Basic Graph Theory

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Problem 1

Given a graph G with

1. $|V(G)| = 24, \quad |E(G)| = 30$
2. Number of vertices of degree 4= 5.
3. Number of vertices of degree 1= 7.
4. Number of vertices of degree 2= 7.
5. All other vertices have degree 3 or 4.

How many vertices of degree 4 are there?

Problem 2

Can there exist a simple graph with the following?

1. $|V(G)| = 13, \quad |E(G)| = 31$
2. Number of vertices of degree 1= 3.
3. Number of vertices of degree 4= 7.

Exercise :

Problem 3

Prove that if G is a simple graph woth n vertices and n edges with no vertices of degree 0 or 1, then the degree of every vertex is 2.

Problem 4

Prove that if G is a simple graph with n vertices and $n-1$ edges, G has atleast one vertex of degree less than 2.

Problem 4

Prove that in any simple graph (with atleast two vertices) there exists two vertices with same degree.

Problem 5

Find all Non-isomorphic self complementary graphs on 5 vertices.

Problem 6

True or False? If true, provide a brief proof. If false, provide a counterexample.

1. If two graphs have the same number of vertices, same degree sequence and same number of cycles, then they are isomorphic.
2. Two isomorphic graphs must have the same number of edges and vertices.
3. The degree sequence of two isomorphic graphs must be the same.
4. $K_{3,2}$ is isomorphic to C_5.
5. $K_{4,2}$ is isomorphic to $K_{2,4}$.
6. If G contains no cycles, all graphs isomorphic to G also have no cycles.

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Problem 7

A tree has 100 pendant vertices, 20 vertices of degree 6, and half of the remaining vertices have degree 4. The left over vertices are of degree 2, how many vertices are of degree 2?

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True or False? If true, provide a brief proof. If false, provide a counterexample.

1. The deletion of any edge of a tree results in a disconnected graph.
2. Any tree with more than two vertices is bipartite.
3. There exists a tree with 10 vertices and the sum of degrees of vertices is 24.

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What is a necessary and sufficient condition for a tree to be a complete bipartite graph? Explain.

Find the maximum number of edges in a bipartite graph with n vertices.

Problem 10

We know that a tree is always bipartite. Show that a tree always has a pendant vertex in its larger partite set.

Problem 11

Show that there are atleast Δ pendant vertices in a tree.

Problem 12

Show that if $G = (V, E)$ is a simple graph in which every vertex has degree at least 2, then G contains a cycle C as a subgraph.

Let T be tree in which every pendant vertex is adjacent to a vertex of degree atleast 3. Show that there are two pendant vertices with a common neighbour.

Problem 14

Show that a tree with no vertex of degree 2 has more pendant vertices than non-pendant vertices.

Problem 15

Show that if a (simple) graph G has 17 vertices and 121 edges, then G is connected.

Problem 16

Let G be a graph with n vertices such that $\delta(G) \geq \frac{(n+2k-4)}{3}$. Prove that if W is a set of vertices of G of size at most $k-1$ then $G-W$ has at most two components.

Problem 17

Prove that if for a graph, G, of order n every pair of distinct vertices $u, v \in V(G)$

$$deg(v) + deg(u) \geq n - 1$$

then G is connected.

Problem 16

For any given graph G, show that either G or G^c is connected.

Problem 17

If G is a connected graph on n vertices, what is the lower bound for the number of edges G?

Problem 18

If G is a graph with n vertices and k components, then G can have at least $n - k$ edges, and at most $\frac{(n-k)(n-k+1)}{2}$ edges.