Cut Vertices

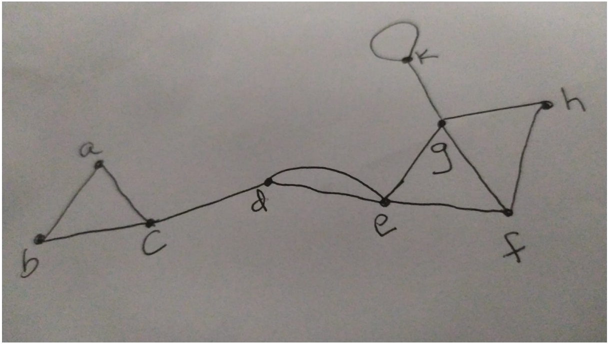
August 26, 2024

Let G be a graph, v be a vertex and e be an edge.

Definition

A vertex v of G is a cut-vertex if the the edge set E can be Partitioned into E_1 and E_2 such that $G[E_1]$ and $G[E_2]$ share only this vertex v.

Example



Cut Vertices : {c, d, e, g, k}

Remark

If G is loopless and non-trivial, the we have the following :

- Cut vertex : v is said to be a cut-vertex if $\omega(G - v) > \omega(G)$.
- If G is connected, then v is a cut-vertex if $G - v$ is disconnected.
- v is a cut-vertex of G if and only if v is a cut-vertex of a component of G:

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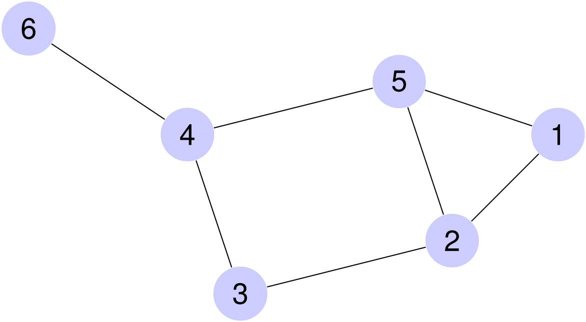
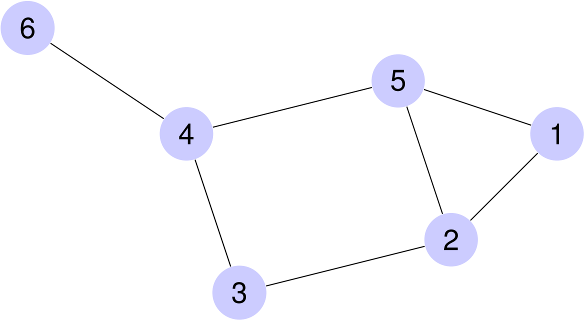


Cut-Vertex : 4

Example



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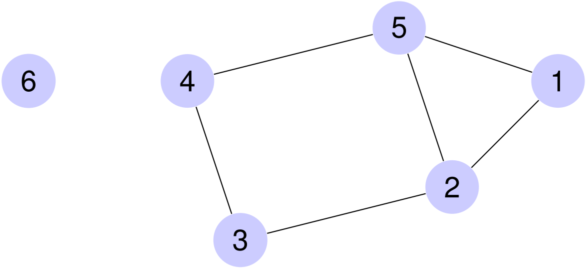

Cut-Vertices : None

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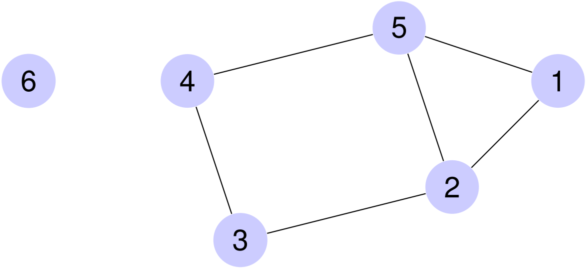

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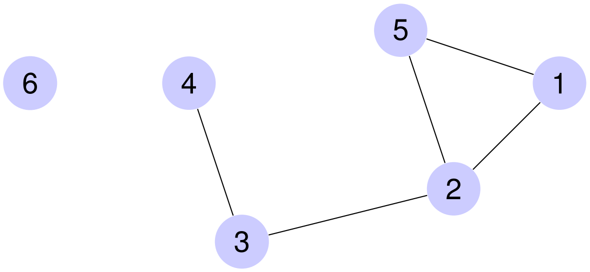
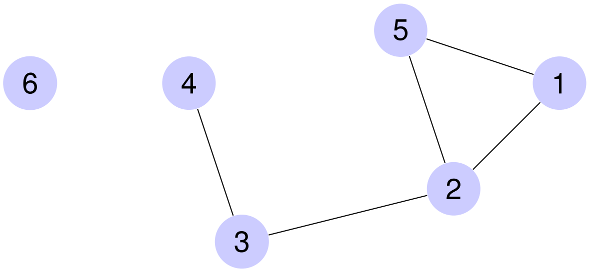


Cut-Vertices : 3 and 2

Example



Cut-Vertices : 3 and 2

Theorem

Let G be a graph with n vertices and $n - 1$ edges, then TFAE:

1. *G is connected*
2. *G is acyclic.*
3. *G is a tree.*

Theorem

A vertex v of a simple connected graph is a cut-vertex if and only if there exist vertices x and y $(\neq v)$ such that every $x - y$-path contains v.

Proof.

" $\Longrightarrow$ " Let $v \in G$ be a cut vertex of G. Need to show existence of vertices x, y such that every $x - y$ path contains v.

1. v is a cut vertex.
2. $G - v$ is disconnected and has atleast two components, say C and D.
3. Let $x \in V(C)$ and $y \in V(D)$.
4. Since there is no $x - y$ path in $G - v$.
5. $\Longrightarrow$ every $x - y$ path in G contains v.

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"$\Leftarrow$" Let there exist vertices x, y such that every $x - y$ path contains v. To show $v \in G$ is a cut vertex of G.

1. Suppose there exist x and y $(\neq v)$ such that every $x - y$ path contains v.
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Theorem

A vertex v is a cut-vertex of a tree if and only if $d(v) > 1$.

Proof.

" $\implies$ "

1. By Previous theorem, if v is a cut vertex, there exists $x, y \neq v$ such that every $x - y$ path contains v.
2. Hence, v is an internal vertex of all $x - y$ paths and $d(v) > 1$.

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Proof.

" $\Leftarrow$ " Given a vertex v such that $d(v) > 1$. To show v is a cut vertex.

1. Given $d(v) > 1$.
2. v is adjacent to atleast two vertices x and y.
3. Note that every $x - y$ path contains v.
4. If not, then P is a $x - y$ path that does not contain v,
5. And $P + xv + vy$ forms a cycle of G a contradiction.
6. Hence, by previous thereom, v is a cut vertex.

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Every non-trivial loopless connected graph has atleast two vertices that are not cut vertices.

Proof.

1. G is a loopless connected graph.
2. G has a spanning tree T.
3. T has atleast two pendant vertices.
4. By previous theorem they are not cut-vertices of T.

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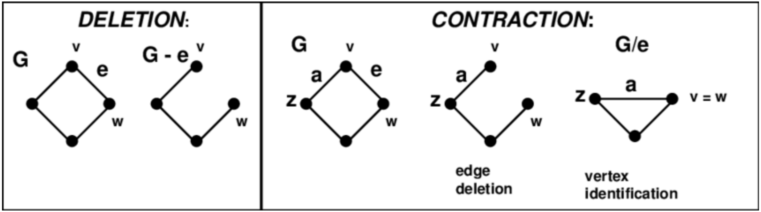
- Counts the number of spanning trees in a given graph.
- Involves two operation on edges : Contraction and Deletion.
- An $e \in G$ is said to be contracted if e is deleted and its ends are identified.
- The resulting graph is denoted by $G \bullet e$ or G/e



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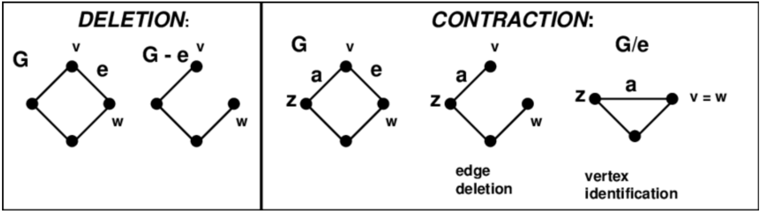

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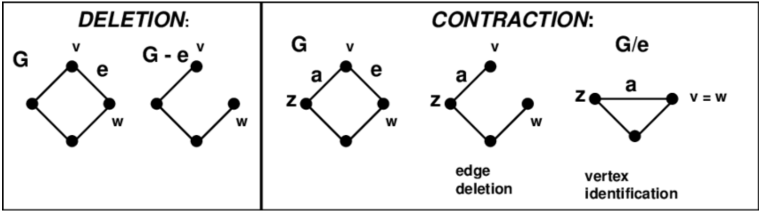

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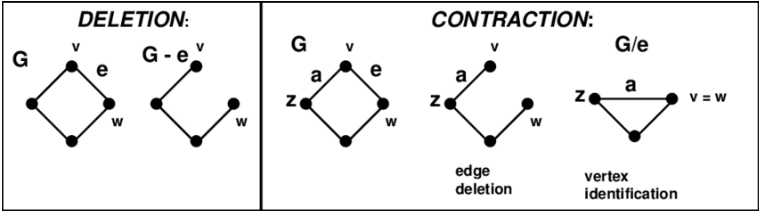

- If e is a link then,
  1. $|V(G \bullet e)| = |V(G)| - 1$
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Denote by $\tau(G)$ the number of spanning trees of G.

Theorem

If e is a link then,

$$\tau(G) = \tau(G - e) + \tau(G \bullet e)$$

Proof.

1. Let A= spanning trees of G containing e.
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Spanning Trees

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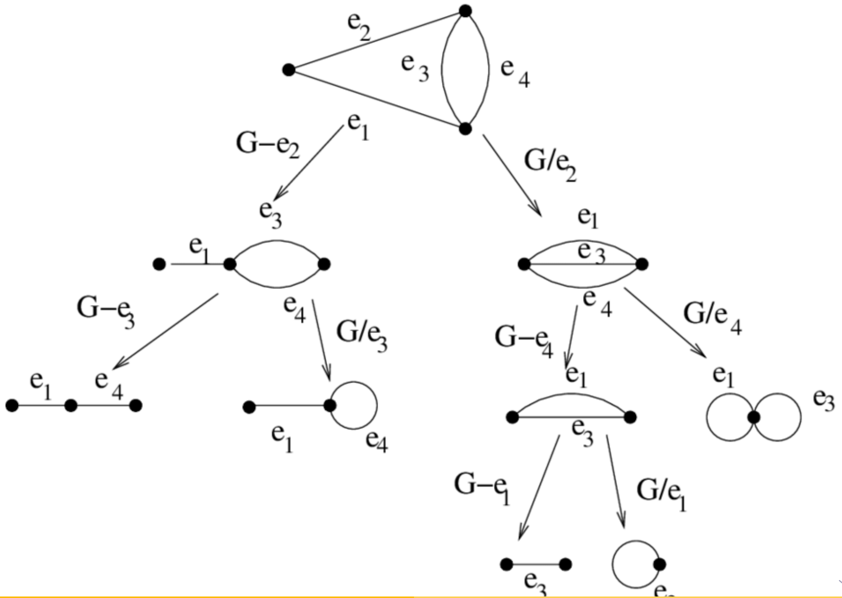

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