Practice problem 3:Optimization in ML

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1. Construct the dual problem of the following LP's:

   (i)

$$
\begin{aligned}
\max\ \ 3x_1 + 4x_2 \\
3x_1 - x_2 \leq 12 \\
7x_1 + 11x_2 \leq 88 \\
x_1, x_2 \geq 0
\end{aligned}
$$

   (ii)

$$
\begin{aligned}
\max\ \ z = x_1 + x_2 + x_3 \\
3x_1 + 2x_2 + x_2 \leq 3 \\
2x_1 + x_2 + 2x_3 \leq 2 \\
x_1, x_2, x_3 \geq 0
\end{aligned}
$$

   (iii)

$$
\begin{aligned}
\max\ \ z = 2x_1 + 3x_2 \\
s.\ t.\ 2x_1 + x_2 \leq 1000 \\
x_1 + x_2 \leq 600 \\
2x_1 + 4x_2 \leq 2000 \\
x_1, x_2 \geq 0
\end{aligned}
$$

(iv)

$$\begin{aligned} \min \quad & 3x_1 + 7x_2 \\ & 2x_1 + x_2 \leq 4 \\ & 3x_1 + 4x_2 \geq 24 \\ & 2x_1 - 3x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(v)

$$\begin{aligned} \min \quad & 5x_1 + 2x_2 \\ & x_1 + 2x_2 \leq 3 \\ & 4x_1 + 3x_2 \geq 6 \\ & 3x_1 + x_2 = 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(vi)

$$\begin{aligned} \min \quad & -3x_1 + x_2 \\ & x_1 + 2x_2 = 0 \\ & 2x_1 - 2x_2 = 9 \\ & x_1, x_2 \geq 0 \end{aligned}$$

2. Justify whether x^* is a KKT point of the following problem or not.

(i)

$$\begin{aligned} \min \quad & (x_1 - 4)^2 + (x_2 - 6)^2 \\ s.\,t. \quad & x_1 \geq x_1^2 \\ & x_2 \leq 4 \end{aligned}$$

$x^* = (2, 4)^T$.

(ii)

$$\min \left(x_1 - \frac{3}{2}\right)^2 + (x_2 - 5)^2$$
$$-x_1 + x_2 \leq 2$$
$$2x_1 + 3x_2 \leq 11$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

$x^* = (1, 3)^T$

(iii)

$$\max \ x_1 + 3x_2$$
$$2x_1 + 3x_2 \leq 6$$
$$-x_1 + 4x_2 \leq 4$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

$x^* = (\frac{12}{11}, \frac{14}{11})^T$

(iv)

$$\max \ (x_1 - 6)^2 + (x_2 - 2)^2$$
$$-x_1 + 2x_2 \leq 4$$
$$3x_1 + 2x_2 \leq 12$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

$x^* = (2, 3)^T$

3. Consider the problem

$$\min x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 - 2x_1 - 4x_2 - 6x_3$$
$$s.t. \ x_1 + x_2 + x_3 \leq 1$$

- Is this problem a convex optimization problem?

- Does this satisfy Slater condition?
- Is $x^* = (-0.5, 0, 1.5)^T$ a KKT point of this problem?

4. Consider the problem

$$\begin{aligned} \min 4x_1^2 + x_2^2 - x_1 - 2x_2 & \\ s.t.\ 2x_1 + x_2 &\leq 1 \\ x_1^2 &\leq 1 \end{aligned}$$

- Is this problem a convex optimization problem?
- Does this satisfy Slater condition?
- Is $x^* = (1/16, 7/8)^T$ a KKT point of this problem?

5. Solve the following problem by substituting one variable and Lagrange multiplier method.

(i)

$$\begin{aligned} \min\ & x_1^2 + 2x_2^2 \\ s.\ t.\ & x_1 + x_2 = 2. \end{aligned}$$

(ii)

$$\begin{aligned} \min\ & (x_1 - 2)^2 + (x_2 - 5)^2 \\ s.\ t.\ & -2x_1 + x_2 = 4. \end{aligned}$$

(iii)

$$\begin{aligned} \min\ & 2x_1^2 + x_2^2 \\ s.\ t.\ & 3x_1 + 2x_2 = 6 \end{aligned}$$

(iv)

$$\begin{aligned} \min\ & 2x_1 + 3x_2 + x_3 \\ s.\ t.\ & x_1^2 + x_2^2 = 5 \\ & x_1 + x_3 = 1 \end{aligned}$$

6. Solve the following problem using Lagrange multiplier method:

   (i)

$$\begin{aligned} \max \ \ & x_1^2 + 4x_1x_2 + x_2^2 \\ s.\,t. \ \ & x_1^2 + x_2^2 = 1. \end{aligned}$$

   (ii)

$$\begin{aligned} \max \ \ & x_1^2 + x_2^2 + 2x_3 \\ s.\,t. \ \ & x_1 + x_2 + x_3 = 6 \\ & -x_1 + x_2 + x_3 = 4 \end{aligned}$$

   (iii) Show that the rectangular parallelepiped with surface area 64 will have maximum volume if it is a cube.

   (iv) Show that the rectangle with perimeter $2R$, where R is the last two digits of your roll number will have maximum diagonal if it is a square.

7. Show that x^* is a Lagrangian saddle point of (P). [Hint: If objective and inequality constraints are convex and equality constrained are affine then KKT point is a Lagrangian Saddle point]

   (i)

$$\begin{aligned} \min \ \ & \left(x_1 - \frac{3}{2}\right)^2 + (x_2 - 5)^2 \\ & -x_1 + x_2 \leq 2 \\ & 2x_1 + 3x_2 \leq 11 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

   $x^* = (1, 3)^T$

(ii)

$$
\begin{aligned}
(P) \quad \min \ & x_1^2 + x_2^2 \\
s.\ t.\ & x_1^2 + x_2^2 \le 5 \\
& x_1 + 2x_2 = 4 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

$x^* = (1,2)^T$.