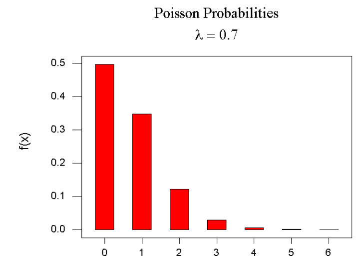
Chapter 3 Continuous Distributions

- Random Variables of the Continuous Type
- Exponential, Gamma, and Chi-Square Distributions
- Normal Distribution
- ~~Additional Models~~

Review

Gamma Distribution

- Consider a Poisson process with parameter λ.
- Let W denote the waiting time until the α-th occurrence occurs.
- The distribution function of W is

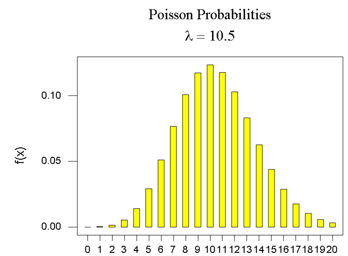


$$F_W(w) = \text{Prob}(W \le w) = 1 - \text{Prob}(W > w)$$

$$= 1 - \text{Prob}(\text{fewer than } \alpha \text{ changes occur in } [0, w])$$

$$= 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k e^{-\lambda w}}{k!} \qquad \text{for } w \ge 0.$$

$$\text{and } \ F_W(w) = 0 \qquad \text{for } w < 0.$$



Recall that, for a r.v. X having a Poisson distribution with parameter λ, its p.m.f. is

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots,$$

$$f_W(w) = \frac{\lambda(\lambda w)^{\alpha-1}}{(\alpha-1)!} e^{-\lambda w}.$$

$$\boxed{\Gamma(n) = (n-1)!}$$

Review

A r.v. X has a gamma distribution if its p.d.f. is defined by

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \qquad 0 \le x < \infty.$$

→ **W,** the waiting time until the α-th change in a Poisson process, has a gamma distribution with parameter α and $\theta = 1/\lambda$.

Chi-Square Distribution

-- a special case of gamma distribution

Gamma: $$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \qquad 0 \le x < \infty.$$

Let $\theta = 2$ and $\alpha = r/2$,

$$\Rightarrow \quad f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, \qquad 0 \le x < \infty.$$

chi-square distribution with r degrees of freedom,

X is $\chi^2(r)$

Remember that
the **moment-generating function** of a *Gamma distribution* with parameters α and $\theta = 1/\lambda$, are

$$M(t) = \frac{1}{(1-\theta t)^{\alpha}}, \qquad t < 1/\theta.$$

For a *chi-square distribution*, let $\theta = 2$ and $\alpha = r/2$,

we have the **moment-generating function** of a chi-square distribution with r degree of freedom:

$$M(t) = (1-2t)^{-r/2}, \qquad t < \frac{1}{2}.$$

Remember that
the **mean** and **variance** of a *Gamma distribution*
with parameters α and $\theta = 1/\lambda$, are

$$\mu = \alpha\theta \quad \text{and} \quad \sigma^2 = \alpha\theta^2$$

For a *chi-square distribution*, let $\theta = 2$ and $\alpha = r/2$,

we have

$$\mu = \alpha\theta = \left(\frac{r}{2}\right)2 = r \qquad \text{and} \qquad \sigma^2 = \alpha\theta^2 = \left(\frac{r}{2}\right)2^2 = 2r.$$

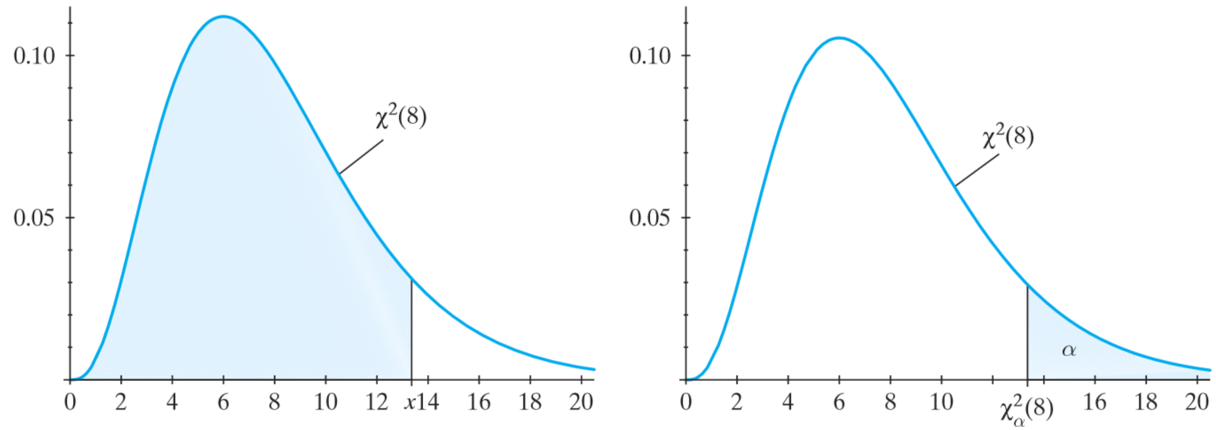
- the **mean** equals the number of degrees of freedom
- the **variance** equals twice the number of degrees of freedom

Because the chi-square distribution is so important in applications, tables have been prepared, giving the values of the cdf,

$$F(x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} w^{r/2-1} e^{-w/2}\, dw,$$

for selected values of r and x. (For an example, see Table IV in Appendix B.)

Table IV The Chi-Square Distribution



$$P(X \le x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} w^{r/2-1} e^{-w/2} dw$$

	$P(X \le x)$							
	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
r	$\chi^2_{0.99}(r)$	$\chi^2_{0.975}(r)$	$\chi^2_{0.95}(r)$	$\chi^2_{0.90}(r)$	$\chi^2_{0.10}(r)$	$\chi^2_{0.05}(r)$	$\chi^2_{0.025}(r)$	$\chi^2_{0.01}(r)$
1	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635
2	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210
3	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.34
4	0.297	0.484	0.711	1.064	7.779	9.488	11.14	13.28
5	0.554	0.831	1.145	1.610	9.236	11.07	12.83	15.09
6	0.872	1.237	1.635	2.204	10.64	12.59	14.45	16.81
7	1.239	1.690	2.167	2.833	12.02	14.07	16.01	18.48
8	1.646	2.180	2.733	3.490	13.36	15.51	17.54	20.09
9	2.088	2.700	3.325	4.168	14.68	16.92	19.02	21.67
10	2.558	3.247	3.940	4.865	15.99	18.31	20.48	23.21

Probabilities like that of Example 3.2-7 are so important in statistical applications that we use special symbols for a and b. Let α be a positive probability (i.e., usually less than 0.5), and let X have a chi-square distribution with r degrees of freedom. Then $\chi^2_\alpha(r)$ is a number such that

$$P[X \ge \chi^2_\alpha(r)] = \alpha.$$

That is, $\chi^2_\alpha(r)$ is the $100(1-\alpha)$th percentile (or upper 100αth percent point) of the chi-square distribution with r degrees of freedom. Then the 100α percentile is the number $\chi^2_{1-\alpha}(r)$ such that

$$P[X \le \chi^2_{1-\alpha}(r)] = \alpha.$$

That is, the probability to the right of $\chi^2_{1-\alpha}(r)$ is $1-\alpha$. (See Figure 3.2-4.)

Example 3.2-8 Let X have a chi-square distribution with five degrees of freedom. Then, using Table IV in Appendix B, we find that $\chi^2_{0.10}(5) = 9.236$ and $\chi^2_{0.90}(5) = 1.610$. These are the points, with $\alpha = 0.10$, that are indicated in Figure 3.2-4. ■

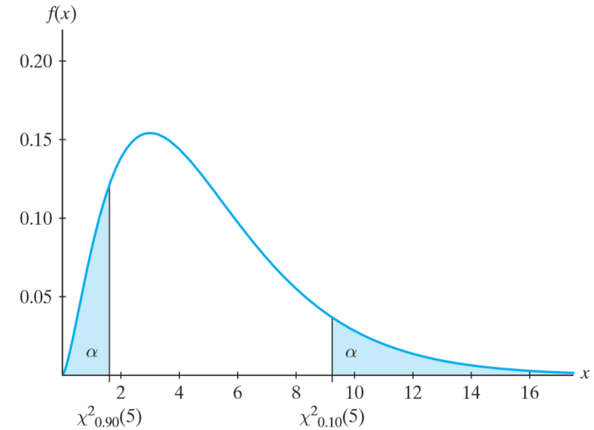


Figure 3.2-4 Chi-square tails, $r = 5, \alpha = 0.10$

Chapter 3 Continuous Distributions

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The Normal Distribution

- A continuous random variable X is said to have a normal distribution, if its p.d.f. is as follows:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} . \quad for \ -\infty < x < \infty$$

and is commonly denoted by

$$N\left(\mu, \sigma^2\right).$$

- We want to prove that $\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ satisfies the condition of a p.d.f.
- It is obvious that $\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ >0.
- Next, prove $\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx = 1$?

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- Next, prove $\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx = 1$?
- Let $c = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx.$

  By substituting $y = \frac{x-\mu}{\sigma}$,

  we have

$$c = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy$$

$$c^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \right) \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(y^2+z^2\right)} dydz$$

Since $e^{-\frac{1}{2}\left(y^2+z^2\right)}$ is a circularly symmetric function on the Y-Z plane,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(y^2+z^2\right)} dydz = \int_{0}^{\infty} 2\pi\gamma e^{-\frac{1}{2}\gamma^2} d\gamma$$

$$= 2\pi e^{-\frac{1}{2}\gamma^2} \Big|_{\infty}^{0} = 2\pi.$$

Therefore, $c^2=1$ and $c=1$. *(proved#)*

The Standard Normal Distribution

- A normal distribution with $\mu = 0$ and $\sigma = 1$ is said to be a standard normal distribution.
- The p.d.f. of a standard normal distribution is $$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

Linear Transformation of the Normal Distribution

- Assume that random variable X has the distribution $N(\mu, \sigma^2)$.

Then, $Y = \dfrac{X-\mu}{\sigma}$ has the standard normal distribution. **(Theorem 3.3-1)**

- Proof：

$$F_Y(y) = \text{Prob}(Y \le y) = \text{Prob}\left(\frac{X-\mu}{\sigma} \le y\right)$$

$$= \text{Prob}(X \le \mu + \sigma y) = \int_{-\infty}^{\mu+\sigma y} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$F_Y(y) = \text{Prob}(Y \le y) = \text{Prob}\left(\frac{X-\mu}{\sigma} \le y\right)$$

$$= \text{Prob}(X \le \mu + \sigma y) = \int_{-\infty}^{\mu+\sigma y} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

- By substituting $t = \dfrac{x-\mu}{\sigma}$,

  we have

$$F_Y(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt.$$

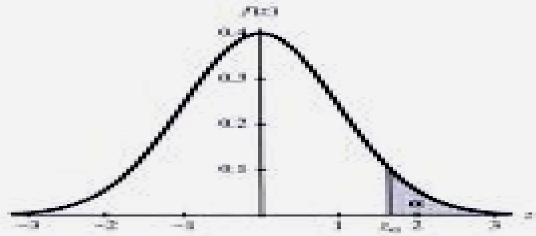
  Therefore, Y is $N(0,1)$.

- Accordingly, if we want to compute $F_X(w)$, we can do that by the following procedure.

$$F_X(w) = \text{Prob}(X \le w) = \text{Prob}\left(\frac{X-\mu}{\sigma} \le \frac{w-\mu}{\sigma}\right) = \text{Prob}\left(Y \le \frac{w-\mu}{\sigma}\right) = F_Y\left(\frac{w-\mu}{\sigma}\right).$$

Table Vb The Normal Distribution

See Table Va for $\Phi(z)$



$P(Z > z_\alpha) = \alpha$

$P(Z > z) = 1 - \Phi(z) = \Phi(-z)$

z_α	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002

The Moment-Generating Function of the Standard Normal Distribution

- Let X be a $N(0,1)$.

Then, the m.g.f. of X is

$$M_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tx} e^{-\frac{1}{2}x^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} e^{\frac{1}{2}t^2} dx$$

$$= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx$$

Since $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2} dw = 1,$

if we substitute（x-t）by w,

we have $M_X(t) = e^{\frac{1}{2}t^2}$.

?

$$M_X(t) = e^{\frac{1}{2}t^2}.$$

$$M_X{}'(t) = te^{\frac{1}{2}t^2}$$

$$M_X{}''(t) = e^{\frac{1}{2}t^2} + t^2 e^{\frac{1}{2}t^2}$$

$$Hence,\ \mu_X = M_X{}'(0) = 0\ and$$

$$\sigma_X{}^2 = M_X{}''(0) - \left(M_X{}'(0) \right)^2 = 1.$$

Two Special Functions of a Normal r.v.

Theorem 3.3-1

Standard normal

1. If X is $N(\mu, \sigma^2)$, then $Z = (X - \mu)/\sigma$ is $N(0, 1)$.
2. If X is $N(\mu, \sigma^2)$, then $V = (X - \mu)^2/\sigma^2$ is $\chi^2(1)$.

Theorem 3.3-2

Chi-square:

- *can be defined* from Gamma as shown above
- *can be derived* from $V=X^2$ as shown below