Chapter 2 Discrete Distributions

- Random Variables of the Discrete Type
  - Uniform Distribution
  - Hypergeometric Distribution
- Mathematical Expectation
- Moment Generating Function
- Bernoulli Trials and the Binomial Distribution
- Geometric and Negative Binomial Distribution
- The Poisson Distribution

Poisson Process

- **Examples:**
  - Number of phone call arriving between 9 and 10 am
  - Number of flaws in 10 feet of wire
  - Number of customers arriving between 2 and 4 pm

Definition 2.6-1

Let the number of occurrences of some event in a given continuous interval be counted. Then we have an **approximate Poisson process** with parameter $\lambda > 0$ if the following conditions are satisfied:

(a) The numbers of occurrences in nonoverlapping subintervals are independent.

(b) The probability of exactly one occurrence in a sufficiently short subinterval of length h is approximately λh.

(c) The probability of two or more occurrences in a sufficiently short subinterval is essentially zero.

Poisson Distribution

- Consider a Poisson Process
- X: number of occurrences in an interval of unit length
- Want to find $P(X = x) = ?$, for $x = 0, 1, 2, \ldots$

- Partition the unit interval into n subintervals of equal length $1/n$
- Suppose $n >> x$
- $P(X = x) = P(\textit{one occurrence in each of exactly } x \textit{ of these } n \textit{ subintervals})$

- By condition (c)
  - *P*(*two or more occurrences occur in any one subinterval*) $\cong$ 0
- By condition (b)
  - *P*(*one occurrence in any one subinterval of length* 1/*n*) $\cong$ $\lambda\,(1/n)$
- By condition (a)
  - We have a sequence of *n* Bernoulli trials with probability *p* approximately equal to $\lambda\,(1/n)$

$$\Longrightarrow \quad P(X = x) \cong \frac{n!}{x!\,(n-x)!}\left(\frac{\lambda}{n}\right)^{x}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

As n increases, p decreases,
(hence, maintain λ = constant)

$\rightarrow p = \lambda/n$ $\rightarrow n\,p = \lambda$

As $n \rightarrow \infty$,

$$P(X = x) = \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!}$$

(see next slide)

The distribution of probability associated with this process has a special name. We say that the random variable X has a **Poisson distribution** if its p.m.f. is of the form

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots,$$

where $\lambda > 0$.

If n increases without bound, we have that

$$\lim_{n\to\infty} \frac{n!}{x!\,(n-x)!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n(n-1)\cdots(n-x+1)}{n^x}\,\frac{\lambda^x}{x!}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}.$$

Now, for fixed x, we have (see Appendix A.3)

$$\lim_{n\to\infty} \frac{n(n-1)\cdots(n-x+1)}{n^x} = \lim_{n\to\infty}\left[(1)\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right] = 1,$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda},$$

(see next slide)

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x} = 1.$$

Thus

$$\lim_{n\to\infty} \frac{n!}{x!\,(n-x)!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} = \frac{\lambda^x e^{-\lambda}}{x!} = P(X=x).$$

Appendix D.3 Limits

Another limit of importance is

$$\lim_{n\to\infty}\left(1+\frac{b}{n}\right)^n = \lim_{n\to\infty} e^{n\ln(1+b/n)},$$

where b is a constant.

Since the exponential function is continuous, the limit can be taken to the exponent. That is,

$$\lim_{n\to\infty} \exp[n\ln(1+b/n)] = \exp[\lim_{n\to\infty} n\ln(1+b/n)].$$

The limit in the exponent is equal to

$$\lim_{n\to\infty}\frac{\ln(1+b/n)}{1/n} = \lim_{n\to\infty}\frac{\dfrac{-b/n^2}{1+b/n}}{-1/n^2} = \lim_{n\to\infty}\frac{b}{1+b/n} = b$$

by L'Hôspital's rule. Since this limit is equal to b, the original limit is

$$\lim_{n\to\infty}\left(1+\frac{b}{n}\right)^n = e^b.$$

As $n \to \infty$,

$$P(X = x) = \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!}\left(\frac{\lambda}{n}\right)^{x}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^{x} e^{-\lambda}}{x!}$$

The distribution of probability associated with this process has a special name. We say that the random variable X has a **Poisson distribution** if its p.m.f. is of the form

$$f(x) = \frac{\lambda^{x} e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots,$$

where $\lambda > 0$.

It is easy to see that $f(x)$ enjoys the properties of a p.m.f. because clearly $f(x) \geq 0$ and, from the Maclaurin's series expansion of e^{λ} (see Appendix A.4), we have

$$\sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1.$$

$\Rightarrow$ Satisfy the properties of a p.m.f.

A function $f(x)$ possessing derivatives of all orders at $x = b$ can be expanded in the following **Taylor's series**:

$$f(x) = f(b) + \frac{f'(b)}{1!}(x-b) + \frac{f''(b)}{2!}(x-b)^2 + \frac{f'''(b)}{3!}(x-b)^3 + \cdots.$$

If $b = 0$, we obtain the special case that is often called **Maclaurin's series**;

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots.$$

For example, if $f(x) = e^x$ so that all derivatives of $f(x) = e^x$ are $f^{(r)}(x) = e^x$, then $f^{(r)}(0) = 1$, for $r = 1, 2, 3, \ldots$ Thus the Maclaurin's series expansion of $f(x) = e^x$ is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Mean and Variance

The moment-generating function of X is

$$M(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}.$$

From the series representation of the exponential function, we have that

$$M(t) = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

for all real values of t. Now

$$M'(t) = \lambda e^t e^{\lambda(e^t - 1)},$$

and

$$M''(t) = (\lambda e^t)^2 e^{\lambda(e^t - 1)} + \lambda e^t e^{\lambda(e^t - 1)}.$$

$$\boxed{M'(t) = \lambda e^t e^{\lambda(e^t - 1)},}$$

$$\boxed{M''(t) = (\lambda e^t)^2 e^{\lambda(e^t - 1)} + \lambda e^t e^{\lambda(e^t - 1)}}$$

The values of the mean and variance of X are

$$\mu = M'(0) = \lambda$$

and

$$\sigma^2 = M''(0) - [M'(0)]^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda.$$

That is, for the Poisson distribution, $\mu = \sigma^2 = \lambda$.

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots,$$

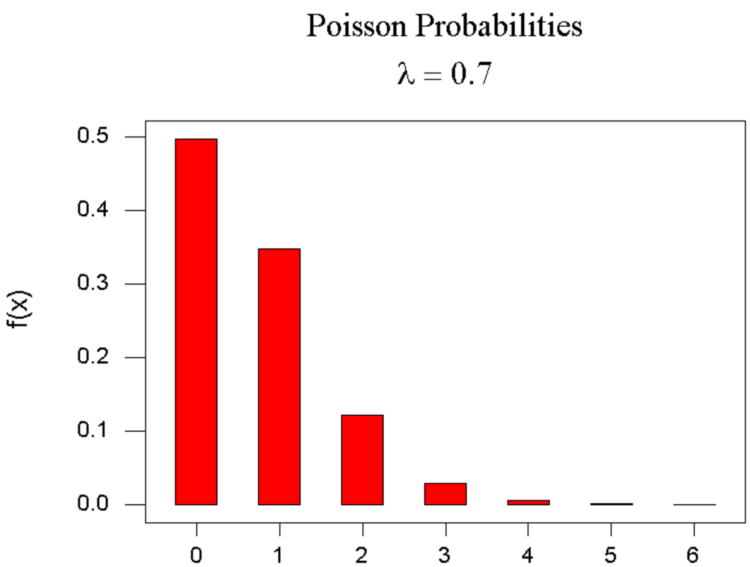
Example 3.5-1 Let X have a Poisson distribution with a mean of $\lambda = 5$. Then using Table III in the Appendix,

$$P(X \leq 6) = \sum_{x=0}^{6} \frac{5^x e^{-5}}{x!} = 0.762,$$

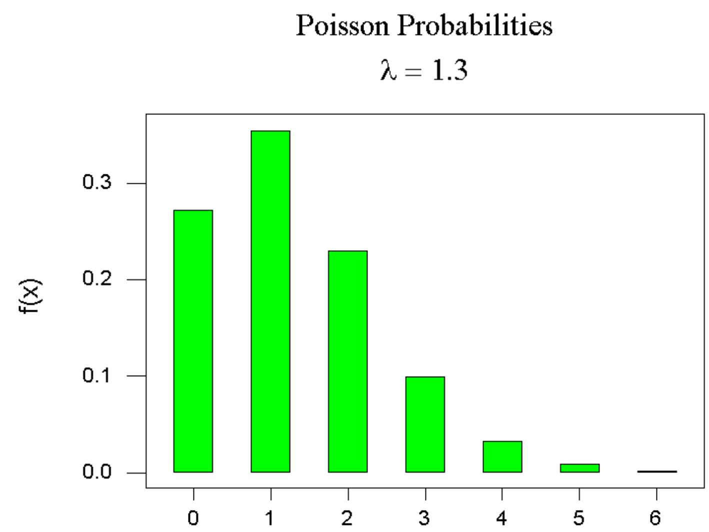
$$P(X > 5) = 1 - P(X \leq 5) = 1 - 0.616 = 0.384,$$

and

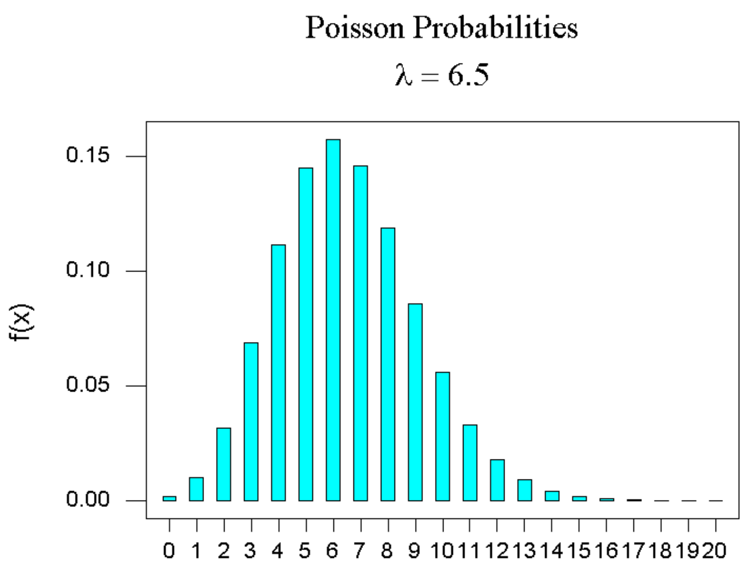
$$P(X = 6) = P(X \leq 6) - P(X \leq 5) = 0.762 - 0.616 = 0.146. \qquad \blacktriangle$$



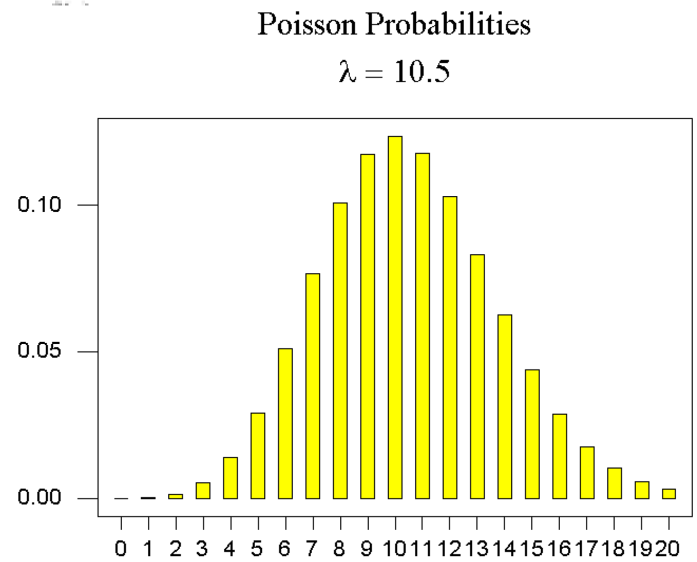
Poisson Probabilities
$\lambda = 0.7$
f(x)
0.5
0.4
0.3
0.2
0.1
0.0
0 1 2 3 4 5 6



Poisson Probabilities
$\lambda = 1.3$
f(x)
0.3
0.2
0.1
0.0
0 1 2 3 4 5 6



Poisson Probabilities
$\lambda = 6.5$
f(x)
0.15
0.10
0.05
0.00
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20



Poisson Probabilities
$\lambda = 10.5$
0.10
0.05
0.00
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

Example 2.6-6 In Figure 2.6-2, Poisson probability histograms have been superimposed on shaded binomial probability histograms so that we can see whether or not these are close to each other. If the distribution of X is $b(n, p)$, the approximating Poisson distribution has a mean of $\lambda = np$. Note that the approximation is not good when p is large (e.g., $p = 0.30$).

比較Poisson以及Bionomial

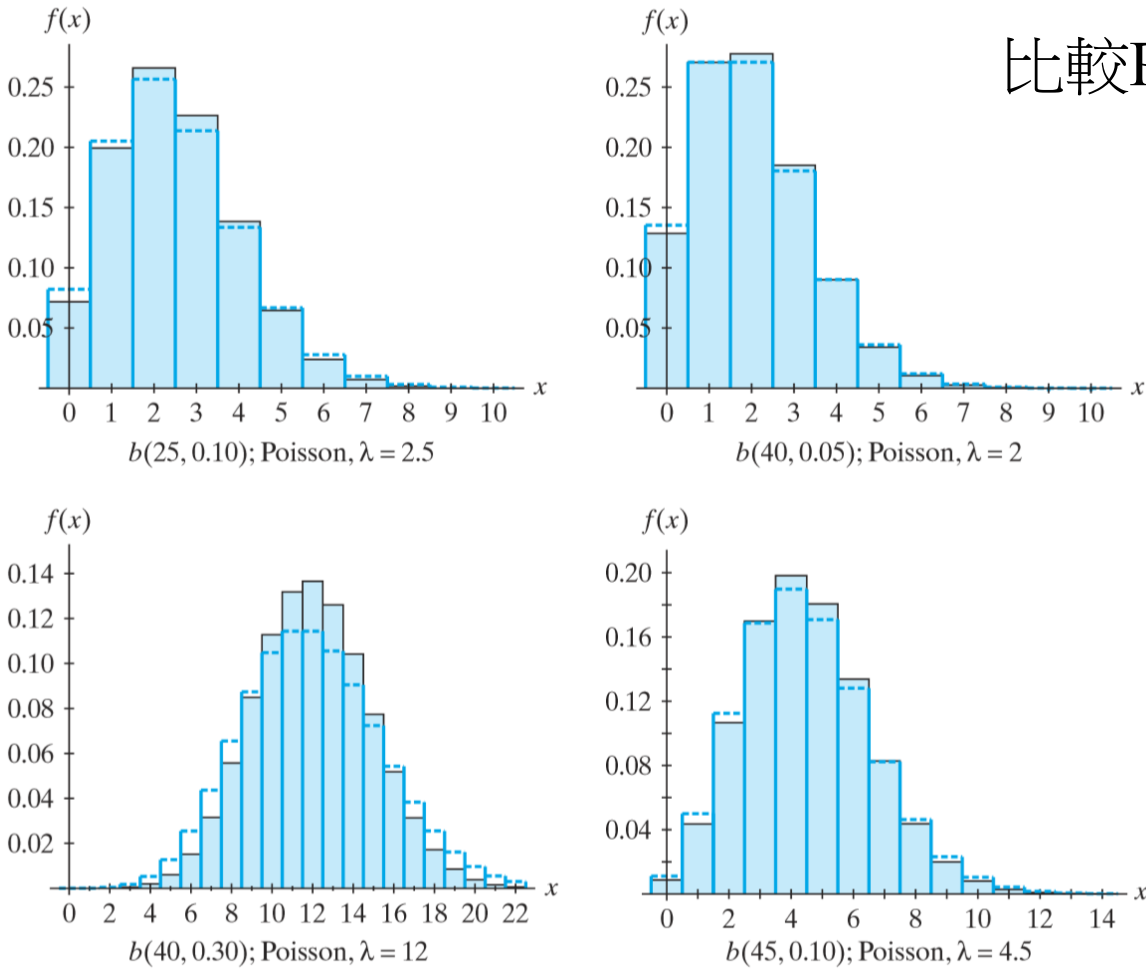


Figure 2.6-2 Binomial (shaded) and Poisson probability histograms