CS2200 : Tutorial 1 Solutions

Wednesday, 29 January 2025

1. The set of all binary strings $\{0,1\}^*$ is countable.

   We can list all binary strings in a systematic way:

   $$\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots$$

   By assigning each string a natural number : $\epsilon \to 1$, $0 \to 2$, $1 \to 3$, $00 \to 4$, ..., we can form a one-to-one correspondence with $\mathbb{N}$. Since we can enumerate all elements, the set $\{0,1\}^*$ is countable.

2. If S is a countable set and $A \subseteq S$ is infinite, then A is countable.

   Since S is countable, there exists a bijection $f : S \to \mathbb{N}$, meaning every element of S can be uniquely mapped to a natural number. Let A be an infinite subset of S. We define a function $g : A \to \mathbb{N}$ by restricting f to A, i.e., for every element $x \in A$, we set $g(x) = f(x)$. We can list the elements of A as $a_1, a_2, a_3, \ldots$ in increasing order of g values, forming a sequence indexed by $\mathbb{N}$.

   Since we have established an bijective function from A to $\mathbb{N}$, it follows that A is countable.

3. (a) **Bijection between $\mathbb{Q}$ and $\mathbb{N}$**

   Every rational number can be written as p/q, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We arrange the rationals as follows:

   $$\frac{0}{1}, \frac{1}{1}, \frac{-1}{1}, \frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}, \frac{1}{3}, \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}, \frac{3}{1}, \frac{3}{2}, \frac{-3}{1}, \frac{-3}{2}, \frac{1}{4}, \frac{3}{4}, \frac{-1}{4}, \frac{-3}{4}, \frac{4}{1}, \frac{4}{3}, \ldots$$

   By following the order above, it is clear that every rational number will appear somewhere in this list. Since we have an explicit enumeration, this shows that $\mathbb{Q}$ is countable, meaning there exists a bijection $f : \mathbb{Q} \to \mathbb{N}$.

   (b) **Bijection between $\mathbb{N}$ and $\mathcal{P}_f(\mathbb{N})$**

   The set $\mathcal{P}_f(\mathbb{N})$ consists of all finite subsets of $\mathbb{N}$. To establish a bijection with $\mathbb{N}$, we enumerate these subsets in a systematic way. We arrange them in increasing order of their largest element.

   For example, we order the subsets as follows:

   $$\emptyset, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{4\}, \{1,4\}, \{2,4\}, \ldots$$

   Here, subsets whose largest element is 1 are listed first, followed by those whose largest element is 2, then 3, and so on. Within each group, subsets are listed in lexicographic order. This ensures that every finite subset appears at a finite position in the sequence.

By assigning a unique natural number to each subset based on this ordering, we establish a bijection $g : \mathbb{N} \to \mathcal{P}_f(\mathbb{N})$, where:

$$\emptyset \to 1, \quad \{1\} \to 2, \quad \{2\} \to 3, \quad \{1,2\} \to 4, \quad \{3\} \to 5, \quad \ldots$$

Since this process lists all finite subsets of $\mathbb{N}$ without repetition and assigns each a unique natural number, it forms a valid bijection.

(c) **Bijection between $\mathbb{R}$ and $\mathcal{P}(\mathbb{N})$**

The power set $\mathcal{P}(\mathbb{N})$ consists of all subsets of $\mathbb{N}$. Each element of $\mathcal{P}(\mathbb{N})$ can be represented as a sequence of 0s and 1s, where a 1 indicates inclusion of that natural number in the subset.

Every real number in the interval $[0, 1]$ has a unique binary expansion:

$$x = \sum_{j=1}^{\infty} a_j \left(\frac{1}{2}\right)^j,$$

where each $a_j \in \{0, 1\}$.

A problem arises: some rational numbers in $[0, 1]$ have two different binary representations. For example:

$$0.100000\ldots_2 = 0.011111\ldots_2 .$$

To fix the ambiguity, we classify problematic sequences into two cases:

- Sequences with a finite number of 0s.
- Sequences with a finite number of 1s.

Both sets are countably infinite, so we can enumerate them as sequences p_n and q_n. We then define a function $f(X)$ to standardize representations:

$$f(X) = \begin{cases} q_{2k}, & X = p_k \\ q_{2k+1}, & X = q_k \\ X, & \text{otherwise.} \end{cases}$$

This ensures that each real number in $[0, 1]$ has a unique binary representation.

Example: Consider the subset $S = \{1, 2\}$, which corresponds to the binary sequence $110000\ldots_2$. This sequence has an alternative representation:

$$0.110000\ldots_2 = 0.101111\ldots_2 .$$

Since this sequence is problematic, it is assigned an index k in the enumeration of such cases. Suppose S is the 5th element in the list of problematic sequences p_k. Then, we apply the function $f(X)$ as follows:

$$f(S) = q_{10}.$$

where q_{10} is the 10th element in the list of uniquely assigned sequences.

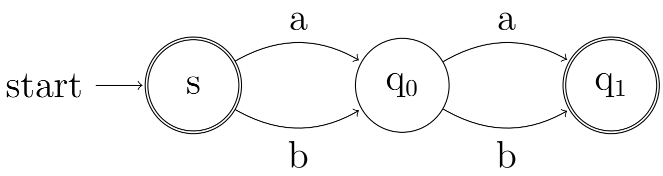
Now, consider another problematic sequence $T = \{3, 4\}$ corresponding to $0.00110000\ldots_2$. If T is the 7th sequence in the problematic list q_k, then:

$$f(T) = q_{15}.$$

Since the mapping $f(X)$ only modifies a countable set of numbers while preserving all others, it does not affect the overall cardinality, thus preserving the bijection between $\mathbb{R}$ and $\mathcal{P}(\mathbb{N})$.

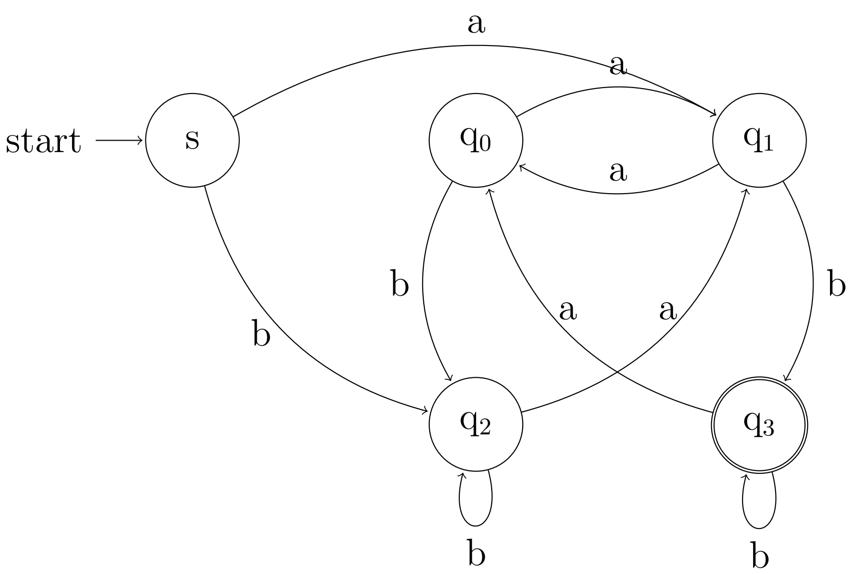
4. Draw state diagrams representing finite automata that will accept the following languages. The alphabet $\Sigma = \{a, b\}$.

(a) $L_1 = \{w \mid w \text{ is any string other than } a \text{ or } b\}$



States s and q_1 are accepting states.

(b) $L_2 = \{w \mid w \text{ has an odd number of } a\text{'s and end with } b\}$

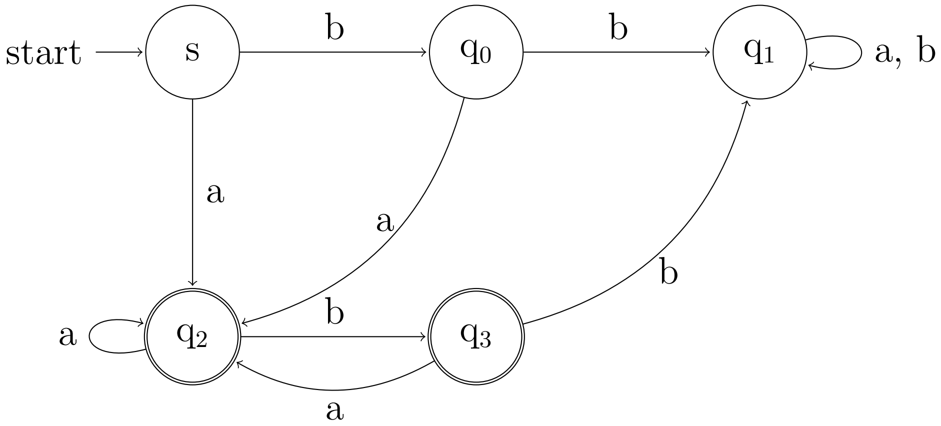


Explanation :

- q_0 represents an even number of a's and the string does not end with b.
- q_1 represents an odd number of a's and the string does not end with b.
- q_2 represents an even number of a's but the string ends with b.
- q_3 represents an odd number of a's and the string ends with b.

States s and q_0 are accepting states.

(c) $L_3 = \{w \text{ where } w \text{ has atleast one } a \text{ and doesn't contain the substring } bb\}$.



Explanation:

- q_0 represents that the string has no a's and has one b.
- q_1 represents that the string has the substring bb and is a dead state.
- q_2 represents that the string has at least one a and ends with a.
- q_3 represents that the string has at least one a and ends with one b.

States q_2 and q_3 are accepting states.

5. The languages accepted by the given DFAs are:

   (a) $L = \{w \mid w$ contains at least three occurrences of a$\}$.

   (b) $L = \{w \mid w$ consists only a's or only b's$\}$.

   (c) $L = \{w \mid w$ is a sequence of at least zero b's, followed by at least one a, followed by at least one b.$\}$