Chapter 3 Continuous Distributions

- Random Variables of the Continuous Type
- Exponential, Gamma, and Chi-Square Distributions
- Normal Distribution
- Additional Models

3.1 Random Variables of the Continuous Type

- A random variable X is a function that maps the possible outcomes of an experiment to real numbers.
  - That is, X: S --> R,

    where S is the set of all outcomes of an experiment, and R is the set of real numbers.

- The space of X is the set of real numbers

  $$S_X = \{x: X(s) = x \ , \ s \in S\}$$

  - S_X was a set of discrete points in Chapter 2
  - S_X is now an interval or a union of intervals here

- In the following, $S_X \rightarrow S$

Distribution Function

- The distribution function of a continuous random variable X is defined as same as that of a discrete random variable, i.e.

$$F_X(t) \equiv \text{Prob}(X \le t).$$

Probability Density Function

- The probability density function（p.d.f.）of a continuous random variable is defined as

$$f_X(t) \equiv \frac{dF_X(t)}{dt}.$$

$$\Rightarrow \quad F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt.$$

Properties of Probability Density Function

- The p.d.f. of a continuous random variable with space $\boldsymbol{S}$ satisfies the following properties:

$(a)\ f_X(x) \geq 0\ \ for\ \ all\ x \in S.$

$(b)\ \int_S f_X(x)\,dx = 1.$

(c) The probability of event A is $\int_A f_X(x)\,dx$.

❖ Uniform Distribution

- Let random variable X correspond to randomly selecting a number in [a,b]. Then,

$$F_X(x) = \text{Prob}(X \le x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x\text{-}a}{b\text{-}a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$$

- Compare with discrete uniform distribution ?

Uniform Distribution

$$F_X(x) = \text{Prob}(X \le x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x\text{-}a}{b\text{-}a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$$

- $$f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} \dfrac{1}{b-a} & \text{if a} \le \text{x} \le \text{b} \\ 0 & \text{otherwise} \end{cases}$$
- X has a uniform distribution if its p.d.f is equal to a constant on its support.
- Mean, Variance, m.g.f.

HW3.1-1

$$\mu = \frac{a+b}{2}, \qquad \sigma^2 = \frac{(b-a)^2}{12},$$

$$M(t) = \begin{cases} \dfrac{e^{tb} - e^{ta}}{t(b-a)}, & t \ne 0, \\ 1, & t = 0. \end{cases}$$

- Pseudo-random number generator: from U(0,1)

Expected Value and Variance

- The expected value of a continuous random variable X is $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx.$
- The variance of X is

$$Var[X] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x)\,dx.$$

Moment-Generating Function

- The moment-generating function of a continuous random variable X is

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)dx.$$

- Note that the moment-generating function, if it is finite for -h<t<h for some h>0, completely determines the distribution. In other words, if two continuous random variables have identical m.g.f., then they have identical probability distribution function.

Example 3.1-1 Let Y be a continuous random variable with p.d.f. $g(y) = 2y$, $0 < y < 1$. The distribution function of Y is defined by

$$G(y) = \begin{cases} 0, & y < 0, \\ \int_0^y 2t\,dt = y^2, & 0 \le y < 1, \\ 1, & 1 \le y. \end{cases}$$

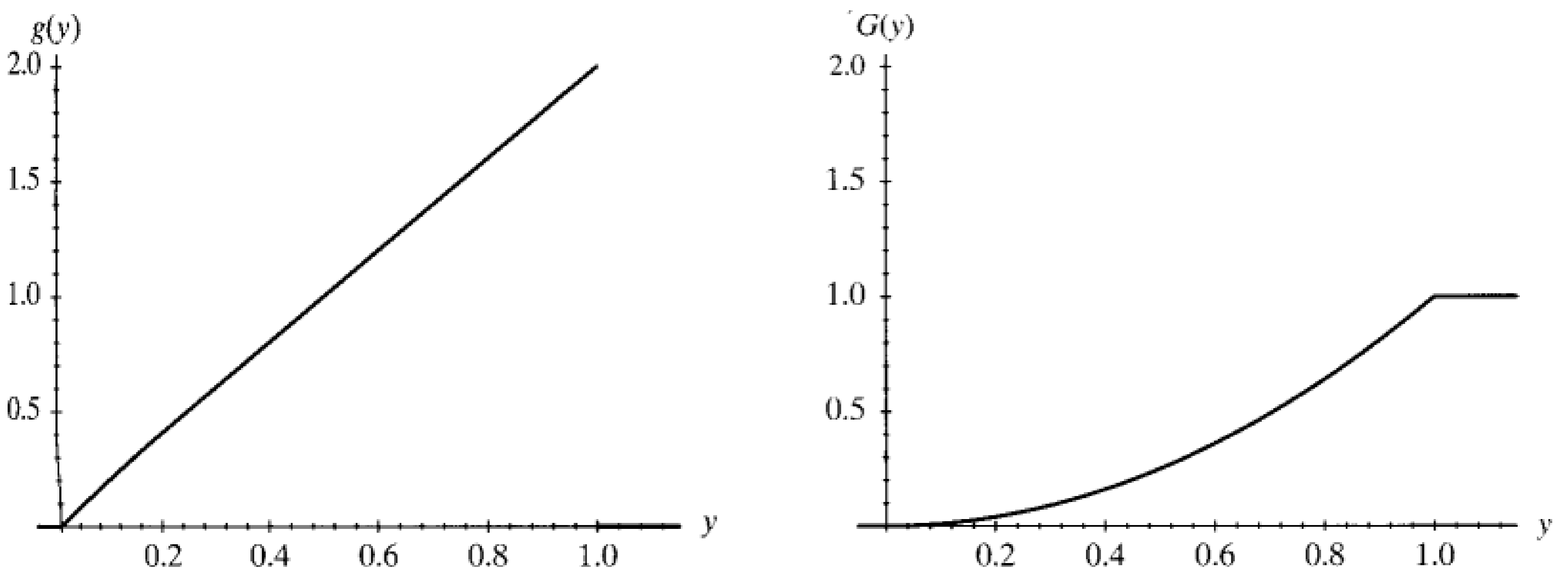

Figure 3.1-2 Continuous distribution p.d.f. and c.d.f.

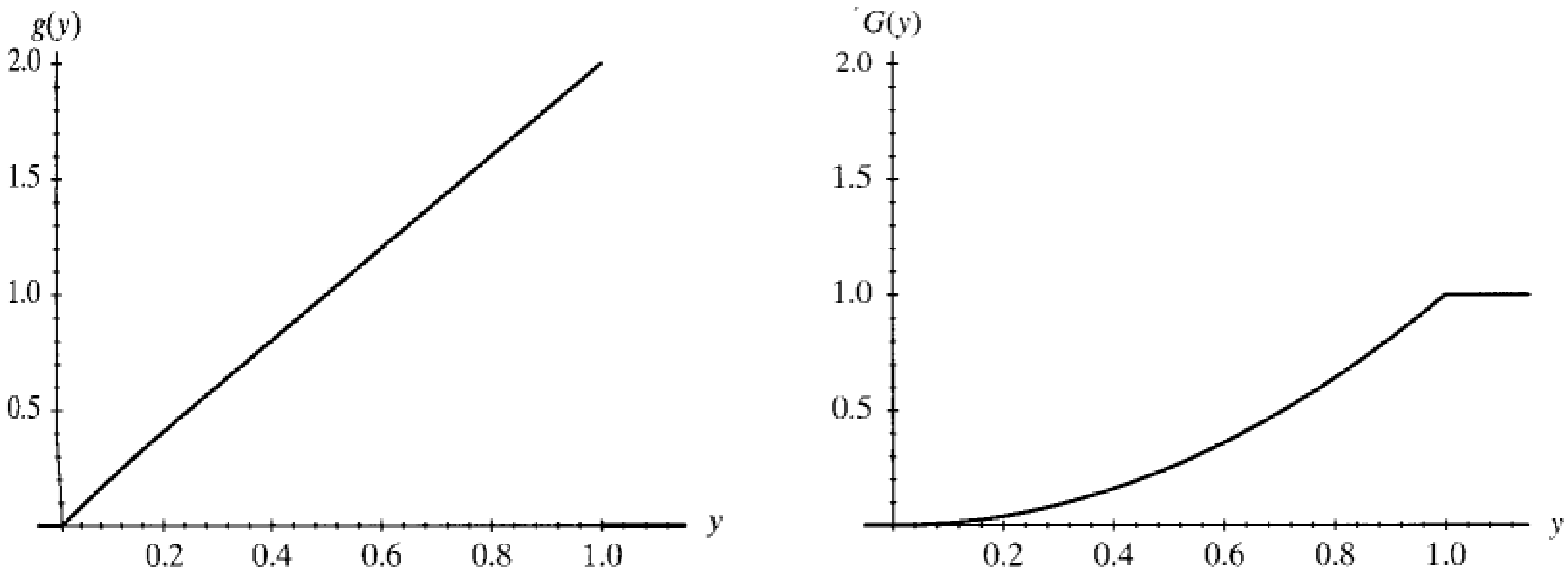

Figure 3.1-2 Continuous distribution p.d.f. and c.d.f.

Figure 3.1-2 gives the graph of the p.d.f. $g(y)$ and the graph of the distribution function $G(y)$. For examples of computations of probabilities, consider

$$P\left(\frac{1}{2} < Y \le \frac{3}{4}\right) = G\left(\frac{3}{4}\right) - G\left(\frac{1}{2}\right) = \left(\frac{3}{4}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{5}{16}$$

and

$$P\left(\frac{1}{4} \le Y < 2\right) = G(2) - G\left(\frac{1}{4}\right) = 1 - \left(\frac{1}{4}\right)^2 = \frac{15}{16}. \qquad \blacktriangle$$

For random variables of the continuous type, the p.d.f. does not have to be bounded [see Exercises **3.1-7(c) and 3.1-8(c)** . The restriction is that the area between the p.d.f. and the x axis must equal one. Furthermore, it should be noted that the p.d.f. of a random variable X of the continuous type does not need to be a continuous function. For example,

$$f(x) = \begin{cases} \dfrac{1}{2}, & 0 < x < 1 \quad \text{or} \quad 2 < x < 3, \\ 0, & \text{elsewhere,} \end{cases}$$

enjoys the properties of a p.d.f. of a distribution of the continuous type, and yet $f(x)$ has discontinuities at $x = 0, 1, 2,$ and 3. However, the distribution function associated with a distribution of the continuous type is always a continuous function.

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❖ Exponential Distribution (v1)

- Consider a Poisson process with parameter λ.
- Let Y be the random variable corresponding to the waiting time of the first occurrence of the event. Then the distribution function of Y is

$$F_Y(y) = \text{Prob}(Y \le y) = 1 - \text{Prob}(Y > y)$$

$$= 1 - \text{Prob}(\textit{no change in } [0, y])$$

$$= 1 - \text{Prob}(X = 0) = 1 - \frac{(\lambda y)^0}{0!} e^{-\lambda y} = 1 - e^{-\lambda y} \qquad \textit{for } y \ge 0$$

$$\textit{and} \qquad F_Y(y) = 0 \qquad \textit{for } y < 0.$$

where X is the number of occurrences of a Poisson process with parameter λy

Recall that, for a r.v. X having a Poisson distribution with parameter λ, its p.m.f. is

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots,$$

❖ Exponential Distribution (v2)

- Consider a Poisson process in which the mean number of occurrences in the unit interval is λ.
- Let W denote the waiting time until the first occurrence *during the observation of the above Poisson process*. Then W is a continuous-type random variable with the cdf derived below.
- Because this waiting time is nonnegative, the cdf

$$F(w) = 0, \quad \text{for } w < 0.$$

- For $w \geq 0$, $F(w) = P(W \leq w) = 1 - P(W > w)$
- What is $P(W > w) = ?$

- What is $P(W > w)$?

- Let the event $A = \{\omega: W(\omega) > w\}$

  i.e., the set of all outcomes such that $W(\omega) > w$,

  i.e., the set of all outcomes such that the waiting time is greater than w,

  i.e., the set of all outcomes such that no occurrences in $[0, w]$.

- Let X be the number of occurrences of a Poisson process with parameter λw
- Let the event $B = \{\omega: X(\omega) = 0\}$

  i.e., the set of all outcomes such that no occurrences in $[0, w]$ *during the observation of a* Poisson process in which the mean number of occurrences in $[0, w]$ is λw.
- Notice that *a* Poisson process in which the mean number of occurrences in $[0, w]$ is λw *is "the same" as a* Poisson process in which the mean number of occurrences in $[0, 1]$ is λ.

- Notice that the event $A = \{\omega: W(\omega) > w\}$

  is equivalent to the event $B = \{\omega: X(\omega) = 0\}$

- Therefore, $P(A) = P(W > w) = P(X = 0) = P(B)$

  where W denote the waiting time until the first occurrence during the observation of a Poisson process with parameter λ,

  and X denote the number of occurrences in the unit interval (i.e., $[0, w]$) during the observation of a Poisson process with parameter λw

- In Section 2.6, it was shown the random variable X has a Poisson distribution with the following pmf:

$$P(X = x) = \frac{(\lambda w)^x e^{-\lambda w}}{x!}, \quad x = 0, 1, 2, \ldots$$

- Therefore, $P(W > w) = P(X = 0) = e^{-\lambda w}$

- Hence, we have

  For $w \geq 0$, $F(w) = P(W \leq w)$

$$= 1 - P(W > w) = 1 - P(X = 0) = 1 - e^{-\lambda w}$$

$$F'(w) = f(w) = \lambda e^{-\lambda w}.$$

That is, we have

$$F_Y(y) = 1 - e^{-\lambda y} \qquad for\ y \geq 0$$
$$and \qquad F_Y(y) = 0 \qquad\qquad for\ y < 0.$$

Therefore

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \lambda e^{-\lambda y} \quad for\ y \geq 0.$$
$$and\ \ f_Y(y) = 0\ \ for\ \ y < 0.$$

Recall that, for a r.v. X having a Poisson distribution with parameter λ, its p.m.f. is

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots,$$

Exponential Distribution

- The p.d.f. of Y is $f_Y(y) = \dfrac{dF_Y(y)}{dy} = \lambda e^{-\lambda y} \;\; for \;\; y \geq 0.$
  $and \;\; f_Y(y) = 0 \;\; for \;\; y < 0.$

- The m.g.f. of Y is

$$
\begin{aligned}
M_Y(t) &= \int_0^\infty e^{ty} \cdot \lambda e^{-\lambda y} dy \\
&= \lambda \int_0^\infty e^{-(\lambda - t)y} dy \\
&= \frac{\lambda}{\lambda - t} e^{-(\lambda - t)y} \Big|_\infty^0 \\
&= \frac{\lambda}{\lambda - t} \qquad\qquad if \;\; \lambda > t.
\end{aligned}
$$

Exponential Distribution

- Then, $M'(t) = \dfrac{\lambda}{(\lambda - t)^2}$ *and* $M''(t) = \dfrac{2\lambda}{(\lambda - t)^3}$,

➔ We have $\mu_Y = M'(0) = \dfrac{1}{\lambda}$ *and* $\sigma_Y{}^2 = M''(0) - (M'(0))^2 = \dfrac{1}{\lambda^2}$.

- Therefore, for a Poisson process with parameter λ, the average waiting time for the first event occurrence is 1/λ.
  - Note that, λ is the expected number of event occurrence during one unit of time interval.

Exponential Distribution

- Based on the above derivation,
  we often let $\lambda = 1/\theta$
  and say that
  the random variable X has an <u>exponential distribution</u>

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \lambda e^{-\lambda y} \ for \ y \geq 0.$$
$$and \ f_Y(y) = 0 \ for \ y < 0.$$

if its pdf is defined by

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 \leq x < \infty$$

where the parameter $\theta > 0$.

Accordingly, the waiting time until the first occurrence in a Poisson process has an exponential distribution with $\theta = 1/\lambda$.

Example 3.2-1

- Let X have an exponential distribution with a mean of θ=20. So the p.d.f. of X is,

$$f(x) = \frac{1}{20} e^{-x/20}, \qquad 0 \le x < \infty.$$

- The probability that X is less that 18 is

$$P(X < 18) = \int_0^{18} \frac{1}{20} e^{-x/20} dx = 1 - e^{-18/20} = 0.593.$$

Example 3.2-2

- Customers arrive in a certain shop according to an approximate Poisson process at a mean rate of 20 per hour.
- What is the probability that the shoekeeper will have to wait more that 5 minutes for the arrival of the first customer?
- Let X denote the waiting time in minutes until the first customer arrives and note that λ=1/3 is the expected number of arrivals per minute. Thus

$$\theta = \frac{1}{\lambda} = 3 \quad \text{and } f(x) = \frac{1}{3} e^{-(1/3)x}, \quad 0 \le \mathrm{x} < \infty.$$

- Hence

$$P(X > 5) = \int_5^\infty \frac{1}{3} e^{-(1/3)x} dx = e^{-5/3} = 0.1889.$$

- The median time until the first arrival is

$$m = -3\ln(0.5) = 2.0794. \quad ?$$

$$\boxed{m = -\theta \ln(0.5)}$$

Median time until the first arrival

Example 3.2-3

- Suppose that the life of a certain type of electronic component has an exponential distribution with a mean life of 500 hours.
- If X denotes the life of this component, then

$$P(X > x) = \int_x^{\infty} \frac{1}{500} e^{-t/500} dt = e^{-x/500}.$$

- Suppose that the component has been in operation for 300 hours. The conditional probability that it will last for another 600 hours is ?

$$P(X > 900 \mid X > 300) = \frac{P(X > 900)}{P(X > 300)} = \frac{e^{-900/500}}{e^{-300/500}} = e^{-6/5}.$$

Example 3.2-3(cont.)

$$P(X > 900 \mid X > 300) = \frac{P(X > 900)}{P(X > 300)} = \frac{e^{-900/500}}{e^{-300/500}} = e^{-6/5}.$$

$$P(X > 600) = \quad ?$$

- That is,

  the probability that it will last an additional 600 hours, given that it has operated for 300 hours,

  is the same as

  the probability that it would last 600 hours when first put into operation.

- Thus, for such components, an old one is as good as a new one, and we say that *the failure rate is constant*.

Example 3.2-3(cont.)

- With constant failure rate, there is no advantage in replacing components that are operating satisfactorily.

- Is it true in practice ? *No!!*

- Obviously, this is not true in practice because most would have an increasing failure rate with time.
- Hence the exponential distribution is probably not the best model for the probability distribution of such a life.
- The exponential distribution has a “forgetfulness” property, or “no memory”.