**DEPARTMENT OF PHYSICS**
**INDIAN INSTITUTE OF TECHNOLOGY, MADRAS**

**PH1020 Physics II** **Problem Set 3** **Feb 2024**

1. A sphere has a radius a and a total charge Q, which is distributed uniformly. Determine the electrostatic energy W of the sphere using

   (a)
   $$W = \frac{1}{2}\int \rho V d\tau.$$

   (b)
   $$W = \frac{\epsilon_0}{2}\int_{\text{all space}} |E|^2 d\tau.$$

   In the above, $\rho$ is the charge density, V is the potential, E is the electric field. Obtain your answer in terms of Q, a and $\epsilon_0$.

2. Find the capacitance per unit length of two coaxial metallic cylindrical tubes of radii a and b, where $b > a$.

3. A *point* dipole moment $\mathbf{p}$ is defined as follows: the separation $2a$ between the positive charge $q$ and the negative charge $-q$ tends to zero, while $q$ tends to infinity, such that $p \equiv 2aq$ is a finite positive number.

   (a) Show that the electric field at any point $\mathbf{r}$ due to a point dipole at the origin is given by
   $$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{3(\mathbf{p}\cdot\hat{e}_r)\hat{e}_r - \mathbf{p}}{r^3} \quad .$$

   (b) Now suppose the direction of the dipole moment $\mathbf{p}$ makes an angle $\theta_p$ with the $z$-axis. Calculate the electric flux through the upper hemisphere of a spherical surface of radius $R$ centred at the origin.

4. The computation of the electrostatic potential $\phi(\mathbf{r})$ reduces to solving Laplace's equation, $\nabla^2\phi = 0$ subject to boundary conditions. Consider the following situations:

   (a) Consider a situation with cylindrical symmetry as well as translational symmetry along the $z$-direction. In such cases, the potential is independent of the coordinates $\varphi$ and $z$ i.e., $\phi = \phi(\varrho)$.. Show that the most general solution to Laplace's equation in this case takes the form $\phi = A\log\varrho + B$ for suitable constants $A$ and $B$. Determine $A$ and $B$ using the following boundary conditions:
   $$\phi(\varrho)\big|_{\varrho=a} = V_0 \quad \text{and} \quad \phi(\varrho)\big|_{\varrho=b} = 0 \quad ,$$
   for $a > b > 0$. Take the region of interest to be $b \leq \varrho \leq a$. Compute the electric field for this electrostatic potential. Plot lines of equipotential as well as electrical lines of force.

(b) Consider the situation where two (infinite) metal plates are parallel to each other and separated by a distance $d$. One plate is grounded and the other is kept at $V_0$. Choose the direction of the separation to be the $y$-axis. Solve Laplace's equation in the region between the two plates subject to the given boundary conditions. Compute the electric field for this electrostatic potential. Plot lines of equipotential as well as electrical lines of force.

(c) Consider a grounded (i.e., at zero potential) infinitely long metal sheet (equipotential surface) lying in the $z = 0$ plane. A point charge $+Q$ is located at $z = a > 0$. Verify that

$$\phi(x, y, z) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{Q}{\sqrt{x^2 + y^2 + (z+a)^2}}\right) ,$$

solves Laplace's equation at all points in the region $z \geq 0$ except for the location of the charge as well as satisfying the boundary conditions. If possible, interpret the second term in the above expression in terms of an image charge located somewhere.

5. (a) Show that if the total charge of a system is zero, the dipole moment does not depend on the location of the origin of the coordinates.

   (b) If a charge distribution has a non-zero monopole moment, show that it is possible to find an origin about which the dipole moment is zero.

   (c) Show that the quadrupole moment $\overleftrightarrow{Q}$ of a charge distribution does not depend on the location of the origin, if the monopole and dipole moments of the charge distribution are both zero.

6. Show that for a spherically symmetric charge distribution, the dipole, quadrupole and all higher moments about the centre of the distribution are identically zero.

7. Three point charges $+q$, $+2q$ and $-4q$ are held fixed at points with Cartesian coordinates $(0, 0, d)$, $(-d, 0, d)$ and $(d, d, d)$ respectively, in a Cartesian coordinate system. Calculate the quadrupole moment of the charge distribution about that particular point about which the dipole moment vanishes.

8. Find the monopole, dipole and quadrupole moments about the origin for the following charge distributions:

   (a) A line charge of constant charge density $\lambda$ lying in the first quadrant of the $xy$-plane with one end at the origin, and making an angle $\varphi_0$ with the positive $x$-axis.

   (b) A solid cylinder of height $H$ and radius $R$ with its geometrical centre at the origin, having a uniform volume charge density $\rho_0$, and its axis along the $z$-axis.