Problems for Practice

November 3, 2024

Problem 1

Prove that every maximal matching in a graph G has at least $\alpha'(G)/2$ edges.

Solution 1

1. Recall $\alpha'(G) =$ size of maximum matching.
2. and $\alpha(G) =$ size of maximum independent set.
3. $\beta(G) =$ size of the minimum vertex cover
4. $\beta'(G) =$ size of the minimum edge cover.

Solution 1

1. Let M be a maximal matching.
2. Let S be the set of vertices saturated by M,
3. Then $|M| = \frac{|S|}{2}$.
4. *Also*, $V(G) - S$ is an independent set.
5. So the smaller M is, the larger of an independent set you get.
6. Then, $2|M| = |S| = |n(G)| - |V(G) - S| \geq n(G) - \alpha(G) = \beta(G) \geq \alpha'(G)$
7. So, $|M| \geq \alpha'(G)/2$

Problem 2

Draw an example graph for each of these.

1. A connected simple planar graph has 5 vertices and 3 faces. How many edges does it have?
2. A connected simple planar graph has 7 edges and 5 faces. How many vertices does it have?

Problem 3

What is the maximum number of edges in a simple connected bipartite planar graph with n vertices?

Problem 4

Draw a graph with chromatic number 6 (i.e., which requires 6 colors to properly color the vertices). Could your graph be planar? Explain.

ANS : For example, K_6. If the chromatic number is 6, then the graph is not planar; the 4-color theorem states that all planar graphs can be colored with 4 or fewer colors.

Problem 5

Give an example of a graph with chromatic number 4 that does not contain a copy of That is, there should be no 4 vertices all pairwise adjacent.

Ans : The wheel graph below has this property. The outside of the wheel forms an odd cycle, so requires 3 colors, the center of the wheel must be different than all the outside vertices.

Problem 6

Classify all 3-critical graphs.

Solution 6

Let G be 3-critical. Since it has chromatic number 3, it cannot be bipartite, and so must have some odd cycles. On the deletion of any vertex, the graph must be bipartite with at least one edge, and so every vertex must belong to every cycle. If G has even two odd cycles, then this is impossible (there must be at least one vertex on one odd cycle, but not on the other). So G is a connected graph, with exactly one odd cycle, and every vertex of the graph must be on this cycle. One possibility: G is an odd cycle.

Problem 7

Show by induction that every planar graph is 6-colorable.

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Hint : Use the existence of a vertex of degree atmost 5.

Problem 8

Given G is a simple connected planar graph such that $\delta(G) \geq 5$. Find the minimum number of vertices in G.

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Hint : Handshaking Lemma and $E \leq 3V - 6$

Problem 9

True or False? Provide an explanation or find a counterexample.

1. If $\chi(G) = 3$ then G contains a triangle.
2. If a planar graph contains a triangle, then $\chi(G) = 3$.
3. Isomorphic graphs have the same chromatic number.
4. Homeomorphic graphs have the same chromatic number.
5. Any Hamiltonian graph with $\chi(G) = 2$ is planar.

Solution 9

1. False : Every odd cycle with length $n \geq 3$ has chromatic number 3 and no triangles.
2. False : K_4 is planar and contains a triangle, but $\chi(K_4) = 4$.
3. True : An isomorphism preserves adjacencies between vertices and hence will also preserve a proper colouring.
4. False : C_4 and C_3 are homeomorphic but $\chi(C_4) = 2$ since it is an even cycle, while $\chi(K_3) = 3$ since it is an odd cycle.
5. False : Example $K_{3,3}$.

Problem 10

Explain why a graph with 8 vertices and 17 edges has chromatic number more than two.

Solution 10

1. $\chi(G) > 1$ since there is an edge.
2. We also know a graph is bipartite if and only if $\chi(G) = 2$,
3. so it will suffice to prove that G is not bipartite.
4. Suppose, that G is a bipartite graph with partite sets V and U. Let $|V| = x$ and $|W| = 8 - x$, with $1 \leq x \leq 7$. If $x = 1$ then $K_{1,7}$ has 7 edges. If $x = 2$ then $K_{2,6}$ has 12 edges. If $x = 3$ then $K_{3,5}$ has 15 edges. If $x = 4$ then $K_{4,4}$ has 16 edges.
5. But G has 17 edges and hence not bipartite.
6. So, $\chi(G) > 2$.

Problem 11

Prove that if every region of a planar graph is bounded by an even number of edges, then there exists a 2-colouring of the graph.

Solution 11

Consider a planar embedding of such a graph. Identify one region, since it is bounded by an even cycle, we can colour this cycle with two colours. We can do the same for every region, and if we have already coloured one of the vertices of the boundary cycle then colour the adjacent vertices the opposite colour.

Problem 12

Use induction to prove that $\chi(G) + \chi(G^c) \leq n + 1$, where n is the number of vertices of G.

Solution 12

Base Case: $n = 1$. Certainly $G = G^c$ with $\chi(G) = 1$, therefore $\chi(G) + \chi(G^c) = 2 \leq 1 + 1$.
Induction Hypothesis: Suppose that $\chi(G) + \chi(G^c) \leq n + 1$ for $n = 1 \cdots, k$.
Induction Step: Consider a graph, G, with $k + 1$ vertices.
Consider some vertex, x, of G. By the induction hypothesis, $\chi(G - x) + \chi((G - x)^c) \leq k + 1$.
Let $G - x$ has a t-coloring and $(G - x)^c$ has an l-coloring such that $t + l = k + 1$.
If $deg_G(x) < t$ we can extend the coloring on $G - x$ to a proper t -coloring of G and use a new color in G^c for a total of at most $k + 2$ colors. Otherwise $deg_G(x) \geq t$, so
$deg_{G^c}(x) \leq k - t = l - 1 < l$. So, we can extend the coloring on $(G - x)^c$ to a proper l-coloring of G^c and use a new color on G as desired.