# Outline of the Course

1. Introduction
2. Discrete Distributions
3. Continuous Distributions
4. Bivariate Distributions
5. Distributions of Functions of Random Variables
6. Point Estimation
7. Interval Estimation
8. Test of Statistical Hypotheses
9. More Tests

# Chapter 1
# Probability

- Basic Concepts
- Mean, Variance, Standard Deviation
- Axioms and Properties of Probability
- Methods of Enumeration
- Conditional Probability
- Independent Events
- Bayes' Theorem

# 1.3 Axioms and Properties of Probability

- Outcome space, Sample space
  - The set of all possible outcomes of an experiment
  - denoted by ***S***
- Event
  - a subset ***A*** of an outcome space ***S***
- The probability that event ***A*** occurs
  - denoted by ***P***(***A***).

# Definition 1.1-1 (Probability Axioms 公理)

- A **probability measure** $P(.)$ is a function
  - that maps **events** in the sample space to **real numbers**

    such that :

    *Axiom 1.* For any event $A \subseteq S$ , $P(A) \geq 0$,

    *Axiom 2.* $P(S) = 1$

    *Axiom 3.* For any countable collection $A_1, A_2, \ldots$

    of mutually exclusive events ($A_i \cap A_j = \varnothing$ for all $i \neq j$),

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

*(axiom of countable additivity)*

# Basic Theorems

- **Theorem 1.1-1** For each event $A$,
  $P(A)=1-P(A')$, where $A'=S-A$.
- **Theorem 1.1-2** $P(\varnothing)=0$.
- **Theorem 1.1-3** $A$ and $B$ are two events.
  If $A \subseteq B$, then $P(A) \leq \mathrm{P}(B)$.
- **Theorem 1.1-4** For each event $A$, $P(A) \leq 1$.
  Therefore, $0 \leq P(A) \leq 1$
- **Theorem 1.1-5** If $A$ and $B$ are any two events.
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- **Theorem 1.1-6**

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - \ldots\ldots$$

# Unordered Sampling with Replacement

- Assume that we take $k$ samples out of $n$ objects with replacement and ignore the order of samples.
- To figure out the number of possible outcomes, we can regard this problem as inserting $(n\text{-}1)$ bars into a list of $k$ objects as follows: (e.g., n=6, k=10)
  - oo | ooo | | o | o | ooo

- Each distinguishable permutation of the string corresponds to an unordered sample.
- Therefore, the number of possible outcomes is

$$C_k^{(n+k-1)} = \frac{(n+k-1)!}{k!(n-1)!}$$

# 總結

Taking ***k* samples** out of *n* samples

— samples of size $k$

| | 重複 | 不重複 |
|---|---|---|
| Ordered 排列 | 重複排列 $n^k$ | 排列 $\frac{n!}{(n-k)!}$ |
| Unordered 組合 | 重複組合 $\frac{(n+k-1)!}{k!(n-1)!}$ | 組合 $\frac{n!}{k!(n-k)!}$ |

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# 1.5 Conditional Probability 條件機率

- **Definition 1.4-1**

  The conditional probability of an event *A* given that event B has occurred is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

provided that $P(B) > 0$.

# Properties of Conditional Probability

- Conditional probability satisfies the axioms for a probability function.

  That is, with $P(B) > 0$,

  (a) $P(A|B) \geq 0$

  (b) $P(B|B) = 1$

  (c) If $A_1, A_2, \ldots$ are mutually exclusive events,

  then $P(A_1 \cup A_2 \cup \ldots | B) = P(A_1|B) + P(A_2|B) + \ldots$.

- Proof of (a)

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Since $P(B) > 0$ and $P(A \cap B) \geq 0$, $P(A \mid B) \geq 0$.

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# 1.6 Independent Events

$P(A \mid B) = P(A)$

$\Updownarrow$ $\longrightarrow$ **independent**

$P(B \mid A) = P(B)$

$P(A \cap B) = P(A)P(B)$

Example 1.4.1

- <u>**Definition 1.4-1**</u> Events A and B are independent if and only if $P(A \cap B) = P(A)P(B)$
  - Statistically independent, stochastically independent, independent in a probability sense

# Example 1.4-1

- Flip a coin twice and observe the sequence of H and T. The sample space is $S = \{HH, HT, TH, TT\}$.
- It is reasonable to assign a probability of ¼ to each.
- Let $A = \{\text{heads on the first flip}\} = \{HH, HT\}$,

  $B = \{\text{tails on the second flip}\} = \{HT, TT\}$,

  $C = \{\text{tails on both flip}\} = \{TT\}$.
- Now P(B)=1/2. However, if we are given C has occurred, then P(B|C)=1 because $C \subset B$. That is, the knowledge of the occurrence of C has changed the probability of B. On the other hand, if we are given that A has occurred,

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{2/4} = \frac{1}{2} = P(B).$$

# Example 1.4-1 (cont.)

$$S = \{HH, HT, TH, TT\}.$$

- So the occurrence of A has not changed the probability of B. Hence, the probability of B does not depend upon knowledge about event A, so we say that A and B are independent events. That is, events A and B are independent if the occurrence of one of them does not affect the probability of the occurrence of the other.
- Another way of saying the independency is

$$P(B \mid A) = P(B) \text{ or } P(A \mid B) = P(A),$$

provided that P(A)>0 or, in the latter case, P(B)>0. With the first of these equalities and the multiplication rule, we have

$$P(A \cap B) = P(A)P(B \mid A) = P(A)P(B).$$

- The second of these equalities, namely P(A|B), give us the same result $P(A \cap B) = P(B)P(A \mid B) = P(B)P(A)$.

# Theorem 1.4-1

If $A$ and $B$ are independent events, then the following pairs of events are also independent

*a)* $A$ and $B'$ ;

*b)* $A'$ and $B$ ;

*c)* $A'$ and $B'$ .

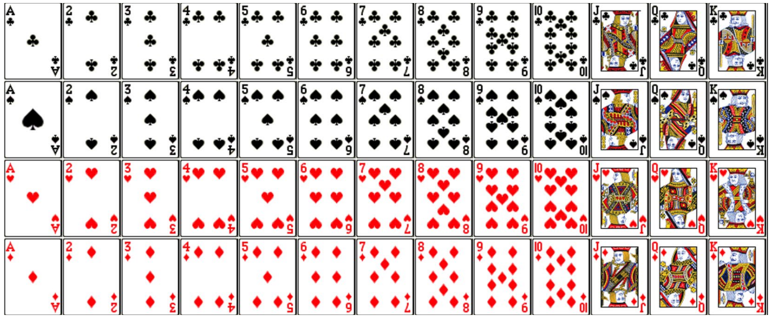
***Proof*** We know that conditional probability satisfies the axioms for a probability function. Hence, if $P(A) > 0$, then $P(B' \mid A) = 1 - P(B \mid A)$. Thus

$$\begin{aligned} P(A \cap B') &= P(A)P(B' \mid A) = P(A)[1 - P(B \mid A)] \\ &= P(A)[1 - P(B)] \\ &= P(A)P(B'), \end{aligned}$$

since $P(B \mid A) = P(B)$ by hypothesis. Thus $A$ and $B'$ are independent events. The proofs for parts (b) and (c) are left as exercises. ∎

# Examples of Independent Events

- Assume that we randomly pick up a bridge card. Let *A* denote the event that the card is a "Heart" and *B* denote the event that the number of the card is 2 or 3. Then, we have



$$P(A) = \frac{1}{4}$$

$$P(B) = \frac{2}{13}$$

$$P(A \cap B) = \frac{2}{52} = P(A) \cdot P(B).$$

- Therefore, *A* and *B* are independent.

# Example 1.4-4: *mutually independent ?*

- Assume that we randomly pick up one ball out of 4 balls that are numbered 1, 2, 3, and 4, respectively.
- Let $A$, and $B$, and $C$ denotes the events {1, 2}, {1, 3}, and {1, 4}, respectively. Then we have

$$P(A \cap B) = \frac{1}{4} = P(A)P(B)$$

$$P(B \cap C) = \frac{1}{4} = P(B)P(C)$$

$$P(A \cap C) = \frac{1}{4} = P(A)P(C).$$

- However, $P(A \cap B \cap C) = \frac{1}{4} \neq \frac{1}{8} = P(A)P(B)P(C)$
- Therefore, we say that $A$, $B$, and $C$ are pairwise independent but not *mutually independent*.

# Definition 1.4-2: Mutually Independent (or simply "independent")

**Definition 1.7.** ***3 Independent Events:*** $A_1$, $A_2$, and $A_3$ are *independent* if and only if

(a) $A_1$ and $A_2$ are independent.

(b) $A_2$ and $A_3$ are independent.

(c) $A_1$ and $A_3$ are independent.

(d) $P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3]$.

**Definition 1.8.** ***More than Two Independent Events:*** If $n \geq 3$, the sets $A_1, A_2, \ldots, A_n$ are independent if and only if

(a) Every set of $n-1$ sets taken from $A_1, A_2, \ldots A_n$ is independent.

(b) $P[A_1 \cap A_2 \cap \cdots \cap A_n] = P[A_1]P[A_2]\cdots P[A_n]$.

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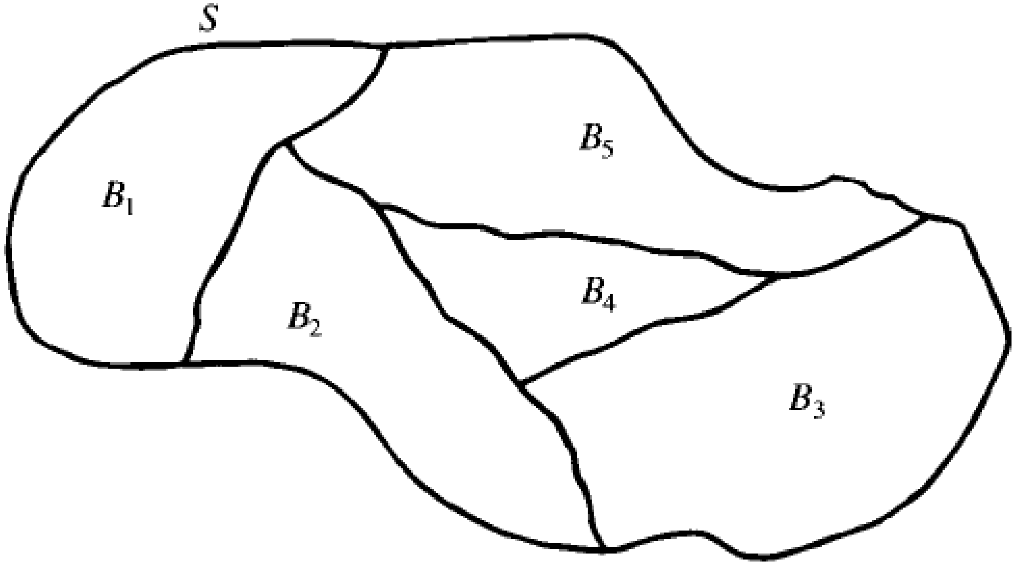
# 1.7 Bayes' Theorem

## • a ***Partition*** of *S*

**Definition** *Let* $\{B_1, B_2, \ldots, B_n\}$ *be a set of nonempty subsets of the sample space* $S$ *of an experiment. If the events* $B_1, B_2, \ldots, B_n$ *are mutually exclusive and* $\bigcup_{i=1}^{n} B_i = S$, *the set* $\{B_1, B_2, \ldots, B_n\}$ *is called a partition of* $S$. ♦

***mutually exclusive and exhaustive***



- Law of Total Probability

**Theorem 3.4 (Law of Total Probability)** *If* $\{B_1, B_2, \ldots, B_n\}$ *is a partition of the sample space of an experiment and* $P(B_i) > 0$ *for* $i = 1, 2, \ldots, n$, *then for any event* $A$ *of* $S$,

$$P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \cdots + P(A \mid B_n)P(B_n)$$

$$= \sum_{i=1}^{n} P(A \mid B_i)P(B_i).$$

*Proof:* Since $B_1, B_2, \ldots, B_n$ are mutually exclusive, $B_iB_j = \emptyset$ for $i \neq j$. Thus $(AB_i)(AB_j) = \emptyset$ for $i \neq j$. Hence $\{AB_1, AB_2, \ldots, AB_n\}$ is a set of mutually exclusive events. Now

$$S = B_1 \cup B_2 \cup \cdots \cup B_n$$

gives

$$A = AS = AB_1 \cup AB_2 \cup \cdots \cup AB_n;$$

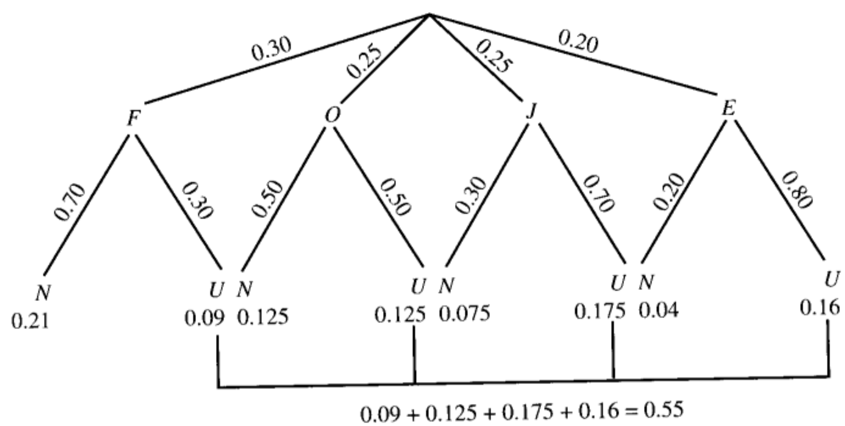
therefore,

$$P(A) = P(AB_1) + P(AB_2) + \cdots + P(AB_n).$$

But $P(AB_i) = P(A \mid B_i)P(B_i)$ for $i = 1, 2, \ldots, n$, so

$$P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \cdots + P(A \mid B_n)P(B_n). \; \blacklozenge$$

**Example 3.15** Suppose that 80% of the seniors, 70% of the juniors, 50% of the sophomores, and 30% of the freshmen of a college use the library of their campus frequently. If 30% of all students are freshmen, 25% are sophomores, 25% are juniors, and 20% are seniors, what percent of all students use the library frequently?




$$
\begin{aligned}
P(A) &= P(A \mid F)P(F) + P(A \mid O)P(O) \\
&\quad + P(A \mid J)P(J) + P(A \mid E)P(E) \\
&= (0.30)(0.30) + (0.50)(0.25) + (0.70)(0.25) + (0.80)(0.20) \\
&= 0.55.
\end{aligned}
$$

## • Bayes' Theorem

**Theorem 3.6 (Bayes' Theorem)** *Let* $\{B_1, B_2, \ldots, B_n\}$ *be a partition of the sample space* $S$ *of an experiment. If for* $i = 1, 2, \ldots, n$, $P(B_i) > 0$, *then for any event* $A$ *of* $S$ *with* $P(A) > 0$,

$$P(B_k \mid A) = \frac{P(A \mid B_k)P(B_k)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \cdots + P(A \mid B_n)P(B_n)}. \blacklozenge$$

$$P(B_k \mid A) = \frac{P(B_k \cap A)}{P(A)}, \qquad k = 1, 2, \ldots, m.$$

$$P(B_k \mid A) = \frac{P(B_k)P(A \mid B_k)}{\sum_{i=1}^{m} P(B_i)P(A \mid B_i)}, \qquad k = 1, 2, \ldots, m.$$

**Example** 1.5-2 In a certain factory, machines I, II, and III are all producing springs of the same length. Of their production, machines I, II, and III produce 2%, 1%, and 3% defective springs, respectively. Of the total production of springs in the factory, machine I produces 35%, machine II produces 25%, and machine III produces 40%. If one spring is selected at random from the total springs produced in a day, the probability that it is defective in an obvious notation equals

$$\begin{aligned} P(D) &= P(I)P(D\,|\,I) + P(II)P(D\,|\,II) + P(III)P(D\,|\,III) \\ &= \left(\frac{35}{100}\right)\left(\frac{2}{100}\right) + \left(\frac{25}{100}\right)\left(\frac{1}{100}\right) + \left(\frac{40}{100}\right)\left(\frac{3}{100}\right) = \frac{215}{10{,}000}. \end{aligned}$$

If the selected spring is defective, the conditional probability that it was produced by machine III is, by Bayes' formula,

$$P(III\,|\,D) = \frac{P(III)P(D\,|\,III)}{P(D)} = \frac{(40/100)(3/100)}{215/10{,}000} = \frac{120}{215}.$$

Note how the posterior probability of III increased from the prior probability of III after the defective spring was observed because III produces a larger percentage of defectives than do I and II. ▲