DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH1020 Physics II | Problem Set 3 (Solutions) | Feb 2024

---

1. To calculate the electrostatic energy we need to first find the electric field. By using Gauss's law this can be easily determined to be

$$E_i = \frac{\rho r}{3\epsilon_0}\hat{e}_r = \frac{Qr}{4\pi\epsilon_0 a^3}\hat{e}_r$$

in the interior of the sphere, and

$$E_O = \frac{Q}{4\pi\epsilon_0 r^2}\hat{e}_r$$

in the exterior. Since, this is the only input needed to calculate part.(b), we will attempt that part first:

(a) Using

$$W = \frac{\epsilon_0}{2}\int_{\text{allspace}} |E|^2 d\tau,$$

we get

$$W = 2\pi\epsilon_0\left[\int_0^a |E_i|^2 r^2 dr + \int_a^\infty |E_O|^2 r^2 dr\right].$$

Substituting for E_i, and E_O, we obtain

$$W = 2\pi\epsilon_0\left[\frac{Q^2}{(4\pi\epsilon_0)^2 a^6}\int_0^a r^4 dr + \frac{Q^2}{(4\pi\epsilon_0)^2}\int_a^\infty \frac{1}{r^2}dr\right].$$

Performing the integrals we get

$$W = \frac{3Q^2}{20\pi\epsilon_0 a}.$$

. Now let us see if our answer using the other formula matches what we have obtained.

(b) To use the formula

$$W = \frac{1}{2}\int \rho \mathrm{V} d\tau,$$

we need to derive V. The value of the potential can be easily found out by solving the differential equation

$$\frac{\partial \mathrm{V_O}}{\partial r} = -\frac{Q}{4\pi\epsilon_0 r^2}$$

. Solving this under the condition that as $r \to \infty$, $\mathrm{V_O} \to 0$ gives

$$\mathrm{V_O} = \frac{\mathrm{Q}}{4\pi\epsilon_0 \mathrm{r}}.$$

To solve for the potential in the interior we use

$$\frac{\partial \mathrm{V_i}}{\partial r} = -\frac{Qr}{4\pi\epsilon_0 a^3},$$

with the boundary condition that $\mathrm{V_O} = \mathrm{V_i}$ at $r = a$. This gives the value of the

$$\mathrm{V_i} = \frac{\mathrm{Q}}{8\pi\epsilon_0 \mathrm{a}}\left[3 - \frac{\mathrm{r}^2}{\mathrm{a}^2}\right].$$

Thus substituting in the formula for the electrostatic energy in terms of the charge density and the potential inside the sphere we get

$$W = \frac{3Q}{8\pi a^3}\int_0^a\int_0^\pi\int_0^{2\pi}\frac{Q}{8\pi\epsilon_0 a}\left[3-\frac{r^2}{a^2}\right]r^2\sin\theta dr d\theta d\phi.$$

Performing the angular integrals we get

$$W = \frac{3Q^2}{16\pi\epsilon_0 a^4}\int_0^a\left[3-\frac{r^2}{a^2}\right]r^2 dr = \frac{3Q^2}{16\pi\epsilon_0 a^4}\left[a^3-\frac{a^3}{5}\right].$$

Once again we get

$$W = \frac{3Q^2}{20\pi\epsilon_0 a}.$$

2. Assume the charge per unit length is given by λ. Exploiting the cylindrical symmetry and using Gauss's law we can write

$$\int E\cdot dS = E2\pi\rho L = \frac{Q_{\text{enc}}}{\epsilon_0} = \lambda L,$$

we get $E_\rho = \dfrac{\lambda}{2\pi\epsilon_0}\dfrac{1}{\rho}\hat{e}_\rho$. To evaluate the potential difference between the cylinders we use

$$V(b)-V(a) = -\int_a^b E\cdot\ell = -\frac{\lambda}{2\pi\epsilon_0}\int_b^a\frac{1}{\rho}d\rho = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right).$$

Thus, the capacitance is

$$C = Q/V = \frac{2\pi\epsilon_0 L}{\ln\left(\dfrac{b}{a}\right)}.$$

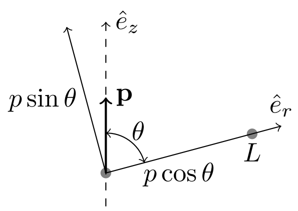

Figure 1: A point dipole aligned with the z-axis

3. (a) With no loss of generality, choose the align the dipole along the z-axis. The electrostatic potential, $\phi(\mathbf{r})$, at a point L which is at a distance r from the point dipole is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^2}\,\boldsymbol{p}\cdot\hat{e}_r = \frac{1}{4\pi\epsilon_0 r^2}\,(p\cos\theta)\ .$$

The electric field is determined from the above potential as follows:

$$\begin{aligned}\boldsymbol{E} &= -\boldsymbol{\nabla}\phi\\ &= \frac{p}{4\pi\epsilon_0 r^3}[2\cos\theta\ \hat{e}_r + \sin\theta\ \hat{e}_\theta]\\ &= \frac{1}{4\pi\epsilon_0 r^3}[3(\boldsymbol{p}\cdot\hat{e}_r)\hat{e}_r - (p\cos\theta\ \hat{e}_r - p\sin\theta\ \hat{e}_\theta)]\\ &= \frac{1}{4\pi\epsilon_0 r^3}[3(\boldsymbol{p}\cdot\hat{e}_r)\hat{e}_r - \boldsymbol{p}]\end{aligned}$$

The last expression is independent of the coordinate system used.

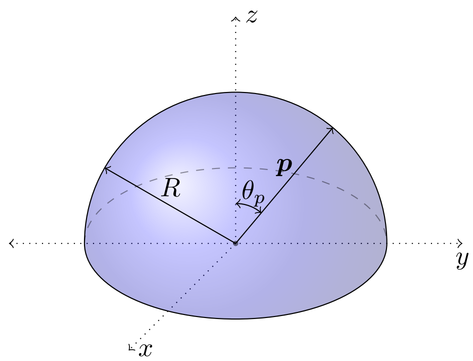

Figure 2: Calculating flux through the curved surface of upper hemisphere of a sphere.

(b) From (3a) above,

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0 r^3}[3(\boldsymbol{p}\cdot\hat{e}_r)\hat{e}_r - \boldsymbol{p}],$$
$$d\boldsymbol{S} = R^2 \sin\theta \; d\theta \; d\varphi \; \hat{e}_r.$$
$$\therefore \Phi = \int \boldsymbol{E}\cdot d\boldsymbol{S} = \int \frac{1}{4\pi\epsilon_0 R}(2\boldsymbol{p}\cdot\hat{e}_r)d\Omega.$$

Note that the limits on θ integration are from 0 to $\pi/2$ while the limits on φ integration are from 0 to 2π.

$$\boldsymbol{p}\cdot\hat{e}_r = p_x \sin\theta\cos\varphi + p_y \sin\theta\sin\varphi + p_z\cos\theta$$

Since

$$\int_0^{2\pi} \cos\varphi = 0 \quad \text{and} \quad \int_0^{2\pi} \sin\varphi = 0,$$

only the p_z term survives in the integration. On performing the integration, the flux is obtained to be

$$\Phi = \frac{p_z}{2\epsilon_0 R} = \frac{p\cos\theta_p}{2\epsilon_0 R}.$$

4. **(a)** This problem is taken from the first experiment in your PH1040 lab manual. Kindly refer to this for the solution! The final answer is (for $b \le \varrho \le a$)

$$\phi(\varrho) = V_0 \; \frac{\log(\varrho/b)}{\log(a/b)} \; , \tag{1}$$
$$\mathbf{E} = -\frac{V_0}{\varrho \; \log(a/b)} \; \hat{e}_\varrho \; . \tag{2}$$

(b) This problem is taken from the first experiment in your PH1040 lab manual. Kindly refer to this for the solution! The final solution (for $0 \le y \le d$) is

$$\phi(y) = V_0 \; \frac{y}{d} \; , \tag{3}$$
$$\mathbf{E} = -\frac{V_0}{d} \; \hat{e}_y \; . \tag{4}$$

(c) It is straightforward to check that the given potential

$$\phi(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{\sqrt{x^2+y^2+(z-a)^2}} - \frac{Q}{\sqrt{x^2+y^2+(z+a)^2}}\right) \; ,$$

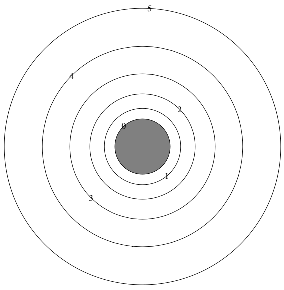

Figure 3: Equipotentials for $\phi = 0, 1, 2, 3, 4, 5V$ when $V_0 = 5V$

satisfies Laplace's equation. The boundary conditions are that the potential vanishes for $z = 0$ and goes to zero as $z \to \infty$. Again these can be checked easily and hence no details are necessary. It is important to note that the above potential is identical to the the potential in all space i.e., in $\mathbb{R}^3$ with two charges, $+Q$ located at $(0, 0, a)$ and another charge $-Q$ located at $(0, 0, -a)$. The location of the second charge is the **mirror** image of the location of the first charge if there were a mirror at $z = 0$. It is important to notice that the image is located outside the region of interest in our problem i.e., $z > 0$. This method of solving Laplace's equation is called the **method of images**. A fun exercise is to ask a similar question (for the potential due to a point charge located say at $(0, a, b)$) but now with an additional grounded plate at $y = 0$ and the region of interest is now $y > 0$ and $z > 0$. How many image charges will you need?

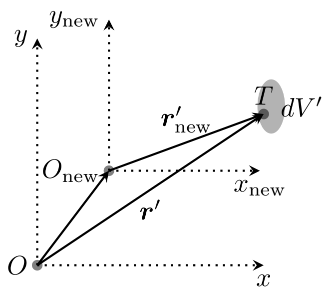

Figure 4: Shift of origin of coordinates and its effect on the dipole moment

5. (a)

$$\begin{aligned}\boldsymbol{p}_{\text{new}} &= \int \varrho(T)\boldsymbol{r}'_{\text{new}} dV' \\ &= \int \varrho \boldsymbol{r}' dV' - a \int \varrho \; dV' \\ &= \boldsymbol{p} - \boldsymbol{a}Q\end{aligned}$$

If total charge $Q = 0$, then $\boldsymbol{p}_{\text{new}} = \boldsymbol{p}$.

(b) $\boldsymbol{p}_{\text{new}}$ in 5a will be zero if

$$\begin{aligned}\boldsymbol{p} &= \boldsymbol{a}Q \\ \text{i.e.,} \int \varrho \boldsymbol{r}' dv' &= \boldsymbol{a} \int \varrho \; dv' \\ \therefore a = |\boldsymbol{a}| &= \frac{\int \varrho r' dv'}{\int \varrho dv'}\end{aligned} \tag{5}$$

Hence, the point about which the new dipole moment is zero is at a distance a from O ('a' as given in equation (5)). So, shift the origin from O to O_{new}. In the discrete case, consider a collection of charges q_i at positions $\boldsymbol{r}_i$ from O. The expression for $\boldsymbol{a}$ now becomes

$$\boldsymbol{a} = \frac{\sum_i q_i \boldsymbol{r}_i}{\sum_i q_i}.$$

Note: The denominator, that is, the total charge, should not be zero. When that happens, the dipole moment is independent of the choice of origin. So if it non-vanishing, it can never be made to vanish by a shift.

(c) Q is the quadrupole moment tensor.

$$Q_{ij} = \int (3r'_i r'_j - (r'^2)\delta_{ij})\rho dV' \tag{6}$$

where δ_{ij} is Kronecker delta (Recall PH1010). $(Q_{xx})_{\text{new}}$ is the xx component of quadrupole moment tensor evaluated about O_{new}. Similarly, other components of Q_{new} are defined.

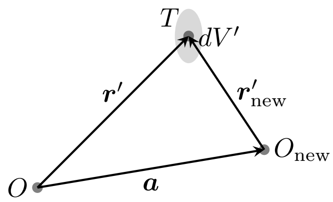

Figure 5: Finding a new origin about which the dipole moment vanishes.

$$\begin{aligned} x'_{\text{new}} &= x' - a_x \\ y'_{\text{new}} &= y' - a_y \\ z'_{\text{new}} &= z' - a_z \end{aligned} \tag{7}$$

$\boldsymbol{a} = a_x\hat{e}_x + a_y\hat{e}_y + a_z\hat{e}_z$ is a constant vector. $(Q_{xx})_{\text{new}}$ (from Equation (6) above)

$$\int \rho(T)(2x'^2_{\text{new}} - y'^2_{\text{new}} - z'^2_{\text{new}})dV'. \tag{8}$$

Substitute (7) in (8) and regroup terms. Then,

$$\begin{aligned} (Q_{xx})_{\text{new}} = &\int \rho(T)(2x'^2 - y'^2 - z'^2)dV' \\ &+ \int \rho(T)(2a_x^2 - a_y^2 - a_z^2)dV' \\ &+ \int (-4x'a_x + 2y'a_y + 2z'a_z)dV' \end{aligned} \tag{9}$$

The first term in (9) is Q_{xx}, the xx component of the quadrupole moment tensor about O. The second term in (9) is

$$(2a_x^2 - a_y^2 - a_z^2)\int \rho\, dV' \tag{10}$$

where $q = \int \rho\, dV'$ is the total charge. The third term is

$$-4a_x \int \rho x' dV' + 2a_y \int \rho y' dV' + 2a_z \int \rho z' dV' = -4a_x p_x + 2a_y p_y + 2a_z p_z$$

This implies that if $\boldsymbol{p} = 0$ and $q = \int \rho\, dV' = 0$, then $(Q_{xx})_{\text{new}} = Q_{xx}$. More generally, one can show that

$$\boxed{(Q_{ij})_{\text{new}} = Q_{ij} - 3a_i p_j - 3a_j p_i + (\mathbf{a}\cdot\mathbf{p})\delta_{ij} + 3qa_i a_j - qa^2\delta_{ij}\ ,} \tag{11}$$

where $\mathbf{p}$ and q are the dipole moment and the total charge respectively in the original coordinate system. Thus, if $\mathbf{p} = 0$ and $q = 0$, we see that $Q_{ij})_{\text{new}} = Q_{ij}$.

6. Consider a spherical distribution of charges with total charge Q. At a point $\boldsymbol{r}$ outside the distribution,

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0 r}\int \rho dV.$$

But this is the monopole term in the multipole expansion. This implies that all higher moments vanish.

7. From Problem (5b), the point about which the dipole moment of the charge distribution is zero, has position vector

$$\frac{\sum_i q_i \boldsymbol{r}_i}{\sum_i q_i}$$

This point is called the **centre of charge**. Therefore, in this problem the x component of the position vector of the center of charge is $6d$. Similarly, the y coordinate is $4d$, and the z coordinate is d. Thus, the position vector for the centre of charge is $(6d, 4d, d)$. With respect to this centre of charge,

(a) charge q has position vector $\boldsymbol{r}_1 = (-6d, -4d, 0)$
(b) charge $2q$ has position vector $\boldsymbol{r}_2 = (-7d, -4d, 0)$
(c) charge $-4q$ has position vector $\boldsymbol{r}_3 = (-5d, -3d, 0)$

$$\therefore Q_{xx} = \sum_{i=1}^{3}(3x_i^2 - r_i^2)$$

Note:

(a) $x_1 = -6d$ and $\boldsymbol{r}_i^2 = 52d^2$ and so on. Substituting and simplifying, we get $Q_{xx} = 56qd^2$. Similarly, Q_{yy} and Q_{zz} can be obtained.

(b) $Q_{zz} = -(Q_{xx} + Q_{yy})$. While $Q_{xy} = Q_{yx} = 60qd^2$, all other components of Q are zero because the z coordinate of all the charges is zero.

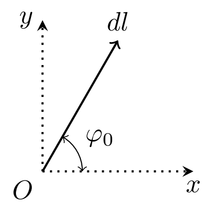

Figure 6: A line charge (of density λ) lying in the first quadrant of the xy plane, making an angle φ_0 with positive x axis.

8. (a) Monopole moment $q = \lambda L$.
Dipole moment about O is $\boldsymbol{p} = \int_0^L \boldsymbol{r}\lambda \, dl$

$$\boldsymbol{r} = x\hat{e}_x + y\hat{e}_y \text{ with } x = l\cos\varphi_0,\ y = l\sin\varphi_0 \ .$$

Substituting for x and y in terms of l

$$\begin{aligned}\boldsymbol{p} &= \lambda\int_0^L (l\cos\varphi_0\hat{e}_x + l\sin\varphi_0\hat{e}_y)dl \\ &= \frac{\lambda L^2}{2}(\cos\varphi_0\hat{e}_x + \sin\varphi_0\hat{e}_y)\end{aligned}$$

$$
\begin{aligned}
Q_{xx} &= \lambda \int_0^L (3x^2 - r^2) dl \\
&= \lambda \int_0^L (3l^2 \cos^2 \varphi_0 - l^2) dl \\
&= (3\cos^2 \varphi_0 - 1)\frac{\lambda L^3}{3} \\
Q_{yy} &= \lambda \int_0^L (3y^2 - r^2)\ dl \\
&= (3\sin^2 \varphi_0 - 1)\frac{\lambda L^3}{3} \\
Q_{zz} &= -(Q_{xx} + Q_{yy}) \\
&= -\frac{\lambda L^3}{3} \\
Q_{xy} &= 3\lambda \cos\varphi_0 \sin\varphi_0 \int_0^L l^2\ dl \\
&= \lambda L^3 \cos\varphi_0 \sin\varphi_0 \\
Q_{yz} &= Q_{xz} = 0\ . \quad (\because z \text{ coordinate is zero})
\end{aligned}
$$

In summary we have $q = \lambda L$,

$$
\boxed{\mathbf{p} = \frac{\lambda L^2}{2}(\cos\varphi_0 \hat{e}_x + \sin\varphi_0 \hat{e}_y) \quad , \quad Q = \lambda L^3 \begin{pmatrix} \cos^2\varphi_0 - \frac{1}{3} & \cos\varphi_0 \sin\varphi_0 & 0 \\ \cos\varphi_0 \sin\varphi_0 & \sin^2\varphi_0 - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}\ .}
$$

(b) Limits of integration: $-H/2 \leq z' \leq H/2$, $0 \leq \varrho' \leq R$ and $0 \leq \varphi' < 2\pi$.

$$
\begin{aligned}
\text{Monopole term} &= \text{total charge } q = \rho_0 \pi R^2 H \\
\text{Dipole moment about origin} &= 0 \quad (\because \text{ reflection symmetry}) \\
Q_{xx} &= Q_{yy} \quad (\because x \text{ and } y \text{ are equivalent}) \\
Q_{zz} &= \int (3z'^2 - r'^2)\rho_0 dV' \qquad r'^2 = x'^2 + y'^2 + z'^2 \\
dV' &= \varrho'\ d\varrho'\ d\varphi\ dz \qquad \varrho'^2 = x'^2 + y'^2 \\
\therefore Q_{zz} &= \rho_0 \int (2z'^2 - \rho'^2)\varrho'\ d\varrho'\ d\varphi'\ dz' \\
&= \frac{\rho_0 \pi H R^2}{6}\left(H^2 - 3R^2\right) \\
&= \frac{q}{6}\left(H^2 - 3R^2\right) \\
&= -(Q_{xx} + Q_{yy}) \\
Q_{yz} &= 0 \qquad (\because \rho(x,y,z) = \rho(x,y,-z)) \\
\text{Similarly, } Q_{xy} &= Q_{xz} = 0\ .
\end{aligned}
$$

In summary we have

$$
\boxed{q = \rho_0 \pi R^2 H \quad , \quad \mathbf{p} = 0 \quad , \quad Q = \frac{q\left(H^2 - 3R^2\right)}{12} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\ .}
$$