# Chapter 3 Continuous Distributions

- Random Variables of the Continuous Type
- Exponential, Gamma, and Chi-Square Distributions
- Normal Distribution
- Additional Models

## ❖ Exponential Distribution (v1 in Textbook)

- Consider a Poisson process with parameter λ.
- Let W be the random variable corresponding to the waiting time of the first occurrence of the event. Then the distribution function of *W* is

$$
\begin{aligned}
F(w) = P(W \le w) &= 1 - P(W > w) \\
&= 1 - P(\text{no occurrences in } [0, w]\,) \\
&\overset{?}{=} 1 - e^{-\lambda w},
\end{aligned}
$$

For $w \ge 0$,

Recall that, for a r.v. X having a Poisson distribution with parameter $\lambda$, its p.m.f. is

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots,$$

# ❖ Exponential Distribution (v2)

- Consider a Poisson process in which the mean number of occurrences in the unit interval is $\lambda$.
- Let $W$ denote the waiting time until the first occurrence *during the observation of the above Poisson process*. Then $W$ is a continuous-type random variable with the cdf derived below.
- Because this waiting time is nonnegative, the cdf

$$F(w) = 0, \qquad \text{for } w < 0.$$

- For $w \geq 0$, $F(w) = P(W \leq w) = 1 - P(W > w)$
- What is $P(W > w)$ = ?

- What is $P(W > w)$ ?

- Let the event A = $\{\omega: W(\omega) > w\}$

  i.e., the set of all outcomes such that $W(\omega) > w$,

  i.e., the set of all outcomes such that the waiting time is greater than $w$,

  i.e., the set of all outcomes such that no occurrences in $[0, w]$ .

- Let X be the number of occurrences of a Poisson process with parameter $\lambda w$
- Let the event B = $\{\omega: X(\omega) = 0\}$

  i.e., the set of all outcomes such that no occurrences in $[0, w]$ *during the observation of a* Poisson process in which the mean number of occurrences in $[0, w]$ is $\lambda w$.

- Notice that *a* Poisson process in which the mean number of occurrences in $[0, w]$ is $\lambda w$ *is "the same" as a* Poisson process in which the mean number of occurrences in $[0, 1]$ is $\lambda$.

- Notice that the event A = $\{\omega: W(\omega) > w\}$

  is equivalent to the event B = $\{\omega: X(\omega) = 0\}$

- Therefore, $P(A) = P(W > w) = P(X = 0) = P(B)$

  where $W$ denote the waiting time until the first occurrence during the observation of a Poisson process with parameter $\lambda$,

  and $X$ denote the number of occurrences in the unit interval (i.e., [0, $w$]) during the observation of a Poisson process with parameter $\lambda w$

- In Section 2.6, it was shown the random variable $X$ has a Poisson distribution with the following pmf:

$$P(X = x) = \frac{(\lambda w)^x e^{-\lambda w}}{x!}, \quad x = 0, 1, 2, \ldots$$

- Therefore, $P(W > w) = P(X = 0) = e^{-\lambda w}$

- Hence, we have

  For $w \geq 0$, $F(w) = P(W \leq w)$

$$= 1 - P(W > w) = 1 - P(X = 0) = 1 - e^{-\lambda w}$$

$$F'(w) = f(w) = \lambda e^{-\lambda w}.$$

That is, we have

$$F_Y(y) = 1 - e^{-\lambda y} \qquad for\ y \geq 0$$
$$and \qquad F_Y(y) = 0 \qquad\qquad for\ y < 0.$$

Therefore

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \lambda e^{-\lambda y} \ \ for\ y \geq 0.$$
$$and\ \ f_Y(y) = 0\ \ for\ \ y < 0.$$

Recall that, for a r.v. X having a Poisson distribution with parameter $\lambda$, its p.m.f. is

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots,$$

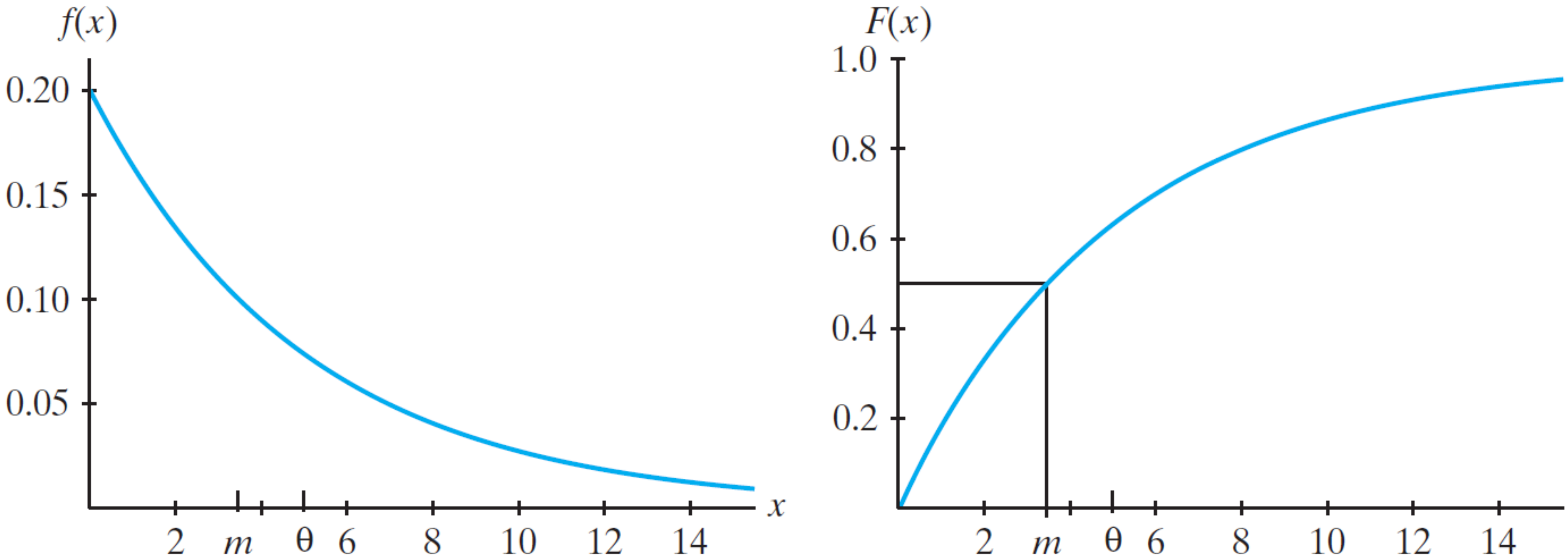


**Figure 3.2-1** Exponential pdf, $f(x)$, and cdf, $F(x)$

# Alternative Way to Derive the Exponential Distribution

$F_Y(y) = \text{Prob}(Y \le y)$

If we divide y into n intervals and let Z be the geometric random variable corresponding to the first occurrence of the event, then

$$\text{Prob}(Y \le y) = \lim_{n\to\infty} \sum_{k=1}^{\text{n}} \text{Prob}(Z = k) = \lim_{n\to\infty} \sum_{k=1}^{\text{n}} \frac{\lambda y}{n}\left(1 - \frac{\lambda y}{n}\right)^{k-1}$$

$$= \lim_{n\to\infty} \frac{\lambda y}{n} \sum_{k=0}^{n-1}\left(1 - \frac{\lambda y}{n}\right)^{k} = \lim_{n\to\infty} \frac{\lambda y}{n} \cdot \frac{1 - \left(1 - \frac{\lambda y}{n}\right)^{n}}{1 - \left(1 - \frac{\lambda y}{n}\right)}$$

$$= \lim_{n\to\infty} 1 - \left(1 - \frac{\lambda y}{n}\right)^{n} = 1 - e^{-\lambda y}$$

# Exponential Distribution

- The p.d.f. of Y is

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \lambda e^{-\lambda y} \;\; for \;\; y \geq 0.$$

$$and \;\; f_Y(y) = 0 \;\; for \;\; y < 0.$$

- The m.g.f. of Y is

$$\begin{aligned} M_Y(t) &= \int_0^\infty e^{ty} \cdot \lambda e^{-\lambda y} dy \\ &= \lambda \int_0^\infty e^{-(\lambda - t)y} dy \\ &= \frac{\lambda}{\lambda - t} e^{-(\lambda - t)y} \Big|_\infty^0 \\ &= \frac{\lambda}{\lambda - t} \qquad if \;\; t < \lambda \end{aligned}$$

$$\mu = M'(0) = \theta$$

$$\sigma^2 = M''(0) - [M'(0)]^2 = \theta^2.$$

# Exponential Distribution

- Then, $M'(t) = \dfrac{\lambda}{(\lambda - t)^2}$ *and* $M''(t) = \dfrac{2\lambda}{(\lambda - t)^3}$,

➔ We have $\mu_Y = M'(0) = \dfrac{1}{\lambda}$ *and* $\sigma_Y{}^2 = M''(0) - (M'(0))^2 = \dfrac{1}{\lambda^2}$.

- Therefore, for a Poisson process with parameter λ, the average waiting time for the first event occurrence is 1/λ.
  - Note that, λ is the expected number of event occurrence during one unit of time interval.

# Exponential Distribution

- Based on the above derivation,
  we often let λ = 1/θ
  and say that
  the random variable *X* has an exponential distribution

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \lambda e^{-\lambda y} \ for \ y \geq 0.$$
$$and \ f_Y(y) = 0 \ for \ y < 0.$$

if its pdf is defined by

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 \leq x < \infty$$

where the parameter $\theta > 0$.

Accordingly, the waiting time until the first occurrence in a Poisson process has an exponential distribution with θ = 1/λ.

# Example 3.2-1

- Let X have an exponential distribution with a mean of θ = 20. So the p.d.f. of X is,

$$f(x) = \frac{1}{20} e^{-x/20}, \qquad 0 \le x < \infty.$$

- The probability that X is less that 18 is

$$P(X < 18) = \int_0^{18} \frac{1}{20} e^{-x/20} dx = 1 - e^{-18/20} = 0.593.$$

# Example 3.2-2

- Customers arrive in a certain shop according to an approximate Poisson process at a mean rate of 20 per hour.
- What is the probability that the shoekeeper will have to wait more that 5 minutes for the arrival of the first customer?
- Let X denote the waiting time in minutes until the first customer arrives and note that λ=1/3 is the expected number of arrivals per minute. Thus

$$\theta = \frac{1}{\lambda} = 3 \quad \text{and } f(x) = \frac{1}{3} e^{-(1/3)x}, \quad 0 \le \mathrm{x} < \infty.$$

- Hence

$$P(X > 5) = \int_5^\infty \frac{1}{3} e^{-(1/3)x} dx = e^{-5/3} = 0.1889.$$

- The median time until the first arrival is

$$m = -3\ln(0.5) = 2.0794. \quad ?$$

$\boxed{m = -\theta \ln(0.5)}$

*Median time until the first arrival*

# Example 3.2-3

- Suppose that the life of a certain type of electronic component has an exponential distribution with a mean life of 500 hours.
- If X denotes the life of this component, then

$$P(X > x) = \int_x^\infty \frac{1}{500} e^{-t/500}\,dt = e^{-x/500}.$$

- Suppose that the component has been in operation for 300 hours. The conditional probability that it will last for another 600 hours is ?

$$P(X > 900 \mid X > 300) = \frac{P(X > 900)}{P(X > 300)} = \frac{e^{-900/500}}{e^{-300/500}} = e^{-6/5}.$$

# Example 3.2-3(cont.)

$$P(X > 900 \mid X > 300) = \frac{P(X > 900)}{P(X > 300)} = \frac{e^{-900/500}}{e^{-300/500}} = e^{-6/5}.$$

$$P(X > 600) = \quad ?$$

- That is,

  *the probability that it will last an additional 600 hours, given that it has operated for 300 hours,*

  *is the same as*

  *the probability that it would last 600 hours*

  *when first put into operation*.

- Thus, for such components, an old one is as good as a new one, and we say that *the failure rate is constant.*

# Example 3.2-3(cont.)

- With constant failure rate, there is no advantage in replacing components that are operating satisfactorily.
- Is it true in practice ? *No!!*
- Obviously, this is not true in practice because most would have an increasing failure rate with time.
- Hence the exponential distribution is probably not the best model for the probability distribution of such a life.
- The exponential distribution has a "forgetfulness" property, or "no memory".

# Comparison - Bernoulli trials and Poisson Process

- Observe a sequence of $n$ Bernoulli trials
  - the number of successes → ***Binomial Distribution***

$$P_X(x) = \text{Prob}(X = x) = \binom{n}{x} p^x (1-p)^{n-x},$$

- Observe a sequence of Bernoulli trials until exactly $r$ successes occur,
  - the number of trials needed to observe the $r$th success
    → ***Negative Binomial Distribution***
  - the number of trials needed to observe the *first* success
    → ***Geometric Distribution***

Review

# Geometric Distribution

Let X denote the number of trial needed to observe the first success.

$$g(x) = p(1-p)^{x-1}, \qquad x = 1, 2, 3, \cdots.$$

Recall that for a geometric series (see Appendix A for a review), the sum is given by

$$\sum_{k=0}^{\infty} ar^k = \sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

when $|r| < 1$. Thus, for the geometric distribution,

$$\sum_{x=1}^{\infty} g(x) = \sum_{x=1}^{\infty} (1-p)^{x-1} p = \frac{p}{1-(1-p)} = 1,$$

→ Satisfy the properties of a p.m.f.

Review

# Negative Binomial Distribution

**Let the r.v. *X* denote the number of trials needed to observe the $r^{th}$ success**

***X: the trial number on which the $r^{th}$ success is observed***

**The p.m.f. of *X* equals the product of the following two probabilities:**

**1. the probability of obtaining exactly *r-1* successes in the first *x-1* trials**

$$\binom{x-1}{r-1} p^{r-1}(1-p)^{x-r} = \binom{x-1}{r-1} p^{r-1} q^{x-r}$$

**2. the probability *p* of a success on the $r^{th}$ trial**

**Thus, the p.m.f. of *X* is**

$$\boxed{g(x) = \binom{x-1}{r-1} p^{r}(1-p)^{x-r} = \binom{x-1}{r-1} p^{r} q^{x-r}, \ x = r, r+1, \ldots .}$$

Review

# Poisson Distribution

- Consider a Poisson Process
- X: number of occurrences in an interval of unit length
- Want to find $P(X = x) = ?$ , for $x = 0, 1, 2, \ldots$

- Partition the unit interval into $n$ subintervals of equal length $1/n$
- Suppose $n >> x$
- $P(X = x) = P($*one occurrence occurs in each of exactly $x$ of these $n$ subintervals*$)$

Review

As $n$ increases, $p$ decreases,
(hence, maintain $\lambda$ = constant)

➔ $p = \lambda/n$ ➔ $n\,p = \lambda$

As $n \to \infty$,

$$P(X = x) = \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!}$$

The distribution of probability associated with this process has a special name. We say that the random variable $X$ has a **Poisson distribution** if its p.m.f. is of the form

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots,$$

where $\lambda > 0$.

# ❖ Exponential Distribution

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- For $w \geq 0$, $F(w) = P(W \leq w) = 1 - P(W > w)$
- What is $P(W > w) = ?$

**Review**

- Notice that the event $A = \{\omega: W(\omega) > w\}$ is equivalent to the event $B = \{\omega: X(\omega) = 0\}$
- Therefore, $P(A) = P(W > w) = P(X = 0) = P(B)$

  where $W$ denote the waiting time until the first occurrence during the observation of a Poisson process with parameter $\lambda$,

  and $X$ denote the number of occurrences in the unit interval (i.e., $[0, w]$) during the observation of a Poisson process with parameter $\lambda w$

- In Section 2.6, it was shown the random variable $X$ has a Poisson distribution with the following pmf:

$$\boxed{P(X = x) = \frac{(\lambda w)^x e^{-\lambda w}}{x!}, \quad x = 0, 1, 2, \ldots}$$

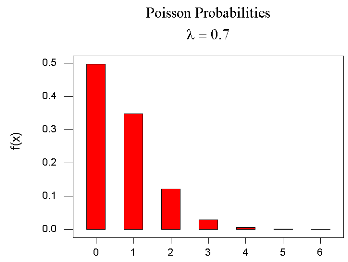
- Therefore, $P(W > w) = P(X = 0) = e^{-\lambda w}$
- Hence, we have

$$\text{For } w \geq 0,\ F(w) = P(W \leq w)$$
$$= 1 - P(W > w) = 1 - P(X = 0) = 1 - e^{-\lambda w}$$

$$F'(w) = f(w) = \lambda e^{-\lambda w}.$$

# Gamma Distribution

- Consider a Poisson process with parameter λ.
- Let W denote the waiting time until the $\alpha$-th occurrence occurs.
- The distribution function of W is




$$F_W(w) = \text{Prob}(W \le w) = 1 - \text{Prob}(W > w)$$

$$= 1 - \text{Prob}(\text{fewer than } \alpha \text{ changes occur in } [0, w])$$

$$= 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k e^{-\lambda w}}{k!} \qquad \text{for } w \ge 0.$$

$$\text{and } F_W(w) = 0 \qquad \text{for } w < 0.$$