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PH1020 Physics II   Problem Set 7 (Solutions)   April 2024

1. We know that the magnetic field due to an infinitely long current carrying conductor is

$$\mathbf{B}(\varrho) = \frac{\mu_0 I}{2\pi\varrho}\ \hat{e}_\varphi\ ,$$

where ϱ and φ are in cylindrical polar coordinates (This follows from Ampère's theorem.) (In the present case, $\varrho = y$)

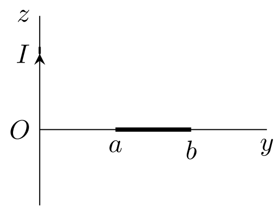

Figure 1: Rod moving in a magnetic field

The induced emf in a length $d\varrho$ of the rod situated at a distance ϱ from the z-axis,

$$\begin{aligned} d\mathcal{E} &= -\text{rate of change of flux} \\ &= -\mathbf{B}(\varrho)\cdot(\mathbf{v}\times\mathbf{dl}), \end{aligned}$$

where $\mathbf{v}$ is the velocity of the moving element and $\mathbf{dl}$ is an element of the conductor.

$$\begin{aligned} d\mathcal{E} &= -\mathbf{B}(\varrho)\cdot(v\hat{e}_z \times d\varrho\hat{e}_\varrho) \\ &= -\frac{\mu_0 I}{2\pi\varrho}\hat{e}_\varphi\cdot(v\ d\varrho\ \hat{e}_\varphi) \\ &= -\frac{\mu_0 I v}{2\pi\varrho}d\varrho \end{aligned}$$

Therefore, emf induced in the rod,

$$\mathcal{E} = \int d\mathcal{E} = -\frac{\mu_0 I v}{2\pi}\int_a^b \frac{d\varrho}{\varrho}$$

$$\boxed{\mathcal{E} = -\frac{\mu_0 I v}{2\pi}\ln\left(\frac{b}{a}\right)\ .}$$

2. (a) By Ampère's theorem, the field at any point $(0, y, z)$ inside the square loop is given by

$$\mathbf{B}(0, y, z) = -\frac{\mu_0 I}{2\pi y}\ \hat{e}_x\ . \tag{1}$$

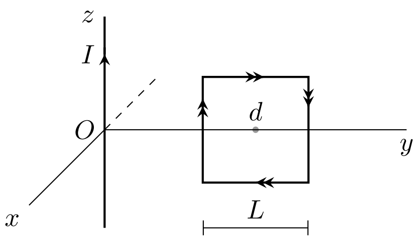

Figure 2: Square loop in a magnetic field

Taking the normal to the square loop to be $-\hat{e}_x$, the flux, Φ_m, through the square loop,

$$\begin{aligned}\Phi_m &= \int \mathbf{B}\cdot\mathbf{da}, \quad \text{where } \mathbf{da} = -dy\ dz\ \hat{e}_x \\ &= \int_{d-\frac{L}{2}}^{d+\frac{L}{2}} dy \int_{-\frac{L}{2}}^{\frac{L}{2}} dz \left(-\frac{\mu_0 I}{2\pi y}\ \hat{e}_x\right)\cdot(-\hat{e}_x) \\ &= \frac{\mu_0 I L}{2\pi}\ln\left(\frac{d+\frac{L}{2}}{d-\frac{L}{2}}\right) \\ &\equiv M\ I\ ,\end{aligned}$$

where

$$\boxed{M = \frac{\mu_0 I}{2\pi}\ln\left(\frac{2d+L}{2d-L}\right)}$$

Observe that if $d \gg \frac{L}{2}$, $M \approx \frac{\mu_0 L^2}{2\pi d}$ on expanding the logarithm in powers of $\frac{L}{2d}$ and retaining the leading term.

(b) The answer found above remains valid in the quasi-static case.[1] (Then Ampére's theorem can still be applied, leading to the relation $\oint \mathbf{B}\cdot\mathbf{dl} = \mu_0 I_{\text{enclosed}}$) Assuming this to be the case, emf induced in the loop,

$$\begin{aligned}\mathcal{E} &= -\frac{d\Phi_m}{dt} = -M\frac{dI}{dt}\ , \quad \text{or} \\ \mathcal{E} &= M\lambda I_0 e^{-\lambda t} \quad \text{(with } M \text{ as given above)}\end{aligned}$$

Now, $\mathbf{B}$ decreases as t increases. Hence, by Lenz's law, the direction of the induced current in the loop will be such that the field it creates will oppose this decrease in flux. Hence, *current is in the direction shown by the double arrows in the figure above.*

3. Since $\mathbf{B}$ is spatially uniform inside the cylindrical region $\varrho \leq a$, but oscillates in time, and $\mathbf{B} = 0$ for $\varrho > a$, we can visualize this field as being produced by an alternating current (AC) flowing in an ideal infinite solenoid of radius a. By cylindrical symmetry, therefore, we may expect $\mathbf{E}$ to be of the form $\mathbf{E}(\varrho,\varphi,z) = E_\varphi(\varrho)\ \hat{e}_\varphi$. That is, it has only an azimuthal component, whose magnitude depends only on the radial distance ρ from the z-axis.

   Integrate over a (planar) circular path of radius ρ about the z-axis to obtain

$$\begin{aligned}\oint_C \mathbf{E}\cdot\mathbf{dl} &= \int_{\varphi=0}^{2\pi} \mathbf{E}_\varphi(\varrho)\hat{e}_\varphi\cdot\varrho d\varphi\ \hat{e}_\varphi \\ &= 2\pi\varrho\ E_\varphi(\varrho)\ .\end{aligned}$$

   But $\oint_C \mathbf{E}\cdot\mathbf{dl} = \int_S \boldsymbol{\nabla}\times\mathbf{E}\cdot\mathbf{da} = -\int_S \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{da}$.

[1] This happens when the term $\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$ in the Maxwell equation, that we are yet to discuss in the class lectures(!), $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$, can be neglected compared to the term $\mu_0\mathbf{J}$.

For $\rho \leq a$, (note that $\mathbf{da}$ is along $\hat{e}_z$)

$$-\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{da} = \omega B_0 \sin(\omega t + \alpha) \pi \varrho^2$$

For $\rho > a$,

$$-\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{da} = \omega B_0 \sin(\omega t + \alpha) \pi a^2.$$

Therefore

$$\boxed{\mathbf{E}(\varrho, \varphi, z) = \begin{cases} \frac{\omega B_0}{2} \varrho \sin(\omega t + \alpha)\ \hat{e}_\varphi\ , & \text{for } \varrho \leq a \\ \frac{\omega B_0}{2} \frac{a^2}{\varrho} \sin(\omega t + \alpha)\ \hat{e}_\varphi\ , & \text{for } \varrho > a \end{cases}} \tag{2}$$

4. The field $\mathbf{H}$ at a distance $\varrho (a \leq \varrho \leq b)$ from the z-axis is $\mathbf{H} = \frac{nI}{2\pi\varrho}\ \hat{e}_\varphi$.

$$\therefore \mathbf{B} = \frac{\mu_0 n I}{2\pi\varrho}\ \hat{e}_\varphi.$$

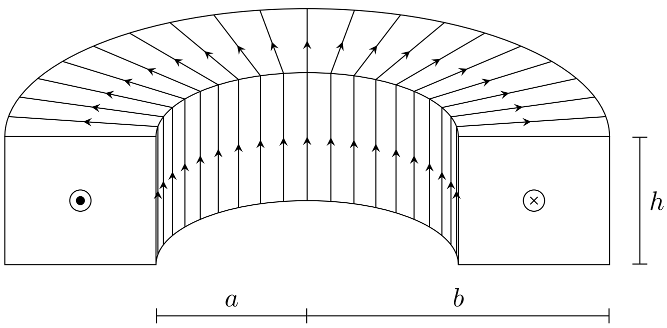

Figure 3: Toroidal Coil

The flux through

$$\begin{aligned} \Phi_m &= n \int \mathbf{B} \cdot \mathbf{da} \\ &= \frac{\mu_0 n^2 h I}{2\pi} \ln\left(\frac{b}{a}\right) \equiv L\ I\ . \end{aligned}$$

Therefore, the self-inductance is given by

$$\boxed{L = \frac{\mu_0 n^2 h}{2\pi} \ln\left(\frac{b}{a}\right)\ .}$$

5. (a) The magnetic field due to the larger coil (call it coil 1) at the location of the small coil (call it coil 2) is

$$\mathbf{B}_1 = \frac{\mu_0 I_1 b^2}{2(b^2 + z^2)^{3/2}}\ \hat{e}_z\ .$$

The flux $\Phi_{2,1}$ through loop 2 due to the current I_1 through loop 1 is

$$\boxed{\Phi_{2,1} = (\pi a^2) \frac{\mu_0 I_1 b^2}{2(b^2 + z^2)^{3/2}} = M_{21}\ I_1\ ,}$$

where $M_{21} = \frac{\mu_0 \pi a^2 b^2}{2(b^2+z^2)^{3/2}}$.

(b) The magnetic dipole moment of the small loop due to a current I_2 flowing through it is

$$\mathbf{m} = \pi a^2 I_2 \ \hat{e}_z \ .$$

The magnetic field due to the dipole located at $(0,0,z)$ at a point $(x,y,0)$ located in the xy-plane is

$$\mathbf{B}_2 = \frac{\mu_0}{4\pi(\varrho^2+z^2)^{3/2}} \left((3\mathbf{m}\cdot\hat{n})\ \hat{n} - \mathbf{m} \right) ,$$

where $\hat{n}$ is the unit vector along $(\varrho\hat{e}_\varrho - z\hat{e}_z)$. The flux $\Phi_{1,2}$ through loop 1 due to the current I_2 through loop 2 is

$$\begin{aligned}
\Phi_{1,2} &= \int_{\text{loop 1}} (\mathbf{B}_2 \cdot \hat{e}_z)\ \varrho d\varrho d\varphi \\
&= \int \frac{\mu_0 m}{4\pi(\varrho^2+z^2)^{3/2}} \left(\frac{3z^2}{(\varrho^2+z^2)} - 1 \right)\ 2\pi\varrho d\varrho \\
&= \frac{\mu_0 m b^2}{2(b^2+z^2)^{3/2}} \ ,
\end{aligned}$$

after carrying out the ϱ integration. We thus obtain

$$\boxed{\Phi_{1,2} = \frac{\mu_0 a^2 b^2}{2(b^2+z^2)^{3/2}}\ I_2 = M_{12}\ I_2 \ ,}$$

where $M_{12} = \frac{\mu_0 \pi a^2 b^2}{2(b^2+z^2)^{3/2}}$.

(c) From the definition of mutual inductance, we see that

$$\boxed{M = M_{12} = M_{21} = \frac{\mu_0 \pi a^2 b^2}{2(b^2+z^2)^{3/2}} \ .}$$

We saw the symmetry from Neumann's formula for mutual inductance. The symmetry $M_{ij} = M_{ji}$ also follows from the definition of energy stored in a system of inductors analogous to what happened to a system of conductors.