M2. Program Correctness

Instructor: Manikandan Narayanan

Weeks 1-2

CS2700 (PDS) Moodle: https://courses.iitm.ac.in/course/view.php?id=4892

Acknowledgment of Sources

- Slides based on content from related
  - Courses:
    - IITM – Profs. **Rupesh**/Krishna(S)/Prashanth/Kartik's PDS (Thy/Lab) offerings (slides, quizzes, notes, lab assignments, etc. for instance from Rupesh's Jul 2019 offering - www.cse.iitm.ac.in/~rupesh/teaching/pds/jul19/)
    - *Most slides are based on Rupesh Nasre's slides – we thank him and acknowledge by marking* ***[RN]*** *in the bottom right of these slides.*
    - *Program correctness slides in turn based on IITB Prof. Supratik Chakraborti's slides:* https://www.cse.iitb.ac.in/~supratik/courses/cs615/msri_ss08.pdf
  - Books:
    - **Main textbook:** *"Data Structures and Algorithm Analysis in C++"* by ***Weiss*** (content, figures, slides, exercises/questions, etc.). – cited as [WeissBook]
    - Additional/optional book: *"Practice of Programming"* by *Kernighan and Pike* (style of programming, programming exercises/questions, etc.) – cited as [KPBook]

Outline for Module M2

- M2 Program Correctness
  - **M2.1 Introduction and Example**
  - M2.2 Pre/Post-Conditions (of assignments and conditionals)
  - M2.3 Loop invariants (and pre/post-conditions of loops)
  - M2.4 Hoare Logic/Triples (for formal proofs)

Program Correctness

- Notions of correctness
  - Proof
  - Testing
- Proofs cover all the cases.
- Testcases are inherently incomplete.
  - except when the number of inputs is finite.
  - In general, testing can prove *presence* of bugs, and not their absence.
- Proofs are guided by computation properties.

[RN]

Example

```c
#include <stdio.h>

int main() {
	int var = 3;
	int flag = 0;
	int *ptr = &var;

	if (flag == 0) ptr = NULL;

	if (*ptr < 5) printf("if\n");
	else printf("else\n");

	return 0;
}
```

```
$ gcc bug.c
$ a.out
Segmentation fault (core dumped)
$
```

[RN]

Example

```c
#include <stdio.h>
#include <assert.h>
int main() {
	int var = 3;
	int flag = 0;
	int *ptr = &var;

	if (flag == 0) ptr = NULL;

	assert(ptr != NULL);

	if (*ptr < 5) printf("if\n");
	else printf("else\n");

	return 0;
}
```

$ gcc bug.c

Error: Undefined reference to **assert**

$ gcc bug.c

$ a.out

a.out: bug.c:12: main: Assertion `ptr != NULL' failed.

Aborted (core dumped)

$

With this knowledge, now the error can be traced backwards.

[RN]

Example

```c
#include <stdio.h>
#include <assert.h>
int main() {
	int var = 3;
	int flag = 0;
	int *ptr = &var;

	assert(ptr != NULL);

	if (flag == 0) ptr = NULL;

	if (*ptr < 5) printf("if\n");
	else printf("else\n");

	return 0;
}
```

```
$ gcc bug.c
$ a.out
Segmentation fault (core dumped)
$
```

With this knowledge, now the error can be traced forward.

[RN]

Outline for Module M2

- M2 Program Correctness
  - M2.1 Introduction and Example
  - **M2.2 Pre/Post-Conditions (of assignments and conditionals)**
  - M2.3 Loop invariants (and pre/post-conditions of loops)
  - M2.4 Hoare Logic/Triples (for formal proofs)

Pre and Post-conditions

- Define constraints that **must** be satisfied at a program point.
- Constraints are simple boolean expressions.
  - We will use C-style conditions.
- Pre-condition (post-condition) identifies constraints prior to (after) a statement.
- Ideally, we want the **most precise** conditions.
- Types of statements:
  - assignments, conditionals, loops.

[RN]

Assignments

```
{x > 0}
x++;
{??}
```

[RN]

Assignments

```
{x > 0}
x++;
{x > 1}
```

Assumptions:

- Values do not overflow (no wrap-around).
- Type information is available.

```
{x > 0}
x = 2 * x;
{ ?? }
x = x + y;
{ ?? }
x = x - y;
{ ?? }
```

Notes

- If x > 0 and it is an int, x is at least 1. Hence, 2 * x > 1. Additionally, x % 2 == 0.
- If y is unknown, x + y can be any value.
- If type of y is known to be unsigned, then x + y would also be above 1.
- If the effect of two statements can be gauged together, we can regain the information that x > 1. Otherwise, we lose track of x's value.

[RN]

Conditionals

```
{x > 0}
if (x > 5) {

    x = 4;

}
```

[RN]

Conditionals

```
{x > 0}
if (x > 5) {
   {x > 5}
   x = 4;
   {x == 4}
}
{ ?? }
```

[RN]

Conditionals

```
{x > 0}
if (x > 5) {
    {x > 5}
    x = 4;
    {x == 4}
}
{x > 0 && x <= 5 ||
 x == 4}
```

After an `if (C) S`

- either the effect of `S` is visible;
- or the effect of the statements prior to this `if` statement falls through. In this case, the condition `C` is guaranteed to be false.

[RN]

Conditionals

```
{x > 0}
if (x > 5) {
   {x > 5}
   x = 4;
   {x == 4}
}
{x > 0 && x <= 5}
```

[RN]

Conditionals

```
{x > 0}
if (x > 5) {
   {x > 5}
   x = 4;
   {x == 4}
} else {

   x -= 10;

}
```

[RN]

Conditionals

```
{x > 0}
if (x > 5) {
    {x > 5}
    x = 4;
    {x == 4}
} else {
    {x <= 5}
    x -= 10;

}
```

[RN]

Conditionals

```
{x > 0}
if (x > 5) {
    {x > 5}
    x = 4;
    {x == 4}
} else {
    {x <= 5 && x > 0}
    x -= 10;
    {x <= -5 && x > -10}
}
```

[RN]

Conditionals

```
{x > 0}
if (x > 5) {
   {x > 5}
   x = 4;
   {x == 4}
} else {
   {x <= 5 && x > 0}
   x -= 10;
   {x <= -5 && x > -10}
}
{x == 4 || x ϵ (-10..-5]}
```

After an if (C) S1 else S2

- On entering a conditional, the conditions get ANDed (at S1, C gets ANDed; at S2, NOT C gets ANDed).
- On exiting a conditional, the conditions get ORed (at the end of if-else).
- Relative updates retain+modify existing conditions (e.g., x -= 10).
- Absolute updates generate new conditions (e.g., x = 4).

Note that the effect of the conditions prior to if-else may not reach end of if-else (unlike in if-without-else)

[RN]

Getting back to the segfault

```c
#include <stdio.h>

int main() {
	int var = 3;
	int flag = 0;
	int *ptr = &var;

	if (flag == 0) ptr = NULL;

	if (*ptr < 5) printf("if\n");
	else printf("else\n");

	return 0;
}
```

Write pre- and post-conditions for this program.

Compare your inference with what happens at program execution time.

[RN]

Conditionals

```
{x > 0}
if (...) {
   {x > 5}
   ...
  {x == 4}
} else {
   {x <= 5 && x > 0}
   ...
   {x <= -5 && x > -10}
}
{x == 4 || x ϵ (-10..-5]}
```

Thought exercise: Given a set of pre and post-conditions, can we automatically generate a program?

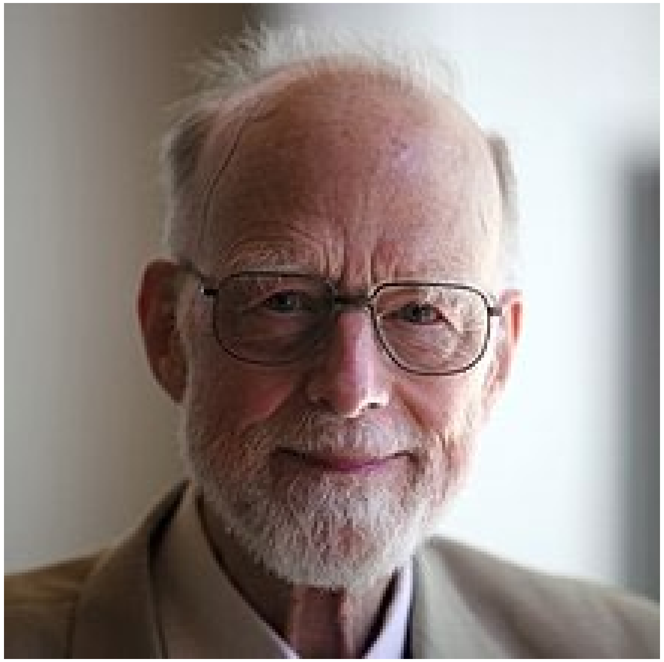
[RN]

Outline for Module M2

- M2 Program Correctness
  - M2.1 Introduction and Example
  - M2.2 Pre/Post-Conditions (of assignments and conditionals)
  - **M2.3 Loop invariants (and pre/post-conditions of loops)**
  - M2.4 Hoare Logic/Triples (for formal proofs)

Tony Hoare

- Inventor of quicksort
- Known for Hoare logic
- Turing Award in 1980
- *"fundamental contributions to the definition and design of programming languages"*



[RN]

Loops

```
while (x >= y) {

    x -= y;

}
```

[RN]

Loops

```
{x >= 0 && y >= 0}
while (x >= y) {

   x -= y;

}
```

We want to check if x becomes negative at the end of the loop.

Loops

```
{x >= 0 && y >= 0}
while (x >= y) {
    {x >= y}
    x -= y;


}
```

[RN]

Loops

```
{x >= 0 && y >= 0}
while (x >= y) {
   {x >= y && y >= 0}
   x -= y;


}
```

[RN]

Loops

```
{x >= 0 && y >= 0}
while (x >= y) {
   {x >= y && y >= 0}
   x -= y;
   {x >= 0 && y >= 0}
}
```

[RN]

Loops

```
{x >= 0 && y >= 0}
while (x >= y) {
   {x >= y && y >= 0}
   x -= y;
   {x >= 0 && y >= 0}
}
{x < y && x >= 0 && y >= 0}
```

For while (C) S

- On entering the loop for the first time, the conditions get ANDed.
- On reaching C in further iterations, the conditions again get ANDed with C, but the iterative conditions get ORed.
- On exiting a loop, NOT C must be true.
- Similar to if, conditions from within the loop and those from outside would get ORed.

[RN]

Loops

```
{x >= 0 && y >= 0}
while (x >= y) {
   {x >= y && y >= 0}
   x -= y;
   {x >= 0 && y >= 0}
}
{x < y && x >= 0 && y >= 0}
```

x cannot be negative at the end of the loop.

[RN]

Classwork

- Write the final post-conditions for the following programs.
  - To find the maximum element in an array
  - To compute the sum of the odd elements in an array
  - To compute the sum of the odd index elements
  - To find an anagram of a string (e.g., structures == trust cures, and data structures == custard stature)
- Find a reasonable anagram of annual grit.

[RN]

Loops

```
{x >= 0 && y >= 0}
while (x >= y) {
   {x >= y && y >= 0}
   x -= y;
   {x >= 0 && y >= 0}
}
{x < y && y >= 0 && x >= 0}
```

x cannot be negative at the end of the loop.

Such a condition is called a **loop-invariant**.

- It is true prior to entering the loop.
- It is true in every iteration of the loop.
- It is true at the end of the loop.

[RN]

Loop Invariants

- Properties to be satisfied prior to the loop, after every iteration, and at the end of the loop.
- Abstractly specifies the loop
  - bears potential to hide implementation details
- Similarity with recursion
  - resembles inductive hypothesis
- Often, finding loop invariants is not difficult, but finding useful (elegant) loop invariants is non-trivial.

https://en.wikipedia.org/wiki/Loop_invariant

[RN]

Loop Invariants

```
i = 0;
while (i < N) {
        sum += i;
        ++i;
}
```

Trivial loop-invariants:

- (i == 0 || i == 1 || … || i == N – 1 || i == N)
- (sum == 0 || sum == 1 || sum == 3 || sum == 6 || sum == arbitrary)
- (i == 0 && sum == 0 || i == 1 && sum == 1 || i == 2 && sum == 3 || … || i == N && sum == arbitrary)

Useful loop-invariant:

- sum == $\sum^{\mathbf{0}}_{\mathbf{i}}$

Now let's check if this invariant is satisfied:

- **After every iteration?**
- The incremented i is not summed up yet. Hence the invariant doesn't hold.
- So the invariant should be sum == $\sum^{\mathbf{0}}_{\mathbf{i-1}}$
- **At the end of the loop?**
- i == N, and the sum contains the summation of 0..N-1.
- **Just prior to the loop?**
- i == 0, but sum is undefined!
- To satisfy the invariant, we need to add this line: sum = 0;
- Thus, loop invariants can help us identify errors in the program.

[RN]

Classwork

```
satya = 1;
anand = pm[0];
while (satya < nobel) {
        if (pm[satya] > anand)
                anand =
pm[satya];
        ++satya;
}
```

- Finding useful / elegant loop invariants need high-level understanding of the code.
- This is not feasible, in general, automatically.
- Even for humans, loop-invariants are often guessed.
- For your own code, you should be able to state those precisely!

```
i = 1;
max = a[0];
while (i < N) {
        if (a[i] > max)
                max = a[i];
        ++i;
}
```

Loop Invariant:

- max contains the maximum value in a[0]..a[i-1].
- This is satisfied just prior to the loop.
- This is satisfied after every iteration.
- This is satisfied at the end of the loop.
- Hence, this program correctly finds maximum element in an array (having at least one element).

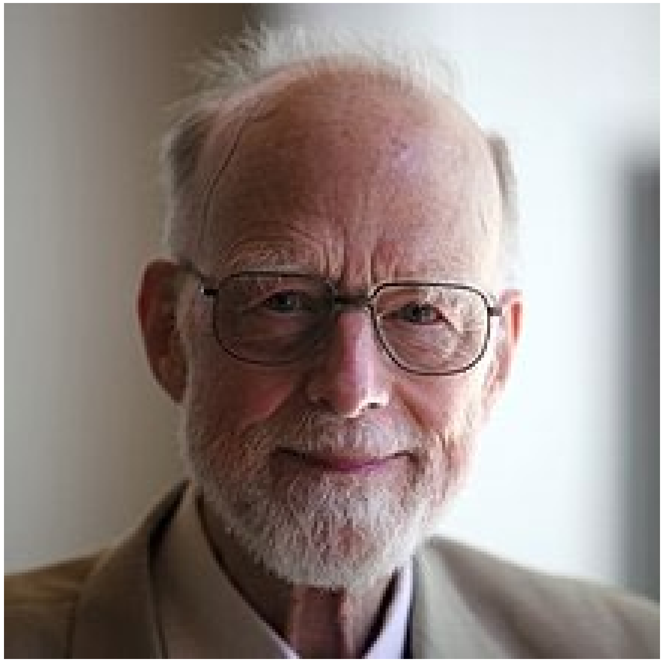
[RN]

Outline for Module M2

- M2 Program Correctness
  - M2.1 Introduction and Example
  - M2.2 Pre/Post-Conditions (of assignments and conditionals)
  - M2.3 Loop invariants (and pre/post-conditions of loops)
  - **M2.4 Hoare Logic/Triples (for formal proofs)**

Tony Hoare

- Inventor of quicksort
- Known for Hoare logic
- Turing Award in 1980
- *"fundamental contributions to the definition and design of programming languages"*



[RN]

Hoare Logic

- To formally reason about correctness of programs
- Originally seeded by Floyd
- Hence, now called Floyd-Hoare Logic
- Dates back to when neither you nor I was born (1969).
- Works with Hoare Triples
  - {pre-condition} Stmt {post-condition}

[RN]

Hoare Triples

- Empty statement / no-op
- Assignment
- Rule of composition
- Conditional
- Consequence
- Loop

$$\frac{}{\{P\}\ \text{no-op}\ \{P\}}$$

$$\frac{}{\{P[x \rightarrow E]\}\ x = E\ \{P\}}$$

Try for:

- x = y + 2 {x > 7}
- x = x + 1 {x > 7}
- x = 3 {x == 3}

$$\frac{\{P\}\ S1\ \{Q\},\ \{Q\}\ S2\ \{R\}}{\{P\}\ S1;\ S2\ \{R\}}$$

[RN]

Classwork

- Use Hoare Logic, prove that the following code correctly swaps two variables.

```
a = a + b;
b = a – b;
a = a – b;
```

[RN]

Classwork

- Use Hoare Logic, prove that the following code correctly swaps two variables.

$\{b == B \land a == A\}$

$\{b == B \land a + b - b == A\}$

a = a + b;

$\{b == B \land a - b == A\}$

$\{a - (a - b) == B \land a - b == A\}$

b = a – b;

$\{a - b == B \land b == A\}$

a = a – b;

$\{a == B \land b == A\}$

For assignment statements, always proceed in the REVERSE direction and see if the proof is consistent!

(i.e., start from the desired post-condition, then work your way back by applying the assignment rule, and check if you get the desired pre-condition;
this implies that Hoare rules can be applied in the forward direction to start from the desired pre-condition to prove the desired post-condtion)

[RN]

Another Classwork

- Use Hoare Logic, prove that the following code correctly swaps two variables.

```
temp = a;
a = b;
b = temp;
```

Another Classwork

- Use Hoare Logic, prove that the following code correctly swaps two variables.

$\{b == B \wedge a == A\}$

`temp = a;`

$\{b == B \wedge temp == A\}$

`a = b;`

$\{a == B \wedge temp == A\}$

`b = temp;`

$\{a == B \wedge b == A\}$

As mentioned before, for assignment statements, always proceed in the REVERSE direction and see if the proof is consistent!

Hoare Triples

- Empty statement / no-op

$$\frac{}{\{P\}\ \text{no-op}\ \{P\}}$$

- Assignment

$$\frac{}{\{P[x \longrightarrow E]\}\ x = E\ \{P\}}$$

- Rule of composition

$$\frac{\{P\}\ S1\ \{Q\},\ \{Q\}\ S2\ \{R\}}{\{P\}\ S1;\ S2\ \{R\}}$$

- Conditional

$$\frac{\{P \wedge C\}\ S1\ \{Q\},\ \{P \wedge \neg C\}\ S2\ \{Q\}}{\{P\}\ \text{if}\ (C)\ S1\ \text{else}\ S2\ \{Q\}}$$

- Loop

General form (not useful and incorrect)

$$\xcancel{\frac{\{P \wedge C\}\ S\ \{Q\}}{\{P\}\ \text{while}\ (C)\ S\ \{Q \wedge \neg C\}}}$$

Loop-invariant form

$$\frac{\{P \wedge C\}\ S\ \{P\}}{\{P\}\ \text{while}\ (C)\ S\ \{P \wedge \neg C\}}$$

[RN]

Classwork

```
int power2(int n) {
    int k, j;

    k = 0;
    j = 1;

    while (k != n) {
        k = k + 1;
        j = 2 * j;
    }

    return j;
}
```

- We want to prove that this function returns 2^n.
  - for n > 0
- Using testing, we can *validate* for certain inputs.
- Using a proof, we can *verify correctness*.

Source: Supratik Chakraborty, IITB

[RN]

Classwork

```
int power2(int n) {
        int k, j;

        k = 0;
        j = 1;

        while (k != n) {
                k = k + 1;
                j = 2 * j;
        }

        return j;
}
```

$\{n > 0\}$

```
k = 0;
j = 1;

while (k != n) {
        k = k + 1;
        j = 2 * j;
}
```

$\{j == 2^n\}$

Hoare Triple

[RN]

Classwork

```
    {n > 0}
k = 0;
    {ϕ1}
j = 1;
    {ϕ2}
while (k != n) {
    k = k + 1;
    j = 2 * j;
}
    {j == 2^n}
```

```
    {n > 0}
k = 0;
j = 1;

while (k != n) {
    k = k + 1;
    j = 2 * j;
}

    {j == 2^n}
```

$\left.\right\}$ **Hoare Triple**

[RN]

Classwork

$\{n > 0\}$
```
k = 0;
```
$\{\phi_1\}$
```
j = 1;
```
$\{\phi_2\}$
```
while (k != n) {
    k = k + 1;
    j = 2 * j;
}
```
$\{j == 2^n\}$

How do we prove

$\{\phi_2\}$
```
while (k != n) {
    k = k + 1;
    j = 2 * j;
}
```
$\{j == 2^n\}$

?

Let’s use the Hoare rule for loops.

But that requires a loop-invariant!

“Guess” the loop-invariant.

Since proof requires $j == 2^n$, we guess that the invariant is $j == 2^k$.

[RN]

Classwork

$$\frac{\{P \wedge C\}\ S\ \{P\}}{\{P\}\ \text{while}\ (C)\ S\ \{P \wedge \neg C\}}$$

$$\frac{\{j == 2^k \wedge k\ != n\}\ k = k + 1;\ j = 2 * j;\ \{j == 2^k\}}{\{j == 2^k \wedge k\ != n\}}$$

```
while (k != n) {

k = k + 1; j = 2 * j;
}
```

$\{j == 2^k \wedge \neg(k\ != n)\}$

To prove the **consequent**, prove the **precedent**.

How do we prove

$\{\phi_2\}$

```
while (k != n) {
    k = k + 1;
    j = 2 * j;
}
```

$\{j == 2^n\}$

?

Let's use the Hoare rule for loops.

But that requires a loop-invariant!

"Guess" the loop-invariant.

Since proof requires $j == 2^n$, we guess that the invariant is $j == 2^k$.

[RN]

Classwork

To Prove:

$$\frac{\{j == 2^k \wedge k \text{ != } n\}\ k = k + 1;\ j = 2 * j;\ \{j == 2^k\}}{\{j == 2^k\}}$$

while (k != n) {

k = k + 1; j = 2 * j;
}

$\{j == 2^k \wedge \neg(k \text{ != } n)\}$

To prove the **consequent**, prove the **precedent**.

Loop-invariant Hoare rule:

$$\frac{\{P \wedge C\}\ S\ \{P\}}{\{P\}\ \text{while (C) S}\ \{P \wedge \neg C\}}$$

Proof:

$\{j == 2^k\}$
$\{2 * j == 2^{k+1}\}$
k = k + 1;
$\{2 * j == 2^k\}$
j = 2 * j;
$\{j == 2^k\}$

We are almost there, except that the precondition is missing k != n.
This is where we use implication:

$(j == 2^k) \wedge (k \text{ != } n) \Rightarrow (j == 2^k)$

In general, we can strengthen the precondition, and weaken the post-condition.

[RN]

Classwork

- So far we have proved:

$\{n > 0\}$

k = 0;

$\{\phi_1\}$

j = 1;

$\{\phi_2: j == 2^k\}$

while (k != n) {

k = k + 1;

j = 2 * j;

}

$\{j == 2^n\}$

$\{n > 0\}$

// strengthening the precondition

$\{1 == 2^0\}$

k = 0

$\{\phi_1: 1 == 2^k\}$

j = 1

$\{\phi_2: j == 2^k\}$

[RN]

Homework

- Prove that the function computes n! .

```
int fact(int n) {
	k = 1;
	f = 1;

	while (k < n) {
		k = k + 1;
		f = f * k;
	}

	return f;
}
```

[RN]