < \rho < 2a) = \dfrac{\sigma_0 a}{\epsilon_0 K_1 \rho}$. Thus, the energy density

$$W = \frac{1}{2}\frac{\sigma_0^2 a^2}{\epsilon_0 K_1 \rho^2}.$$

- We have

$$P_\rho = D_\rho - \epsilon_0 E_\rho.$$

Thus, for $0 < \rho < 2a$, we have

$$P_\rho = (1 - 1/K_1)\,\frac{\sigma_0 a}{\rho},$$

and for $\rho > 2a$, we get

$$P_\rho = (1 - 1/K_2)\,\frac{\sigma_0 a}{\rho}.$$

The ratio of polarization at the boundary is thus

$$= \frac{K_1 - 1}{K_1}\frac{K_2}{K_2 - 1}.$$

- From above we see that $|\mathbf{E}(a < \rho < 2a)| = \dfrac{\sigma_0 a}{\epsilon_0 K_1 \rho}$, and $|\mathbf{E}(2a < \rho < 4a)| = \dfrac{\sigma_0 a}{\epsilon_0 K_2 \rho}$.