Bayesian statistics with R
3. Analyses by hand

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Back to Bayes

A simple example

- Let us take a simple example to fix ideas.
- 120 deer were radio-tracked over winter.
- 61 close to a plant, 59 far from any human activity.
- Question: is there a treatment effect on survival?

	Released	Alive	Dead	Other
treatment	61	19	38	4
control	59	21	38	0

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- Of interest is the probability of over-winter survival, call it θ, for the general population within the treatment area.
- The obvious estimate is simply to take the ratio $k/n = 19/57$.
- How would the classical statistician justify this estimate?

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- K the number of alive individuals at the end of the winter, so that $P(K = k) = \binom{n}{k}\theta^k(1-\theta)^{n-k}$.
- The classical approach is to maximise the corresponding likelihood with respect to θ to obtain the entirely plausible MLE:

$$\hat{\theta} = k/n = 19/57$$

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- What is the Beta distribution?

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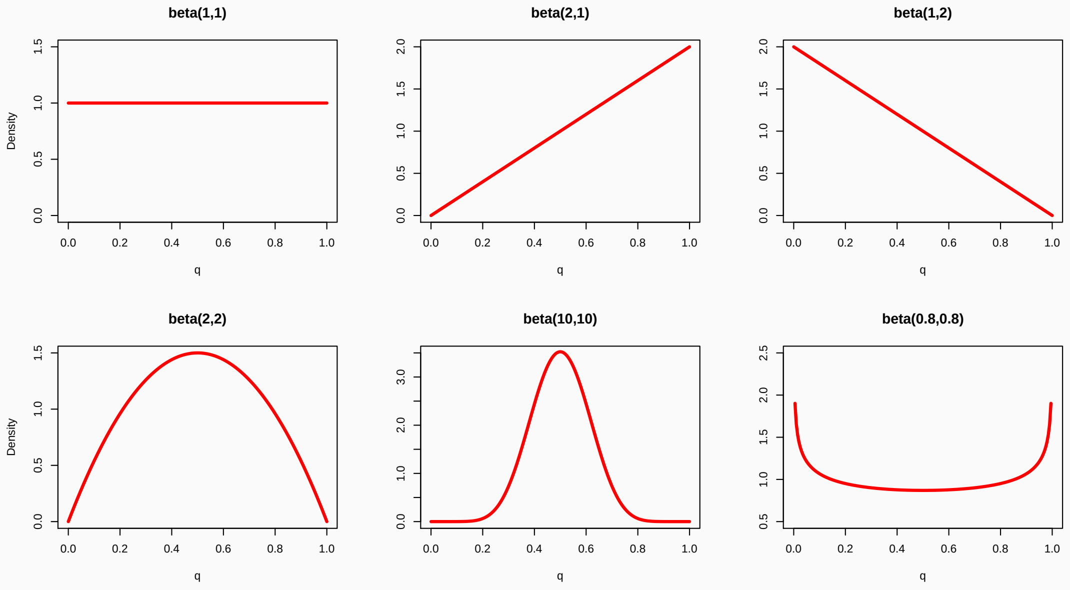
$$q(\theta \mid \alpha, \beta) = \frac{1}{\text{Beta}(\alpha, \beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

with $\text{Beta}(\alpha, \beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ and $\Gamma(n) = (n-1)!$


beta(1,1)
beta(2,1)
beta(1,2)
beta(2,2)
beta(10,10)
beta(0.8,0.8)
Density
q
0.0
0.2
0.4
0.6
0.8
1.0
0.5
1.5
2.0
2.5
3.0

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- Take a Beta prior with a Binomial likelihood, you get a Beta posterior (conjugacy)

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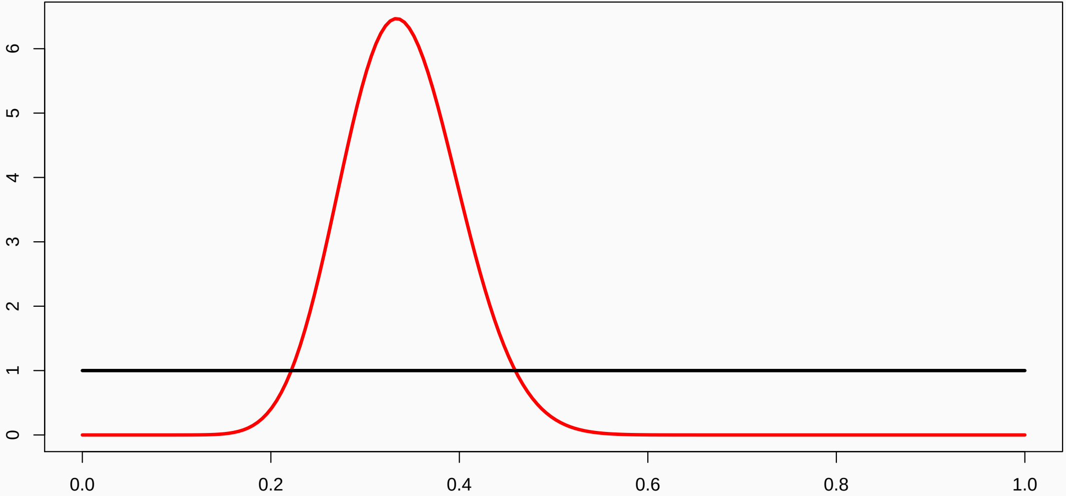
- Note that in this specific situation, the posterior has an explicit expression, easy to manipulate.
- In particular, $E(Beta(a,b)) = \dfrac{a}{a+b} = 20/59$ to be compared with the MLE $19/57$.

A general result

This is a general result, the Bayesian and frequentist estimates will always agree if there is sufficient data, so long as the likelihood is not explicitly ruled out by the prior.

Prior $Beta(1,1)$ and posterior survival $Beta(20,39)$



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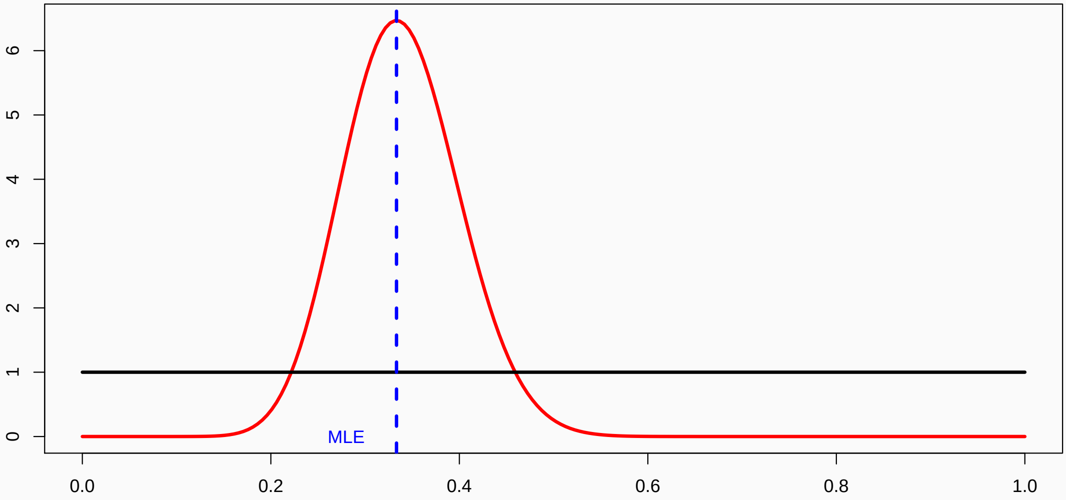
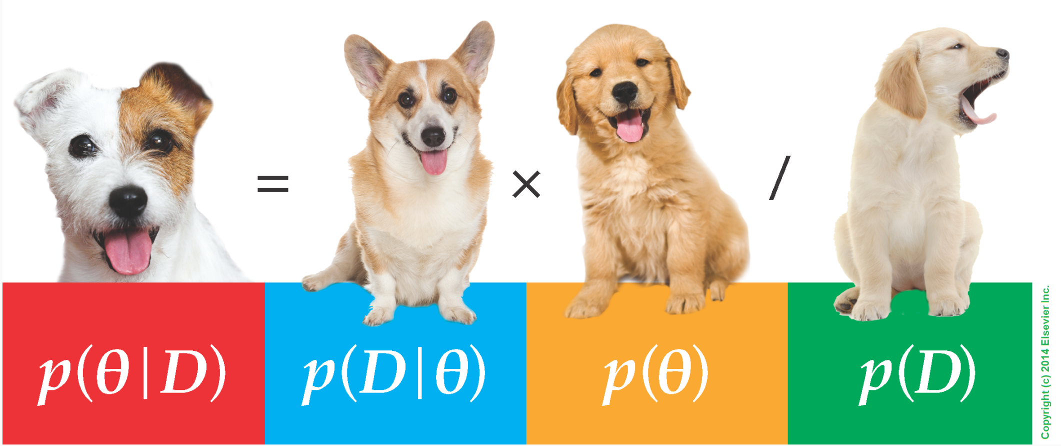


Notation

Our model so far

$$
\begin{aligned}
y &\sim \text{Binomial}(N, \theta) && \text{[likelihood]} \\
\theta &\sim \text{Beta}(1, 1) && \text{[prior for } \theta\text{]}
\end{aligned}
$$


$p(\theta|D) = p(D|\theta) \times p(\theta) / p(D)$
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