Bayesian statistics with R
5. Markov chains Monte Carlo (MCMC)

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Get posteriors with Markov chains Monte Carlo (MCMC) methods

Back to the Bayes' theorem

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- $\Pr(\text{data}) = \int L(\text{data} \mid \theta)\Pr(\theta) d\theta$ is a N-dimensional integral if $\theta = \theta_1, \ldots, \theta_N$
- Difficult, if not impossible to calculate!

Brute force approach via numerical integration

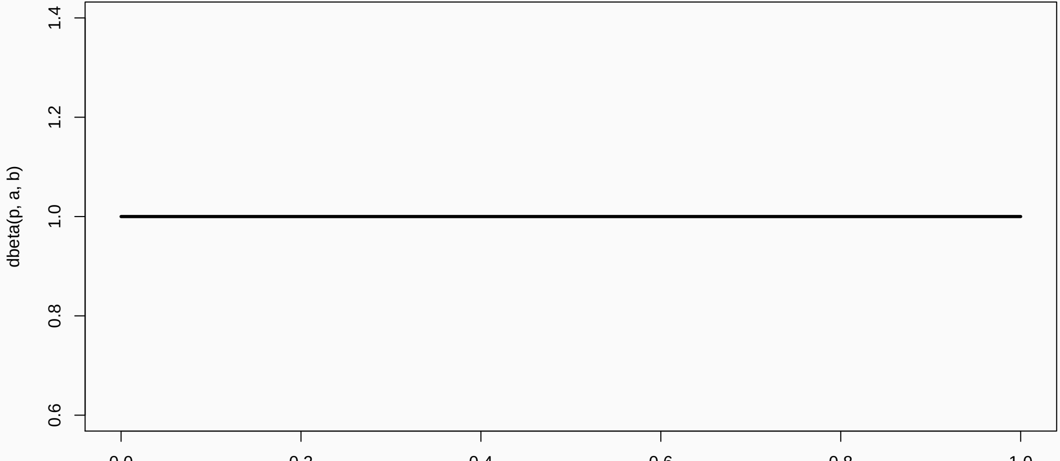
- Deer data

```r
y <- 19 # nb of success
n <- 57 # nb of attempts
```

- Likelihood Binomial$(57, \theta)$
- Prior Beta$(a = 1, b = 1)$

Beta prior

```r
a <- 1; b <- 1; p <- seq(0,1,.002)
plot(p, dbeta(p,a,b), type='l', lwd=3)
```



Apply Bayes theorem

- Likelihood times the prior: $\Pr(\text{data} \mid \theta)\ \Pr(\theta)$

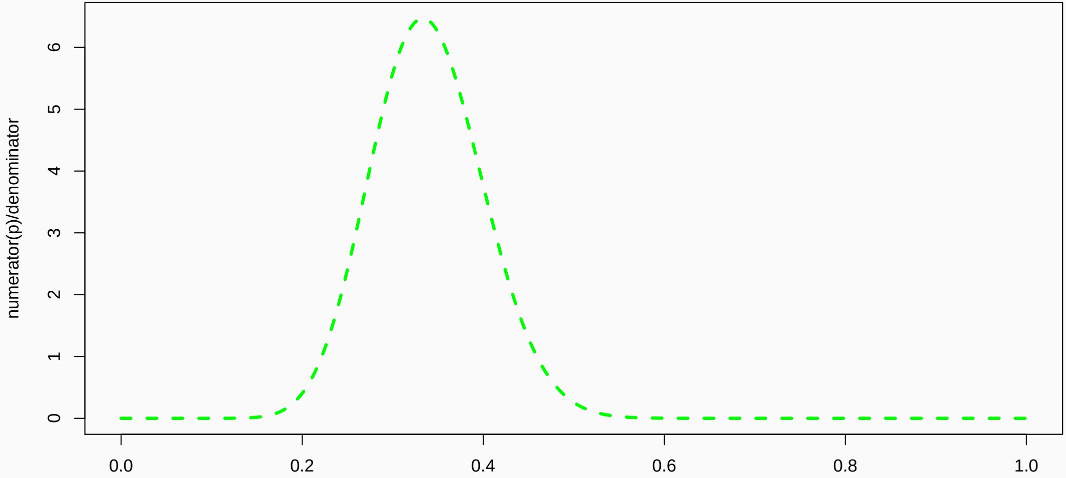
```
numerator <- function(p) dbinom(y,n,p)*dbeta(p,a,b)
```

- Averaged likelihood: $\Pr(\text{data}) = \int L(\theta \mid \text{data})\ \Pr(\theta) d\theta$

```
denominator <- integrate(numerator,0,1)$value
```

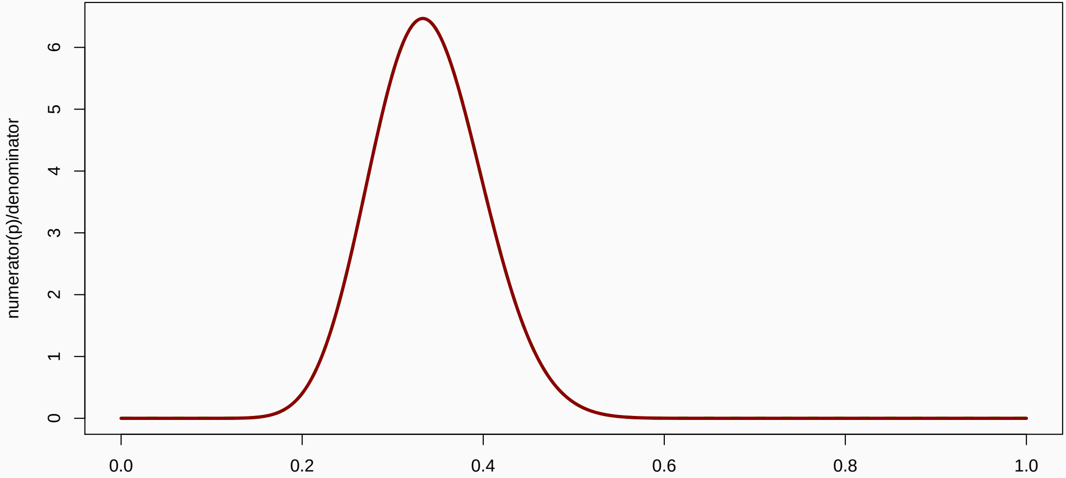
Posterior inference via numerical integration

```
plot(p, numerator(p)/denominator,type="l", lwd=3, col="green", lty=2)
```



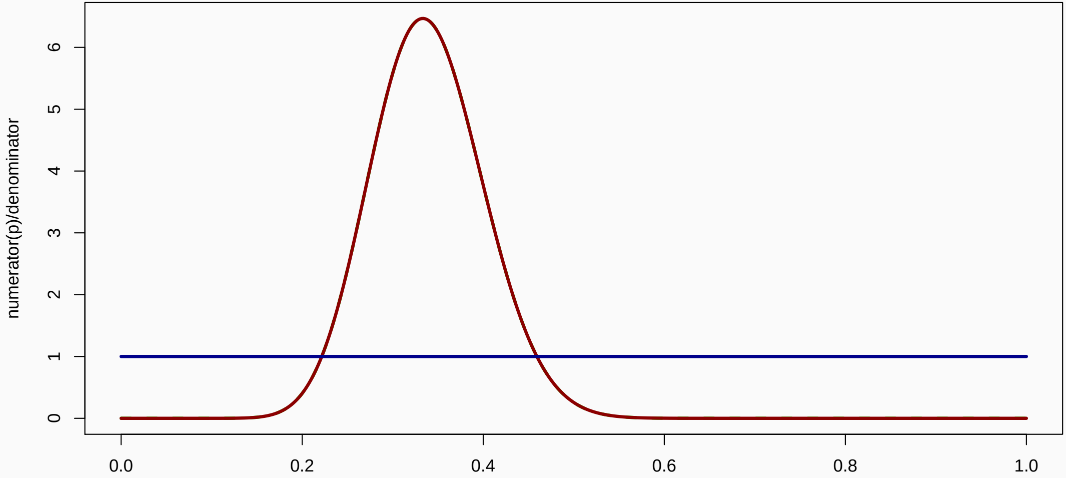
Superimpose explicit posterior distribution (Beta formula)

```
lines(p, dbeta(p,y+a,n-y+b), col='darkred', lwd=3)
```



And the prior

```
lines(p, dbeta(p,a,b), col='darkblue', lwd=3)
```



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- Do we really wish to calculate a 3D integral?

Bayesian computation

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Equation of State Calculations by Fast Computing Machines

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(Received March 6, 1953)

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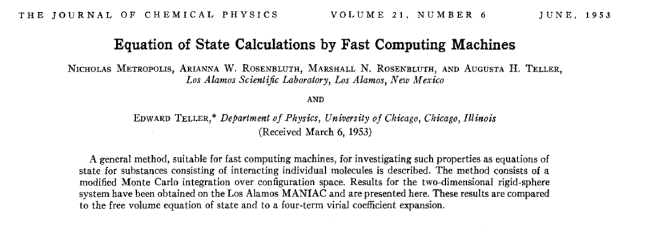

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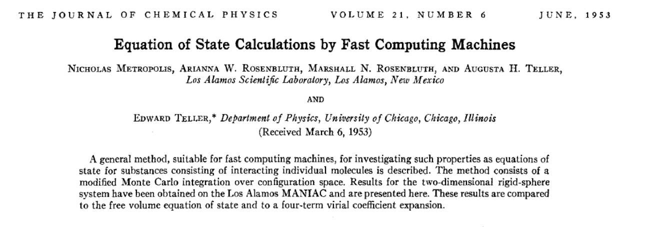

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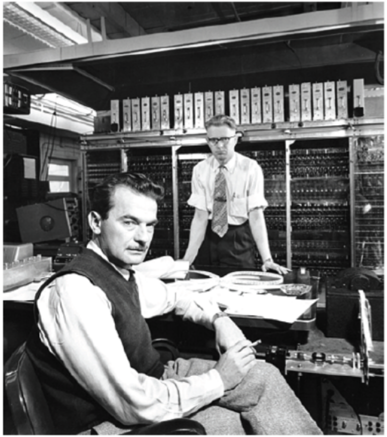
- Avoid explicit calculation of integrals in Bayes formula.
- Instead, approximate posterior to arbitrary degree of precision by drawing large sample.
- Markov chain Monte Carlo = MCMC; boost to Bayesian statistics!

MANIAC

MANIAC:
Mathematical Analyzer, Numerical Integrator, and Computer



MANIAC:
1000 pounds
5 kilobytes of memory
70k multiplications/sec

Your laptop:
4–7 pounds
2–8 million kilobytes
Billions of multiplications/sec

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- Equilibrium distribution is the desired posterior distribution!
- Several ways of constructing these chains: e.g., Metropolis-Hastings, Gibbs sampler, Metropolis-within-Gibbs.
- How to implement them in practice?!

The Metropolis algorithm

- We illustrate sampling from a discrete distribution. Suppose we define a discrete probability distribution on the integers $1, \ldots, K$.

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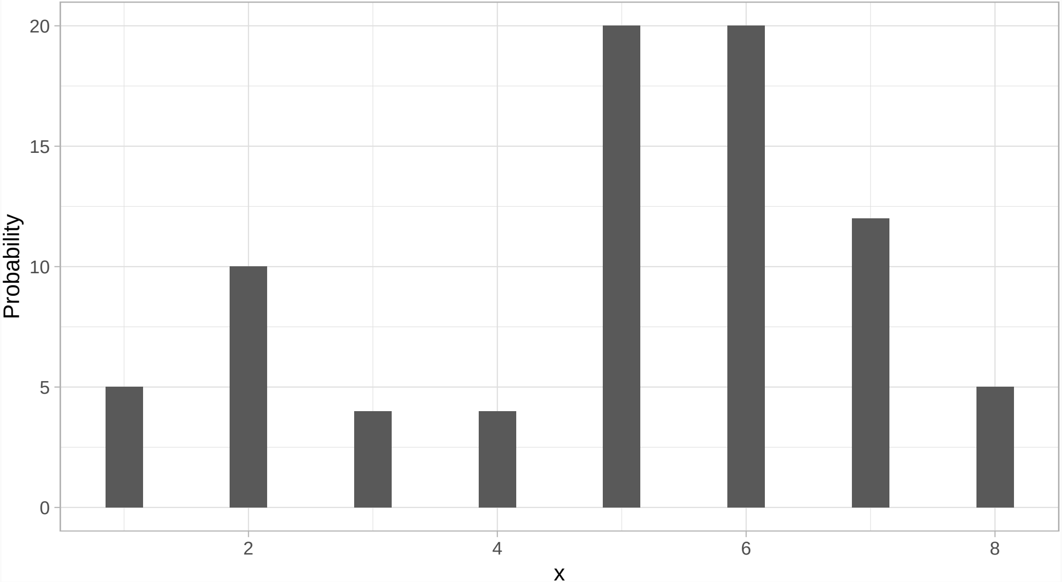
- We write a short function `pd()` in `R` taking on the values $1, \ldots, 8$ with probabilities proportional to the values 5, 10, 4, 4, 20, 20, 12, and 5.

```r
pd <- function(x){
  values <- c(5, 10, 4, 4, 20, 20, 12, 5)
  ifelse(x %in% 1:length(values), values[x], 0)
}
prob_dist <- data.frame(x = 1:8, prob = pd(1:8))
prob_dist
#>   x prob
#> 1 1    5
#> 2 2   10
#> 3 3    4
#> 4 4    4
#> 5 5   20
#> 6 6   20
#> 7 7   12
#> 8 8    5
```


Probability
20
15
10
5
0
2
4
6
8
x

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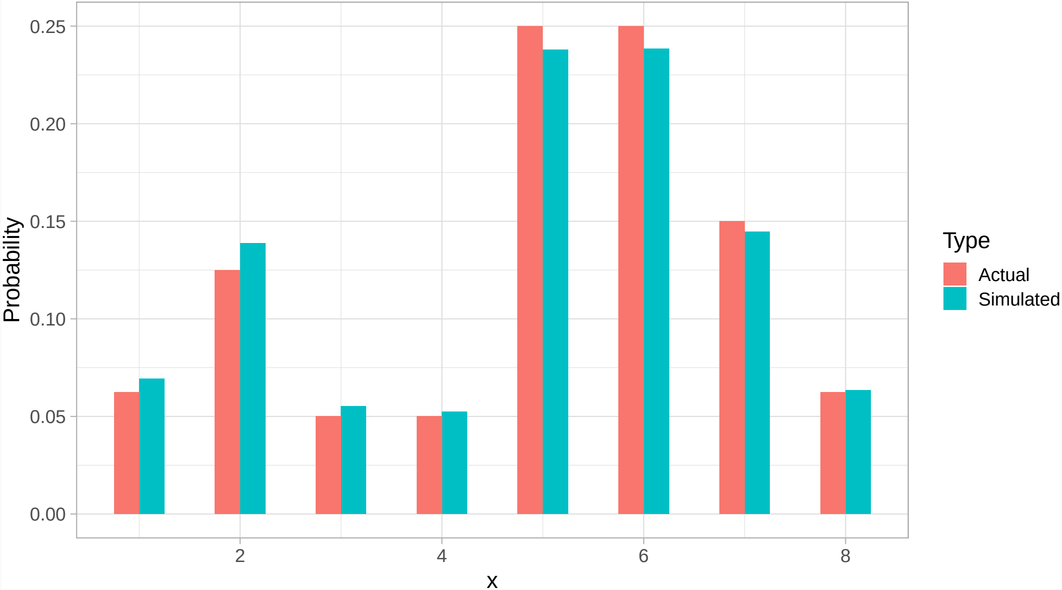
5. We repeat 2-4 a number of times called **steps** (many steps).

```r
random_walk <- function(pd, start, num_steps){
  y <- rep(0, num_steps)
  current <- start
  for (j in 1:num_steps){
    candidate <- current + sample(c(-1, 1), 1)
    prob <- pd(candidate) / pd(current)
    if (runif(1) < prob) current <- candidate
    y[j] <- current
  }
  return(y)
}
```

Starting at the value $X = 4$ and running the algorithm for $s = 10,000$ iterations.

```
out <- random_walk(pd, 4, 10000)
head(out)
#> [1] 3 2 2 2 1 1
tail(out)
#> [1] 4 5 5 5 6 7
```


0.25
0.20
0.15
0.10
0.05
0.00
Probability
2
4
6
8
x
Type
Actual
Simulated

Animating the Metropolis algorithm - 2D example

https://mbjoseph.github.io/posts/2018-12-25-animating-the-metropolis-algorithm/

The Markov-chain Monte Carlo Interactive Gallery

https://chi-feng.github.io/mcmc-demo/