PH1020 Physics II | Problem Set 6 (Solutions) | April 2024

1. Recall that the formula for the magnetic dipole moment of a current loop is given by

$$\mathbf{m} = \frac{I}{2}\oint_C (\mathbf{r}' \times d\mathbf{r}') \ .$$

Carrying out the usual replacement

$$I d\mathbf{r}' \longrightarrow q\,\mathbf{u}$$

in the above formula, we obtain that the magnetic dipole moment of a charged particle (located at $\mathbf{r}'$) is given by

$$\mathbf{m} = \frac{q}{2}\Big(\mathbf{r}' \times \mathbf{u}\Big) = \frac{q}{2m}\,\mathbf{L} \ .$$

Thus the magnetic dipole moment of a moving particle is proportional to its orbital angular momentum. **Remark:** The quantity $\mu_B := \frac{e\hbar}{2m_e}$ has the dimensions of magnetic dipole moment and is called the **Bohr magneton**. (Here $\hbar = h/2\pi$ and m_e is the mass of the electron.)

2. We are given $\mathbf{M} = M_0 \frac{\varrho^2}{a^2}\ \hat{e}_\varphi$ inside the cylinder. Inside the cylinder, we have

$$\mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M} = \frac{3M_0\varrho}{a^2}\ \hat{e}_z \ .$$

Outside the cylinder, $\mathbf{J}_b = 0$. The bound surface current is given by

$$\mathbf{K}_b = \mathbf{M} \times \hat{e}_\varrho\Big|_{\varrho=a} = -M_0\ \hat{e}_z \ .$$

Since $\mathbf{J}_f = 0$ in this problem, we can determine $\mathbf{B}$ using the bound currents that we just determined. Symmetry considerations imply that we can choose $\mathbf{B} = B(\varrho)\ \hat{e}_\varphi$. To determine, $B(\varrho)$, we choose as an Amperian loop a circle of radius R centered about the z-axis and lying in a plane given by $z =$ constant. Using Ampère's law, for $R < a$, we get

$$\begin{aligned}\int_{C_R} \mathbf{B}\cdot \mathbf{dl} = (2\pi R)\ B(R) &= \mu_0 \int_0^R (\mathbf{J}_b(\varrho)\cdot\hat{e}_z)\ 2\pi\varrho d\varrho \ , \\ &= \frac{2\pi\mu_0 M_0 R^3}{a^2} \ .\end{aligned}$$

This implies that $B(\varrho) = \frac{\mu_0 M_0 \varrho^2}{a^2}$ for $\varrho < a$. For $R > a$, a similar computation gives

$$\begin{aligned}\int_{C_R} \mathbf{B}\cdot \mathbf{dl} = (2\pi R)\ B(R) &= \mu_0 \int_0^a \mathbf{J}_b(\varrho)\cdot\hat{e}_z\ 2\pi\varrho d\varrho + \mu_0(2\pi a)(-M_0), \\ &= \frac{2\pi\mu_0 M_0 a^3}{a^2} + \mu_0(2\pi a)(-M_0) \\ &= 0 \implies B(R) = 0 \ .\end{aligned}$$

We thus obtain

$$\boxed{\mathbf{B} = \begin{cases} \frac{\mu_0 M_0 \varrho^2}{a^2}\ \hat{e}_\varphi & \text{for } \varrho < a\ , \\ 0 & \text{for } \varrho > a\ . \end{cases}} \tag{1}$$

The discontinuity in $\mathbf{B}$ at $\varrho = a$ is expected due to the presence of a non-zero surface current. The student is asked to check that the discontinuity is as expected i.e. $\hat{n} \times (\mathbf{B}_2 - \mathbf{B}_1) = \mu_0\ \mathbf{K}_b$. We determine $\mathbf{H}$ using the relation $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$ to obtain

$$\boxed{\mathbf{H} = 0 \text{ everywhere.}}$$

The vanishing of $\mathbf{H}$ is related to the absence of free currents.

3. Recall that for a toroidal coil, there is a constant auxiliary field $\mathbf{H}$ (and $\mathbf{B}$) directed along $\hat{e}_\varphi$ in its interior. The magnitude is determined in terms of the current and the total number of coils. We assume that the magnetic field $\mathbf{H}$ (and $\mathbf{B}$) is along $\hat{e}_\varphi$ – this is a good approximation if the wedge angle ψ is small. At the boundary of the wedge, the normal component of $\mathbf{B}$ must be continuous. Since the normal to the wedge is along $\hat{e}_\varphi$, we have $\mathbf{B}_{\text{air}} = \mathbf{B}_{\text{toroid}}$. Thus, the auxiliary field is given by

$$\mathbf{B}_{\text{toroid}} = \mu_0(1+\chi_m)\ \mathbf{H}_{\text{toroid}} \quad , \quad \mathbf{B}_{\text{air}} = \mu_0 \mathbf{H}_{\text{air}}\ . \tag{2}$$

The continuity of the normal component of $\mathbf{B}$ implies that

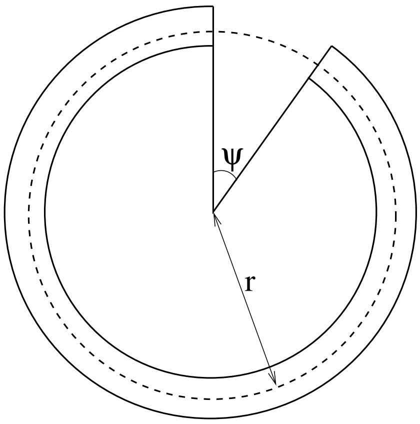

Figure 1: Top view of the toroid.

$$\mathbf{H}_{\text{air}} = (1+\chi_m)\ \mathbf{H}_{\text{toroid}}\ .$$

In order to determine $\mathbf{H}$, we choose an Amperian loop of radius r (indicated by a dashed line in Figure 1) passing through the interior of the toroid. Carrying out the line integral over $\mathbf{H}$, we get

$$\begin{aligned} NI &= H_{\text{toroid}}(2\pi - \psi)\ r + H_{\text{air}}\psi\ r\ , \\ &= H_{\text{toroid}}\ r\,(2\pi - \psi + (1+\chi_m)\psi) \\ &= H_{\text{toroid}}\ r\,(2\pi + \chi_m\psi) \end{aligned}$$

$$\boxed{\implies \mathbf{H}_{\text{toroid}} = \frac{NI}{(2\pi + \chi_m\psi)r}\ \hat{e}_\varphi\ .}$$

4. Translation invariance in the x and y directions implies that

$$\mathbf{H} = \mathbf{H}(z) = H_1(z)\ \hat{e}_x + H_2(z)\ \hat{e}_y + H_3(z)\ \hat{e}_z\ .$$

The boundary conditions on $H(z)$ is

$$\lim_{|z|\to\infty} H(z) = \frac{B_0}{\mu_0}\ .$$

But $(\boldsymbol{\nabla}\times\mathbf{H}) = 0$ implies that $dH_i(z)/dz = 0$ or $H_i(z) = \text{constant}$ for $i = 1, 2$ with no condition on $H_3(z)$. We thus obtain

$$\mathbf{H} = \frac{B_0}{\mu_0}\ \hat{e}_y + H_3(z)\ \hat{e}_z \quad \text{everywhere}\ .$$

To fix $H_3(z)$, we need to impose $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$. Outside the strip it implies $H_3(z)$ is a a constant which vanishes due to the boundary conditions as $|z| \to \infty$. This implies that

$$\mathbf{H} = \frac{B_0}{\mu_0}\ \hat{e}_y \quad \text{outside the strip.}$$

Inside the strip, $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ implies that $\mu(z)H_3(z)$ is a constant. Continuity of the z component of $\mathbf{B}$, at $z = 0$, forces this constant to vanish. Thus, we obtain that $H_3(z) = 0$ everywhere.

$$\boxed{\mathbf{H} = \frac{B_0}{\mu_0}\ \hat{e}_y \quad \text{everywhere}\ .}$$

From this we obtain the magnetic field to be

$$\mathbf{B} = \begin{cases} B_0\left(1+\frac{z}{d}\right)^2\ \hat{e}_y\ , & \text{for } 0 < z < d\ , \\ B_0\ \hat{e}_y\ , & \text{outside the magnetic sheet}\ . \end{cases}$$

The magnetization is determined using $\mathbf{M} = \frac{\mathbf{B}}{\mu_0} - \mathbf{H}$ and we get

$$\mathbf{M} = \begin{cases} \frac{B_0}{\mu_0}\left(2\frac{z}{d} + \frac{z^2}{d^2}\right)\ \hat{e}_y\ , & \text{for } 0 < z < d\ , \\ 0\ , & \text{outside the magnetic sheet}\ . \end{cases}$$

We can determine the current densities using the formulae $\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M}$ and $\mathbf{K}_b = (\mathbf{M}\times\hat{n})$. We get that volume charge density for $0 < z < d$ is given by

$$\boxed{\mathbf{J}_b = -\frac{2B_0}{\mu_0 d}\left(1+\frac{z}{d}\right)\ \hat{e}_x\ .}$$

Using ($\mathbf{M} = 0$, $\hat{n} = -\hat{e}_z$) at $z = 0$ and ($\mathbf{M} = (3B_0/\mu_0)\ \hat{e}_y$, $\hat{n} = \hat{e}_z$) at $z = d$, the surface current densities are

$$\boxed{\mathbf{K}_b = \begin{cases} 0\ , & \text{at } z = 0\ , \\ \frac{3B_0}{\mu_0}\hat{e}_x\ , & \text{at } z = d\ . \end{cases}}$$

Again, the student is asked to check the consistency of the expressions for $\mathbf{B}$ and $\mathbf{K}_b$ by checking that the discontinuity at the two interfaces are consistent with $\hat{n}\times(\mathbf{B}_2 - \mathbf{B}_1) = \mu_0\ \mathbf{K}_b$.