CS1200 — Discrete Mathematics for Computer Science

Practice questions

Part A

1. Let p and q be the propositions:
   $p :$ A restaurant has a long queue outside.
   $q :$ The food offered is not delicious.
   Express the following in an English sentence:

   1. $\neg p \land q$
   2. $q \lor (p \land \neg q)$
   3. $\neg q \to p$

2. Show that the following are logically equivalent:

   1. $(p \lor q) \land (p \lor \neg q)$ and p.
   2. $(p \land \neg q) \lor (\neg p \land q)$ and $p \leftrightarrow \neg q$.

3. Express the following statements in terms of predicates and quantifiers:

   (a) Birds that do not live on honey are dull in color.

   (b) For every anime fan, there exists at least one favorite character.

   (c) For every decision, there exists at least one alternate universe where a different choice was made.

4. Express the following statements in terms of predicates and quantifiers. Also check if the following statements be concluded from the given premises:

   1. Premises:
      - Some American football fans support Taylor Swift.
      - No American football fan was allowed to vote in the Grammys.

      Conclusion : Taylor Swift got zero votes at the Grammys.
   2. Premises:
      - All dogs have spots on their body.
      - No small animals eat biscuits.
      - Animals that do not eat biscuits are spotless.

      Conclusion : Dogs are large animals.

5. Charlie is given a quadratic equation $f(x) = x^2 - 8x + 16$ and he believes that if α is a root of $f(x)$ then α is an integer, in other words, if $p : \alpha$ is a root of $x^2 - 8x + 16$ and $q : \alpha \in \mathbb{Z}$ he proposes that $p \to q$. Write an implication proposition $p \to q$ that you believe in for propositions p, q of your choice. Follow up with the converse, contra positive and inverse.

Part B

6. Five friends - Amy, Brian, Chloe, David, and Ethan are playing a board game together. Can you determine who is playing the game if people who are playing are more in number than people who aren't and the following information is known?

    1. Either Amy or Brian, or both, are playing the game.
    2. Either Chloe or David, but not both, are playing the game.
    3. If Ethan is playing the game, then so is David.
    4. Chloe and Amy are either both playing the game or neither is.
    5. If Brian is playing the game, then so are Chloe and Amy.

7. An ancient Sicilian legend says that the barber in a remote town who can be reached only by traveling a dangerous mountain road shaves those people, and only those people, who do not shave themselves. Can there be such a barber?

8. Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x (P(x) \vee Q(x))$ are not logically equivalent.

9. Fill in the missing explanations in the first version of the proof of negating universal quantifier that was sent to class.

10. Let D(x, y) be the statement "The driver x drives a car from team y", where the domain for x consists of all the drivers and the domain for y consists of all the constructors in F1 2023. Let W(x, y) be the statement "The driver x got a podium finish at the track y", where the domain for x consists of all the drivers and the domain for y consists of all the tracks in F1 2023. For each of the following quantification, write the corresponding sentence in English:

    1. $x =$ Max, $\forall y \ . \ W(x, y)$
    2. $y =$ Mclaren, $\exists x \ . \ D(x, y)$
    3. $\forall x \ . \ (\exists y \ . \ W(x, y))$
    4. $y =$ Mercedes, $\forall x \ . \ (D(x, y) \to (\exists z \ . \ (W(x, z))))$
    5. $y =$ Ferrari, $\exists x \ . \ ((D(x, y)) \wedge (\forall z \ . \ W(x, z)))$
    6. $\exists y \ . \ (\forall z \ . \ (\exists x \ . \ (D(x, y) \wedge W(x, z)))$

11. Show that the following are logically equivalent:

    1. $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$

2. $(p \to r) \land (q \to r)$ and $(p \lor q) \to r$
3. $p \to (q \to r)$ and $q \to (\neg p \lor r)$
4. $p \leftrightarrow q$ and $(p \to q) \land (q \to p)$
5. $\neg p \leftrightarrow \neg q$ and $p \leftrightarrow q$

12. Vaishali and Pragg are playing a game (say of chess) where the moves alternate. If C is a game configuration and a series of valid moves $m_1, m_2, .., m_n$ are taken, we denote the new game configuration by $C(m_1, m_2, .., m_n)$
    (a) Let V denote the proposition 'Vaishali wins the game'. After a game configuration $\mathcal{C}$, it is Pragg's turn to play and $M(\mathcal{C})$ denotes the set of valid moves he has at that configuration. However Vaishali believes that she is definitely going to win no matter what move Pragg makes at that point. Being a math fanatic she wants to assert it using quantifiers, help her with a right assertion.
    (b) Let us now denote Vaishali winning at a certain configuration as $A(\mathcal{C})$, which can only occur after one of her moves according to the rules (this need not be expressed in the assertion). Now Vaishali is certain that she is going to win in the next move no matter what move Pragg has made. How do you assert this?
    (c) Suppose Vaishali can win in exactly k number of steps from the current configuration. How can this be expressed using quantifiers?
    (d) In what is called a 'winning strategy', at certain configuration C of Pragg, Vaishali knows that no matter what moves (valid) Pragg makes from now on, she can win the game. How can a 'winning strategy' be expressed using quantifiers?
    (e) Describe one of your favourite game (tic-tac-toe is an option :)) configurations where you can have a winning strategy.

13. One of the well studied problems in theoretical computer science: the SAT or Satisfiability problem is to check if a compound proposition is satisfiable, or in other words, if there exists an assignment of truth values to its variables that makes it true. For example, the proposition $(p \lor \neg q) \land (q \lor \neg p)$ is satisfiable as setting p and q to true satifies the proposition. Determine the satisfiability of the below propositions:
    (a) $(p \lor q \lor \neg r) \land (p \lor \neg q \lor \neg s) \land (p \lor \neg r \lor \neg s) \land (\neg p \lor \neg q \lor \neg s) \land (p \lor q \lor \neg s)$
    (b) $(p \lor q \lor r) \land (p \lor \neg q \lor \neg s) \land (q \lor \neg r \lor s) \land (\neg p \lor r \lor s) \land (\neg p \lor q \lor \neg s) \land (p \lor \neg q \lor \neg r) \land (\neg p \lor \neg q \lor s) \land (\neg p \lor \neg r \lor \neg s)$

14. The #-SAT problem is the problem of counting the number of satisfying assignments for a given proposition, i.e., the number of assignments of truth values to its variables that make it true. For example, the proposition $p \land q$ has 1 satisfying assignment (out of the 4 possible assignments). Compute the #-SAT of the following propositions:
    (a) $(p \to q) \lor (p \land q)$
    (b) $(\neg q \to \neg p) \land (\neg p \lor r)$
    (c) $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg s) \land (s \lor \neg p)$

(d) $s \wedge [(p \to s) \vee (((p \to q) \wedge (q \to r)) \wedge (r \to p))]$

Part C

15. Hundred prisoners in jail are standing in a queue facing in one direction. Each prisoner is wearing a hat of color either black or white. A prisoner can see hats of all prisoners in front of him in the queue, but cannot see his hat and hats of prisoners standing behind him. The jailer is going to ask color of each prisoner's hat starting from the last prisoner in queue. If a prisoner tells the correct color of his own hat, then is saved, otherwise executed. How many prisoners can be saved at most if they are allowed to discuss a strategy before the jailer starts asking colors of their hats.

16. Hundred tigers and one sheep are put on a magic island that only has grass. Tigers can live on grass, but they want to eat sheep. If a Tiger bites the Sheep then it will become a sheep itself. If 2 tigers attack a sheep, only the first tiger to bite converts into a sheep. Tigers don't mind being a sheep, but they have a risk of getting eaten by another tiger. All tigers are intelligent and want to survive. Will the one sheep survive?