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PH1020 Physics II | Problem Set 1 (Solutions) | January 2024

The Electric Field and its Flux

1. There are quite a few ways that we can approach this problem. Let us assume that the square is made of rods which are infinitesimal thickness and of length a. Let us assume an infinitesimal charge element placed at $r' = (x', y', 0)$ and the electric field felt due to the element at the observation point given by $r = (0, 0, z)$. Thus, the field due to a rod of length a at the observation point r can be found as:

$$dE_{\mathbf{rod}} = \frac{1}{4\pi\epsilon_0}\int \lambda \frac{-x'\hat{e}_x - y'\hat{e}_y + z\hat{e}_z}{(z^2 + x'^2 + y'^2)^{3/2}} dx'.$$

By symmetry one can argue that the x and y components vanish and one is left to do the integral

$$dE_{\mathbf{rod}} = \frac{1}{4\pi\epsilon_0}\int \lambda \frac{z\hat{e}_z}{(z^2 + x'^2 + y'^2)^{3/2}} dx'.$$

This is the same integral we have seen in Griffiths, with the modification that $L \to a/2$, and $z^2 \to (y'^2 + z^2)$. Thus, we get

$$dE_{\mathbf{rod}} = \frac{1}{4\pi\epsilon_0}\frac{\lambda a z}{(y'^2 + z^2)\sqrt{y'^2 + z^2 + a^2/4}}\hat{e}_z.$$

Now, to get the result for the entire sheet, I integrate the above expression over the width of the plate. Thus, we get

$$E_z = \frac{1}{4\pi\epsilon_0}\int_{-a/2}^{a/2} \frac{\sigma a z}{(y'^2 + z^2)\sqrt{y'^2 + z^2 + a^2/4}} dy'.$$

Performing the integral we get

$$E_z = \frac{\sigma}{\pi\epsilon_0}\arctan\frac{a^2}{2z\sqrt{2a^2 + 4z^2}}. \tag{1}$$

Now, let us look at limits: In the limit $z \gg a$, we can expand the $\sqrt{2a^2 + 4z^2}$ in the argument of the arctan above giving

$$E_z \approx \frac{\sigma}{\pi\epsilon_0}\arctan\left[\frac{a^2}{4z^2}\left(1 - a^2/4\right)\right] \approx \frac{\sigma}{\pi\epsilon_0}\arctan\left[a^2/4z^2\right].$$

Now, expand the $\arctan x \approx x$ giving

$$E_z = \frac{\sigma a^2}{4\pi\epsilon_0 z^2}.$$

Similarly, we can look at the limit where $a \to \infty$, giving

$$E_z = \frac{\sigma}{2\epsilon_0}$$

Just a note of caution: The above derived expression looks different from the solution given in Griffiths to problem 2.45. This is not so as we argue below.
Consider

$$\arctan\frac{x^2-1}{2x} = \frac{\pi}{2} - \arctan\frac{2x}{x^2-1} = \frac{\pi}{2} + \arctan\frac{2x}{1-x^2}$$

Where in the first equality we have used $\arctan x + \arctan 1/x = \pi/2$, and the second equality we have used $\arctan x = -\arctan -x$. Now using the fact the addition formula for the arctan, we get $\arctan 2x/(1-x^2) = 2\arctan x \pmod{\pi}$. This implies that

$$\arctan\frac{x^2-1}{2x} = \pi/2 + 2\arctan x$$

Now, for us we can substitute $x^2 = 1 + a^2/2z^2$ in Eq. 1 and use the above relation for the arctan to arrive at the same answer given in Griffiths. Note that to get the factor of $\pi/4$ to work out you have to add enough factors of $n\pi$

2. (a)
   - With no loss of generality, choose $\boldsymbol{K}$ along $\hat{e}_z$. Then the charge density is

     $$\sigma(\boldsymbol{r}) = Kr\cos\theta,$$

     which is positive in the northern ($0 \le \theta \le \frac{\pi}{2}$) hemisphere and negative in the southern hemisphere ($\frac{\pi}{2} \le \theta \le \pi$) and zero on the equator ($\theta = \frac{\pi}{2}$).
   - Using cylindrical symmetry, the electric field at O has no x or y components.
   - The z-component field due to the surface element at (θ, φ) with area $R^2 d\Omega$ at O is

     $$\begin{aligned} \boldsymbol{dE}\cdot\hat{e}_z &= \left(\frac{R^2 d\Omega}{4\pi\epsilon_0 R^2}\sigma(\boldsymbol{r})(-\hat{e}_r)\right)\cdot\hat{e}_z \\ &= -\frac{d\Omega}{4\pi\epsilon_0}\sigma(\boldsymbol{r})\cos\theta \\ &= -\frac{RK}{4\pi\epsilon_0}\, d\theta\ \sin\theta\ (\cos\theta)^2\ d\varphi \end{aligned}$$

Integrating over the surface of the sphere, we get

$$\begin{aligned}
\hat{e}_z \cdot \boldsymbol{E} &= -\frac{K}{4\pi\epsilon_0}\int_0^{\pi} d\theta \ \sin\theta \ (\cos\theta)^2 \int_0^{2\pi} d\varphi \\
&= -\frac{RK}{3\epsilon_0} \\
&\boxed{\text{or } \boldsymbol{E} = -\frac{R\boldsymbol{K}}{3\epsilon_0}}
\end{aligned}$$

(b) Again, choose the z-axis along $\boldsymbol{K}$. Think of the solid sphere as a collection of spherical shells of thickness Δr and

$$\sigma(\boldsymbol{r}) = \rho(\boldsymbol{r})\Delta r = Kr\cos\theta \ \Delta r.$$

From part (a), the shell at radius r contributes

$$\hat{e}_z \cdot \Delta\boldsymbol{E}|_{\text{shell}} = -\frac{\boldsymbol{K}r\Delta r}{3\epsilon_0} \cdot \hat{e}_z.$$

Integrating (summing over all shells), we get

$$\begin{aligned}
\hat{e}_z \cdot \boldsymbol{E} &= \int_0^R -\frac{\boldsymbol{K}r dr}{3\epsilon_0} \cdot \hat{e}_z \\
&\boxed{\boldsymbol{E} = -\frac{\boldsymbol{K}R^2}{6\epsilon_0}}
\end{aligned}$$

Note: $\boldsymbol{K}$ has different physical dimensions in part (a) and part (b).

3. (a)

$$\rho_{\text{charge}} = \begin{cases} \frac{\beta\varrho}{a} & 0 < \varrho \leq a \\ 0 & a < \varrho < \infty \end{cases}$$

Choose the Gaussian surface to be a cylinder of height L and radius ϱ, co-axial with the z-axis.

For $0 < \varrho \leq a$

$$\begin{aligned}
\oint_S \boldsymbol{E} \cdot d\boldsymbol{S} &= \frac{1}{\epsilon_0} Q_{\text{enclosed}} \\
&= \frac{1}{\epsilon_0}\int_0^L dz \int_0^{2\pi} d\varphi \int_0^{\varrho} d\varrho' \varrho' \rho_{\text{charge}} \\
&= \frac{2\pi L}{\epsilon_0}\int_0^{\varrho} d\varrho' \varrho' \frac{\beta\varrho'}{a} \\
&= \frac{2\pi L\beta}{a\epsilon_0}\frac{\varrho^3}{3} \qquad (2)
\end{aligned}$$

The direct flux obtains contribution from the 'curved' part of the cylinder and the two caps. The two caps have $d\boldsymbol{S}$ along $\pm\hat{e}_z$ while the 'curved part' has $d\boldsymbol{S}$ along $+\hat{e}_z$. Since $\boldsymbol{E} = E(\varrho)\,\hat{e}_\varrho$, we see that the caps have *no* contribution.

$$\begin{aligned}
\oint_S \boldsymbol{E}\cdot\boldsymbol{S} &= \int_{\substack{\text{`curved'}\\ \text{part}}} \boldsymbol{E}\cdot\boldsymbol{S} \\
&= \int_0^L dz \int_0^{2\pi} d\varphi\varrho E(\varrho) \\
&= (2\pi L)\varrho E(\varrho). \qquad (3)
\end{aligned}$$

Equating (2) to (3), we get

$$(2\pi L)\varrho E(\varrho) = (2\pi L)\frac{\beta}{a\epsilon_0}\frac{\varrho^3}{3}.$$

$$\boxed{E(\varrho) = \frac{\beta\varrho^2}{3a\epsilon_0}} \quad \text{for } 0 \le \varrho \le a. \qquad (4)$$

where $\boldsymbol{E} = E(\varrho)\hat{e}_\varrho$.

For $\varrho > a$, we need to redo the computation done in (2). Since $\rho_{\text{charge}} = 0$ for $\rho > a$,

$$\begin{aligned}
Q_{\text{enclosed}} &= \int_0^L dz \int_0^{2\pi} d\varphi \int_0^a d\varrho'\varrho'\frac{\beta\varrho'}{a} \\
&= (2\pi L)\frac{\beta a^2}{3}. \\
\frac{Q_{\text{enclosed}}}{\epsilon_0} &= (2\pi L)\frac{\beta a^2}{3\epsilon_0}. \qquad (5)
\end{aligned}$$

Equation (3) continues to hold. Now equate (5) to (3) to get

$$(2\pi L)\ \varrho E(\varrho) = (2\pi L)\ \frac{\beta a^2}{3\epsilon_0}$$

$$\boxed{E(\varrho) = \frac{\beta a^2}{3\epsilon_0\varrho}} \quad \text{for } \rho > a \qquad (6)$$

where $\boldsymbol{E} = E(\varrho)\hat{e}_\varrho$. Combining the two results we get

$$\boxed{\boldsymbol{E} = \begin{cases} \frac{\beta\varrho^2}{3a\epsilon_0}\ \hat{e}_\varrho & 0 \le \varrho \le a \\ \frac{\beta a^2}{3\epsilon_0\varrho}\ \hat{e}_\varrho & \varrho \ge a \end{cases}} \qquad (7)$$

Note: Check that $\boldsymbol{E}$ is continuous at $\varrho = a$.

(b) The calculation proceeds in a fashion similar to the previous one. We do the computation in two parts, one for $r < a$ and another for $r > a$.

$$\begin{aligned}
\int_{S_r} \boldsymbol{E}\cdot d\boldsymbol{S} &= \int_{S_r} E(r) r^2 d\Omega \\
&= 4\pi r^2 E(r),
\end{aligned}$$

where we have chosen a Gaussian surface that is a sphere of radius r centered at the origin.

$$
\begin{aligned}
\left.\frac{Q_{\text{enclosed}}}{\epsilon_0}\right|_{r\leq a} &= \frac{1}{\epsilon_0}\int_0^r 4\pi r'^2\beta\left(1-\frac{r'^2}{a^2}\right)dr' \\
&= \frac{4\pi\beta}{\epsilon_0}\left(\frac{r^3}{3}-\frac{r^5}{5a^2}\right) \\
\left.\frac{Q_{\text{enclosed}}}{\epsilon_0}\right|_{r>a} &= \frac{1}{\epsilon_0}\int_0^a 4\pi r'^2\beta\left(1-\frac{r'^2}{a^2}\right)dr' \\
&= \frac{4\pi\beta}{\epsilon_0}\left(\frac{a^3}{3}-\frac{a^5}{5a^2}\right) \\
&= \frac{8\pi\beta a^3}{15\epsilon_0}
\end{aligned}
$$

So we obtain

$$
\boxed{\boldsymbol{E} = \begin{cases} \frac{\beta r}{\epsilon_0}\left(\frac{1}{3}-\frac{r^2}{5a^2}\right)\hat{e}_r & 0\leq r\leq a \\ \frac{2\beta a^3}{15\epsilon_0 r^2}\hat{e}_r & r\geq a \end{cases}} \tag{8}
$$

Note: Check that $\boldsymbol{E}$ is continuous at $r=a$.

4. We need to compute the force on the charge on the x-axis close to the origin and show that it is of the form (for small x)

$$F(x) = -k\ x \quad \text{for some } k>0\ .$$

Then we can use the 'standard' formula for the time-period i.e., $T = 2\pi\sqrt{\frac{m}{k}}$.

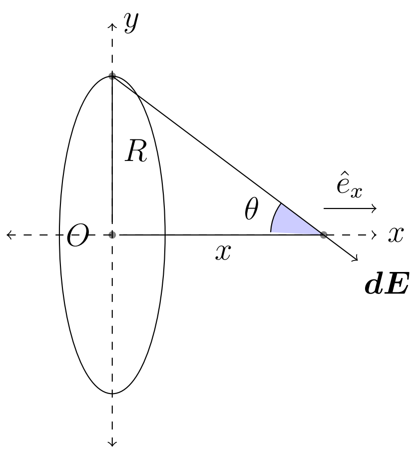

Figure 1: The contribution of an infinitesimal segment of the circular line charge at $(x,0,0)$.

The x-component of the force on the charge is (see Figure 1)

$$
\begin{aligned}
F(x) &= \hat{e}_x\cdot(-Q\ \boldsymbol{E}(x)) \\
&= (-Q)\ \frac{2\pi R\lambda}{4\pi\epsilon_0}\frac{\cos\theta}{(x^2+R^2)}\ ,
\end{aligned}
$$

where we see that x component of $\boldsymbol{E}$ for each infinitesimal segment of the ring *add* up with the y and z components canceling out.

$$
\begin{aligned}
F(x) &= -\frac{QR\lambda}{2\epsilon_0}\frac{x}{(x^2+R^2)^{3/2}} \\
&\approx -\left(\frac{Q\lambda}{2\epsilon_0 R^2}\right)x \\
\implies T &= 2\pi\sqrt{\frac{2m\epsilon_0 R^2}{Q\lambda}}
\end{aligned}
$$

5. The electric field $\vec{E}$ due to the two point charges placed at $\pm\ell$ is given by the superposition principle to be $E_x(x,y)\hat{e}_x + E_y(x,y)\hat{e}_y$, where the components $E_x(x,y)$, and $E_y(x,y)$ of the electric field is given by

$$
E_x(x,y) = \frac{q}{4\pi\epsilon_0}\left[\frac{(x+\ell)}{\left((x+\ell)^2+y^2\right)^{3/2}} - \frac{(\ell-x)}{\left((x-\ell)^2+y^2\right)^{3/2}}\right], \tag{9}
$$

and the componet $E_y(x,y)$ is given by

$$
E_y(x,y) = \frac{q}{4\pi\epsilon_0}\left[\frac{y}{\left((x+\ell)^2+y^2\right)^{3/2}} + \frac{y}{\left((x-\ell)^2+y^2\right)^{3/2}}\right], \tag{10}
$$

(a) The problem asks us to find an approximation for $E_x(x, y=0)$ for points close to the x axis and in the limit $x \ll \ell$. To do so Taylor expand $E_x(x, y=0$

$$
E_x(x, y=0) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(x+\ell)^2} - \frac{1}{(\ell-x)^2}\right] \tag{11}
$$

giving

$$
\begin{aligned}
E_x(x, y=0) &= \frac{q}{4\pi\epsilon_0\ell^2}\left[\frac{1}{(1+x/\ell)^2} - \frac{1}{(1-x/\ell)^2}\right] \\
&\approx \frac{q}{4\pi\epsilon_0\ell^2}\left[(1-\frac{2x}{\ell}) - (1+\frac{2x}{\ell})\right] \\
&= -\frac{qx}{\pi\epsilon_0\ell^3}.
\end{aligned}
$$

In a similar manner at points close to the origin on the y axis, the component E_y can be approximated using Eq. 10 to be

$$
\begin{aligned}
E_y(y) &= \frac{q}{4\pi\epsilon_0}\left[\frac{2y}{(\ell^2+y^2)^{3/2}}\right] \\
&\approx \frac{qy}{2\pi\epsilon_0\ell^3}.
\end{aligned}
$$

(b) The second part essentially has us confirming Gauss's divergence theorem. There is a very small cylinder with length $2x_0$ and radius r_0 placed along the x axis. There is an inward flux through the two circular faces which is given $\Phi_{\textbf{caps}} = E_x(x_0)2\pi r_0^2$, (here the factor of 2 comes from the fact that there two circular caps at the end of the cylinder each giving an infinitesimal area πr_0^2). Furthermore, there is a curved surface that gives a contribution $\Phi_{\textbf{curved}} = E_y(r_0)2\pi r_0 2x_0$, (here the term $2\pi r_0 2x_0$, is the area of the curved surface of a cylinder of radius $2r_0$, and length $2x_0$). Thus, the total flux

$$\Phi = \Phi_{\textbf{caps}} + \Phi_{\textbf{curved}} = -\frac{qx_0}{\pi\epsilon_0\ell^3}2\pi r_0^2 + \frac{qr_0}{2\pi\epsilon_0\ell^3}2\pi r_0 2x_0 = 0.$$

Thus, confirming Gauss's law.

6. The two perpendicular line charges are depicted by skewer rods in Figure 6. O is the point of intersection of the 2 lines and is the center of the cube of length L. The field due to the line charge along with the x-axis has components along $\hat{e}_y$ and $\hat{e}_z$ (since in this case radially outward direction lies in the $y-z$ plane.). Thus, the flux due to this line charge passes through only four of the faces that **do not intersect** the line charge and vanishes for the other two faces since $\hat{e}_x \cdot \mathbf{E} = 0$. A rotation about the x-axis by angle $\pi/2$ is a symmetry that permutes these four faces. Thus, the total flux through any of these faces equals $\frac{1}{4}$ of the total flux. The total flux due to this line charge is $\frac{\lambda L}{4\epsilon_0}$ and thus the contribution to each of the four faces is $\frac{\lambda L}{4\epsilon_0}$. Let us denote the contribution from the line charge along x-axis by flux$_x$. Similarly, the field due to the line charge along the y-axis has components along $\hat{e}_x$ and $\hat{e}_z$. Repeating the argument, we see that the faces $y = \pm L/2$ have no flux passing through them. The other four share the total flux equally. Let us denote the contribution from the line charge along y-axis by flux$_y$. The total flux due to both line charges is the sum of these two fluxes and the result is summarized in the following table.

Face	flux$_x$	flux$_y$	Total flux
$x = +\frac{L}{2}$	0	$\frac{\lambda L}{4\epsilon_0}$	$\frac{\lambda L}{4\epsilon_0}$
$x = -\frac{L}{2}$	0	$\frac{\lambda L}{4\epsilon_0}$	$\frac{\lambda L}{4\epsilon_0}$
$y = +\frac{L}{2}$	$\frac{\lambda L}{4\epsilon_0}$	0	$\frac{\lambda L}{4\epsilon_0}$
$y = -\frac{L}{2}$	$\frac{\lambda L}{4\epsilon_0}$	0	$\frac{\lambda L}{4\epsilon_0}$
$z = +\frac{L}{2}$	$\frac{\lambda L}{4\epsilon_0}$	$\frac{\lambda L}{4\epsilon_0}$	$\frac{\lambda L}{2\epsilon_0}$
$z = -\frac{L}{2}$	$\frac{\lambda L}{4\epsilon_0}$	$\frac{\lambda L}{4\epsilon_0}$	$\frac{\lambda L}{2\epsilon_0}$