

Bayesian statistics with R

3. Analyses by hand

Olivier Gimenez

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Back to Bayes

A simple example

- Let us take a simple example to fix ideas.
- 120 deer were radio-tracked over winter.
- 61 close to a plant, 59 far from any human activity.
- Question: is there a treatment effect on survival?

	Released	Alive	Dead	Other
treatment	61	19	38	4
control	59	21	38	0

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- Of interest is the probability of over-winter survival, call it θ , for the general population within the treatment area.
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- How would the classical statistician justify this estimate?

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- K the number of alive individuals at the end of the winter, so that $P(K = k) = \binom{n}{k} \theta^k (1 - \theta)^{n-k}$.
- The classical approach is to maximise the corresponding likelihood with respect to θ to obtain the entirely plausible MLE:

$$\hat{\theta} = k/n = 19/57$$

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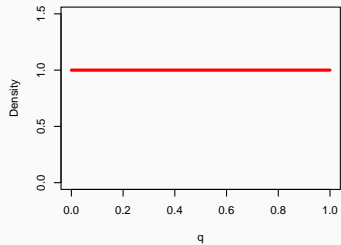
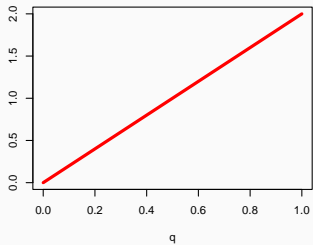
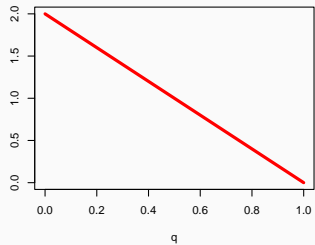
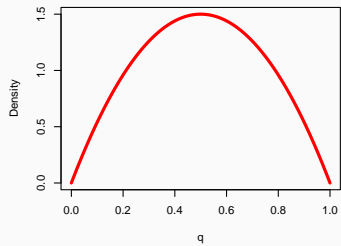
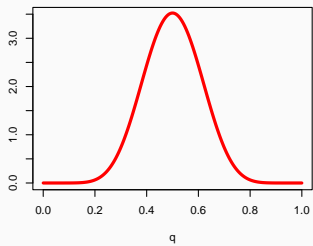
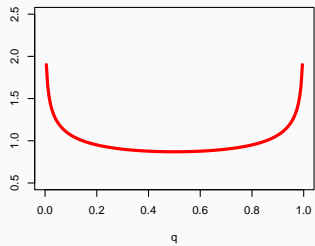
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$$q(\theta \mid \alpha, \beta) = \frac{1}{\text{Beta}(\alpha, \beta)} \theta^{\alpha-1} (1 - \theta)^{\beta-1}$$

with $\text{Beta}(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$ and $\Gamma(n) = (n - 1)!$

beta(1,1)**beta(2,1)****beta(1,2)****beta(2,2)****beta(10,10)****beta(0.8,0.8)**

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- Take a Beta prior with a Binomial likelihood, you get a Beta posterior (conjugacy)

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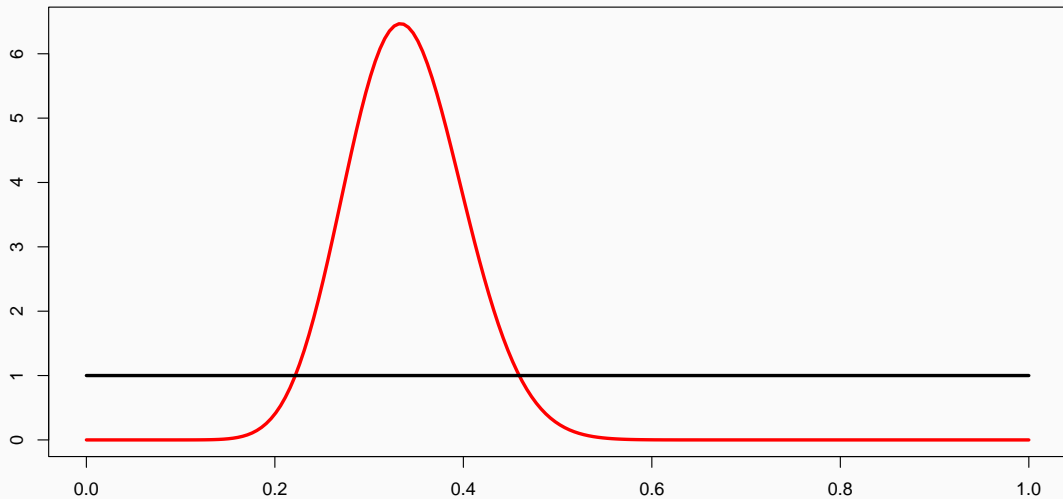
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- Note that in this specific situation, the posterior has an explicit expression, easy to manipulate.
- In particular, $E(\text{Beta}(a, b)) = \frac{a}{a + b} = 20/59$ to be compared with the MLE $19/57$.

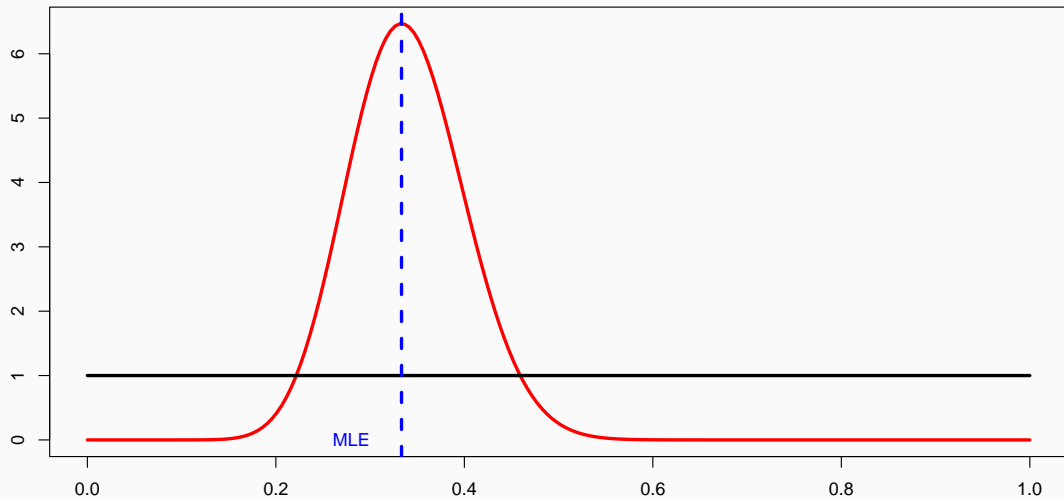
A general result

This is a general result, the Bayesian and frequentist estimates will always agree if there is sufficient data, so long as the likelihood is not explicitly ruled out by the prior.

Prior $Beta(1, 1)$ and posterior survival $Beta(20, 39)$



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Our model so far

$$y \sim \text{Binomial}(N, \theta)$$

[likelihood]

$$\theta \sim \text{Beta}(1, 1)$$

[prior for θ]

$p(\theta | D) = p(D | \theta) \times p(\theta) / p(D)$

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