Bayesian statistics with R
4. A detour with priors

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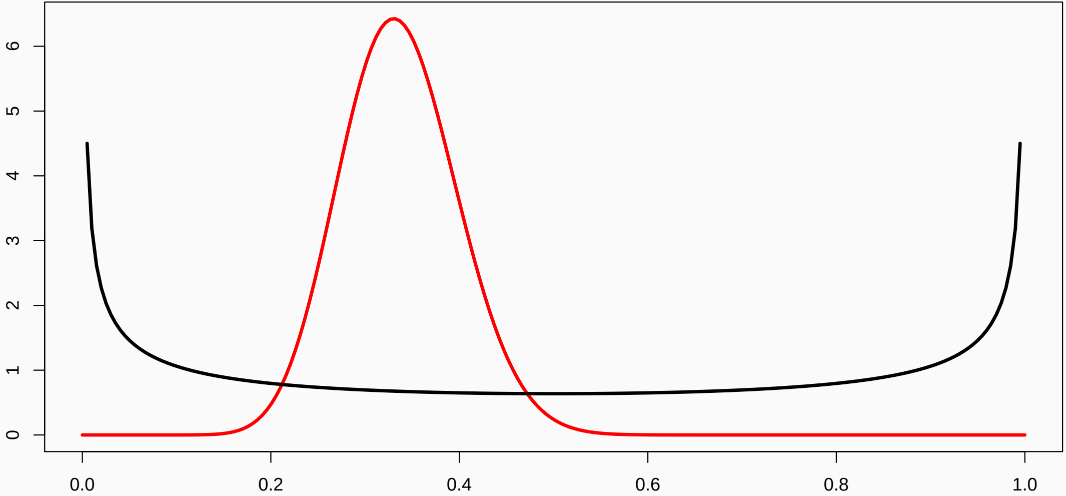
March 2021

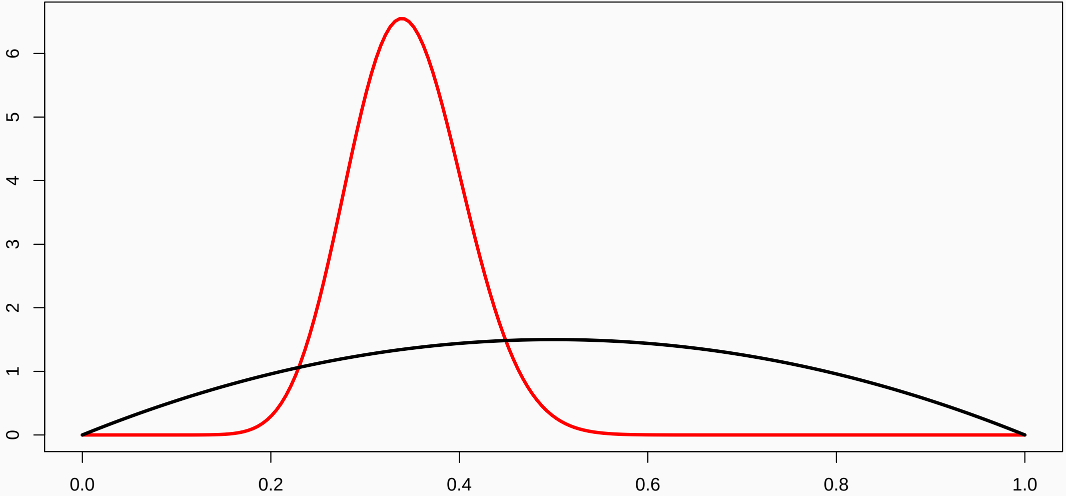
A detour to explore priors

Influence of the prior

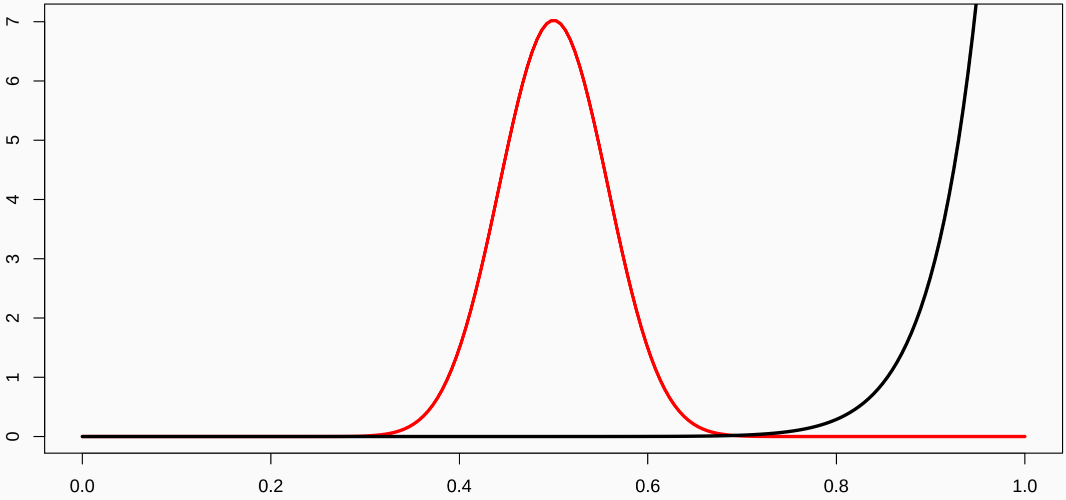
Prior $Beta(0.5, 0.5)$ and posterior survival $Beta(19.5, 38.5)$



Prior $Beta(2, 2)$ and posterior survival $Beta(21, 40)$



Prior $Beta(20, 1)$ and posterior survival $Beta(39, 49)$



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- Always perform a sensitivity analysis.

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WE ARE ALL BAYESIANS,...
Based on my priors
I will be just fine

How to incorporate prior information?

Estimating survival using capture-recapture data

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- We assume a vague prior:

$$\phi_{prior} \sim \text{Beta}(1, 1) = \text{Uniform}(0, 1)$$

Notation

- $y_{i,t} = 1$ if individual i detected at occasion t and 0 otherwise
- $z_{i,t} = 1$ if individual i alive between occasions t and $t+1$ and 0 otherwise

$$
\begin{aligned}
y_{i,t} \mid z_{i,t} &\sim \text{Bernoulli}(p \, z_{i,t}) && \text{[likelihood (observation eq.)]} \\
z_{i,t+1} \mid z_{i,t} &\sim \text{Bernoulli}(\phi \, z_{i,t}) && \text{[likelihood (state eq.)]} \\
\phi &\sim \text{Beta}(1,1) && \text{[prior for } \phi \text{]} \\
p &\sim \text{Beta}(1,1) && \text{[prior for p]}
\end{aligned}
$$

European dippers in Eastern France (1981-1987)

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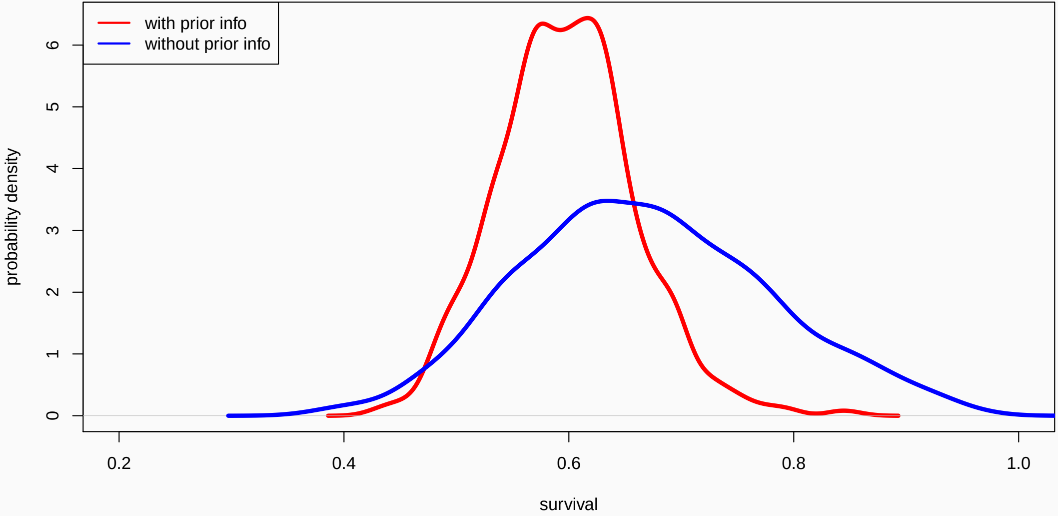
- Now if you had only the three first years of data, what would have happened?
- Width of credible interval is 0.47 (vague prior) vs. 0.30 (informative prior).
- Huge increase of precision in posterior inference (40% gain)!

Compare vague vs. informative prior



Prior elicitation via moment matching

Remember the Beta distribution

- Recall that the Beta distribution is a continuous distribution with values between 0 and 1. Useful for modelling survival or detection probabilities.

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- If $X \sim \mathit{Beta}(\alpha, \beta)$, then the first and second moments of X are:

$$\mu = \mathsf{E}(X) = \frac{\alpha}{\alpha + \beta}$$

$$\sigma^2 = \mathsf{Var}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

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- Parameters μ and σ^2 are seen as the moments of a *Beta*(α, β) distribution.
- Now we look for values of α and β that match the observed moments of the Beta distribution (μ and σ^2).
- We need another set of equations:

$$\alpha = \left(\frac{1-\mu}{\sigma^2} - \frac{1}{\mu}\right)\mu^2$$

$$\beta = \alpha\left(\frac{1}{\mu} - 1\right)$$

- For our model, that means:

```r
(alpha <- ( (1 - 0.57)/(0.073*0.073) - (1/0.57) )*0.57^2)
#> [1] 25.64636
(beta <- alpha * ( (1/0.57) - 1))
#> [1] 19.34726
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- Now use $\phi_{prior} \sim \text{Beta}(\alpha = 25.6, \beta = 19.3)$ instead of $\phi_{prior} \sim \text{Normal}(0.57, 0.073^2)$

Your turn

Question

Use simulations to check that our estimates are correct.

Solution

R code

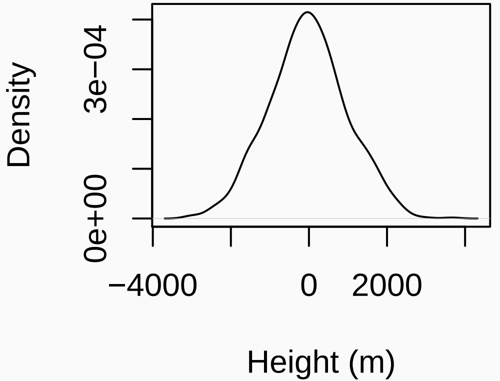
```r
alpha <- ( (1 - 0.57)/(0.073*0.073) - (1/0.57) )*0.57^2
beta <- alpha * ( (1/0.57) - 1)
n <- 10000
samp <- rbeta(n, alpha, beta)
(mu <- mean(samp))
(sigma <- sqrt(var(samp)))
```

Prior predictive checks

Linear regression

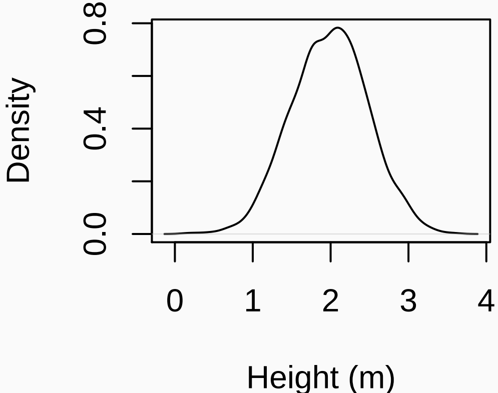
Unreasonable prior $\beta \sim N(0, 1000^2)$

```r
plot(density(rnorm(1000, 0, 1000)),
     main="", xlab="Height (m)")
```



Reasonable prior $\beta \sim N(2, 0.5^2)$

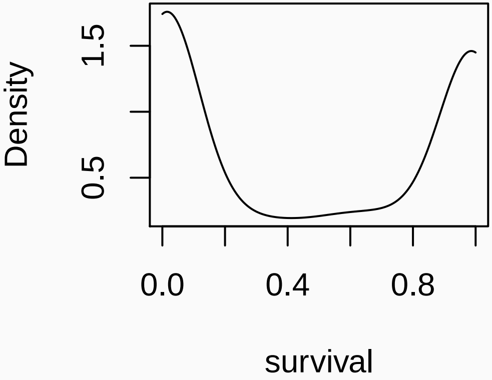
```r
plot(density(rnorm(1000, 2, 0.5)),
     main="", xlab="Height (m)")
```



Logistic regression

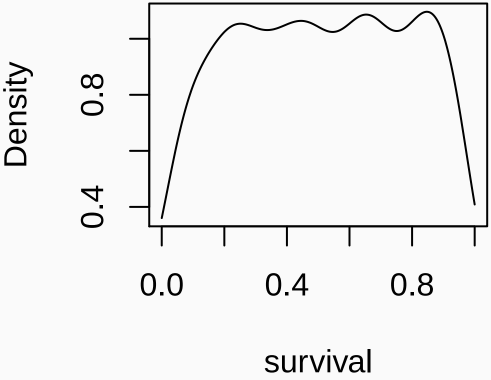
Unreasonable $\text{logit}(\phi) = \beta \sim N(0, 10^2)$

```
plot(density(plogis(rnorm(1000,0,10)),
from = 0, to = 1), main='', xlab='surv
```



Reasonable $\text{logit}(\phi) = \beta \sim N(0, 1.5^2)$

```
plot(density(plogis(rnorm(1000,0,1.5))
from = 0, to = 1), main='', xlab='surv
```



Dynamic updating

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If you obtain more data, no need to redo all of the analysis. Your posterior from the first analysis simply becomes your prior for the next analysis (and so on).

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- Stage 0. Prior $p(\theta) \sim \text{Beta}(1, 1)$.
- Stage 1. Observe $y_1 = 22$ successes from $n_1 = 29$ trials.
    - Likelihood is $p(y_1|\theta) \sim \text{Binomial}(n_1 = 29, \theta)$.
    - Posterior is $p(\theta|y_1) \sim \text{Beta}(23, 8)$ with mean $23/31 = 0.74$.

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- Stage 2. Observe $y_2 = 5$ successed from $n_2 = 10$ new trials.
  - Likelihood is $p(y_2|\theta) \sim \text{Binomial}(n_2 = 10, \theta)$.
  - Prior is $p(\theta) \sim \text{Beta}(23, 8)$ from stage 1.
  - Posterior is $p(\theta|y_1 \text{ and } y_2) \propto p(\theta|y_1)p(y_2|\theta) = \text{Beta}(28, 13)$ with mean $28/41 = 0.68$.