Chapter 3 Continuous Distributions

- Random Variables of the Continuous Type
- Exponential, Gamma, and Chi-Square Distributions
- Normal Distribution
- ~~Additional Models~~

Two Special Functions of a Normal r.v.

Theorem 3.3-1

Standard normal

1. If X is $N(\mu, \sigma^2)$, then $Z = (X - \mu)/\sigma$ is $N(0, 1)$.
2. If X is $N(\mu, \sigma^2)$, then $V = (X - \mu)^2/\sigma^2$ is $\chi^2(1)$.

Theorem 3.3-2

Chi-square:

- *can be defined* from Gamma as shown above
- *can be derived* from $V=X^2$ as shown below

Review on integration

If $F'(t) = f(t)$, $a \le t \le b$, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Thus if $u(x)$ is such that $u'(x)$ exists and $a \le u(x)$, then

$$\int_a^{u(x)} f(t)\,dt = F[u(x)] - F(a).$$

Taking derivatives of this latter equation, we obtain

$$D_x\left[\int_a^{u(x)} f(t)\,dt\right] = F'[u(x)]u'(x) = f[u(x)]u'(x).$$

Theorem 3.3-2

If X is $N(\mu, \sigma^2)$, then $V = (X - \mu)^2/\sigma^2$ is $\chi^2(1)$.

- Assume that X is N（0,1）.
- In statistics, it is common that we are interested in

$$\text{Prob}(|X| \le x) \; or \; \text{Prob}(|X| \ge x).$$

Therefore, we define $Z = X^2$.

- The distribution function of Z is

$$F_Z(z) = \text{Prob}(Z \le z) = \text{Prob}(X^2 \le z) = \text{Prob}(-\sqrt{z} \le X \le \sqrt{z})$$

$$= \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx, \qquad \text{for } z \ge 0.$$

$$= 2\int_0^{\sqrt{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{z}} e^{-\frac{1}{2}x^2} dx$$

$$F_Z(z) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{z}} e^{-\frac{1}{2}x^2} dx$$

$$D_x\left[\int_a^{u(x)} f(t)\,dt\right] = F'[u(x)]u'(x) = f[u(x)]u'(x).$$

- The p.d.f. of Z is

$$\frac{dF_Z(z)}{dz} = \sqrt{\frac{2}{\pi}} \cdot \frac{d\int_0^{\sqrt{z}} e^{-\frac{1}{2}x^2} dx}{dz}$$

?

$$F_Z(z) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{z}} e^{-\frac{1}{2}x^2}\, dx$$

$$D_x\left[\int_a^{u(x)} f(t)\,dt\right] = F'[u(x)]u'(x) = f[u(x)]u'(x).$$

- The p.d.f. of Z is

$$\frac{dF_Z(z)}{dz} = \sqrt{\frac{2}{\pi}} \cdot \frac{d\int_0^{\sqrt{z}} e^{-\frac{1}{2}x^2}\, dx}{dz}$$

?

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{d\int_0^{t} e^{-\frac{1}{2}x^2}\, dx}{dt} \cdot \frac{dt}{dz}, \text{ where } t = \sqrt{z}$$

$$= \frac{1}{\sqrt{2\pi}} z^{-\frac{1}{2}} e^{-\frac{1}{2}z}, \ for\ z \geq 0.$$

$$\boxed{f_Z(z) = \frac{1}{\sqrt{2\pi}} z^{-\frac{1}{2}} e^{-\frac{1}{2}z}, \ for\ z \geq 0.}$$

Chi-Square Distribution

-- a special case of gamma distribution

Gamma: $f(x) = \dfrac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \qquad 0 \le x < \infty.$

Let $\theta = 2$ and $\alpha = r/2$,

$$\Rightarrow \quad f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, \qquad 0 \le x < \infty.$$

X is $\chi^2(r)$ **chi-square distribution with *r* degrees of freedom,**

$$\boxed{f_Z(z) = \frac{1}{\sqrt{2\pi}} z^{-\frac{1}{2}} e^{-\frac{1}{2}z}, \; for \; z \ge 0.}$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} z^{-\frac{1}{2}} e^{-\frac{1}{2}z}, \; for \; z \geq 0.$$

Derived, rather than defined

- Z is typically said to have the chi-square distribution of 1 degree of freedom and denoted by $\chi^2(1)$.

Theorem 3.3-2 (9e, p.119) proved **?**

If X is $N(\mu, \sigma^2)$, then $V = (X - \mu)^2/\sigma^2$ is $\chi^2(1)$.

$$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, \qquad 0 \leq x < \infty.$$

Not really! Need to know $\Gamma(1/2) = ?$

-- by using the above result obtained from $Z=X^2$

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y}\, dy, \qquad 0 < t.$$

Given X is standard normal

if $Z=X^2$ $\rightarrow$ $f_Z(z) = \frac{1}{\sqrt{2\pi}} z^{-\frac{1}{2}} e^{-\frac{1}{2}z}$, *for* $z \geq 0$. ?

$$\int_0^\infty \frac{1}{\sqrt{\pi}\sqrt{2}} v^{1/2-1} e^{-v/2}\, dv = 1.$$

The change of variables $x = v/2$

$$\rightarrow \quad 1 = \frac{1}{\sqrt{\pi}} \int_0^\infty x^{1/2-1} e^{-x}\, dx = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right).$$

$\rightarrow$ $\Gamma(1/2) = \sqrt{\pi}$

代入下式、讓$r=1$
→與上式一致

$$\boxed{f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, \qquad 0 \leq x < \infty.}$$

$\rightarrow$ $\chi^2(1)$