Bayesian statistics with R
2. The likelihood

Olivier Gimenez

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Likelihood

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- In the real world, it is usually the other way around.

A question of interest might be for example:

We have observed 3 births by a female during her 10 breeding attempts. What does this tell us about the true probability of getting a successful breeding attempt for this female? For the population?

- We don't know what the probability of a birth is.
- But we can see what the probability of getting our data would be for different values:

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#> [1] 0.05739563
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dbinom(x=3,size=10,prob=0.25)
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So we would be more likely to observe 3 births if the probability is 0.25 than 0.1 or 0.9.

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- **The likelihood** is the probability of observing the data under a certain model.
- The data are known, we usually consider the likelihood as a function of the model parameters $\theta_1, \theta_2, \ldots, \theta_p$

$$L = P(\theta_1, \theta_2, \ldots, \theta_p \mid \text{data})$$

Likelihood functions

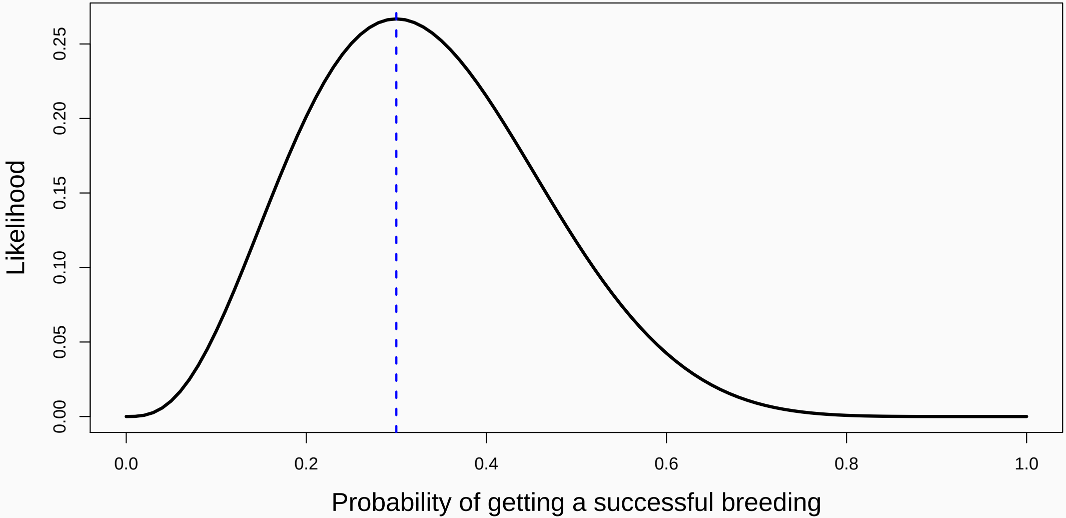
We may create a function to calculate a likelihood:

```
lik.fun <- function(parameter){
  ll <- dbinom(x=3, size=10, prob=parameter)
  return(ll)
}

lik.fun(0.3)
#> [1] 0.2668279

lik.fun(0.6)
#> [1] 0.04246733
```

Maximize the likelihood (3 successes ot of 10 attempts)



The *maximum* of the likelihood is at value 0.3

Maximum likelihood estimation

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- These can be calculated using:
  - Trial and error (not efficient!).
  - Compute the maximum of a function by hand (rarely doable in practice).
  - An iterative optimization algorithm: `?optim` in `R`.

By hand: compute MLE of p from $Y \sim \textbf{Bin}(N = 10, p)$ with $k = 3$ successes

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- Here, the MLE is the proportion of observed successes.

Using a computer: MLE of p from $Y \sim \text{Bin}(N = 10, p)$ with $k = 3$ successes

```r
lik.fun <- function(parameter) dbinom(x=3, size=10, prob=parameter)
# ?optimize
optimize(lik.fun,c(0,1),maximum=TRUE)
#> $maximum
#> [1] 0.3000157
#>
#> $objective
#> [1] 0.2668279
```

Use `optim` when the number of parameters is > 1.

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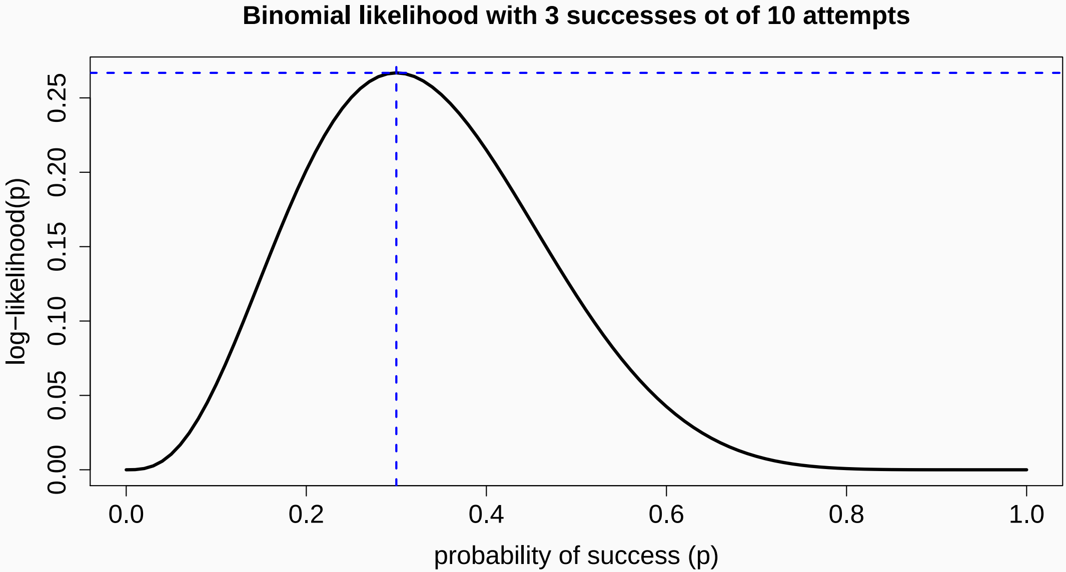

Your turn: Practical 2