On Takens Best Estimator

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Classical Approach Inaccurate on Small Scales

Attractor

"bulk"

$$\text{correlation dimension}(A) = \lim_{\varepsilon \to 0} \frac{\ln C(\varepsilon)}{\ln \varepsilon}$$

"size"

- Generate large number of random pairs for various values of ε, estimate slope of $\ln C(\varepsilon)$ as function of ε
- **Problem:** For small ε, the estimate becomes more and more inaccurate, it is difficult to estimate the accuracy

Takens Best Estimator

- Idea: take finite sequence of distances $\rho_1, \rho_2, \ldots, \rho_m$ of randomly chosen pairs. If correlation dimension exists, then

$$C(\varepsilon) = c \cdot \varepsilon^\alpha + \text{higher order terms}$$

- **Assumption:** $\exists\, \varepsilon_0 > 0$ fixed, so that $\forall\, 0 < \varepsilon \leq \varepsilon_0$

$$C(\varepsilon) = c \cdot \varepsilon^\alpha \qquad \text{exactly.}$$

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- Normalize leftover distances so that $r_i = \rho_i / \varepsilon_0$.

Maximum Likelihood Estimation

- **Problem:** Given $r_1, r_2, \ldots, r_m \in [0,1]$, a random sample from probability distribution, *what is the most likely value of α?*[1]
- If $C(\varepsilon) = c \cdot \varepsilon^{\alpha}$, then the distribution of $r_i = \rho_i/\varepsilon_0$ is given by

$$P(0 \leq r_i \leq t) = t^{\alpha} \qquad \text{for } t \in [0,1]$$

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- Then $y_i = -\ln r_i$ is distributed exponentially with parameter α proof and the MLE for α is given by

$$\hat{\alpha}_{\mathrm{MLE}} = \frac{m}{\sum_{i=1}^m y_i} = -\frac{m}{\sum_{i=1}^m \ln r_i}$$

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The MLE of the Exponential Distribution is biased

$$
\begin{aligned}
\langle \hat{\alpha}_{MLE} \rangle &= \left\langle \frac{m}{\sum_{i=1}^{m} y_i} \right\rangle \\
&= m \left\langle \frac{1}{\sum_{i=1}^{m} y_i} \right\rangle \\
&= m \, \frac{\alpha}{m-1} \neq \alpha
\end{aligned}
$$

$$
\begin{aligned}
& y_i \sim \Gamma(1, \alpha) \\
\Rightarrow \; & \frac{1}{\sum_{i=1}^{m} y_i} \sim \Gamma^{-1}(m, \alpha)
\end{aligned}
$$

Solution:

$$
\hat{\alpha}^* = \frac{m-1}{m} \hat{\alpha}_{MLE} = \frac{m-1}{\sum_{i=1}^{m} y_i}
$$

is the uniformly minimum variance unbiased (UMVU) estimator.

Extension of Takens Estimator

- Define

$$E_{pk}(\varepsilon_0) = \mathbb{E}[(r/\varepsilon_0)^p \,|\ln(r/\varepsilon_0)|^k : 0 < r < \varepsilon_0]$$

$$\lim_{\varepsilon_0 \to 0} E_{pk}(\varepsilon_0) = \int_0^1 x^p \,|\ln x|^k \alpha x^{\alpha-1} \mathrm{d}x = \frac{\alpha\Gamma(k+1)}{(\alpha+p)^{k+1}}$$

$$\Rightarrow \boxed{\alpha_{pk}(\varepsilon_0) = \frac{\sum_{r_i<\varepsilon_0} \left(\frac{r_i}{\varepsilon_0}\right)^p \left|\ln\left(\frac{r_i}{\varepsilon_0}\right)\right|^{k-1}}{\sum_{r_i<\varepsilon_0} \left(\frac{r_i}{\varepsilon_0}\right)^p \left|\ln\left(\frac{r_i}{\varepsilon_0}\right)\right|^{k}} - p}$$

- We get Takens estimator if we set $p = 0, k = 1$

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- We get Takens estimator if we set $p = 0, k = 1$
- Noise dominates: α_{pp} and α_{p0} (improving with increasing p) are best estimators
- Lacunarity dominates: α_{0k} (improving with increasing k) are best estimators

To do

- Test Takens estimator
- Test Takens estimator with lower cutoff, as suggested by Borovkova et al.
- Test extended version of Takens estimator introduced by Shirer et al.

Transformation of Random Variable

Let X be a RV with cumulative distribution function (CDF)

$$P(0 \leq X \leq t) = \underbrace{t^{\alpha}}_{CDF} = \int_0^t \underbrace{\alpha s^{\alpha-1}}_{\text{PDF}} \mathrm{d}s.$$

For $Y = f(X) = -\ln X$, we find

$$\begin{aligned} P(Y \leq y) &= P(-\ln X \leq y) \\ &= P(X \geq e^{-y}) \\ &= \int_{e^{-y}}^{1} \alpha s^{\alpha-1} \mathrm{d}s \\ &= s^{\alpha}\Big|_{e^{-y}}^{1} = \underbrace{1 - e^{-\alpha y}}_{\text{CDF of exp. dist.}} \end{aligned}$$

MLE of Exponential Distribution

- For independent measurements, the likelihood function is the product of their probabilities according to the PDF:

$$L(\alpha; x_1, \dots, x_m) = \prod_{j=1}^{m} \mathrm{PDF}(x_j, \alpha) = \prod_{j=1}^{m} \alpha \exp(-\alpha x_j)$$

$$\Rightarrow l(\alpha; x_1, \dots, x_m) = \ln L(\alpha; x_1, \dots, x_m) = m \ln \alpha - \alpha \sum_{j=1}^{m} x_j$$

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- Now we estimate the α for which this is maximal:

$$\hat{\alpha}_{MLE} = \arg\max_{\alpha} l(\alpha; x_1, \ldots, x_m)$$

$$\rightsquigarrow 0 = \frac{\mathrm{d}}{\mathrm{d}\alpha} l(\alpha; x_1, \ldots, x_m)\bigg|_{\alpha = \hat{\alpha}_{MLE}} = \frac{m}{\hat{\alpha}_{MLE}} - \sum_{j=1}^{m} x_j$$

$$\Rightarrow \hat{\alpha}_{MLE} = \frac{m}{\sum_{j=1}^{m} x_j} = \frac{1}{\bar{x}}$$

sample mean

Estimation of Standard Error

- The variance of an exponentially distributed RV with parameter α is

$$\sigma^2(X) = \frac{1}{\alpha},$$

thus the standard error for m independent measurements is

$$s_X = \frac{1}{\sqrt{m}}\frac{1}{\alpha}$$

Standard Error of α

- We can use the standard error of $\ln R$ to estimate the standard error of α:

$$
\begin{aligned}
\alpha &= \frac{m-1}{m}\overline{(-\ln R)}^{-1} \\
\Rightarrow \frac{\partial \alpha}{\partial \overline{\ln R}} &= \frac{m-1}{m}\frac{1}{\overline{\ln R}^2} \\
\Rightarrow s_\alpha &= s_{\ln R} \cdot \frac{m-1}{m}\frac{1}{\overline{\ln R}^2} \\
&= \frac{1}{\sqrt{m}\alpha}\frac{m}{m-1}\alpha^2 \\
&= \frac{\sqrt{m}}{m-1}\alpha
\end{aligned}
$$

Lacunarity

In lacunar systems, we have

$$C(r) = \underline{\Phi(r)} \cdot r^{\alpha}$$
$$\alpha(\varepsilon_0) = \frac{\Phi(\varepsilon_0)\varepsilon_0^{\alpha}}{\int_0^{\varepsilon_0} \Phi(r) r^{\alpha-1} \mathrm{d}r},$$

- Take $z = -\ln(r/\varepsilon_0) \Rightarrow r = \varepsilon_0 e^{-z}$, then Takens estimator converges if

$$\lim_{\varepsilon_0 \to 0} \int_0^{\infty} e^{-\alpha}[\Phi(\varepsilon_0 e^{-z}/\Phi(\varepsilon_0)) - 1]\mathrm{d}z = 0$$

Literature

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