Matchings

September 22, 2024

Motivation : Job Assignment Problem.

Job Assignment Problem

There are s persons and t jobs. Each person is capable of handling certain jobs. Under what conditions we can employ each of the p persons with a job he/she is capable of handling? The rule of one-person-one-job is assumed.

Definition

Let $G = (V, E)$ be a a given graph.

1. A subset $M \subseteq E(G)$ is a matching of G, if M contains no adjacent edges.
2. The two ends of an edge $e \in M$ are matched under M.

A matching is also called as an edge-independent set.

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1. A matching M saturates a vertex v, and v is said to be M saturated, if some edge of M is incident with v; otherwise, v is M unsaturated.
2. If every vertex of G is M -saturated, the matching M is perfect.
3. M is a maximum matching if G has no matching M' with $|M'| > |M|$.
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Example

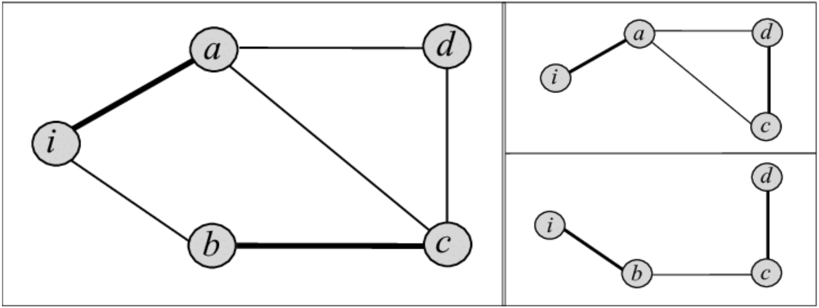

Figure: Matching

Matching : Graph 1 : $\{ia, bc\}$

Saturated vertices : Graph 1: i, a, b, c.

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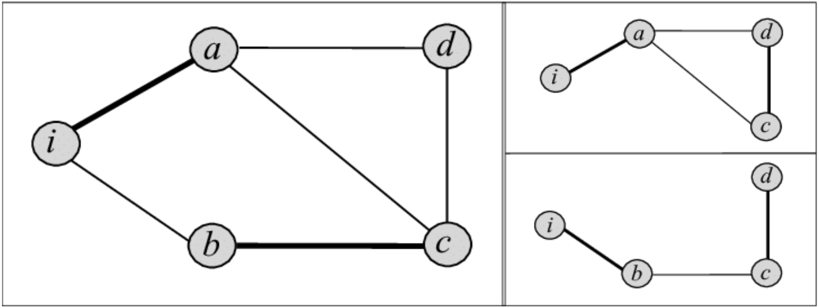

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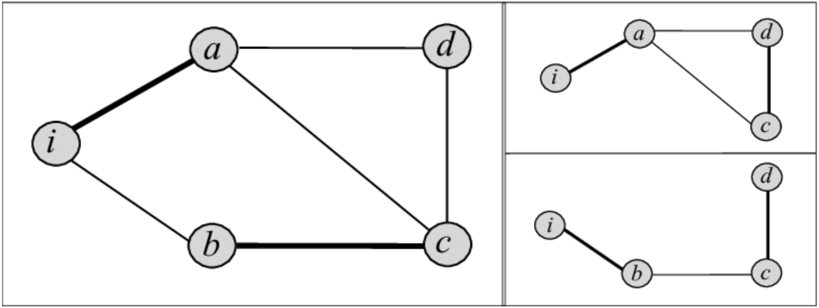

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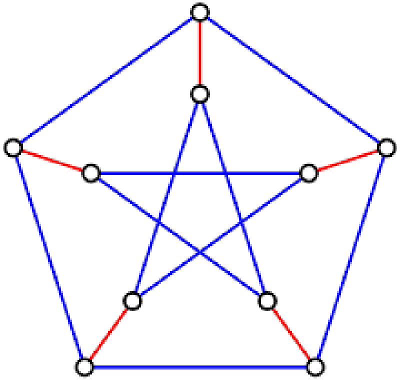
Example-Perfect Matching



Edges in red form a perfect matching.

Let M be a matching in G.

1. An M-alternating path in G is a path whose edges are alternately in $E \setminus M$ and M.
2. An M-augmenting path is an M -alternating path whose origin and terminus are M-unsaturated.

Alternating path

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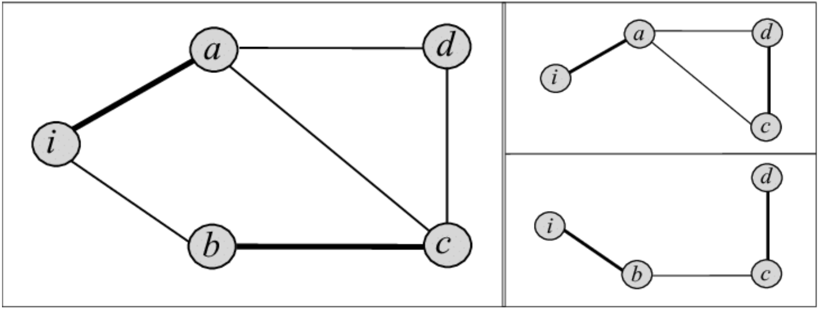

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M-Alternating Path: Graph 1: *i*, *a*, *b*, *c*.

M-Augmenting Path: Graph 1: None.

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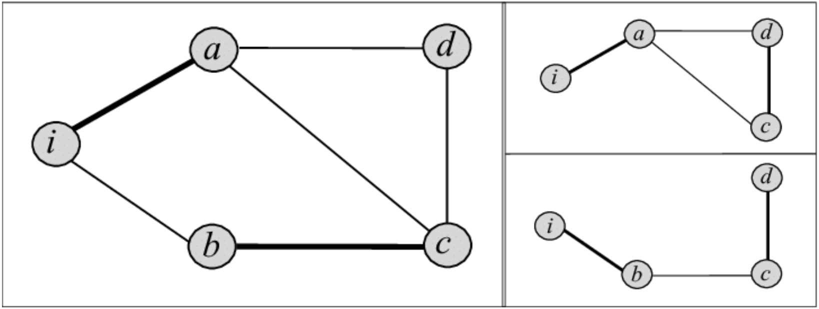

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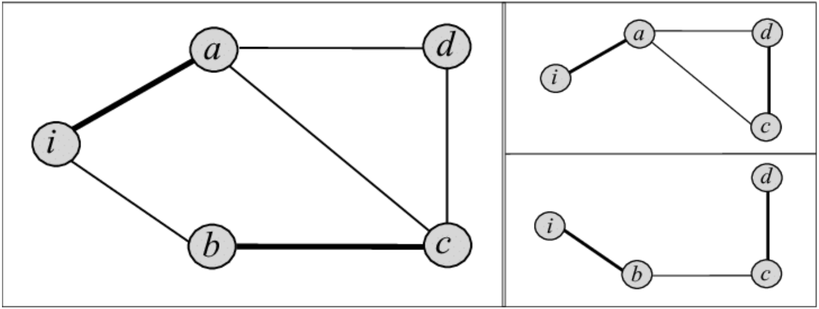

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Theorem

A matching M in G is a maximum matching if and only if G contains no M-augmenting path.

Proof.

" $\Rightarrow$ " By contradiction.

1. Let M be a matching in G.
2. suppose that G contains an M-augmenting path

$$v_0 e_0 v_1 e_1 \cdots e_{2m} v_{2m+1},$$

3. Define $M' \subseteq E(G)$ by

$$M' = M \setminus \{e_1, e_3, e_5, \cdots, e_{2m-1}\} \cup \{e_0, e_2, \cdots, e_{2m}\}$$

4. Then M' is a matching in G, and $|M'| = |M| + 1$.

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Converse: By contradiction.

1. Let G be a graph with no M- Augmenting path for a given matching M.
2. Suppose M is not maximum.
3. There exists M' such that $|M'| > |M|$.
4. Define $H = [M \Delta M']$, the symmetric difference.
5. $1 \leq deg_H(v) \leq 2$.
6. Each component of H is either an even cycle with edges alternately in M and M'.
7. or else a path with edges alternately in M and M'.

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2. P must start and end with edges of M'.
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2. Given a bipartite graph with bipartition (X, Y), one wishes to find a matching that saturates every vertex in X.
3. Necessary and sufficient conditions for the existence of such a matching were first given by Hall (1935).

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Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if

$$|N_G(S)| \geq |S| \quad \text{for all } S \subseteq X$$

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Converse-By contradiction

Proof.

1. Let G be a bipartite graph satisfying the given condition.
2. but G contains no matching saturating all vertices in X.
3. Let M^* be a maximum matching in G.
4. By assumption, M^* does not saturate all vertices in X.
5. Let $u \in X$ be M^*-unsaturated vertex
6. Let Z denote the set of all vertices connected to u by M^*-alternating paths.

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Converse-By contradiction-Contd.

Proof.

1. By Previous theorem, there is no M^*- augmenting path.
2. Hence, u is the only M^* unsaturated vertex in Z.
3. Let $S = Z \cap X$ and $T = Z \cap Y$.
4. Vertices in $S \setminus \{u\}$ are matched under M^* with vertices in T.
5. $\implies |T| = |S| - 1$ and $T \subseteq N_G(S)$.
6. Every vertex in $N_G(S)$ is connected to u by an M^*-alternating path hence we have
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Corollary

If G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.

Proof.

1. Observe that, if X and Y are bipartitions of G then $|X| = |Y|$ as G is k-regular.
2. Let $S \subseteq X$
3. Let E_1 = the sets of edges incident with vertices in S.
4. Let and E_2 = the sets of edges incident with vertices in $N(S)$.
5. $E_1 \subseteq E_2$ and $k|N(S)| = |E_2| \geq |E_1| = k|S|$
6. $\implies |N(S)| \geq |S|$ (Hall's condition).
7. Hence, G has matching M saturating X and since $|X| = |Y|$, M is perfect.

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7. Hence, G has matching M saturating X and since $|X| = |Y|$, M is perfect.

Corollary to Hall's Theorem

Corollary

If G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.

Proof.

1. Observe that, if X and Y are bipartitions of G then $|X| = |Y|$ as G is k-regular.
2. Let $S \subseteq X$
3. Let E_1 = the sets of edges incident with vertices in S.
4. Let and E_2 = the sets of edges incident with vertices in $N(S)$.
5. $E_1 \subseteq E_2$ and $k|N(S)| = |E_2| \geq |E_1| = k|S|$
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□

Definition

1. A covering of a graph G is a subset K of V such that every edge of G has at least one end in K.
2. A covering K is a minimum covering if G has no covering K' with $|K'| < |K|$.
3. If K is a covering of G, and M is a matching of G, then K contains atleast one end of each of the edges in M.
4. For any matching M and any covering K, $|M| \leq |K|$.
5. Indeed, if M^* is a maximum matching and K' is a minimum covering, then $|M^*| \leq |K'|$.

Lemma

Let M be a matching and K be a covering such that $|M| = |K|$. Then, M is a maximum matching and K is a minimum covering.

Proof.

If M^* is a maximum matching and K a minimum covering then,

$$|M| \leq |M^*| < |K'| \leq |K|$$

Since $|M| = |K|$, it follows that $|M| = |M^*|$ and $|K| = |K'|$. $\square$ $\square$

Konig's Theorem

Theorem

In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.

Proof.

1. Given G- bipartite with bipartition (X, Y).
2. Let $M^* \subseteq E(G)$- Maximum matching of G.
3. Let $K' \subseteq V(G)$- Minimum vertex cover of G.
4. To show that $|M^*| = |K'|$.

□

Proof

1. M^*- a maximum matching of G.
2. U = the set of M^*-unsaturated vertices in X,
3. Z= set of all vertices connected by M^*-alternating paths to vertices of U.
4. Set $S = Z \cap X$ and $T = Z \cap Y$.
5. Similar to proof of Hall's theorem,
6. All vertices of T are M^* saturated.
7. So, $N(S) = T$.

1. Define $K' = (X \setminus S) \cup T$
2. Every edge of G must have at least one of its ends in K'.
3. If not, there would be an edge with one end in S and one end in $Y \setminus T$, contradicting $N(S) = T$.
4. Thus K' is a covering of G and $|M^*| = |K'|$.
5. By previous Lemma, K' is a minimum vertex cover.
6. Hence, the proof. $\square$

Definition

1. A component of a graph is odd or even according as it has an odd or even number of vertices.
2. We denote by $o(G)$ the number of odd components of G.
3. If a simple graph G has a perfect matching, then $|V(G)|$ is even.
4. Converse not true.

Tutte's perfect matching condition

Theorem

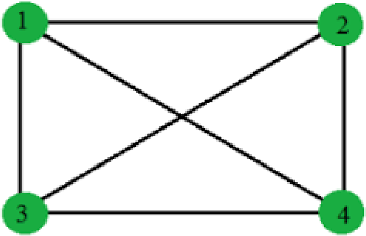
G has a perfect matching if and only if

$$o(G \setminus S) \leq |S|, \ \forall \ S \subseteq V(G)$$

No proof..Complicated at this level.

Cubic graph G with a Perfect Matching :

G satisfies Tutte's condition.



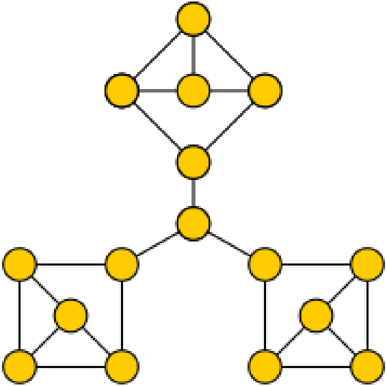
3 Regular

Cubic graph G without a Perfect Matching :

G does not satisfy Tutte's condition.

G has a cut vertex (the middle one) and

$$o(G \setminus \{v\}) = 3 > 1$$



1. Tutte's theorem characterizes the graphs with a perfect matching,
2. It is hard to verify Tutte's condition and conclude that a given graph G has a perfect matching, because we have to verify (Tutte) for 2^n subsets of $V(G)$.
3. Hence, there have been several results proved by various mathematicians which say that a given graph G has a perfect matching if G satisfies a certain property P (where P is easily verifiable).
4. In fact, the first result on perfect matchings was obtained by Petersen (1891) which preceded Tutte's theorem.
5. However, we can easily deduce Petersen's result using Tutte's theorem..

Theorem

If G is a 2-edge-connected 3-regular graph, then G has a perfect matching.

Proof.

1. Given G is 2-connected.
2. G has no cut-edges. (no bridges).
3. We simply show that G satisfies Tutte's condition.
4. Let $S \subseteq V$.
5. Let $G_1, G_2, \cdots, G_t$ be the odd components of $G - S$.
6. Denote by m_i the number of edges with one end in G_i and the other in S.

□

Proof.

1. $|m_i| \neq 1$ as G has no cut-edges.
2. $|m_i| = 2k$ is not even, if not,

$$\sum_{v_i \in S} deg(v_i) = 3|V(G_i)| - 2k = \text{ odd -even } \neq \text{ even}$$

   as $|V(G_i)|$ is odd and each vertex is of degree 3 in G.
3. Hence, $|m_i|$ is odd and $|m_i| \geq 3$.
4. Also, $\sum_{i=1}^{t} m_i \leq 3|S|$
5. Among these $3|S|$ edges, there are atleast three edges per each odd component.
6. Therefore there are at most $|S|$ odd components.

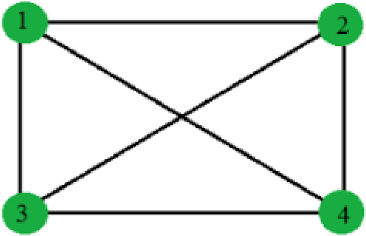
□

Proof.

1. Thus, $o(G \setminus S) \leq |S|$.
2. G satisfies Tutte's condition.
3. G has a perfect matching.

$\square$

Cubic graph (without cut egdes) with a Perfect Matching :



3 Regular

Cubic graph (with cut egdes) without a Perfect Matching :