Chapter 3 Continuous Distributions

- Random Variables of the Continuous Type
- Exponential, Gamma, and Chi-Square Distributions
- Normal Distribution
- Additional Models

Review

Poisson Process

- **Examples:**
  - **Number of phone call arriving between 9 and 10 am**
  - **Number of flaws in 10 feet of wire**
  - **Number of customers arriving between 2 and 4 pm**

Definition 2.6-1

Let the number of occurrences of some event in a given continuous interval be counted. Then we have an **approximate Poisson process** with parameter $\lambda > 0$ if the following conditions are satisfied:

(a) The numbers of occurrences in nonoverlapping subintervals are independent.

(b) The probability of exactly one occurrence in a sufficiently short subinterval of length h is approximately λh.

(c) The probability of two or more occurrences in a sufficiently short subinterval is essentially zero.

Review

Poisson Distribution

- Consider a Poisson Process
- X: number of occurrences in an interval of unit length
- Want to find $P(X = x) = ?$,   for $x = 0, 1, 2, \ldots$

- Partition the unit interval into n subintervals of equal length $1/n$
- Suppose $n >> x$
- $P(X = x) = P($*one occurrence occurs in each of exactly* x *of these* n *subintervals*$)$

Review

- By condition (c)
  - P(*two or more changes occur in any one subinterval*) $\cong$ 0
- By condition (b)
  - P(*one changes occurs in any one subinterval of length* $1/n$) $\cong$ $\lambda\,(1/n)$
- By condition (a)
  - We have a sequence of n Bernoulli trials with probability p approximately equal to $\lambda\,(1/n)$

$$\Rightarrow \quad P(X = x) \cong \frac{n!}{x!\,(n-x)!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

Review

As n increases, p decreases,
(hence, maintain λ = constant)

➔ $p = \lambda/n$ ➔ $n\,p = \lambda$

As $n \to \infty$,

$$P(X = x) = \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!}\left(\frac{\lambda}{n}\right)^{x}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^{x} e^{-\lambda}}{x!}$$

The distribution of probability associated with this process has a special name. We say that the random variable X has a **Poisson distribution** if its p.m.f. is of the form

$$f(x) = \frac{\lambda^{x} e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots,$$

where $\lambda > 0$.

Review

Geometric Distribution

Consider a sequence of independent Bernoulli trials.

Let X denote the number of trial needed to observe the first success.

$$g(x) = p(1-p)^{x-1}, \qquad x = 1, 2, 3, \cdots.$$

Recall that for a geometric series (see Appendix A for a review), the sum is given by

$$\sum_{k=0}^{\infty} ar^k = \sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

when $|r| < 1$. Thus, for the geometric distribution,

$$\boxed{\sum_{x=1}^{\infty} g(x) = \sum_{x=1}^{\infty} (1-p)^{x-1} p = \frac{p}{1-(1-p)} = 1,}$$

→ Satisfy the properties of a p.m.f.

Review

❖ Exponential Distribution

- Consider a Poisson process with parameter λ.
- Let W be the random variable corresponding to the waiting time of the first occurrence of the event. Then the distribution function of *W* is

For $w \geq 0$,

$$F(w) = P(W \leq w) = 1 - P(W > w)$$

$$= 1 - P(\text{no occurrences in } [0, w])$$

$$\overset{?}{=} 1 - e^{-\lambda w},$$

Recall that, for a r.v. X having a Poisson distribution with parameter λ, its p.m.f. is

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots,$$

That is, we have

$$F_Y(y) = 1 - e^{-\lambda y} \quad for\ y \geq 0$$
$$and \quad F_Y(y) = 0 \quad for\ y < 0.$$

Therefore

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \lambda e^{-\lambda y} \ \ for\ y \geq 0.$$
$$and\ f_Y(y) = 0\ for\ y < 0.$$

Recall that, for a r.v. X having a Poisson distribution with parameter λ, its p.m.f. is

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots,$$

Review

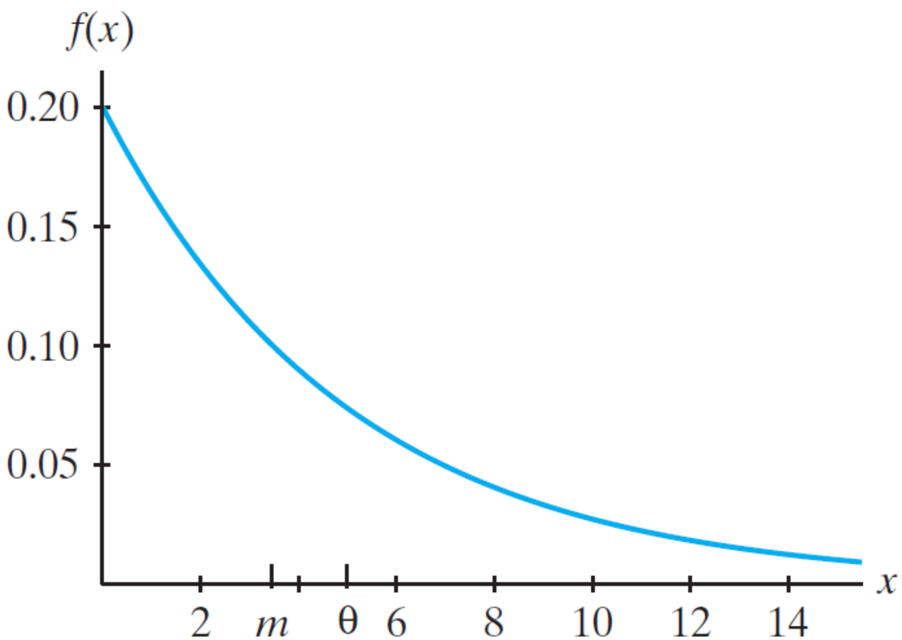


Figure 3.2-1 Exponential pdf, $f(x)$, and cdf, $F(x)$

Exponential Distribution

Review

- The p.d.f. of Y is

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \lambda e^{-\lambda y} \ for\ y \geq 0.$$
$$and\ f_Y(y) = 0\ for\ y < 0.$$

- The m.g.f. of Y is

$$M_Y(t) = \int_0^\infty e^{ty} \cdot \lambda e^{-\lambda y} dy$$
$$= \lambda \int_0^\infty e^{-(\lambda - t)y} dy$$
$$= \frac{\lambda}{\lambda - t} e^{-(\lambda - t)y} \Big|_\infty^0$$
$$= \frac{\lambda}{\lambda - t} \qquad if\ \ t < \lambda$$

?

$$\mu = M'(0) = \theta$$

$$\sigma^2 = M''(0) - [M'(0)]^2 = \theta^2.$$

Review

Exponential Distribution

- Based on the above derivation,
  we often let $\lambda = 1/\theta$
  and say that
  the random variable X has an exponential distribution

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \lambda e^{-\lambda y} \ for\ y \geq 0.$$
$$and\ f_Y(y) = 0\ for\ y < 0.$$

if its pdf is defined by

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 \leq x < \infty$$

where the parameter $\theta > 0$.

Accordingly, the waiting time until the first occurrence in a Poisson process has an exponential distribution with $\theta = 1/\lambda$.

Review

Comparison - Bernoulli trials and Poisson Process

- Observe a sequence of n Bernoulli trials
  - the number of successes → ***Binomial Distribution***

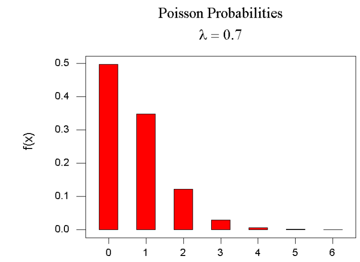
$$P_X(x) = \text{Prob}(X = x) = \binom{n}{x} p^x (1-p)^{n-x},$$

- Observe a sequence of Bernoulli trials until exactly r successes occur,
  - the number of trials needed to observe the rth success
    → ***Negative Binomial Distribution***
  - the number of trials needed to observe the *first* success
    → ***Geometric Distribution***

Review

Gamma Distribution

- Consider a Poisson process with parameter λ.
- Let W denote the waiting time until the α-th occurrence occurs.
- The distribution function of W is

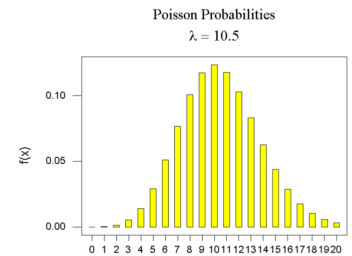


$$F_W(w) = \text{Prob}(W \le w) = 1 - \text{Prob}(W > w)$$

$$= 1 - \text{Prob}(\text{fewer than } \alpha \text{ changes occur in } [0, w])$$

$$= 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k e^{-\lambda w}}{k!} \qquad \text{for } w \ge 0.$$

$$\text{and } F_W(w) = 0 \qquad \text{for } w < 0.$$



Recall that, for a r.v. X having a Poisson distribution with parameter λ, its p.m.f. is

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots,$$

$$F_W(w) = 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k e^{-\lambda w}}{k!} \quad for\ w \geq 0.$$

$$F'(w)$$

$$= \frac{\lambda(\lambda w)^{\alpha-1}}{(\alpha - 1)!} e^{-\lambda w}.$$

$$F_W(w) = 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k e^{-\lambda w}}{k!} \quad for\ w \geq 0.$$

$$\begin{aligned} F'(w) &= \lambda e^{-\lambda w} - e^{-\lambda w} \sum_{k=1}^{\alpha-1} \left[\frac{k(\lambda w)^{k-1}\lambda}{k!} - \frac{(\lambda w)^k \lambda}{k!} \right] \\ &= \lambda e^{-\lambda w} - e^{-\lambda w} \left[\lambda - \frac{\lambda(\lambda w)^{\alpha-1}}{(\alpha-1)!} \right] \\ &= \frac{\lambda(\lambda w)^{\alpha-1}}{(\alpha-1)!} e^{-\lambda w}. \end{aligned}$$

$$f_W(w) = \frac{\lambda(\lambda w)^{\alpha-1}}{(\alpha-1)!} e^{-\lambda w}.$$

"Gamma" distribution (?)

What if $\alpha = 1$? $f(w) = \lambda\, e^{-\lambda w}$.

Gamma Function

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y}\, dy, \qquad 0 < t.$$

→ Generalized factorial

If t>1, ➔ $\Gamma(t) = \left[-y^{t-1}e^{-y}\right]_0^\infty + \int_0^\infty (t-1) y^{t-2} e^{-y}\, dy$

$$= (t-1)\int_0^\infty y^{t-2} e^{-y}\, dy = (t-1)\Gamma(t-1).$$

for example $\Gamma(6) = 5\Gamma(5)$ $\qquad \Gamma(3) = 2\Gamma(2) = (2)(1)\Gamma(1).$

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\cdots(2)(1)\Gamma(1).$$

$$\Gamma(1) = \int_0^\infty e^{-y}\, dy = 1.$$

Thus, when n is a positive integer, we have that

$$\Gamma(n) = (n-1)!;$$

$$f_W(w) = \frac{\lambda(\lambda w)^{\alpha-1}}{(\alpha-1)!} e^{-\lambda w}.$$

$$\boxed{\Gamma(n) = (n-1)!}$$

A r.v. X has a gamma distribution if its p.d.f. is defined by

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \qquad 0 \le x < \infty.$$

W, the waiting time until the α-th change in a Poisson process, has a gamma distribution with parameter α and $\theta = 1/\lambda$.

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \qquad 0 \le x < \infty.$$

properties of a p.d.f.,

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y}\, dy, \qquad 0 < t.$$

1. $f(x) \ge 0$

2. $$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^\infty \frac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)\theta^\alpha}\, dx$$

let $y = x/\theta$

$$\int_0^\infty \frac{(\theta y)^{\alpha-1} e^{-y}}{\Gamma(\alpha)\theta^\alpha} \theta\, dy = \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} e^{-y}\, dy = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1.$$

The moment-generating function of X is (Exercise 3.2-7)

$$M(t) = \frac{1}{(1-\theta t)^{\alpha}}, \qquad t < \theta.$$

The mean and variance are (Exercise 3.2-10)

$$\mu = \alpha\theta \qquad \text{and} \qquad \sigma^2 = \alpha\theta^2.$$

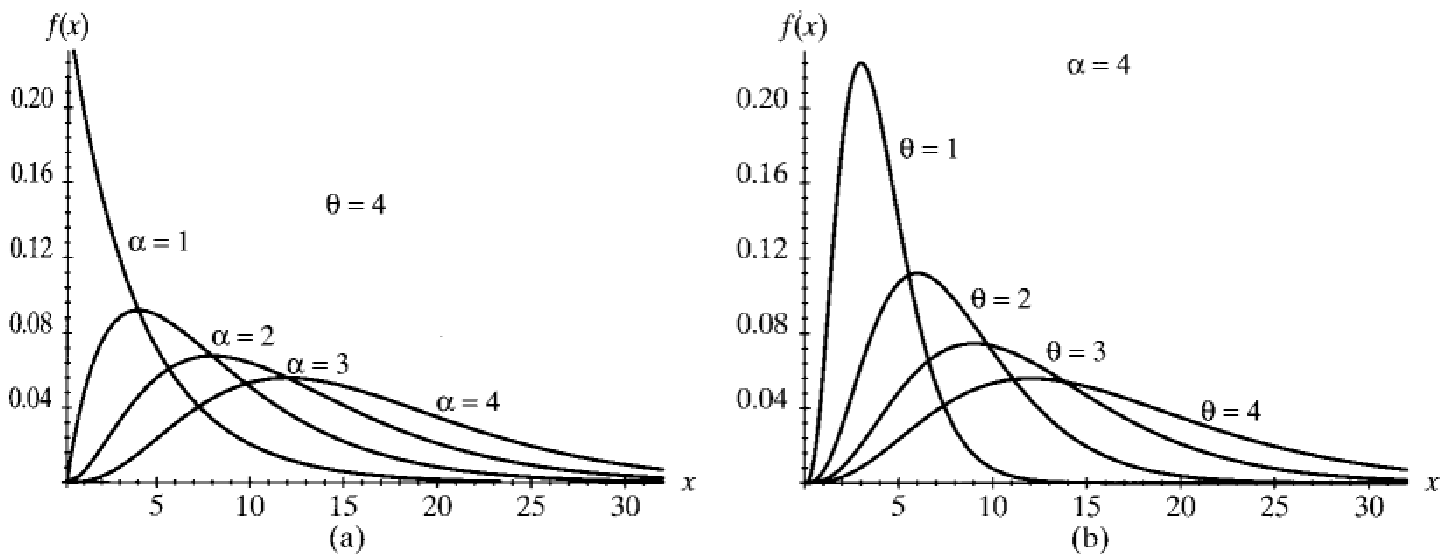


Figure 3.2-2 Gamma p.d.f.s: (a) $\theta = 4$ with $\alpha = 1, 2, 3, 4$; (b) $\alpha = 4$ with $\theta = 1, 2, 3, 4$

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \qquad 0 \le x < \infty.$$

Chi-Square Distribution

-- a special case of gamma distribution

Gamma: $$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \qquad 0 \leq x < \infty.$$

Let $\theta = 2 \text{ and } \alpha = r/2,$

$$\Rightarrow \quad f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, \qquad 0 \leq x < \infty.$$

chi-square distribution with r degrees of freedom,

X is $\chi^2(r)$