Planarity

October 29, 2024

Motivation :

1. Planar graphs are a major link between graph theory and geometry/-topology.
2. There are three easily identifiable milestones in planar graph theory.
3. A formula of Euler that $V - E + F = 2$ for any convex polyhedron with V vertices/corners, E edges and F faces.
4. A deep characterization of planar graphs due to Kuratowski.
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1. A graph G is said to be planar or embeddable in the plane if it can be drawn in the plane so that no two edges intersect except (possibly) at their end vertices; otherwise it is said to be a nonplanar graph.
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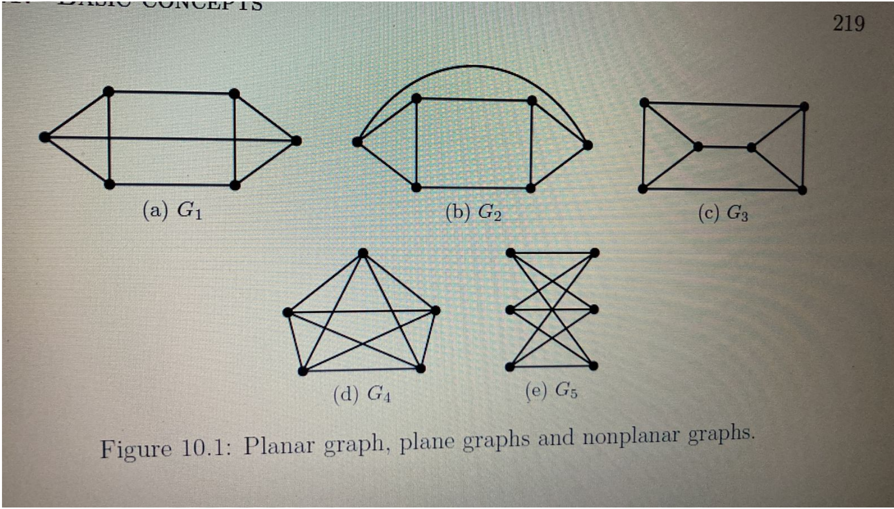
Example



BASIC CONCEPTS 219

(a) G_1 (b) G_2 (c) G_3

(d) G_4 (e) G_5

Figure 10.1: Planar graph, plane graphs and nonplanar graphs.

Questions

1. Find necessary and sufficient conditions for a graph to be planar.
2. How to test a given graph for planarity?
3. Design a (polynomial time) algorithm to draw a given planar graph as a plane graph.

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Possible Solutions

1. The first problem was solved by Kuratowski in 1930.
2. His characterization uses the hereditary nature of planar graphs.
3. A graph theoretic property P is said to be hereditary if a graph has property P then all its subgraphs too have property P.
4. Clearly, acyclicity, bipartiteness and planarity are hereditary properties.
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1. Given any two points a and b in the plane, any non-self-intersecting continuous curve from a to b is called a Jordan curve and it is denoted by $J[a, b]$.

2. If $a = b$, then J is called a closed Jordan curve.

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Jordan Curve Theorem

Theorem

1. *Any closed Jordan curve J partitions the plane into 3 parts namely, interior of J (int J), exterior of J (ext J) and J.*
2. *If J is a closed Jordan curve, $s \in \text{int } J$ and $t \in \text{ext } J$, then any Jordan curve $J'[s, t]$ contains a point of J (that is, J' intersects J).*

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2. Similarly, any cycle is identified with a closed Jordan curve.
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Definition

Let G be a plane graph.

1. G partitions the plane into several regions. These regions are called the faces of G. The set of all faces of G is denoted by $F(G)$.
2. Except one face, every other face is a bounded region. The exceptional face is called the exterior face and other faces are called interior faces of G.
3. The exterior face is unbounded and interior faces are bounded (:= area is finite).

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Faces, Boundary and Degree

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Definition

Let G be a plane graph.

1. The boundary of a face f is the set of all edges of G which are incident with f. It is denoted by $b_G(f)$ or $b(f)$.
2. The degree of a face f in a plane graph G is the number of edges in the boundary of f with cut-edges counted twice.
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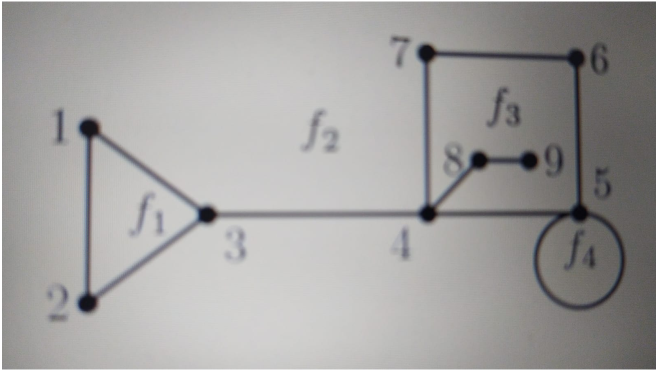

f_1, f_2, f_3, f_4- Faces of G
$b(f_1) = \{12, 23, 13\}$, $d(f_1) = 3$, $b(f_1)$ is a cycle.
$b(f_2) = \{12, 23, 13, 34, 45, 55, 56, 67, 74\}$, $d(f_2) = 10$.
Note that 34 is counted twice and that $b(f_2)$ does not form a cycle.

Consequences of Jordan Curve Theorem

1. A cyclic edge belongs to two faces.
2. A cut-edge belongs to only one face.
3. A plane graph G is acyclic if and only if $|F(G)| = 1$

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Euler's Formula

Theorem

For a connected plane graph G,

$$V - E + F = 2$$

Proof.

We discuss two possible cases:

1. G is acyclic.
2. G is cyclic.

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Case 1 : G is acyclic.

1. G is connected, acyclic and has n vertices.
2. G is a tree and has $n-1$ edges.
3. G has only one face (unbounded one).
4. $V - E + F = n - (n-1) + 1 = 2.$

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Case 2 : G is cyclic.

1. Proof by induction on the number of edges in the graph.
2. Base Case : $E = 3$ (K_3).
3. Then, $v = 3$ and $F = 2$.
4. Hence, $V - E + F = 2$.
5. Induction Hypothesis : Assume true for all connected graphs with m edges.
6. Now consider a graph G with $m + 1$ edges.

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6. Note that $V(G - e) - E(G - e) + F(G - e) = 2$ (By induction if $G - e$ is cyclic, and by Case 1 if $G - e$ is a tree).
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Theorem

If G is a simple connected planar graph with $|V| \geq 3$, then $\frac{3}{2}F \leq E \leq 3V - 6$.

Proof.

We split into two cases.

1. G has no bounded face.
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Case 1 : G has no bounded face.

1. Since G is connected, G is a tree.
2. Then, $|E| = |V| - 1$
3. Since $|V| \geq 3$, $|E| = |V| - 1 \leq 2|V| - 4 \leq 3|V| - 6$.
4. Also, $|F| = 1$, $|E| \geq 2$.
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Proof Contd. Case 2

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Case 2 : G has a bounded face.

1. G contains a cycle.
2. Then, $deg(f) \geq 3$ (Number of edges in the boundary of a face)
3. Each edge is in the boundary of two faces.
4. $\sum deg(f) = 2|E|$
5. Combining the above two, we have, Since $3|F| \leq 2|E| \;\Rightarrow \frac{3}{2}F \leq E$.
6. By Euler's formula, $F = 2 - V + E \;\Rightarrow 3F = 6 - 3V + 3E \leq 2E$.
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1. G contains a cycle.
2. Then, $deg(f) \geq 3$ (Number of edges in the boundary of a face)
3. Each edge is in the boundary of two faces.
4. $\sum deg(f) = 2|E|$
5. Combining the above two, we have, Since $3|F| \leq 2|E| \;\Rightarrow \frac{3}{2}F \leq E$.
6. By Euler's formula, $F = 2 - V + E \;\Rightarrow 3F = 6 - 3V + 3E \leq 2E$.
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Corollary 2

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If G is a connected planar graph, G has a vertex of degree less than six.

Proof.

1. This is clearly true if $|V(G)| = 1$ or 2.
2. Let $|V| \geq 3$.
3. By contradiction, suppose $deg(v) \geq 6$ for all $v \in V(G)$.
4. $6|V| \leq \sum deg(v) = 2|E| \leq 2(3v - 6) = 6V - 12$.
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Let G be a connected planar triangle-free graph with $|V| \geq 3$, then $|E| \leq 2|V| - 4$.

Proof.

1. We have already proved the case when G has no bounded face earlier.
2. Let G be triangle-free and has atleast one bounded face.
3. $\Rightarrow \; deg(f) \geq 4$.
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5. $2|E| = \sum deg(f) \geq 4|F| \;\Rightarrow\; 2|F| \leq |E|$.
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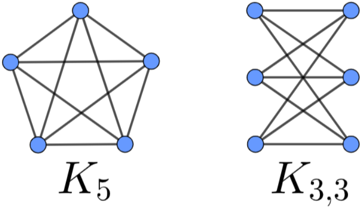

Figure: Kuratowski's graph

Theorem

K_5 and $K_{3,3}$ are non-planar.

Proof.

1. K_5 has 5 vertices and 10 edges.
2. $3|V| - 6 = 15 - 6 = 9 < 10 = |E|$.
3. Hence, K_5 in non-planar.
4. $K_{3,3}$ has 6 vertices, 9 edges and triangle-free.
5. But, $|E| = 9 \not\leq 2|V| - 4 = 12 - 4 = 8$.
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Observations

1. Both are regular graphs.
2. Both are nonplanar.
3. Removal of one edge or a vertex makes each a planar graph.
4. Kuratowski's first graph is the nonplanar graph with the smallest number of vertices.
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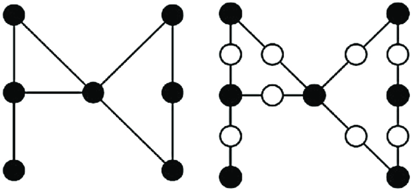
Subdivision

Definition

1. The subdivision of an edge $e(u, v) \in E(G)$ is the operation of replacing e by a path (u, w, v), where w is a new vertex.
2. So, to get a subdivision of e introduce a new vertex w on e.
3. A graph H is said to be a subdivision of G if H can be obtained from G by a sequence of edge subdivisions. (By definition, G is a subdivision of G.)



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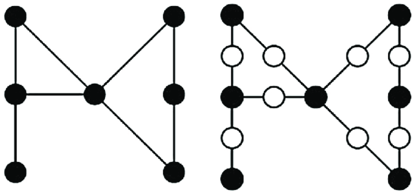

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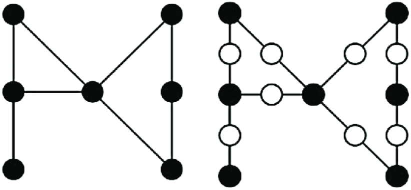

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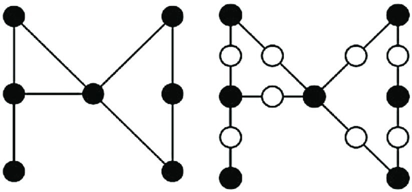

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Kuratowski's Theorem: A graph is nonplanar if and only if it contains a subgraph that is a subdivision of $K_{3,3}$ or K_5.

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