CS1200 - Discrete Mathematics for Computer Science

Ungraded Tutorial 4

Jan - May 2024

Part A

1. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17$$

where x_1, x_2, x_3, x_4 are non negative integers?

2. Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in A_1, 1000 in A_2, and 10,000 in A_3 if
   (a) $A_1 \subseteq A_2$ and $A_2 \subseteq A_3$.
   (b) the sets are pairwise disjoint
   (c) there are two elements common to each pair of sets and one element in all three sets.

3. In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if
   (a) the bride must be in the picture?
   (b) both the bride and groom must be in the picture?
   (c) exactly one of the bride and the groom is in the picture?

4. Count the number of 6 digit decimal numbers such that each digit of the number is greater than the earlier one. [For example 125789 is a valid number according to the above rule but 356748 is invalid].

5. How many bit strings of length 12 contain
   (a) exactly three 1s?
   (b) at most three 1s?
   (c) at least three 1s?
   (d) an equal number of 0s and 1s?

6. How many numbers will you need to select from the set $\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$ to guarantee that at least two of these numbers add up to 20?

7. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$? Give a combinatorial reasoning.

8. Give a formula for the coefficient of the expression x^k in the expansion of the following expressions, when k is an integer.

   (a) $(x + \frac{1}{x})^{100}$

   (b) $(x + \frac{1}{x})^n$ where n is a positive integer

9. A South Indian breakfast joint has the following popular variety of dosas: Plain dosa, Onion dosa, Paneer dosa, Masala dosa and Podi dosa. How many ways are there to choose:

   (a) A dozen dosas?

   (b) A dozen dosas with at least 2 of each variety?

   (c) 2 dozen dosas with at least 5 masala dosas and at most 5 plain dosas?

Part B

1. Sam, from the IIT Madras athletics team, is planning on running a marathon. Over a 30 day period, he pledges to train at least once per day, and 45 times in all. Show that there will be a period of consecutive days where he trains exactly 14 times.

2. Count the number of possible strings of length n that has a sub-string S of length m ($n \geq m$). (You can assume that S does not have a repeated letter and the elements of the string being the lowercase letters of the English Alphabet).

3. There are 201 seats in the assembly of a particular Indian state. This state has only three political parties and it does not allow any independent candidate. Count the number of ways these seats can be divided among these three parties such that no single party gets a majority?

4. An e-sport magazine is conducting a survey to rank the top 5 Rocket League teams according to fans. The teams are - NRG, Cloud9, G2 Esports, Spacestation Gaming, and Vitality.

   Each participant has to provide a ranking amongst the 5 teams, and no two teams can have the same rank. How many participants must be there to ensure that at-least 10,001 fans have selected the same ranking?

5. Prove the binomial theorem using mathematical induction.

6. Prove the **hockeystick identity**

$$\binom{n+r+1}{r} = \sum_{k=0}^{r} \binom{n+k}{k}$$

   whenever n and r are positive integers

(a) using a combinatorial argument

(b) using Pascal's identity

The name stems from the graphical representation of the identity on Pascal's triangle: when the addends represented in the summation and the sum itself are highlighted, the shape revealed is reminiscent of a hockey stick.

7. Suppose that n and k are integers with $1 \leq k \leq n$. Prove the following **hexagon identity**, which relates terms in Pascal's triangle that form a hexagon, in a combinatorial way
$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$
Hint: Consider choosing elements from 3 different sets.

8. While watching a lecture, Fermat, Pascal, Descartes, Mercator, Torricelli, and Mersene sat in that order in a row of six chairs. During a break, they leave the hall to debate the merits of the lecture. When they came back, they sat on those six chairs in such a way that if two of them sat next to each other before the break, then they did not sit next to each other after the break. Find the number of possible seating orders they could have chosen after the break.

Part C

1. An important area of combinatorics, the **Ramsey Theory** deals with distributions of subsets of elements of a set. A central theorem, the Ramsey's theorem talks about the Ramsey number $R(m, n)$ where m, n are positive numbers greater than or equal to 2, denotes the minimum number of people at a party such that there are either m mutual friends or n mutual enemies, assuming that every pair of people at the party are either friends or enemies.

   (a) Show that Ramsey numbers are symmetric, that is, $R(m, n) = R(n, m)$

   (b) Show that $\forall n \geq 2, R(2, n) = R(n, 2) = n$

   (c) In the next two parts we get the exact value of $R(3, 3)$. Show that $R(3, 3) \geq 6$. In other words, a party of 6 people, where each pair consists of 2 friends or 2 enemies will have either 3 mutual enemies or 3 mutual friends.

   (d) Show that $R(3, 3) > 5$ by giving an example of a party with 5 people where there are neither 3 mutual enemies nor 3 mutual friends. Hence we can conclude that $R(3, 3) = 6$.

   (e) Show that $R(m, n) \leq R(m - 1, n) + R(n - 1, m)$ if both m, n are greater than 2.

   While it is possible to get properties like the above, for most part it is very difficult to get exact values of the Ramsey numbers. For $m \geq n \geq 3$, there are only *nine* Ramsey numbers whose exact values are known!

2. Aditya is a 2nd year computer science student. He learnt Python programming while doing his first semester course work. To get a good fluency in Python programming he decided that for next one year he will write at least one Python program every day. So he started writing at least one Python program every day, but no more than 500 programs in last one year(365 days). Prove that there is a period of some consecutive days during which Aditya wrote exactly 100 Python programs.

3. Prove the principle of inclusion-exclusion using mathematical induction.