DEPARTMENT OF PHYSICS
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PH1020 Physics II — Problem Set 2 (Solutions) — FEB 2024

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1. (a) **Potential at P:** We wish to compute the potential at a point P on the circumference of the sheet. The general expression for the potential at a point $\mathbf{r}$ due to the surface charge distribution is (the coordinate $\mathbf{r}'$ runs over the surface charge distribution)

$$\phi(\mathbf{r}) = k \int dS' \, \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \ . \tag{1}$$

Using the coordinate system suggested in the hint, i.e., (ϱ', φ') with $-\pi/2 \leq \varphi' \leq \pi/2$ and for a given φ', one has $0 \leq \varrho' \leq 2a\cos\varphi'$. Thus, we have

$$\int dS' = \int_{-\pi/2}^{\pi/2} d\varphi' \int_0^{2a\cos\varphi'} d\varrho' \varrho' \ .$$

It is useful to check that the limits are fine by just carrying out the integral with integrand equal 1 gives the area to be πa^2. In this coordinate system, the point P has $\mathbf{r} = 0$. Using $|\mathbf{r}-\mathbf{r}'| = |\mathbf{r}'| = \varrho'$, we obtain

$$\begin{aligned}
\phi(P) &= k \int_{-\pi/2}^{\pi/2} d\varphi' \int_0^{2a\cos\varphi'} d\varrho' \varrho' \ \frac{\sigma}{\varrho'} = k\sigma \int_{-\pi/2}^{\pi/2} d\varphi' \int_0^{2a\cos\varphi'} d\varrho' \\
&= (2ak\sigma) \int_{-\pi/2}^{\pi/2} d\varphi' \cos\varphi' \\
&\boxed{\phi(P) = (4ak\sigma)} \ .
\end{aligned}$$

Alternate method: A second way of doing problem 1 is to exchange the order of integration of (ϱ, φ). (Below $\arccos(x) = \cos^{-1}(x)$)

$$\int dS' = \int_0^{2a} d\varrho' \varrho' \int_{-\arccos(\varrho'/(2a))}^{+\arccos(\varrho'/(2a)} d\varphi \ ,$$

Then we can repeat the computation for $\phi(P)$

$$\begin{aligned}
\phi(P) = \int dS' \frac{k\sigma}{\varrho'} &= k\sigma \int_0^{2a} d\varrho \int_{-\arccos(\varrho'/(2a))}^{+\arccos(\varrho'/(2a)} d\varphi \\
&= k\sigma \int_0^{2a} d\varrho' \times 2\arccos\left(\tfrac{\varrho'}{2a}\right) \\
&= (4ak\sigma) \int_0^1 dw \arccos w \quad (\text{defining } w = \varrho'/2a) \\
&= (4ak\sigma) \ .
\end{aligned}$$

It can be shown that the integral $I = \int_0^1 dw \arccos w = 1$ for instance, by integrating by parts to obtain $I = \int_0^1 \frac{w dw}{\sqrt{1-w^2}}$.

(b) **Potential at C:** Now it makes sense to work with a coordinate system centered at C. Again $|\mathbf{r}-\mathbf{r}'| = |\mathbf{r}'| = \varrho'$. The limits of the area integral are suitably adjusted to this coordinate system.

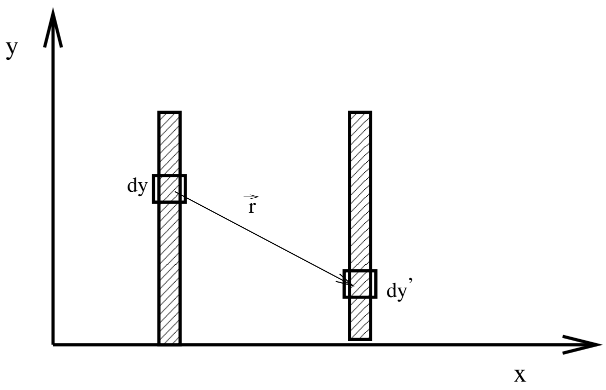
$$\begin{aligned}
\phi(C) &= k \int_0^{2\pi} d\varphi' \int_0^a d\varrho' \varrho' \ \frac{\sigma}{\varrho'} = k\sigma \int_0^{2\pi} d\varphi' \int_0^a d\varrho' \\
&\boxed{\phi(C) = (2\pi ak\sigma)} \ .
\end{aligned}$$

2. Consider two charge elements dy, and dy'. The force on dy' due to dy is given as, (see figure below)

$$dF = \frac{\lambda^2 dy dy'}{4\pi\epsilon_0 \big(a^2 + (y'-y)^2\big)^{3/2}} \big[a\hat{e}_x + (y'-y)\hat{e}_y\big].$$

By symmetry the force in the y direction vanishes. This can be checked by interchanging y and y' resulting in the same magnitude of force in the y-direction but with a reversal in sign. Thus we evaluate

$$F_x = dF_x = \frac{\lambda^2 a}{4\pi\epsilon_0} \int_0^L \int_0^L \frac{dy dy'}{\big(a^2 + (y'-y)^2\big)^{3/2}}.$$



We use

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$$

, to solve for the force above, and we obtain on doing the y integral

$$F_x = \frac{\lambda^2 a}{4\pi\epsilon_0 a^2} \Big[\int_0^L \frac{(L-y')dy'}{\sqrt{(L-y')^2 + a^2}} + \int_0^L \frac{y' dy'}{\sqrt{y'^2 + a^2}} \Big]$$

Doing the y' integral gives

$$F_x = \frac{\lambda^2 2a}{4\pi\epsilon_0 a} \big[\sqrt{1 + L^2/a^2} - 1\big].$$

Thus the force is given by

$$F = \frac{\lambda^2}{2\pi\epsilon_0} \big[\sqrt{1 + L^2/a^2} - 1\big]\hat{e}_x.$$

3. Let us look at

$$\vec{E} = \frac{k}{\epsilon_0 a^2}(x\hat{e}_x + y\hat{e}_y + z\hat{e}_z),$$

which can be obviously written as

$$\vec{E} = \frac{k}{\epsilon_0 a^2}\vec{r}.$$

From what we have learned from PH1010, for such forces $\vec{\nabla} \times \vec{r} = 0$, which implies a conservative field. To compute the potential, we have

$$\Phi = -\int \vec{E} \cdot d\vec{\ell} = -\frac{k}{\epsilon_0 a^2} \int r dr + \mathrm{C} = -\frac{kr^2}{2\epsilon_0 a^2} + \mathrm{C}.$$

To find ρ use Gauss's law

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} = \frac{k}{\epsilon_0 a^2}\vec{\nabla} \cdot \vec{r} = \frac{3k}{\epsilon_0 a^2}.$$

Thus,

$$\rho = \frac{3k}{a^2}.$$

4. Let the sphere S_R be centered at the point $\mathbf{r}_0 = (x_0, y_0, z_0)$. Let

$$\mathbf{r}(R,\theta,\varphi) = (x_0 + R\sin\theta\cos\varphi)\ \hat{e}_x + (y_0 + R\sin\theta\sin\varphi)\ \hat{e}_y + (z_0 + R\cos\theta)\ \hat{e}_z\ .$$

Then, as we vary (θ,φ), the vector $\mathbf{r}(R,\theta,\varphi)$ we obtain the surface of the sphere of radius r centered at (x_0, y_0, z_0).

The average potential, $\langle\phi\rangle$, which is a function of R, is given by

$$\langle\phi\rangle = \frac{1}{4\pi R^2}\int dS\ \phi\left(\mathbf{r}(R,\theta,\varphi)\right)\ , \tag{2}$$

$$= \frac{1}{4\pi}\int d\Omega\ \phi\left(\mathbf{r}(R,\theta,\varphi)\right)\ , \tag{3}$$

where the R^2 gets cancelled by the factor of R^2 in dS. In order to study the R dependence of $\langle\phi\rangle$, we take the R derivative to obtain[1]

$$\frac{d}{dR}\langle\phi\rangle = \left(\frac{1}{4\pi}\int d\Omega\ \frac{\partial}{\partial R}\phi\left(\mathbf{r}(R,\theta,\varphi)\right)\right) \tag{4}$$

$$= \frac{1}{4\pi}\int d\Omega\ \hat{e}_r\cdot\boldsymbol{\nabla}(\phi)\ \text{which is the electric flux up to numerical factors} \tag{5}$$

$$= 0 \quad \text{for a charge-free region.} \tag{6}$$

Thus, we see that $\langle\phi\rangle$ is a constant. The value of the constant is fixed by observing that the limit $R\to 0$ is nice and we get

$$\langle\phi\rangle = \lim_{R\to 0}\frac{1}{4\pi}\int d\Omega\ \phi\left(\mathbf{r}(R,\theta,\varphi)\right) = \phi(\mathbf{r}_0)\times\frac{1}{4\pi}\int d\Omega = \phi(\mathbf{r}_0)\ , \tag{7}$$

completing the required proof.

5. We compute the force by first computing the potential energy of the spherically symmetric charge distribution. Let $\phi(\mathbf{r})$ denote the potential due to charges external to the spherically symmetric charge distribution. Let the centre of the spherical symmetry be at the point $\mathbf{r}_0$. Then, we obtain that the potential energy is

$$U(\mathbf{r}_0) = \int dV\rho(\mathbf{r})\phi(\mathbf{r}_0+\mathbf{r}) \tag{8}$$

$$= \int dr r^2\rho(r)\int d\Omega\ \phi(\mathbf{r}_0+\mathbf{r}) \tag{9}$$

$$= \int dr r^2\rho(r) 4\pi\phi(\mathbf{r}_0)\ , \tag{10}$$

where we use the mean-value theorem to obtain the last line. Thus, we see that $U(\mathbf{r}_0) = Q\phi(\mathbf{r}_0)$ where Q is the total charge of the spherical distribution. The net force, $\mathbf{F}$, on the spherically symmetric charge distribution is then

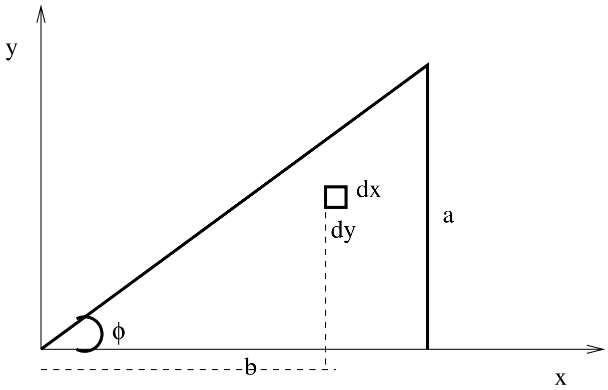
$$\mathbf{F} = -\boldsymbol{\nabla}_0 U(\mathbf{r}_0) = -Q\boldsymbol{\nabla}_0\phi(\mathbf{r}_0) = Q\ \mathbf{E}(\mathbf{r}_0)\ , \tag{11}$$

where $\boldsymbol{\nabla}_0$ is the gradient for the coordinate $\mathbf{r}_0$. This proves the statement.

[1] We need to use the identity

$$\frac{\partial}{\partial R}(\phi(\mathbf{r}(R,\theta,\varphi)) = \boldsymbol{\nabla}\phi(\mathbf{r})\cdot\frac{\partial\mathbf{r}}{\partial R} = \boldsymbol{\nabla}\phi(\mathbf{r})\cdot\hat{e}_r\ .$$

6.



The potential due to the area element "$dxdy$" at the point P is given as

$$dV = \frac{\sigma dxdy}{4\pi\epsilon_0\sqrt{x^2+y^2}}$$

, with $y = x\tan\phi$. Integrating the above we obtain,

$$V = \int dV = \frac{\sigma}{4\pi\epsilon_0}\int_0^b\int_0^{x\tan\phi}\frac{dxdy}{\sqrt{x^2+y^2}} \tag{12}$$

The above integral can be solved by putting $y = x\tan t$. This implies that

$$\int\frac{dy}{\sqrt{x^2+y^2}} = \int\sec t.$$

Now, consider

$$\sec t dt = \int\sec t\frac{\sec t+\tan t}{\sec t+\tan t}dt.$$

One can easily check that the above integral is of the form $\frac{du}{u}$, which can be evaluated to give $\ln|u|$. Thus, re-writing in terms of the variable t, we get

$$\int\sec t = \ln|\sec t+\tan t|$$

Using this result in Eq. **??**, we get

$$V = \frac{\sigma}{4\pi\epsilon_0}\int_0^b dx\ln\big(\tan\phi+\sec\phi\big) = \frac{\sigma b}{4\pi\epsilon_0}\ln\big(\tan\phi+\sec\phi\big)$$

Now, substituting $\tan\phi = \frac{a}{b}$ and $\sec\phi = \frac{\sqrt{a^2+b^2}}{b}$ in the above gives

$$V = \frac{\sigma b}{4\pi\epsilon_0}\ln\big(\frac{a+\sqrt{a^2+b^2}}{b}\big)$$