DEPARTMENT OF PHYSICS
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PH1020 Physics II | Problem Set 8 (Solutions) | April, 2024

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1. (a) We begin with the Maxwell equation curl $\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and compute the curl of both sides of the equation.

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \simeq -\nabla^2 \mathbf{E} \ ,$$
$$-\nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial}{\partial t}\left(4\pi J_f + \frac{\partial \mathbf{D}}{\partial t}\right) \simeq -\mu\sigma \frac{\partial \mathbf{E}}{\partial t} \ ,$$

where we have implemented the approximations that were suggested in the statement of the problem. Equating the expressions at the end of the two lines above, we get

$$\boxed{\nabla^2 \ \mathbf{E} = \mu\sigma \ \frac{\partial \mathbf{E}}{\partial t} \ .}$$

(b) Substituting for $\mathbf{E} = \mathbf{E}_0 \exp i(kz - \omega t)$ in the above equation[1], we get

$$-k^2 \ \mathbf{E}_0 = (-i\omega\mu\sigma) \ \mathbf{E_0} \ .$$

Clearly for the above equation to make sense, we need to solve the above equation for complex k. We get

$$k = \pm \frac{1+i}{\sqrt{2}} \ \sqrt{\omega\mu\sigma} \ .$$

We think of the region $z > 0$ to be a conducting region with a plane wave being incident at $z = 0$. The electric field takes the form, for $z > 0$, for which we choose the positive sign above

$$\mathbf{E} = \mathbf{E}_0 \ \exp\left[i\left(\sqrt{\frac{\mu\sigma\omega}{2}} \ z - \omega t\right)\right] \ \exp\left[-\sqrt{\frac{\mu\sigma\omega}{2}} \ z\right] \ .$$

Taking the real part of the above equation, we obtain

$$\boxed{\mathbf{E} = \mathbf{E}_0 \ \cos\left[\left(\sqrt{\frac{\mu\sigma\omega}{2}} \ z - \omega t\right)\right] \ \exp\left[-\sqrt{\frac{\mu\sigma\omega}{2}} \ z\right] \ .}$$

(c) We thus see that the amplitude of the electric field decreases exponentially as z increases. The depth δ, called the **skin depth**, at which the field decays to $1/e$ of its value at $z = 0$ is therefore

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \ .$$

Thus, in a good conductor, high frequency electromagnetic currents are restricted to a thin layer (the "skin") at the surface of the conductor. For a typical metal, taking $\sigma \sim 10^7 (\Omega m)^{-1}$, $\mu \sim 10^{-6} N/A^2$, we get $\delta = 10^{-8} m$ for $\omega = 10^{15} s^{-1}$ (optical frequencies).

2. The electric field $\mathbf{E}$ between the plates is

$$\mathbf{E}(t) = \frac{q}{Cd} \ \hat{e}_z = \frac{V}{d}\left(1 - e^{-t/RC}\right) \ \hat{e}_z \ .$$

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[1] Let E_0 be real for simplicity. We keep in mind that we will take the real part the end of the calculation.

The corresponding displacement vector is

$$\mathbf{D}(t) = \epsilon_0\kappa\mathbf{E}(t) = \frac{\epsilon_0\kappa V}{d}\left(1 - e^{-t/RC}\right)\ \hat{e}_z\ .$$

Since, we have a time-varying displacement vector, this behaves as a source for the auxiliary field through curl $\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$ as $\mathbf{J}_f = 0$ insider the dielectric. We can solve for the auxiliary field using the same method that we could have done for a current source. Consider a planar Amperian loop, C, of radius ϱ centred on the z-axis lying completely inside the dielectric bounding a surface S with $\hat{n} = \hat{e}_z$ at all points. Cylindrical symmetry implies that we can choose, $\mathbf{H} = H\ \hat{e}_\varphi$, we get

$$\oint_C \mathbf{H}\cdot\mathbf{dl} = \int_S \frac{\partial \mathbf{D}}{\partial t}\cdot\mathbf{dS}\ .$$

This implies that

$$2\pi\varrho H = \frac{\epsilon_0\kappa V}{dRC}\ e^{-t/RC}\ \pi\varrho^2 \implies \boxed{\mathbf{H} = \frac{\epsilon_0\kappa V\varrho}{2dRC}\ e^{-t/RC}\ \hat{e}_\varphi}\ .$$

3. Inside the conductor, the magnetic field is

$$\mathbf{B} = \frac{\mu_0 I\varrho}{2\pi a^2}\ \hat{e}_\varphi\ .$$

The electric field is obtained using $\mathbf{J} = \sigma\mathbf{E}$. We obtain

$$\mathbf{E} = \frac{I}{\pi a^2\sigma}\ \hat{e}_z\ .$$

The Poynting vector $\mathbf{S}$ at the surface of the conductor, i.e., $\varrho = a$, is given by

$$\mathbf{S} = \mathbf{E}\times\frac{\mathbf{B}}{\mu_0} = -\frac{I^2}{2\pi^2a^3\sigma}\ \hat{e}_\rho$$

The energy flux *out* of the surface of the conductor of length L is (with $\mathbf{da} = dzad\varphi\ \hat{e}_\varrho$)

$$\int \mathbf{S}\cdot\mathbf{da} = -\frac{I^2}{2\pi^2a^3\sigma}\times 2\pi aL = -\frac{I^2L}{\pi a^2\sigma} = -I^2R\ ,$$

where $R = \frac{L}{\pi a^2\sigma}$ is the resistance of the conductor. Therefore, the energy flow *into* unit length of the conductor is $+\frac{I^2}{\pi a^2\sigma}$.

4. We need to compute the electric and magnetic fields due to the beam of protons. Choose the beam to be aligned with the z-axis. Then, we have that $\mathbf{H} = \frac{I}{2\pi\varrho}\ \hat{e}_\varphi$ (the magnetic field due to infinitely long wire carrying current I) and $\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0\varrho}\hat{e}_\varrho$, with $\lambda = I/v$ (the electric field due to an infinite line charge λ). The Poynting vector is

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \frac{I^2}{4\pi^2\epsilon_0 v\varrho^2}\ \hat{e}_z = \frac{I^2}{4\pi\epsilon_0(\pi v^2\varrho^2)}\ \mathbf{v}\ .$$

Thus, we see that $\mathbf{S}$ is parallel to $\mathbf{v}$ consistent with the (other) interpretation of $(1/c^2)$ times the Poynting vector as a momentum density. Check that the pre-factor has the dimension of momentum per unit volume.