HOUGH

transform

MACHINE ANALYSIS OF BUBBLE CHAMBER PICTURES

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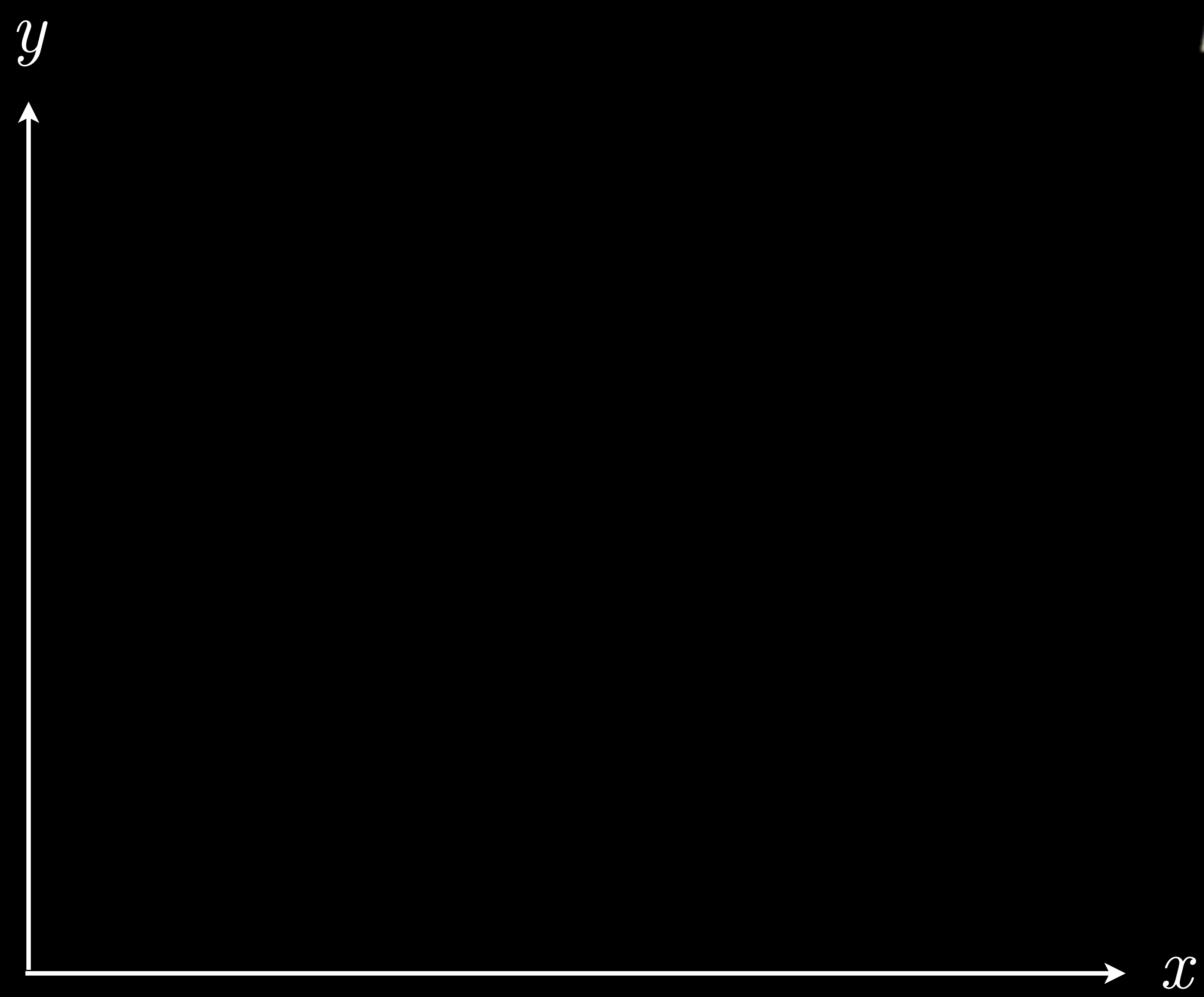
1. AREA ELEMENTS VERSUS LINE SEGMENTS IN PICTURE ANALYSIS

Many people have suggested that a modern digital computer should be able to recognize a fairly complex pattern of tracks in a bubble chamber photograph such as that shown in Fig. 1a (*). Concrete schemes for such recognition generally assume that information is available about the presence or absence of bubbles in area elements covering the pictures and of a size appropriate to the resolution of the chamber. However, rough investigation of the time to read such information into a computer and to conduct a search for linear correlations among bubbles has so far led

important reason, the slope of each line segment provides the computer with a good prediction of the location of the adjoining line segment and so reduces enormously the search time in recognizing a track.

It will be shown below that the tracks in one picture may be recognized in a time of the order of 1-2 s, and therefore a stereo pair of pictures may be analyzed in less than 5 s. It seems that an analysis time of this order of magnitude is a reasonable goal since it matches the cycle time of large accelerators.

Hough Lines



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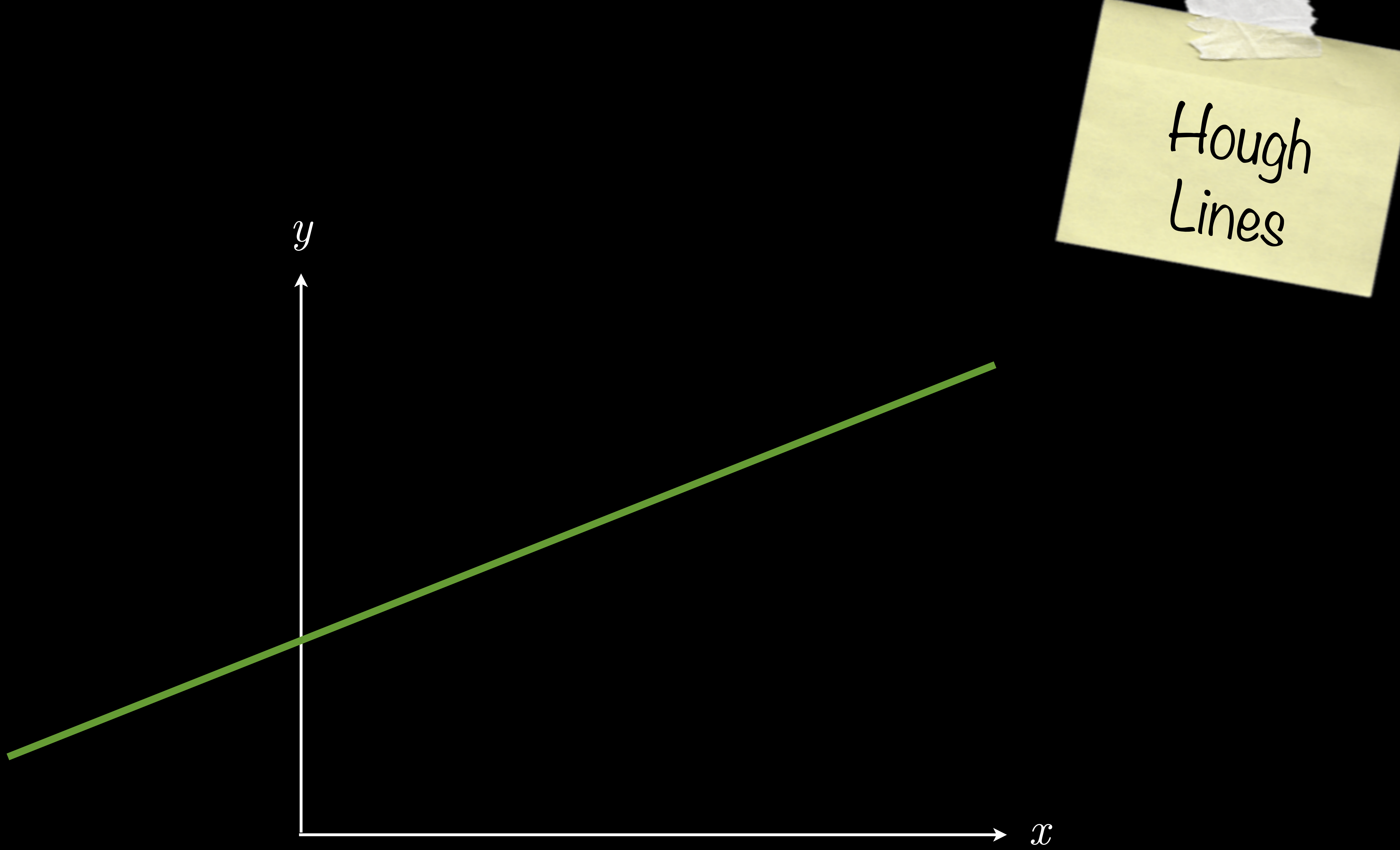

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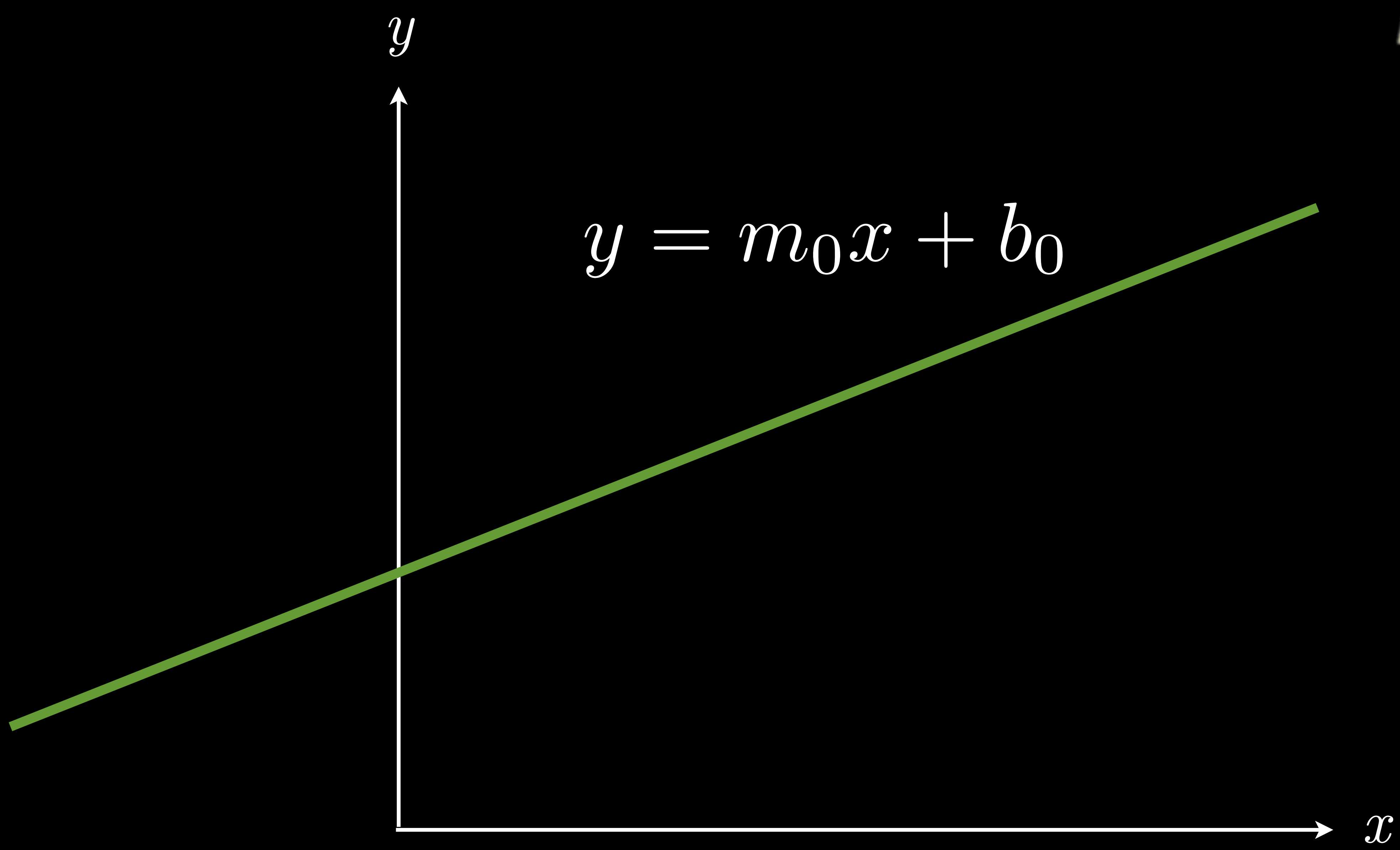

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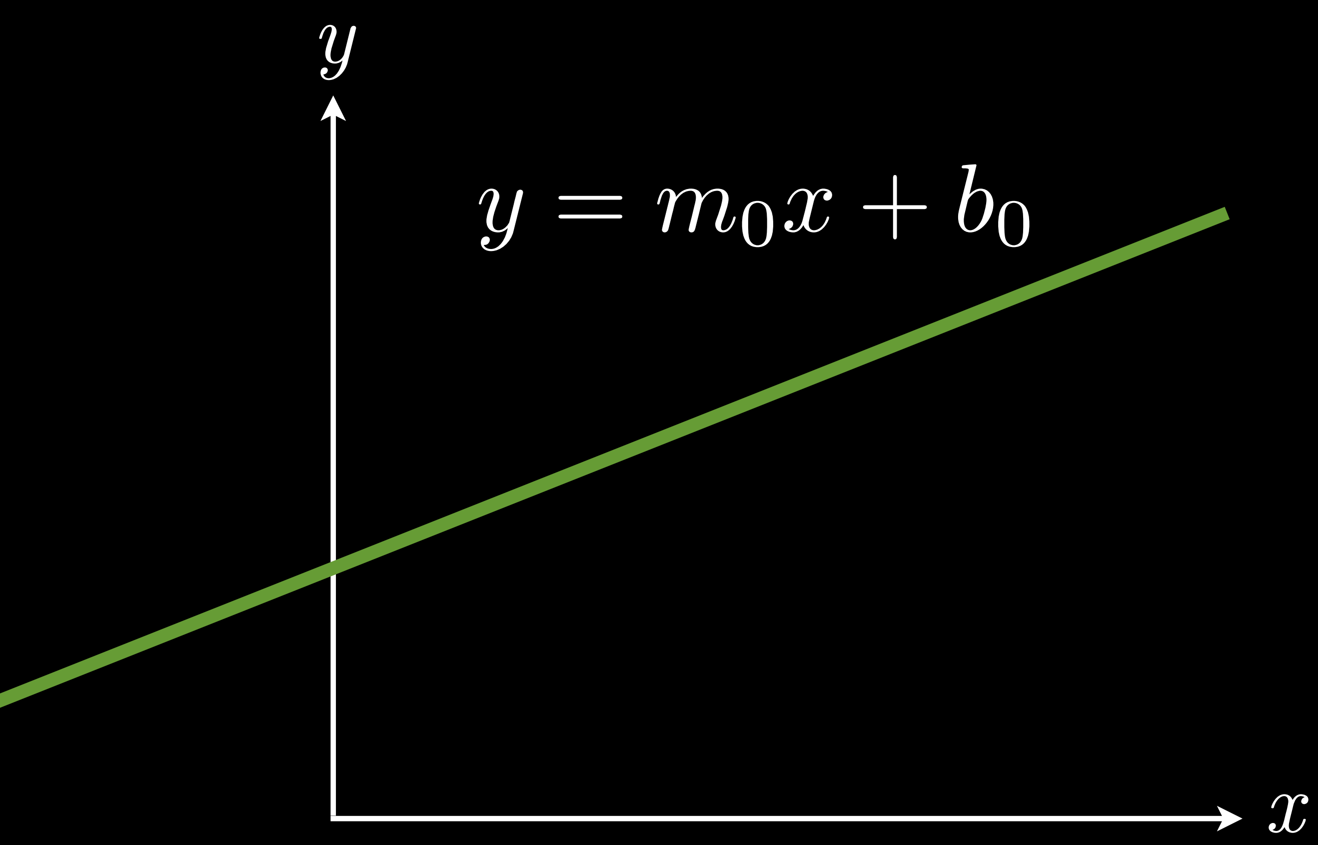

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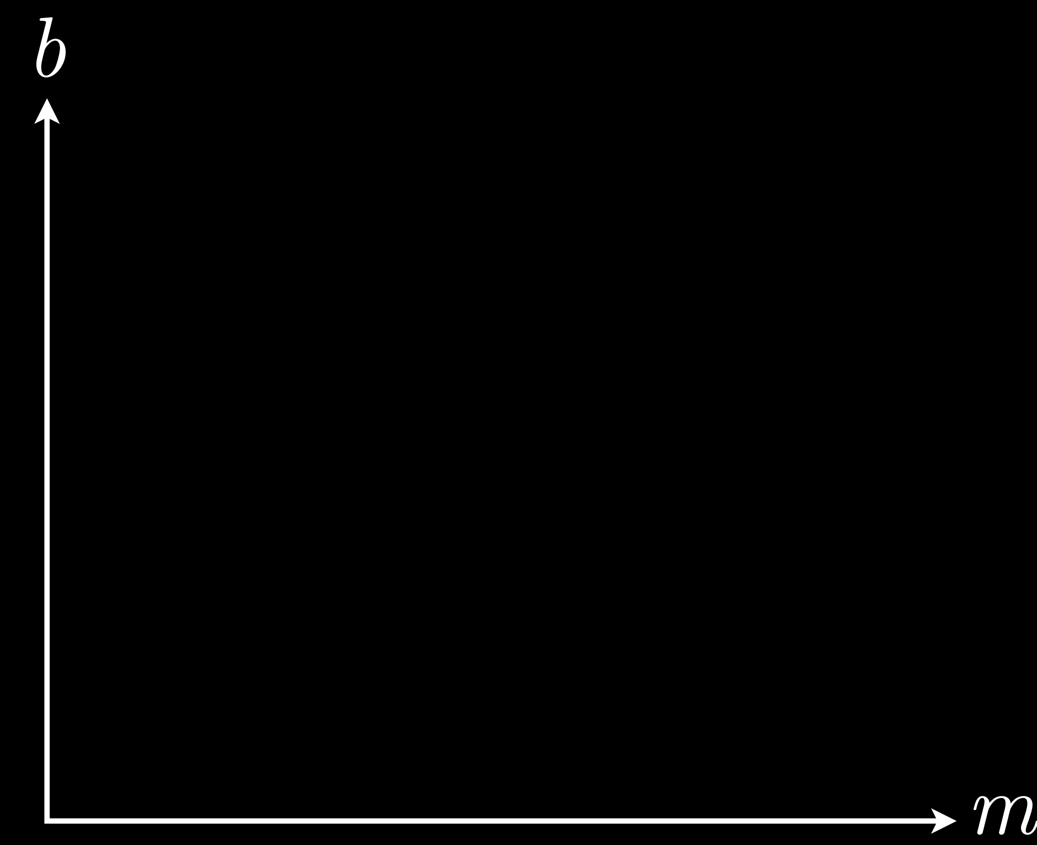
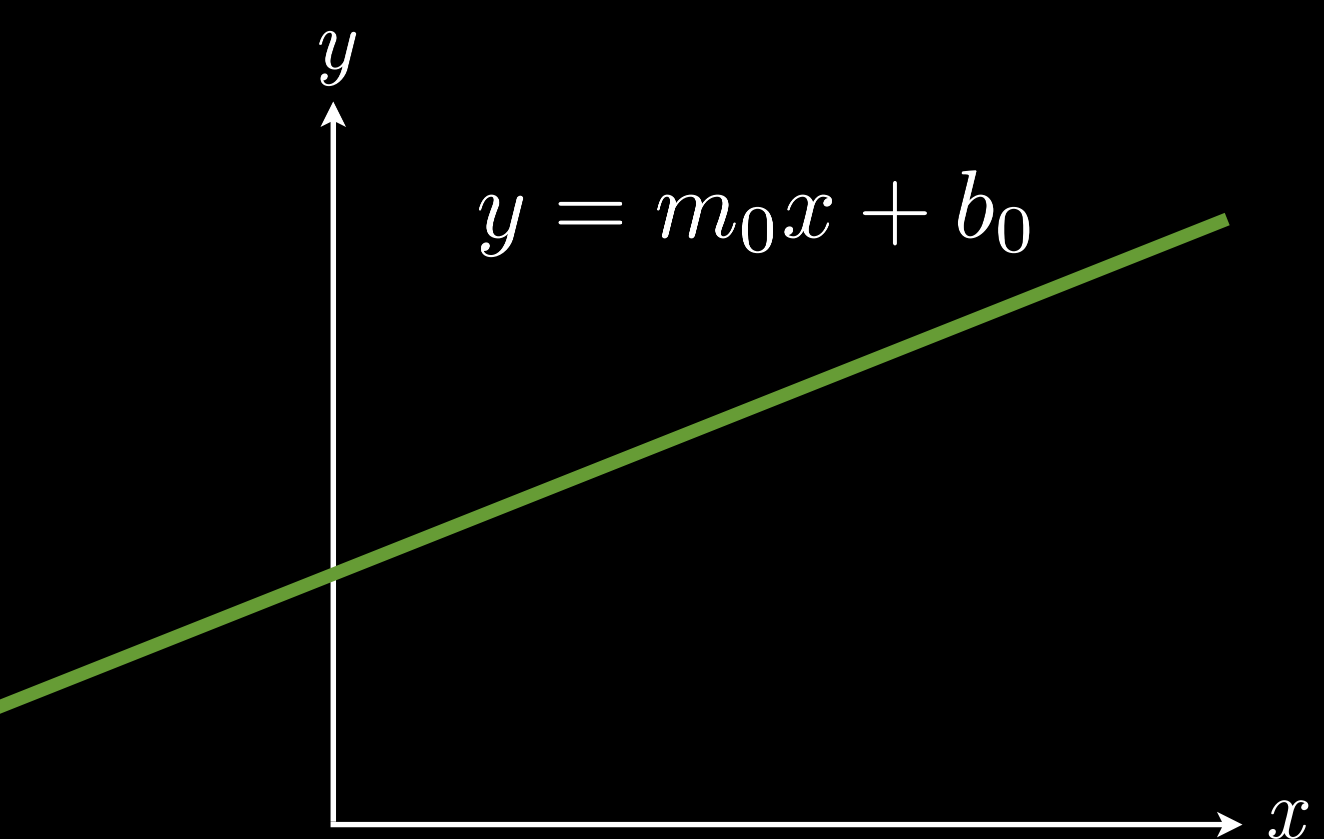


Hough (parameter) space

Hough Lines



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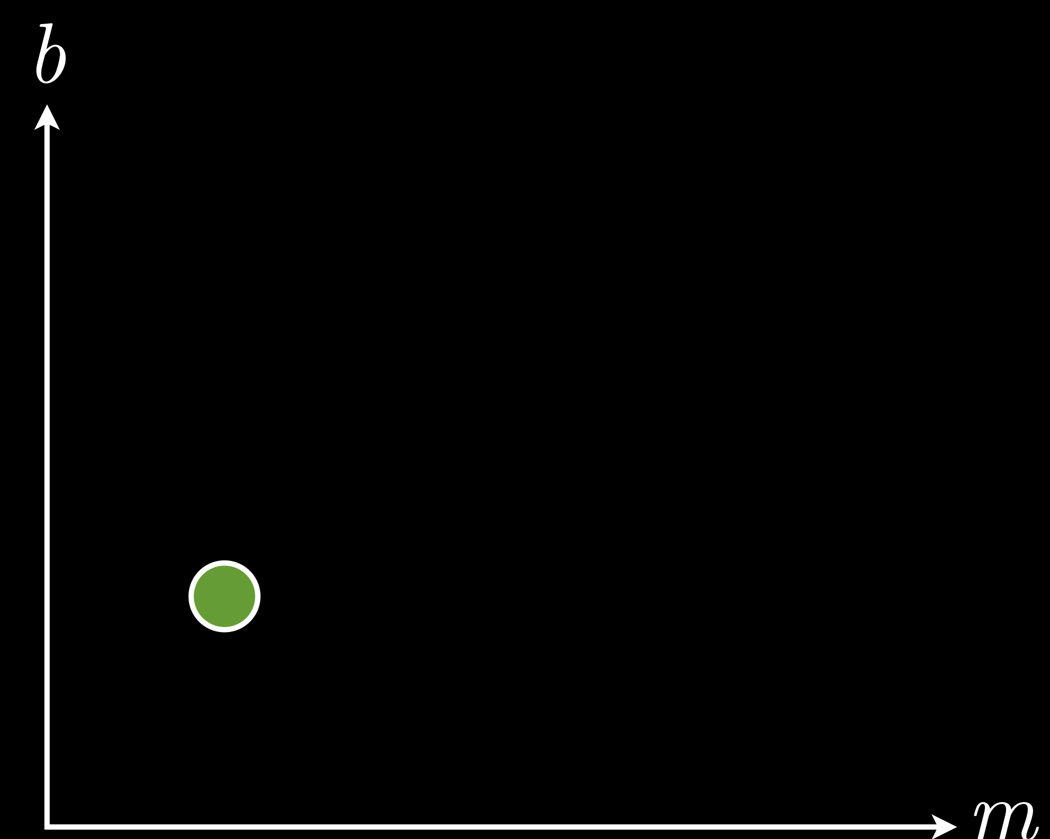

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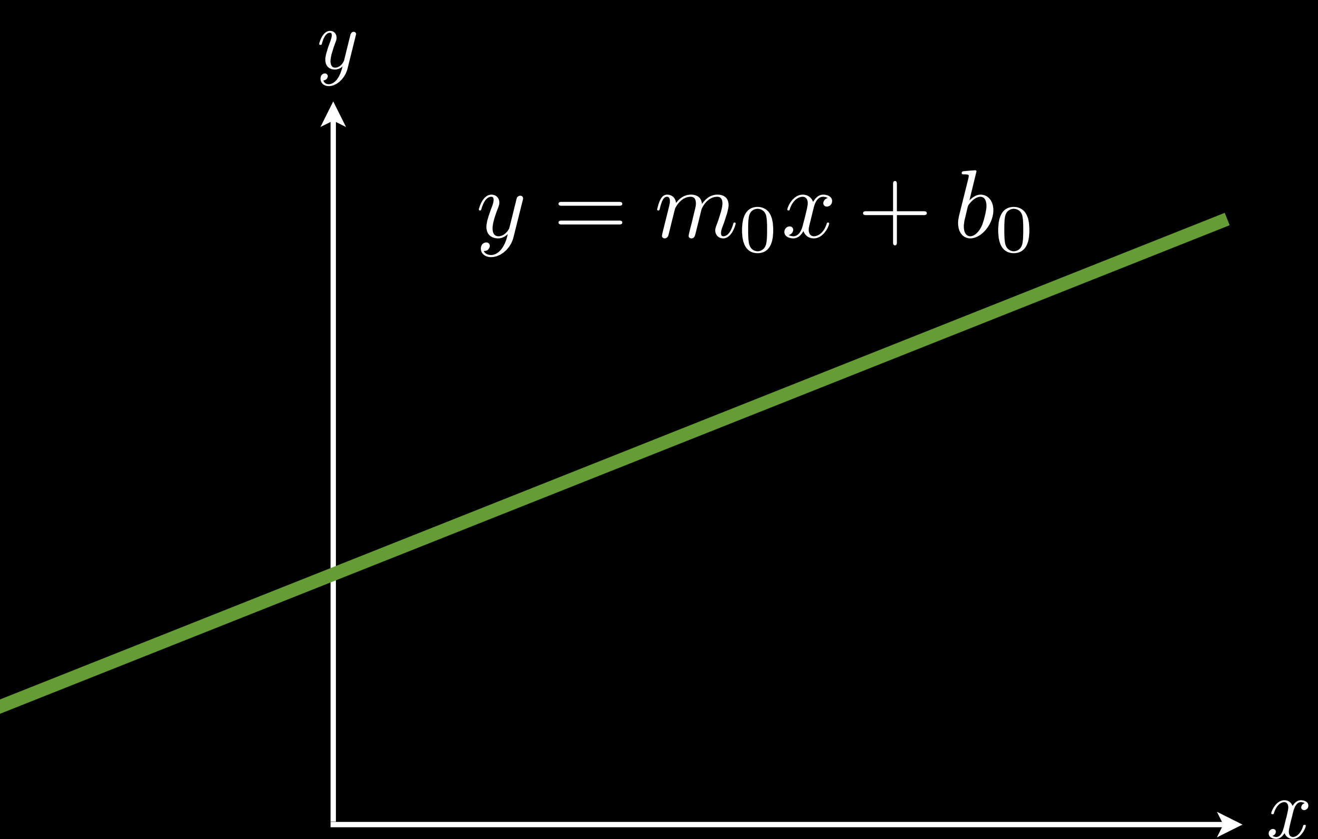

image space

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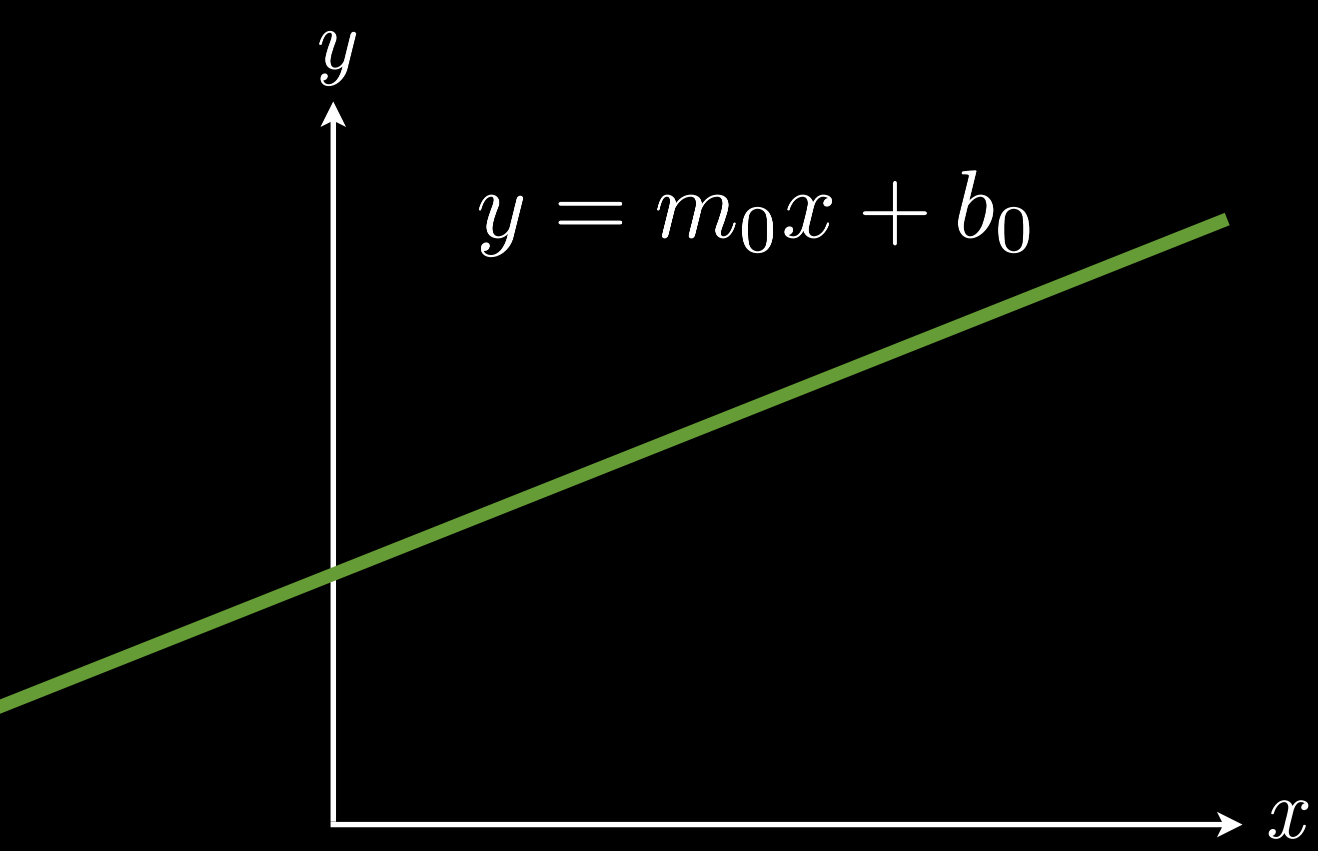

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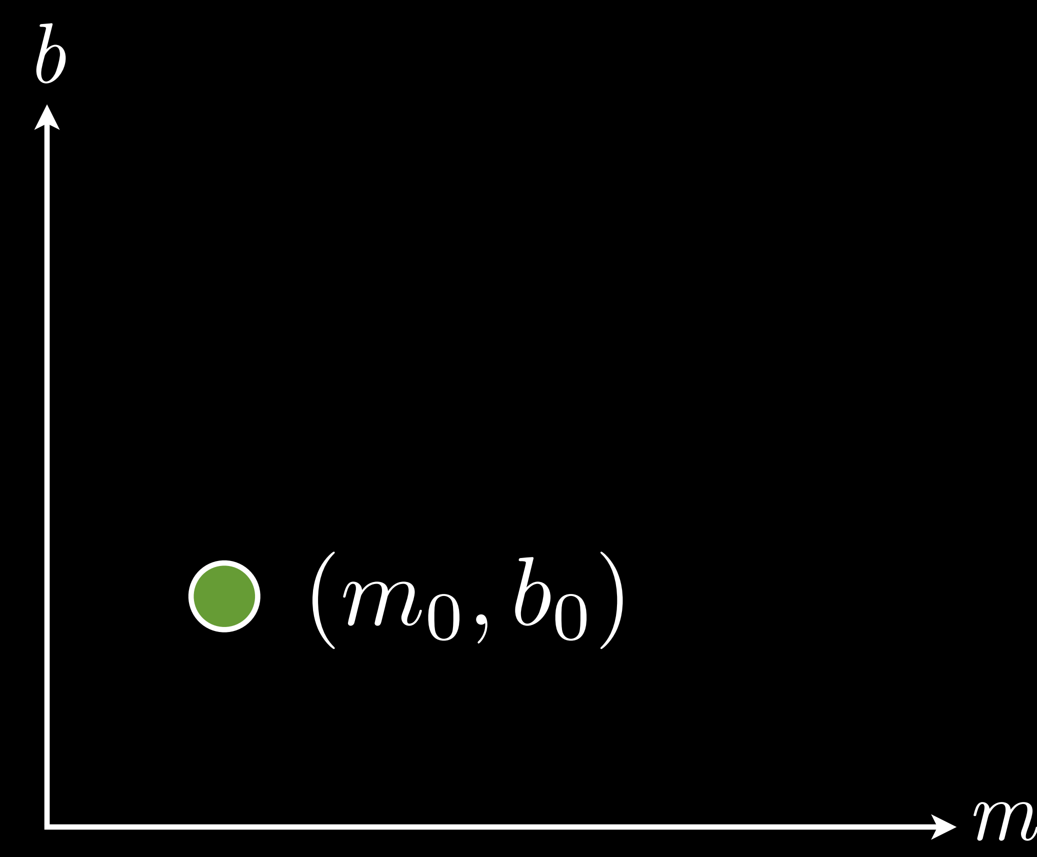
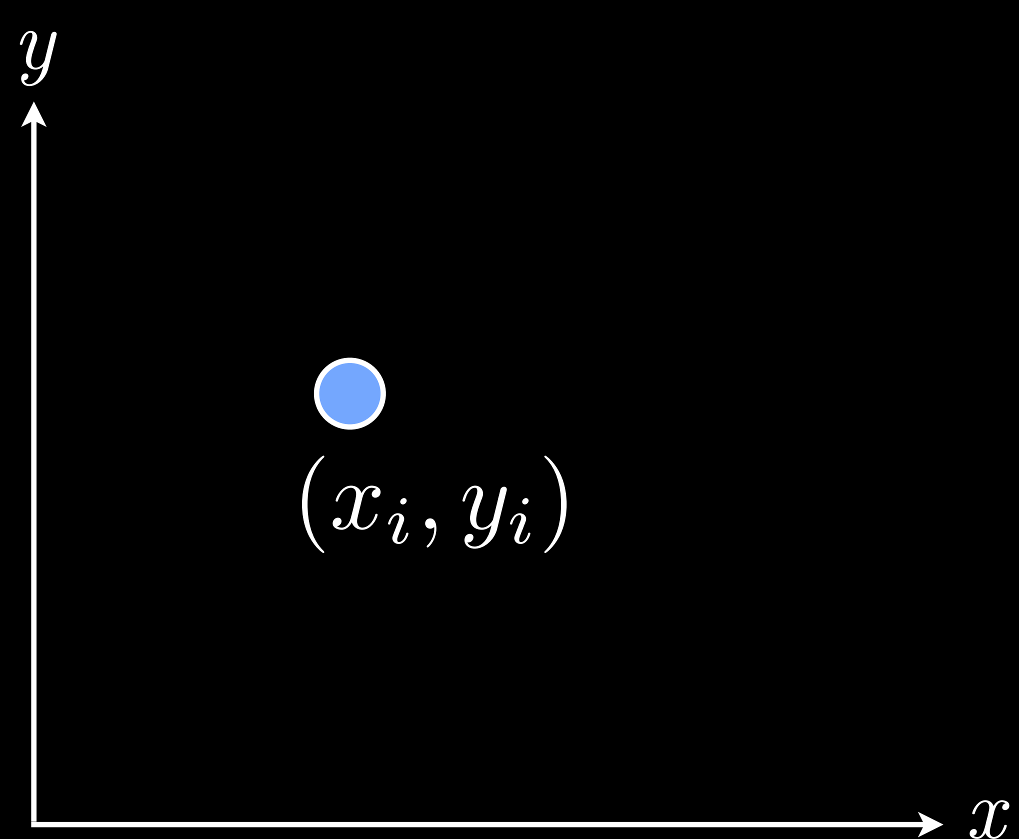


Hough (parameter) space

Point-Line Duality



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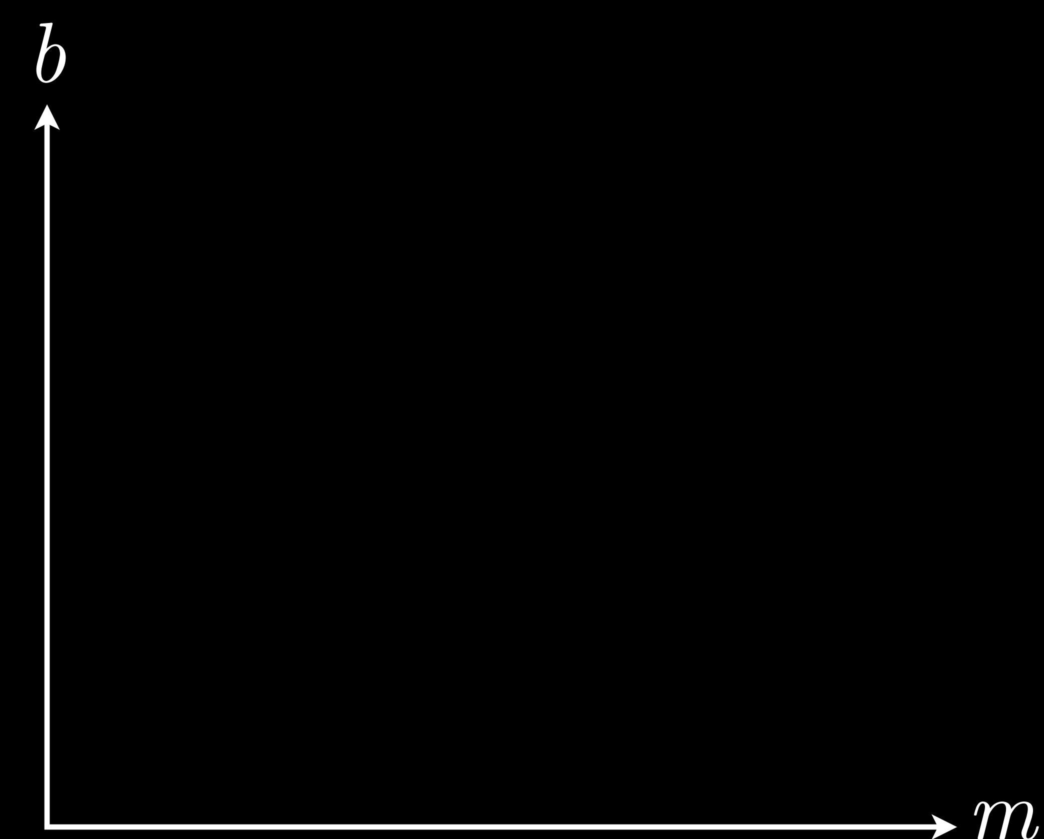

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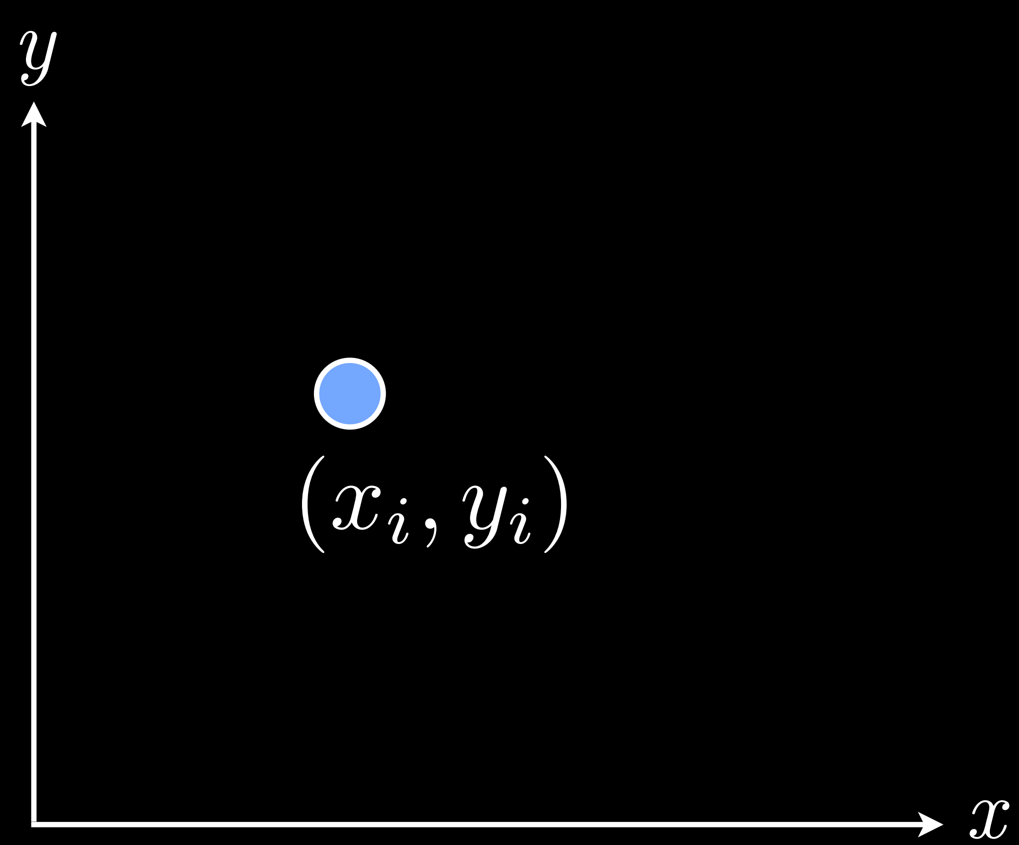

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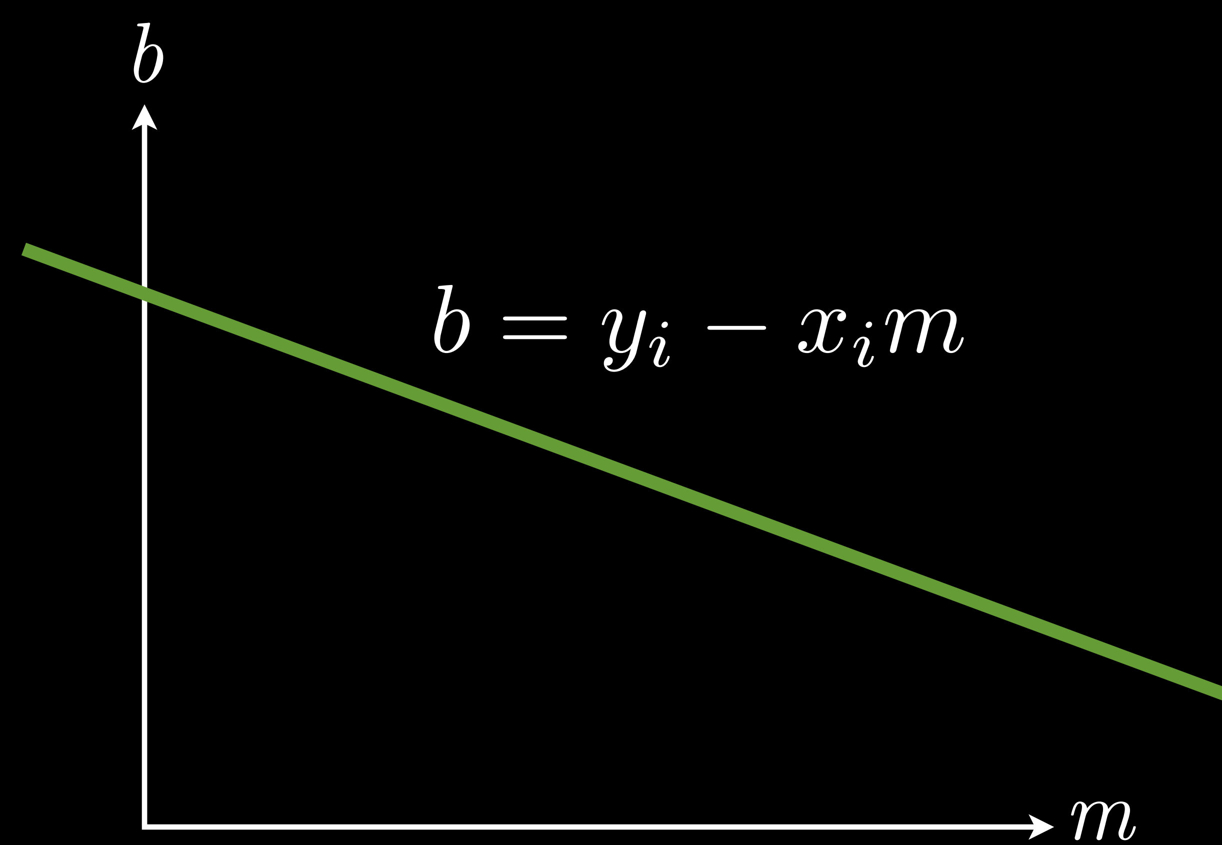

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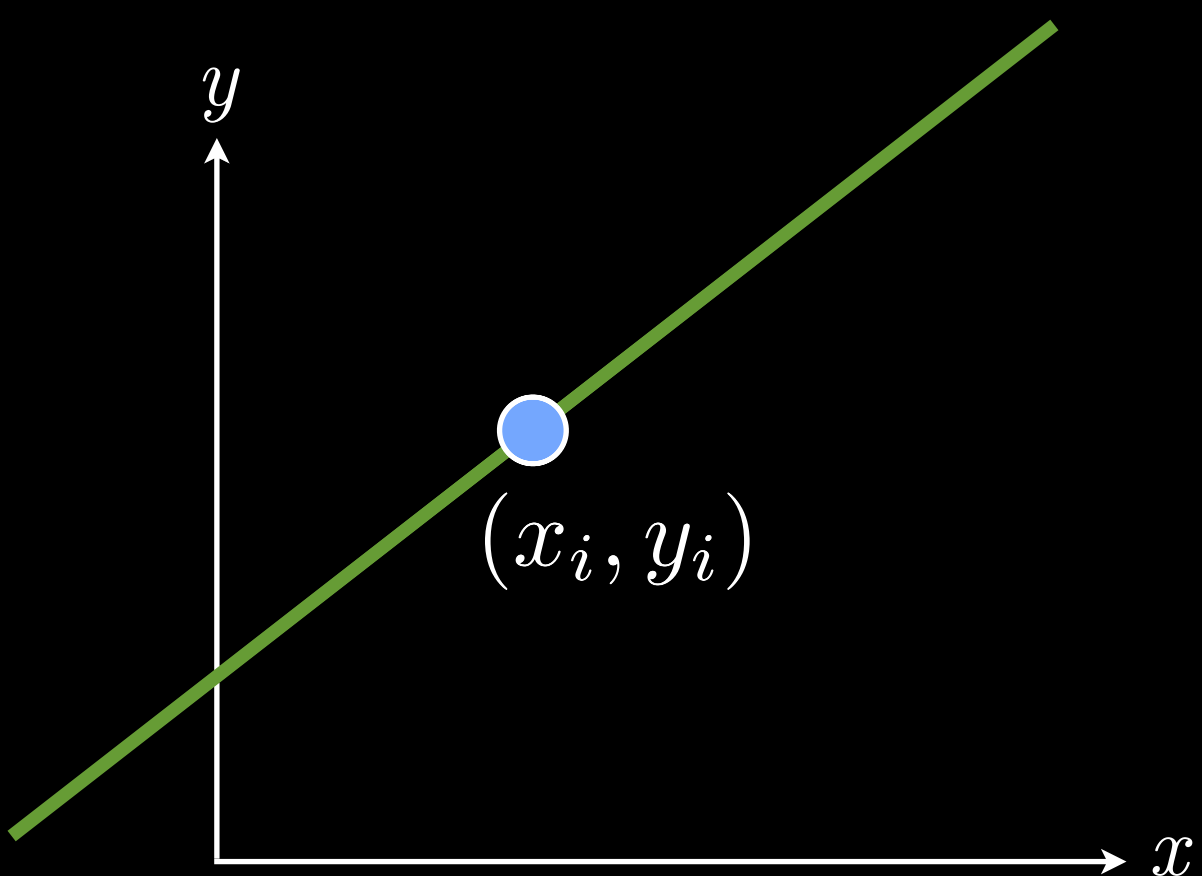

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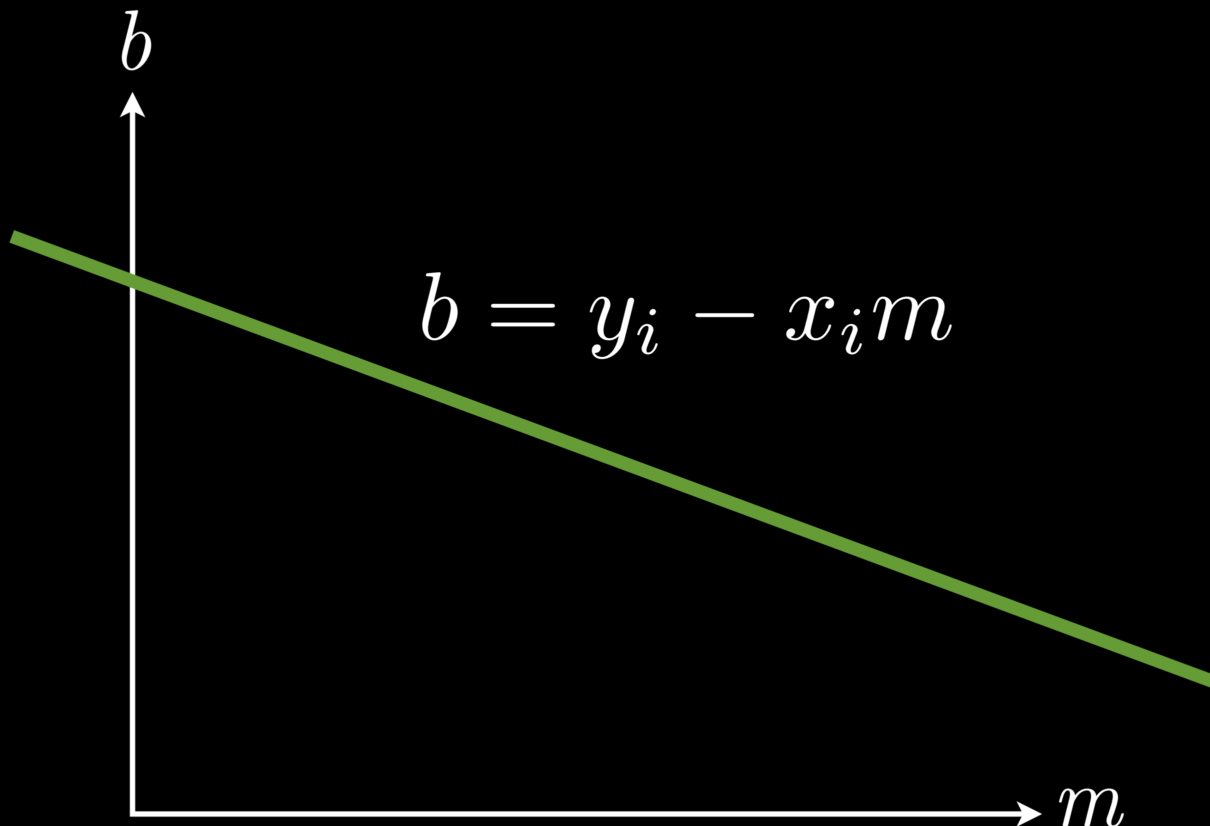

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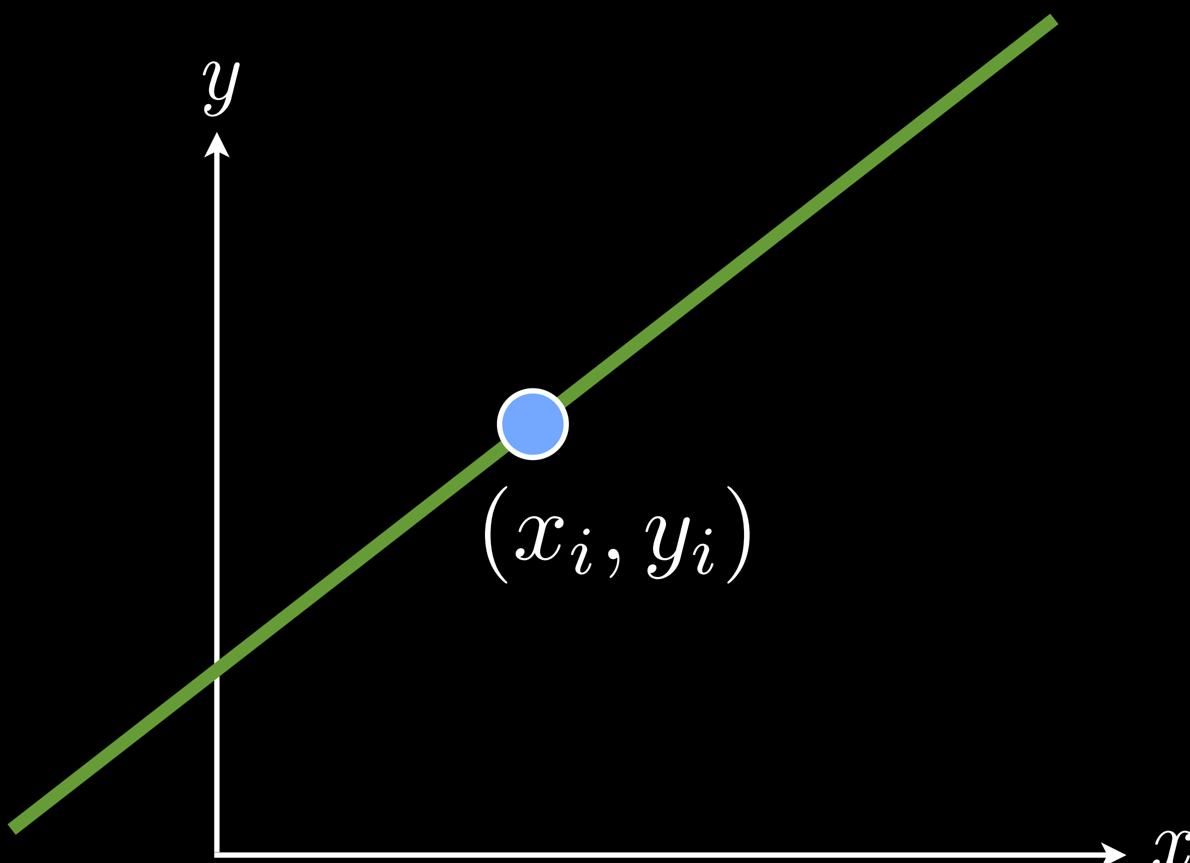

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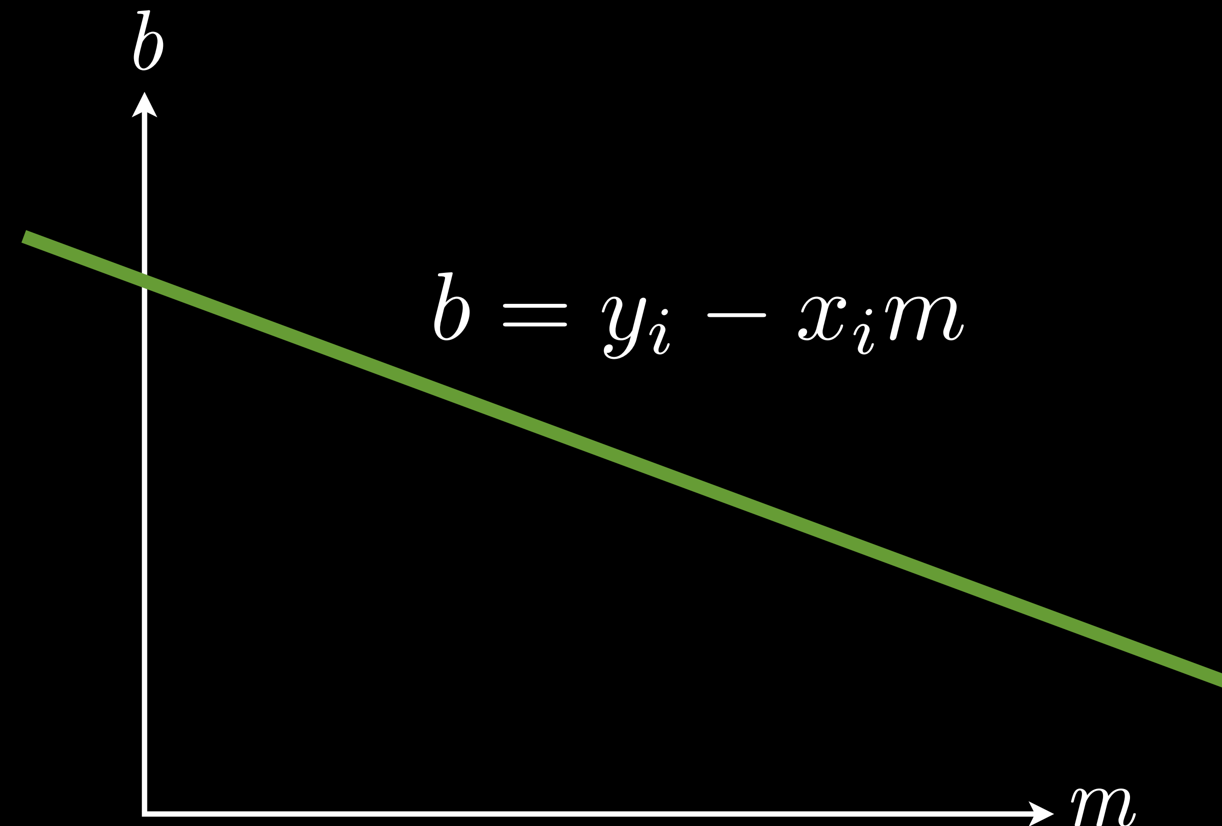

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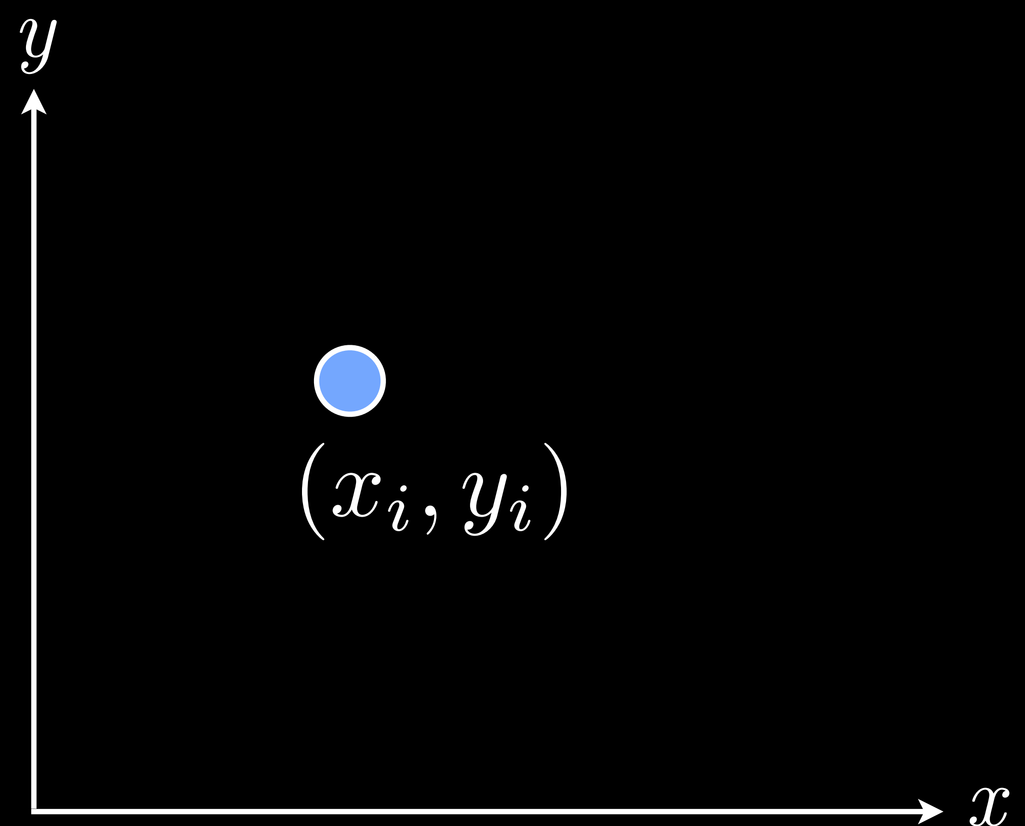

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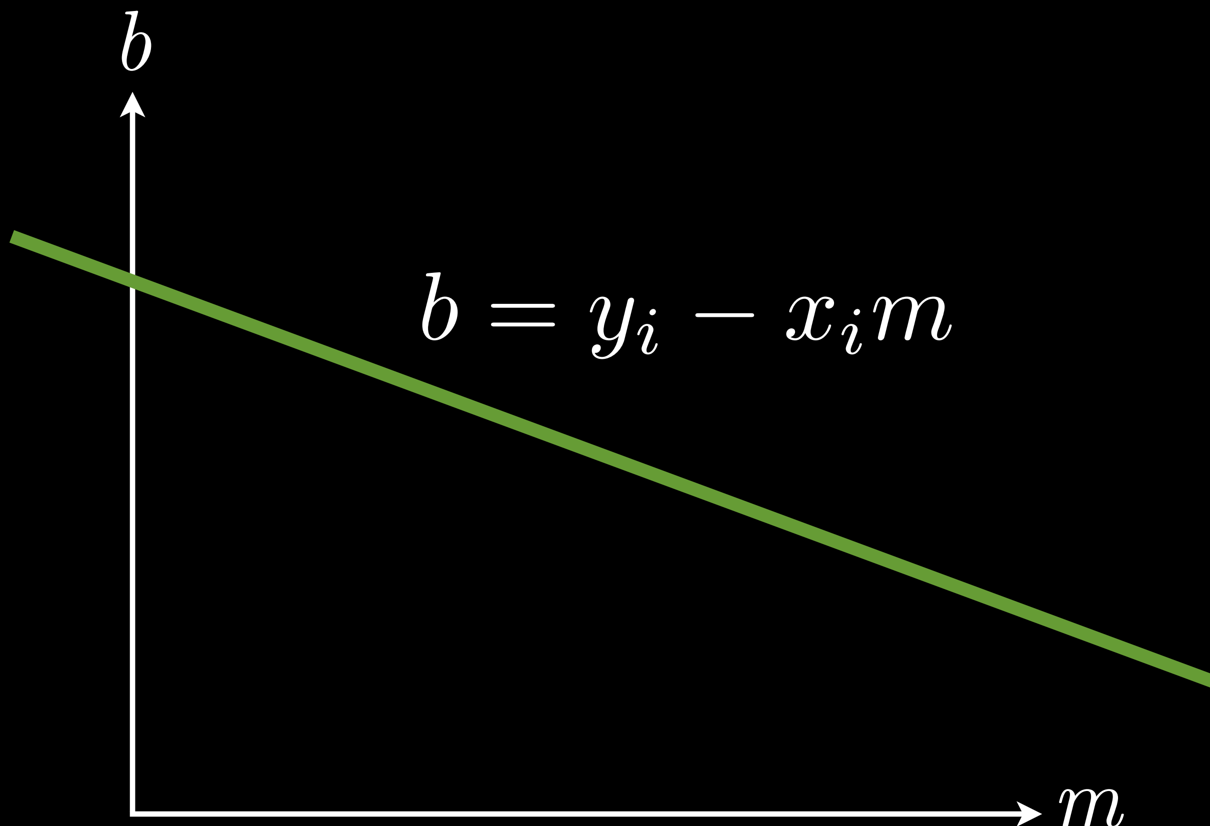

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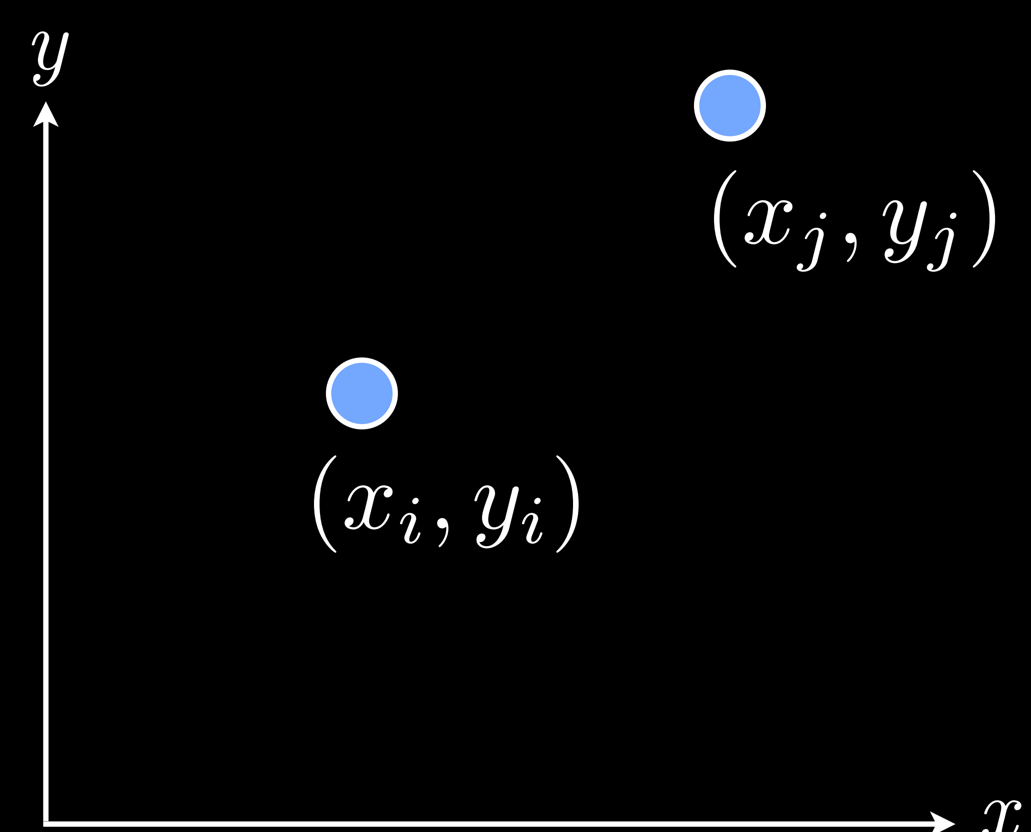

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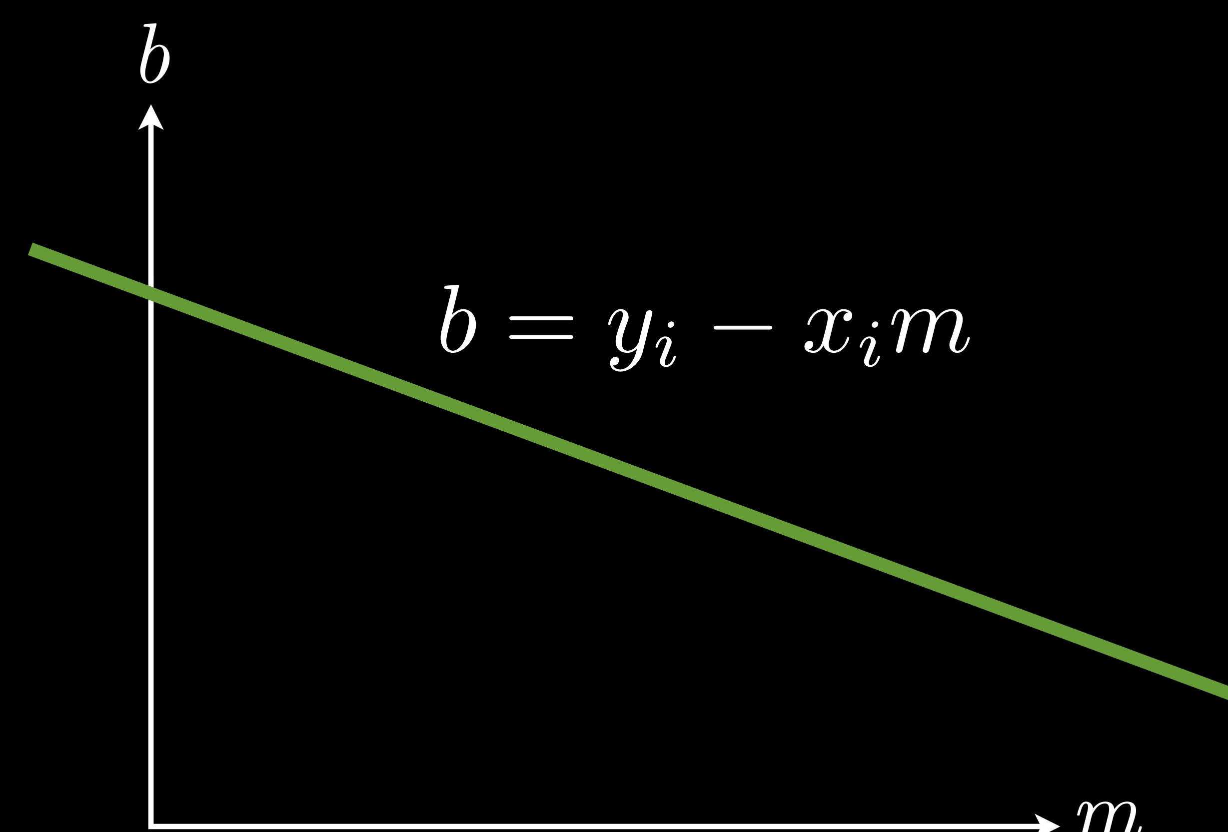

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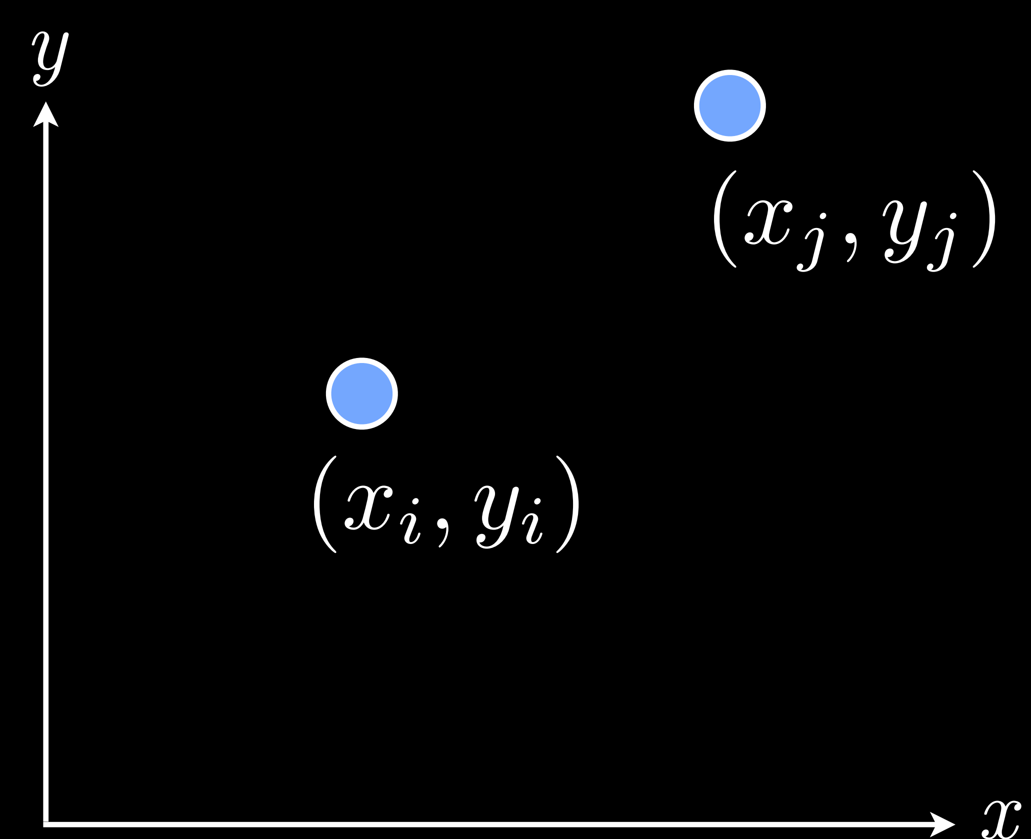

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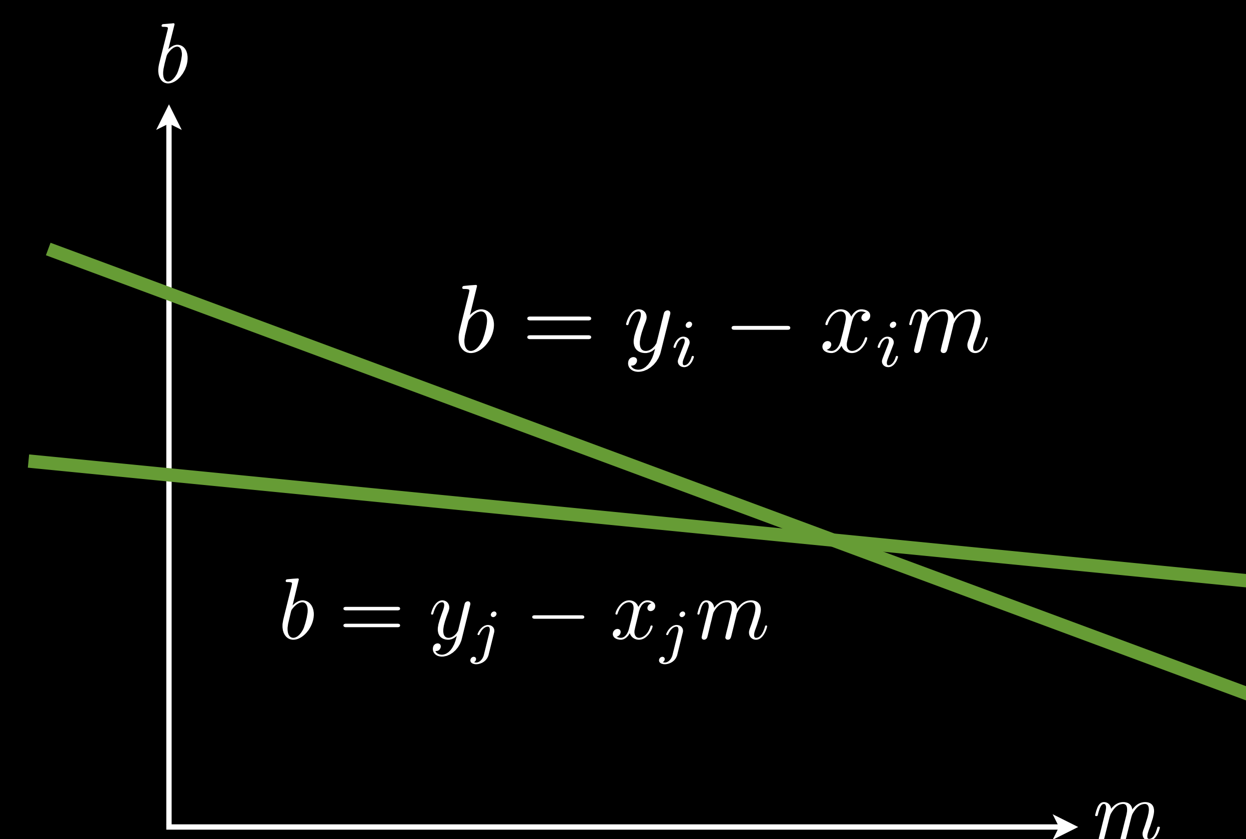

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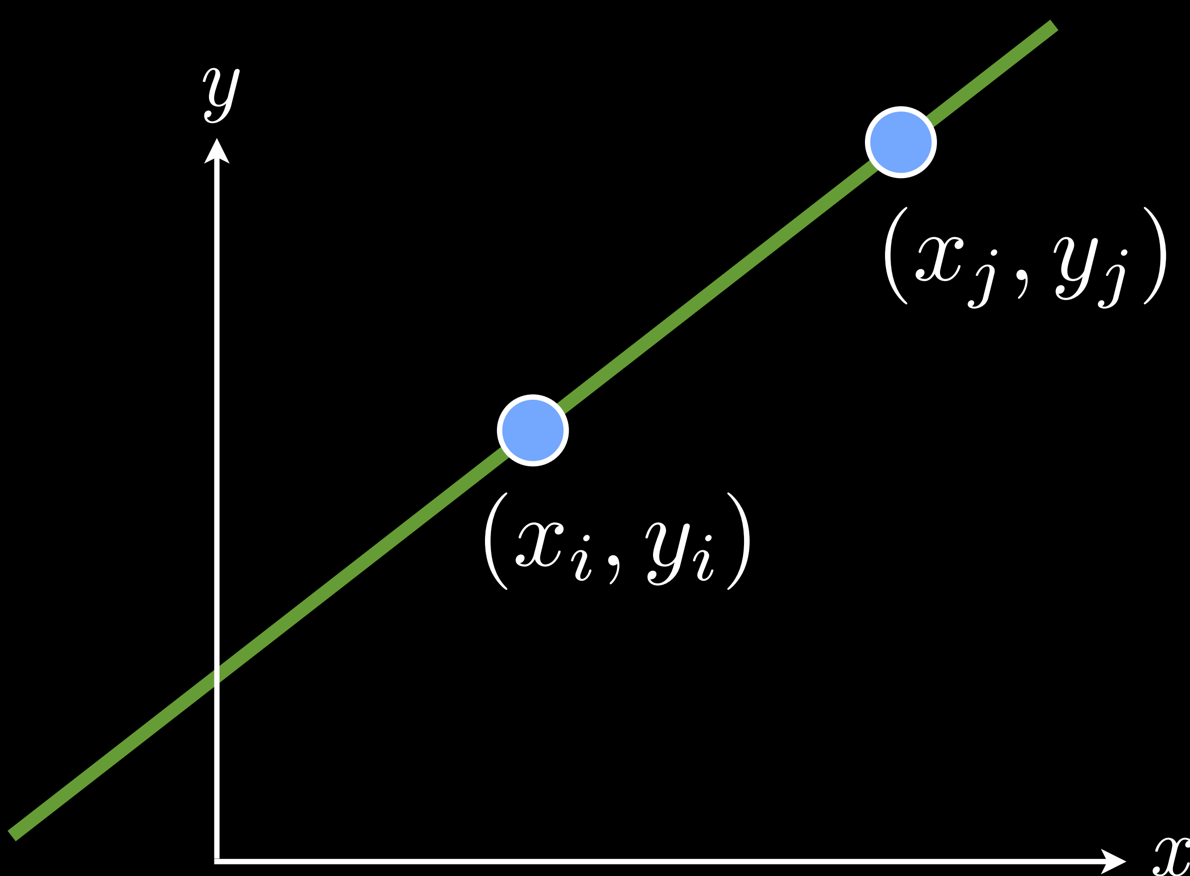

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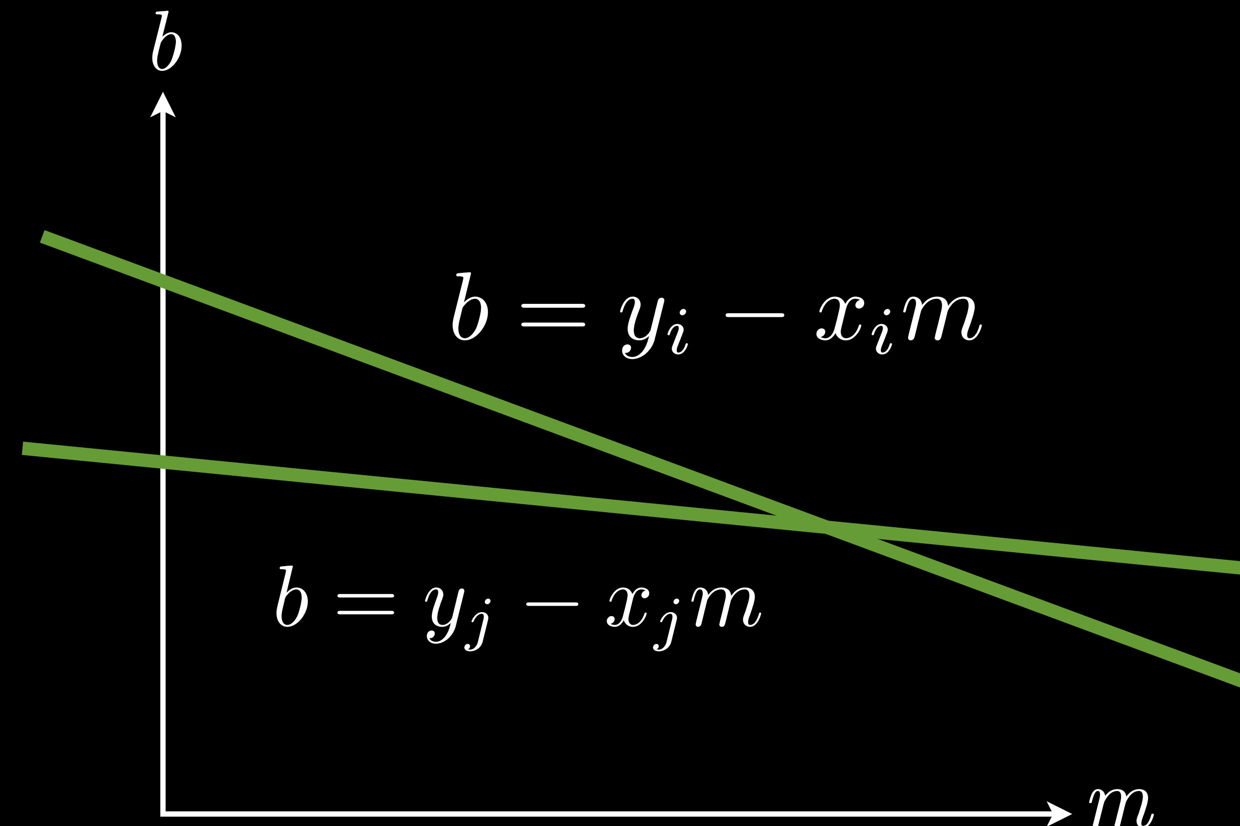

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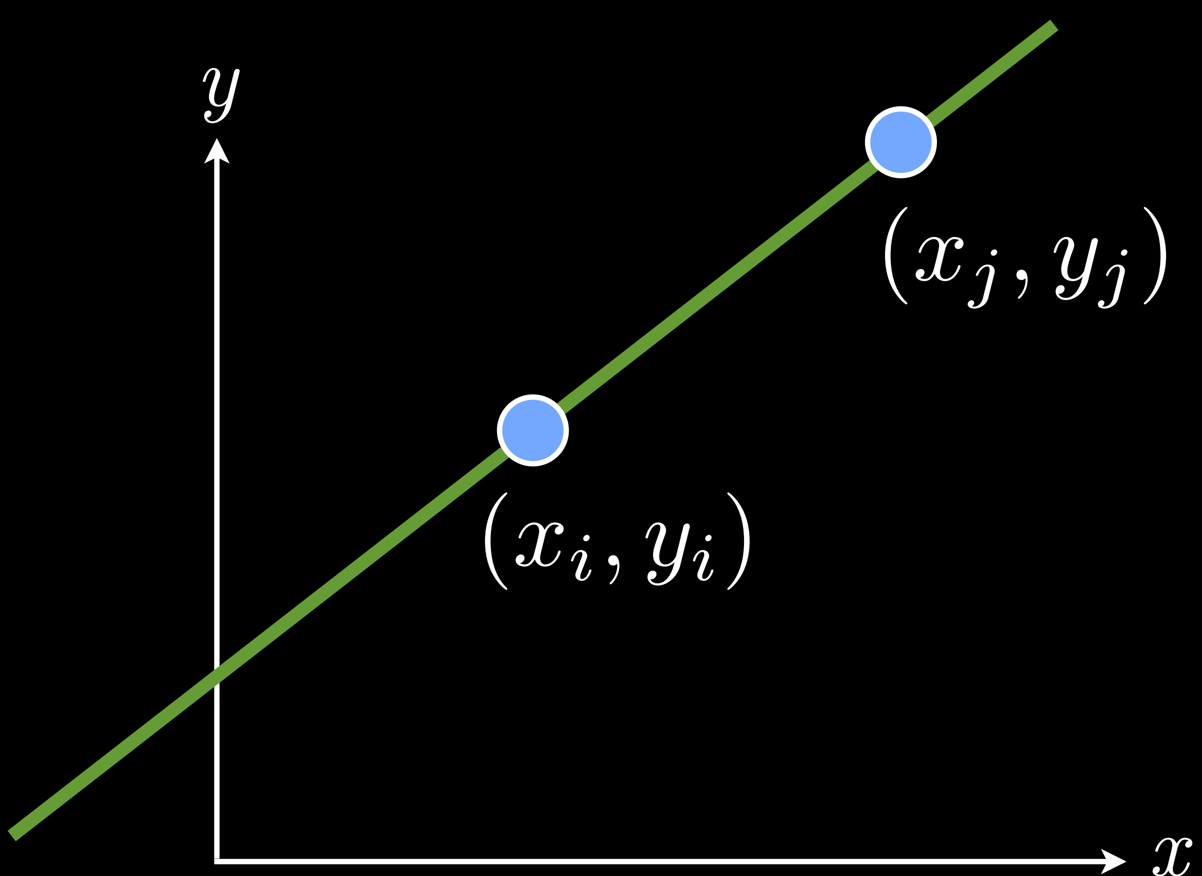

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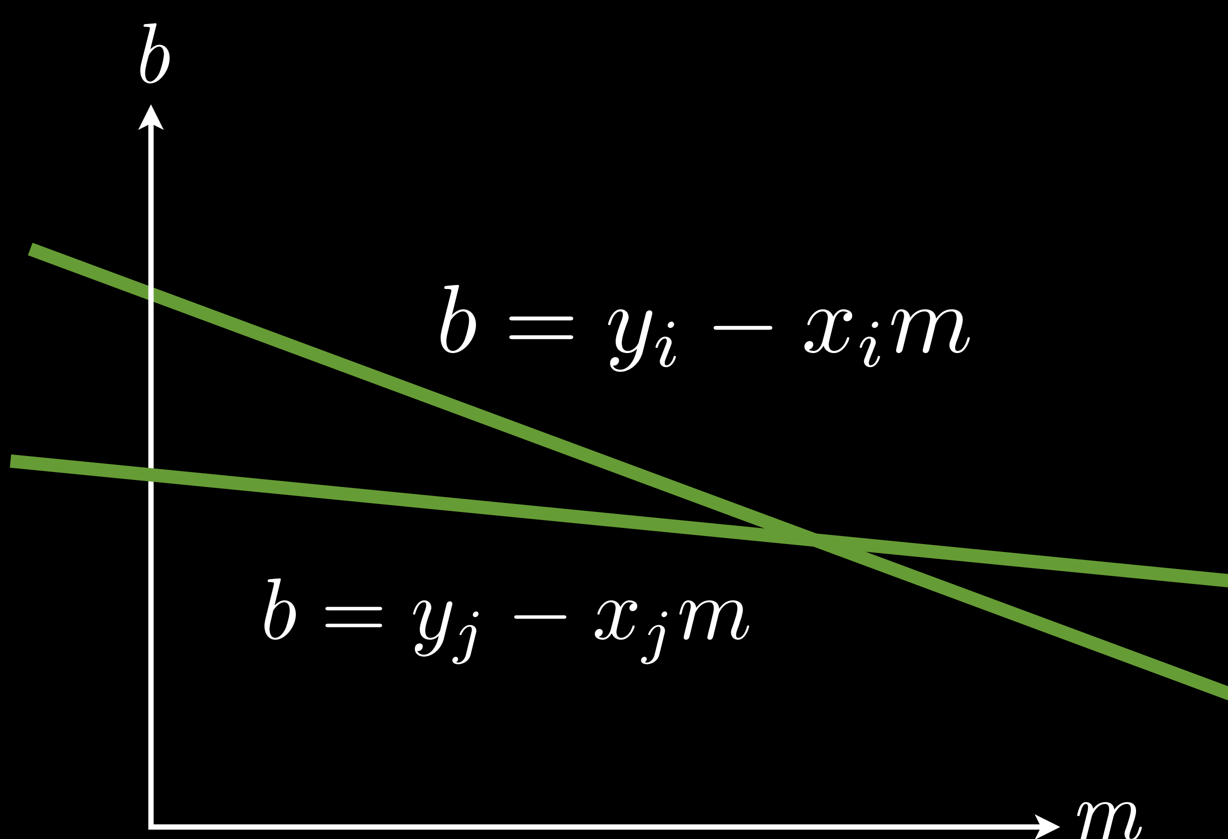

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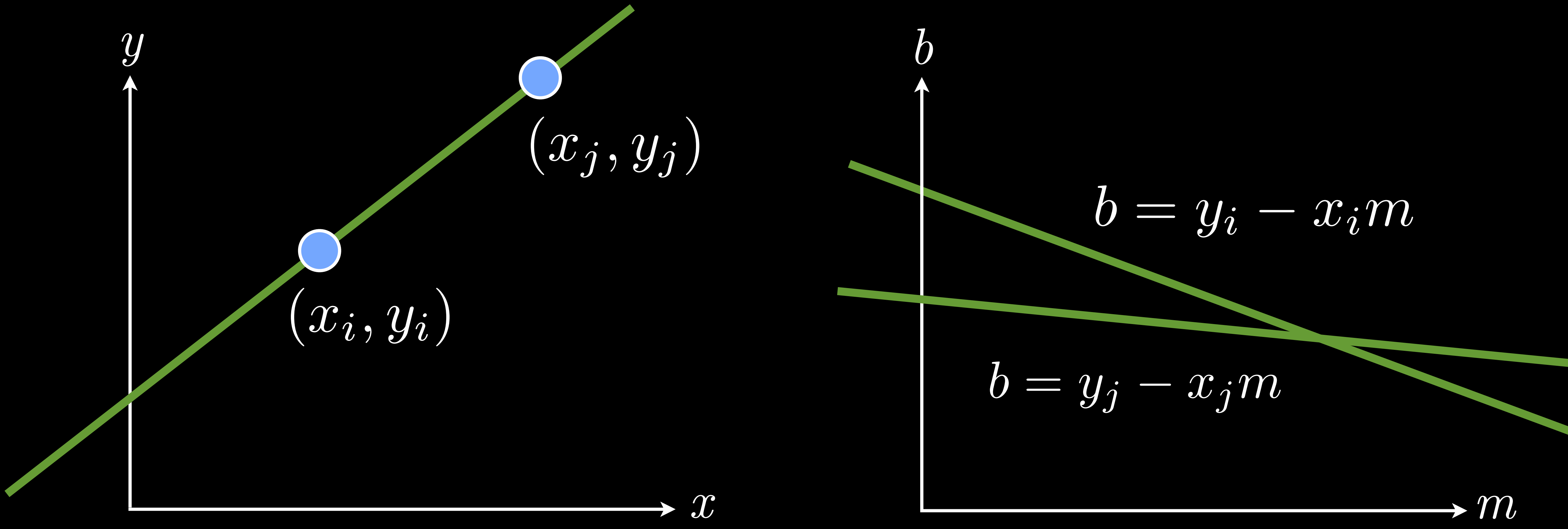

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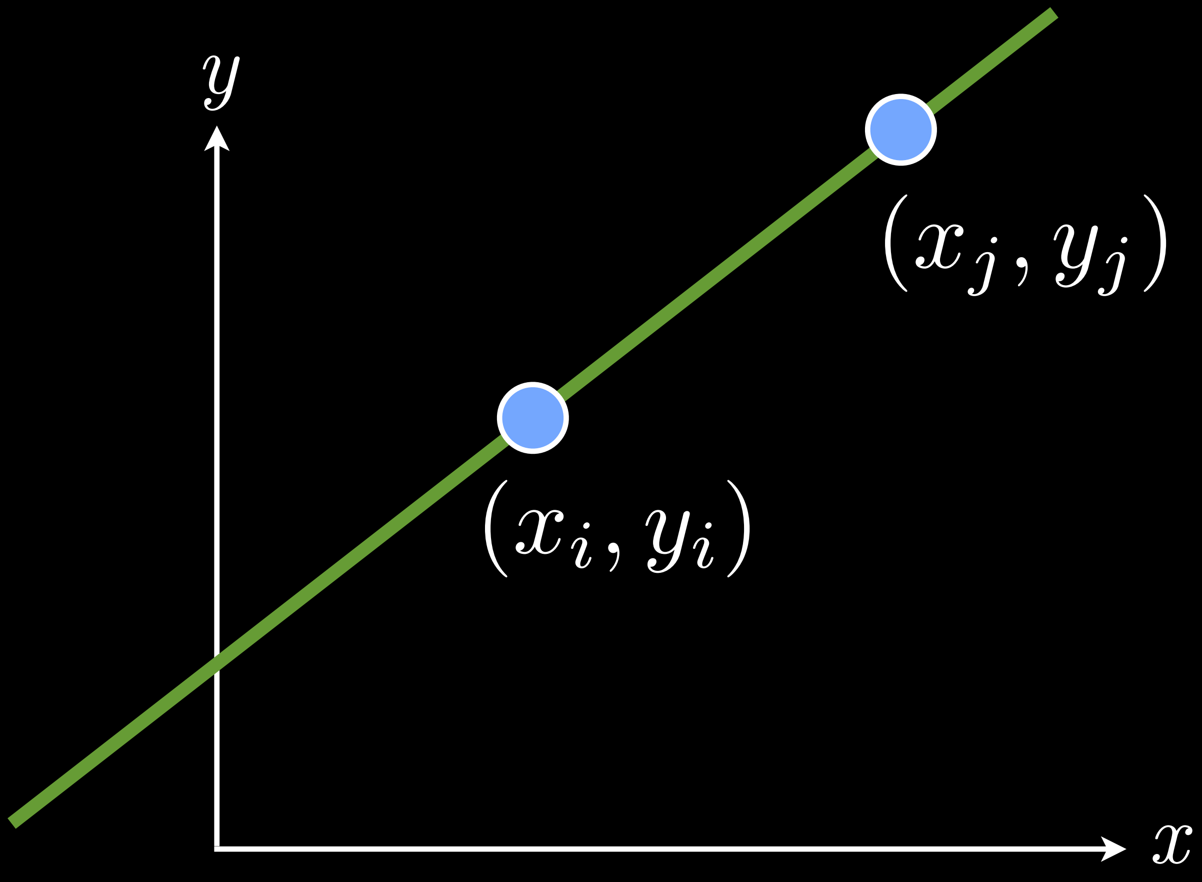
Hough (parameter) space

What are the parameters of the line that contains both points?



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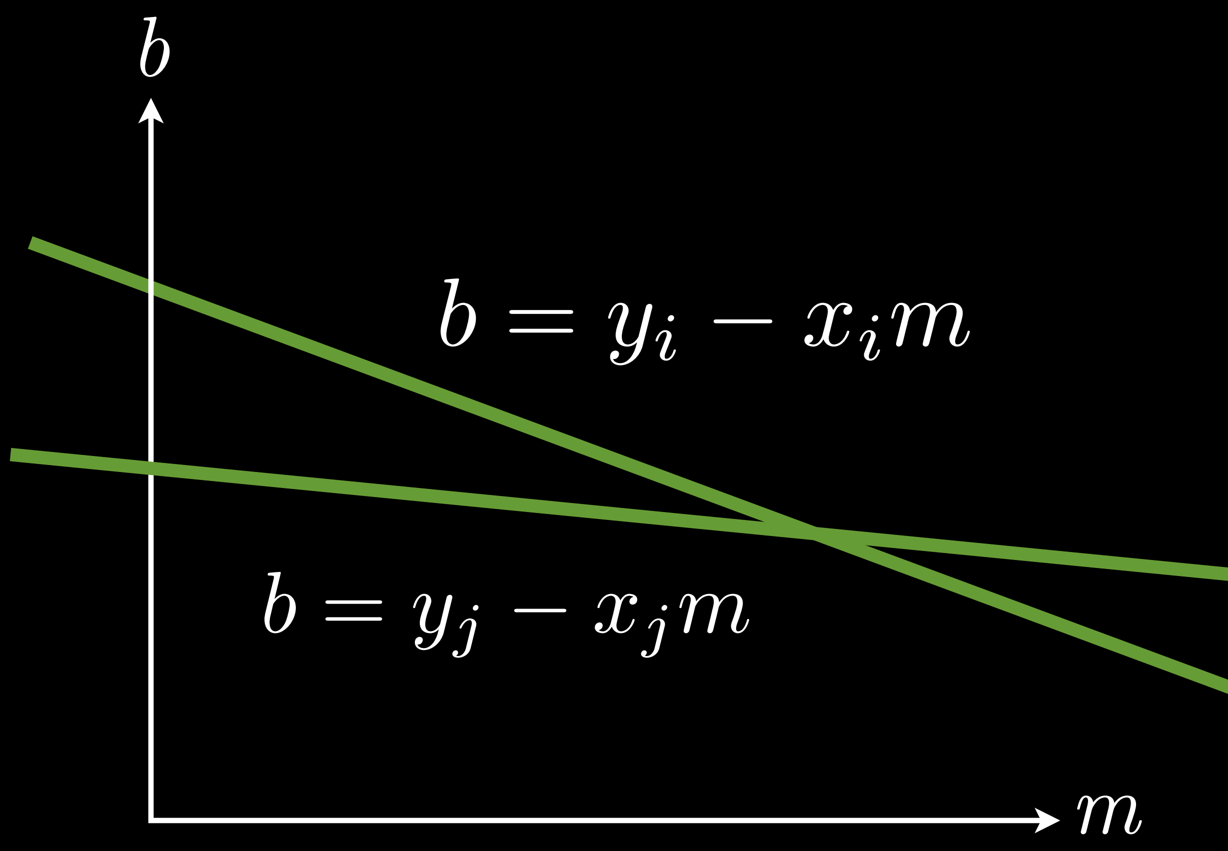
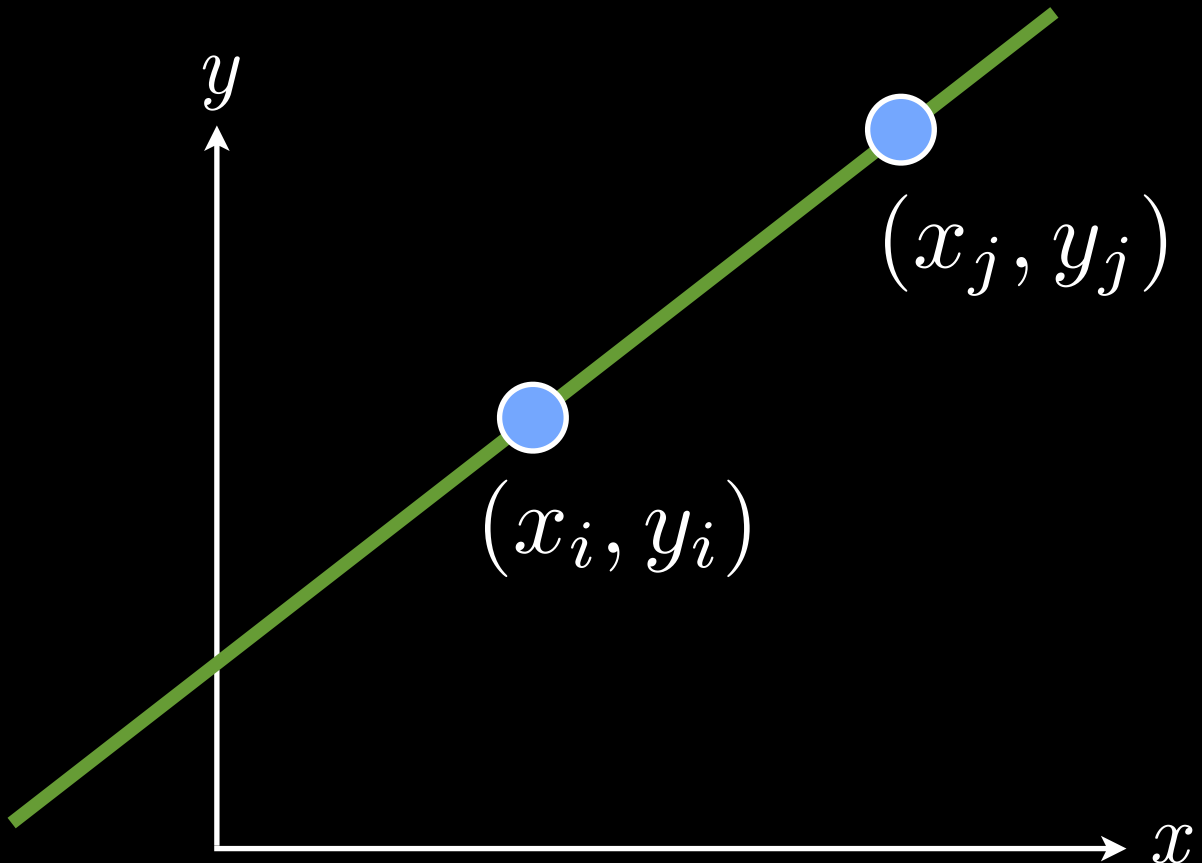


Hough (parameter) space

What are the parameters of the line that contains both points?

Intersection of the two lines in Hough space



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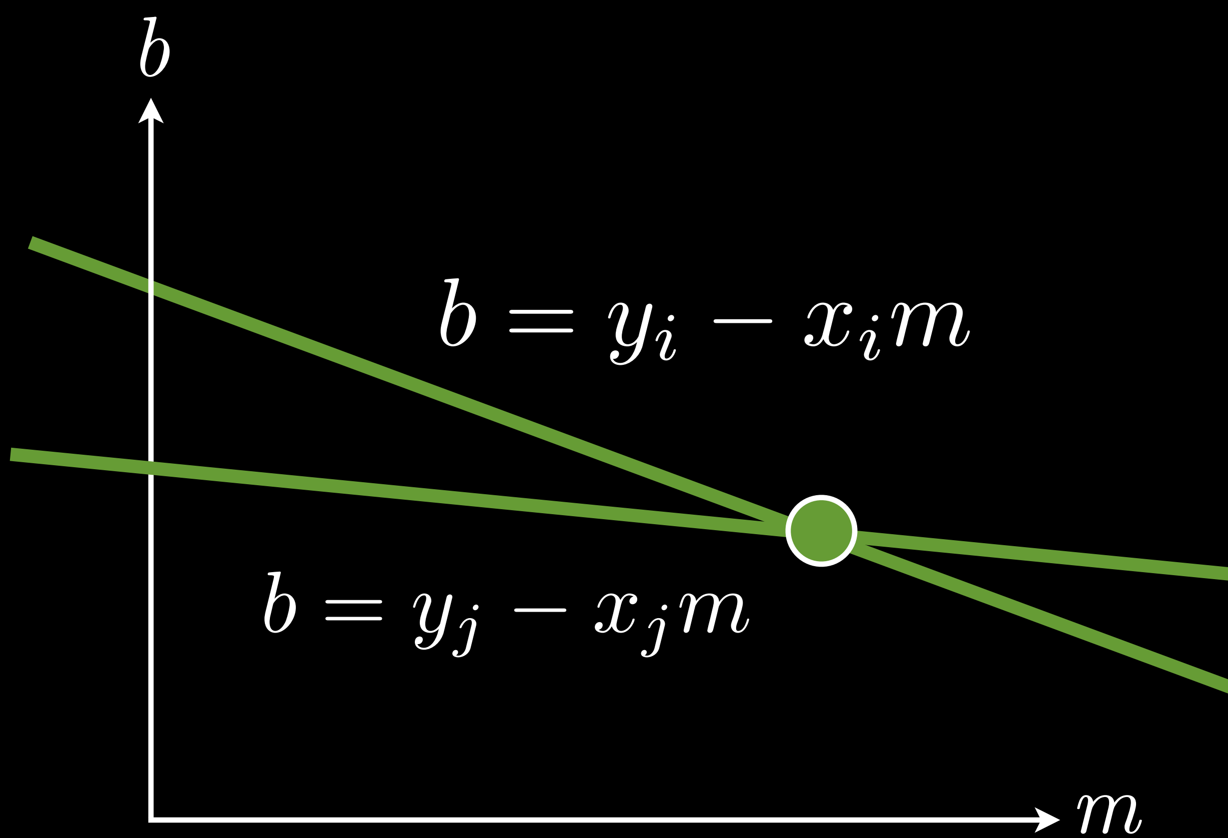

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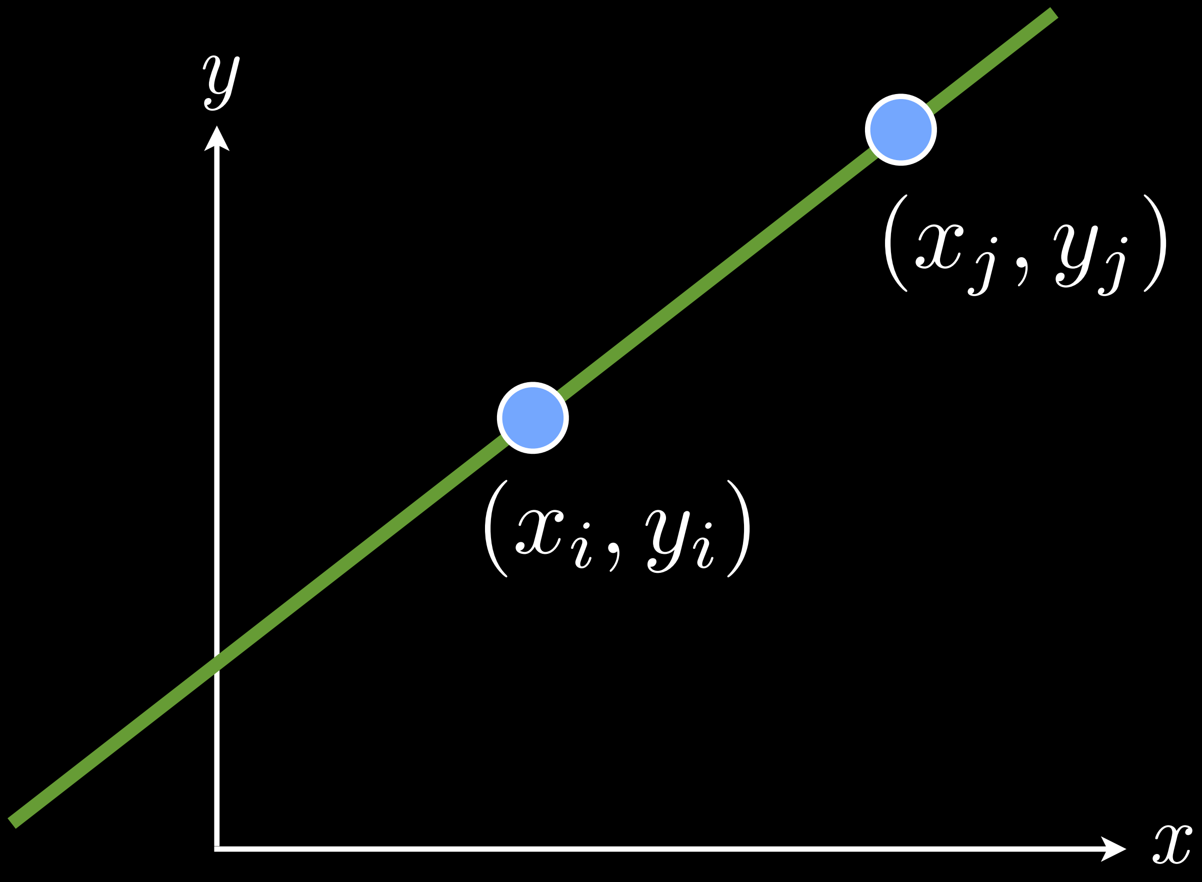

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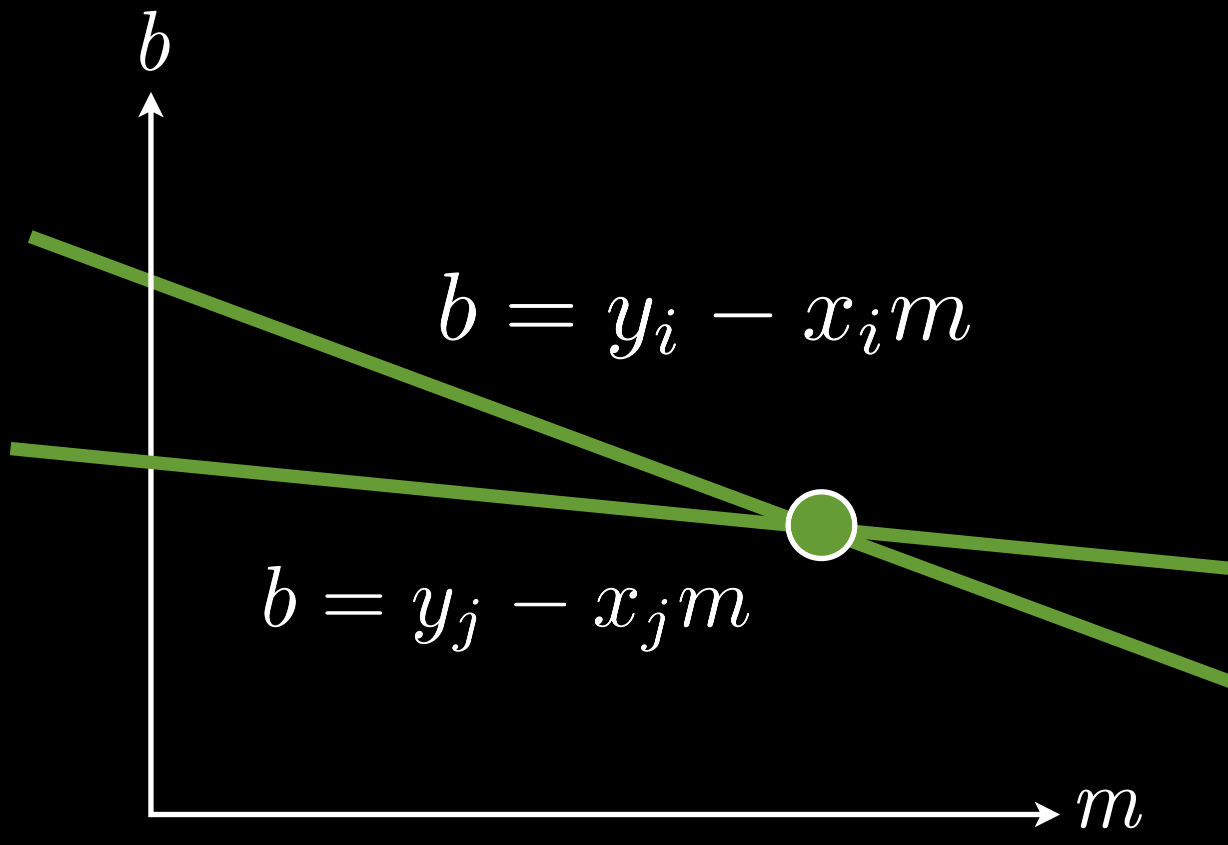

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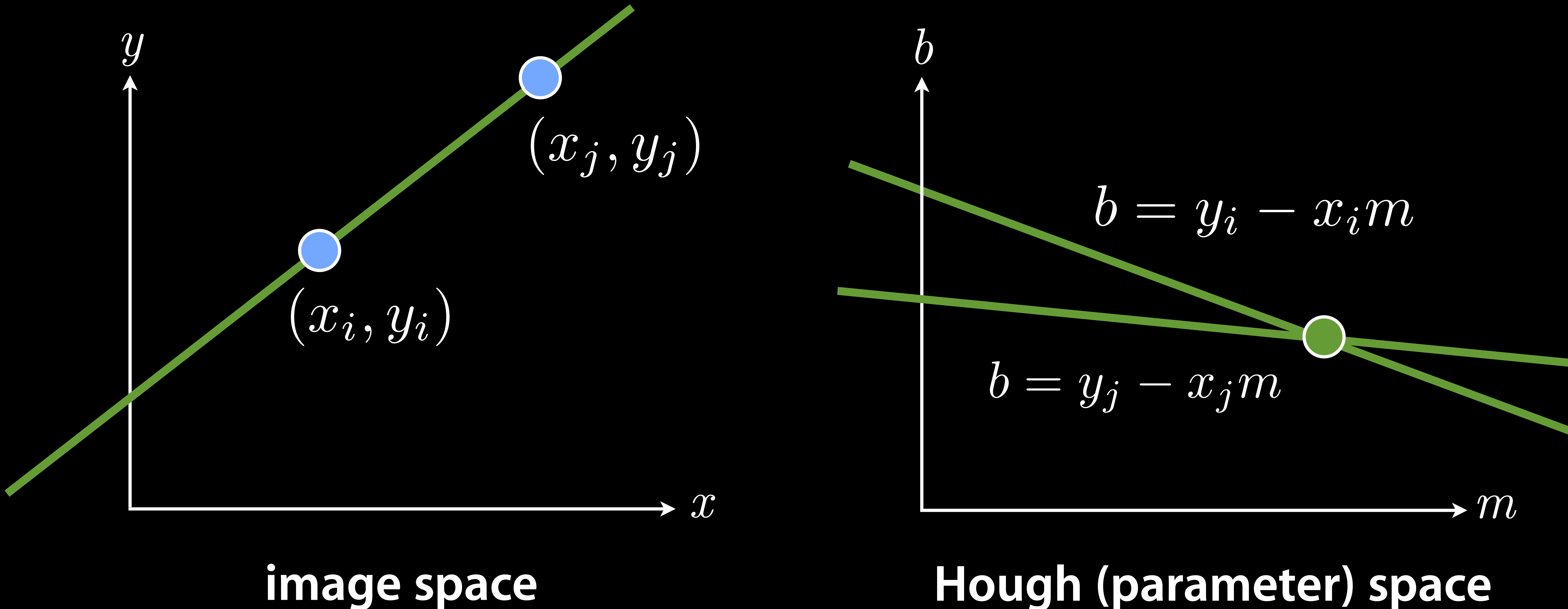
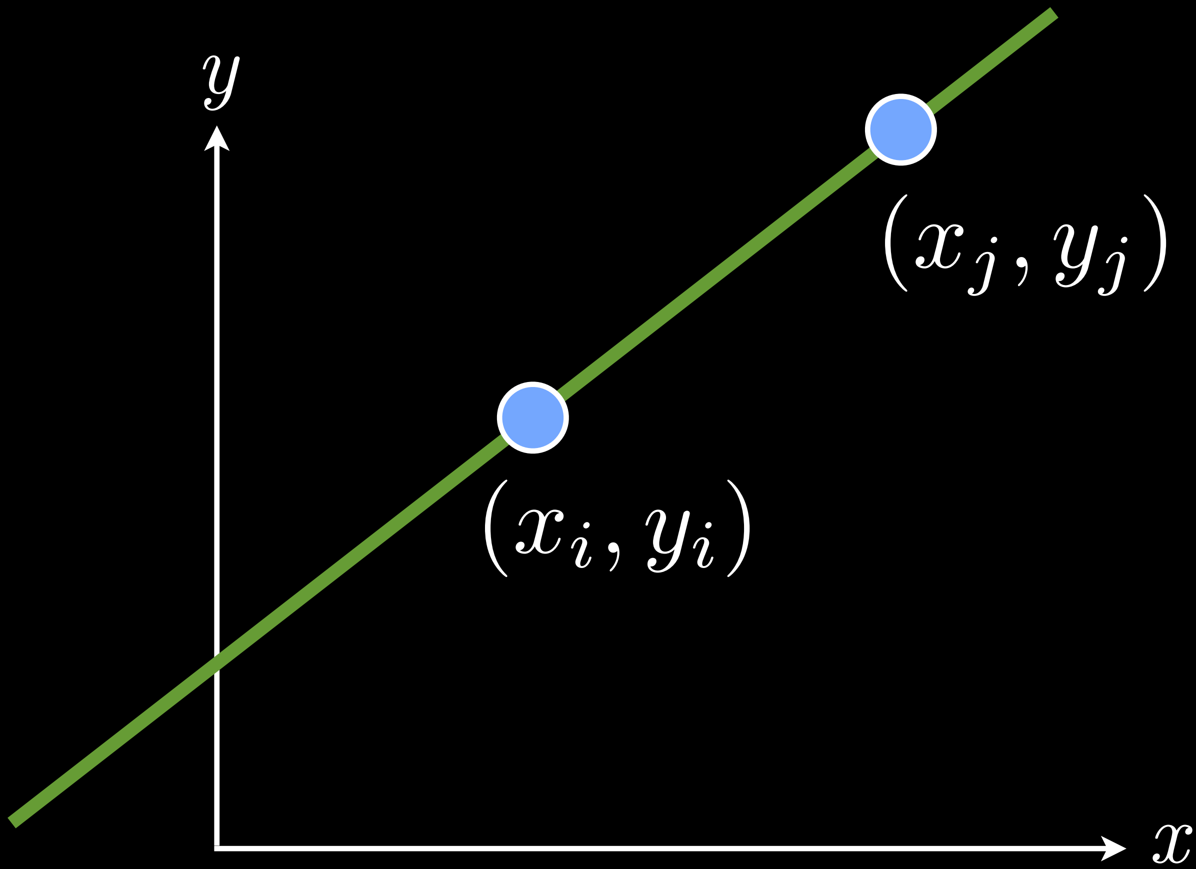


The parameter space is continuous!
How can we handle this?



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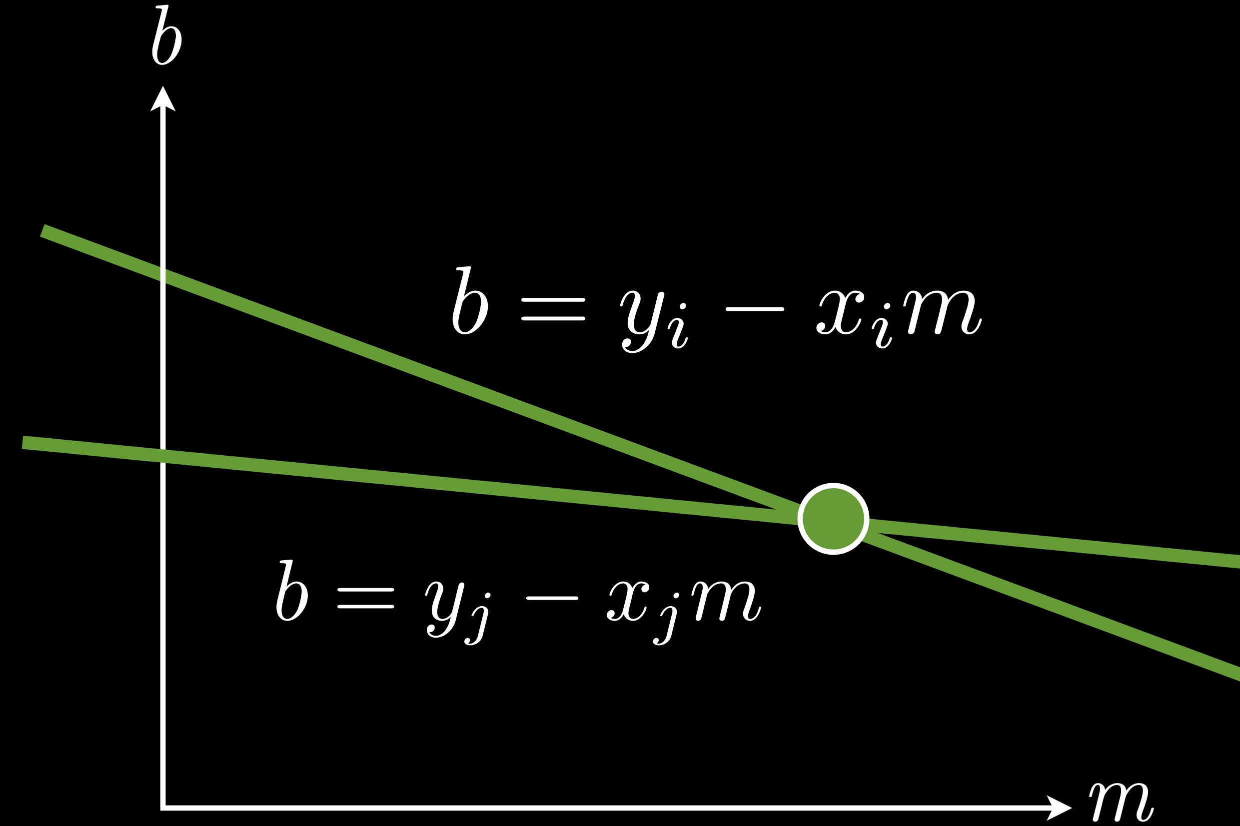
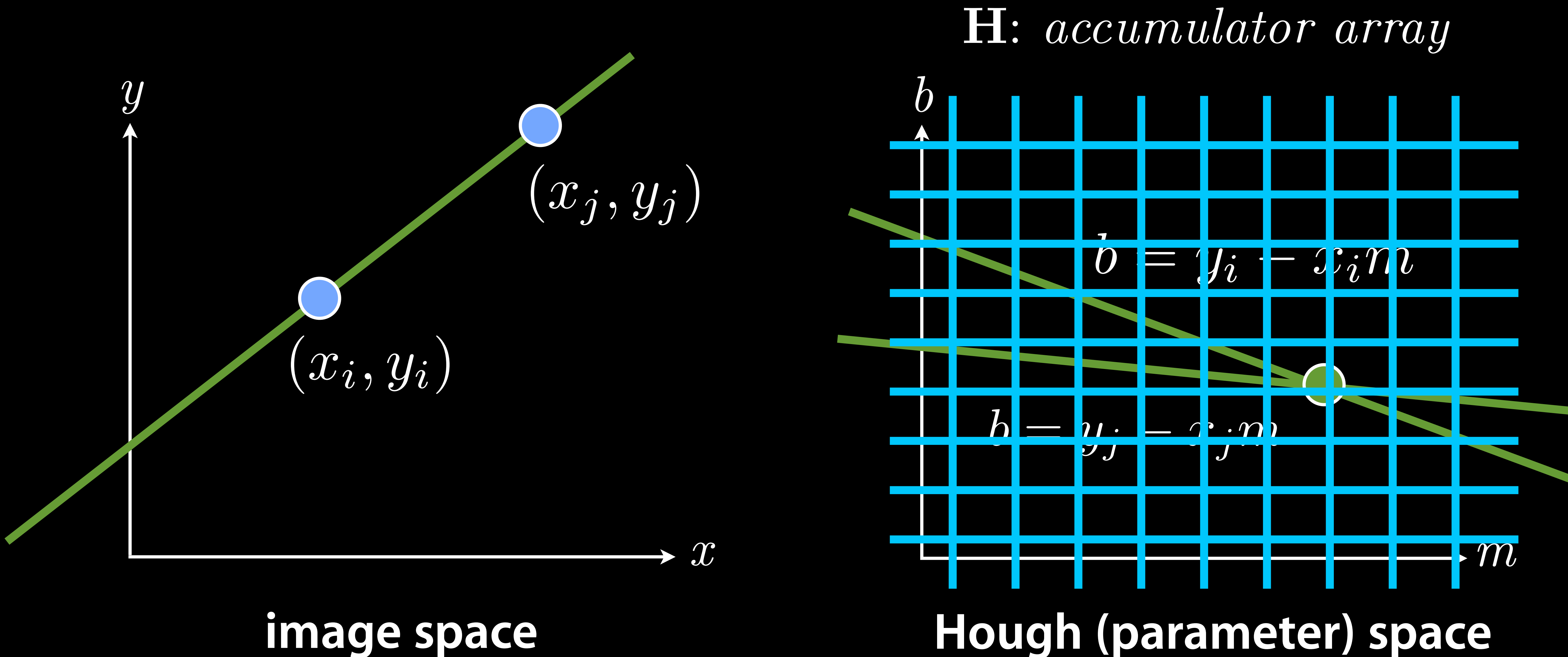


Hough (parameter) space

The parameter space is continuous!
How can we handle this?

Accumulate votes in discrete set of bins



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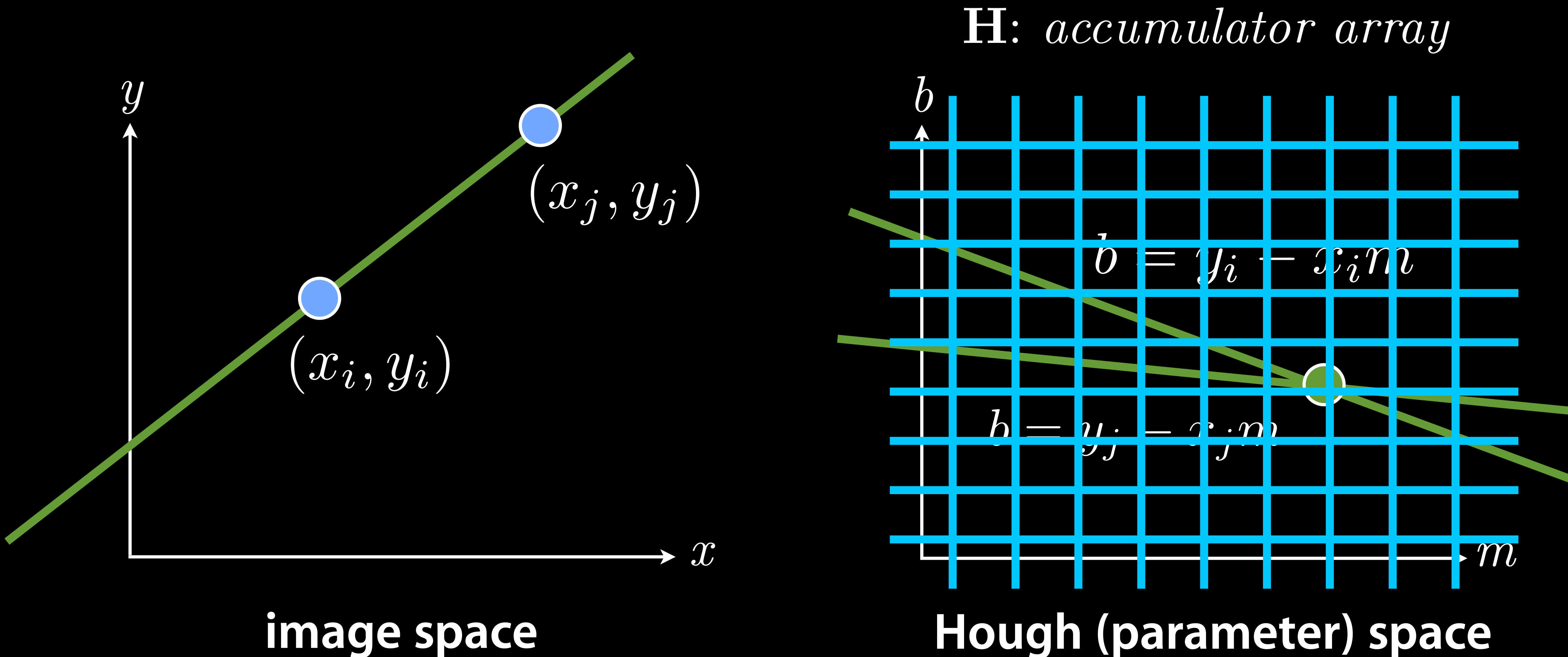


Local peaks correspond to detected lines

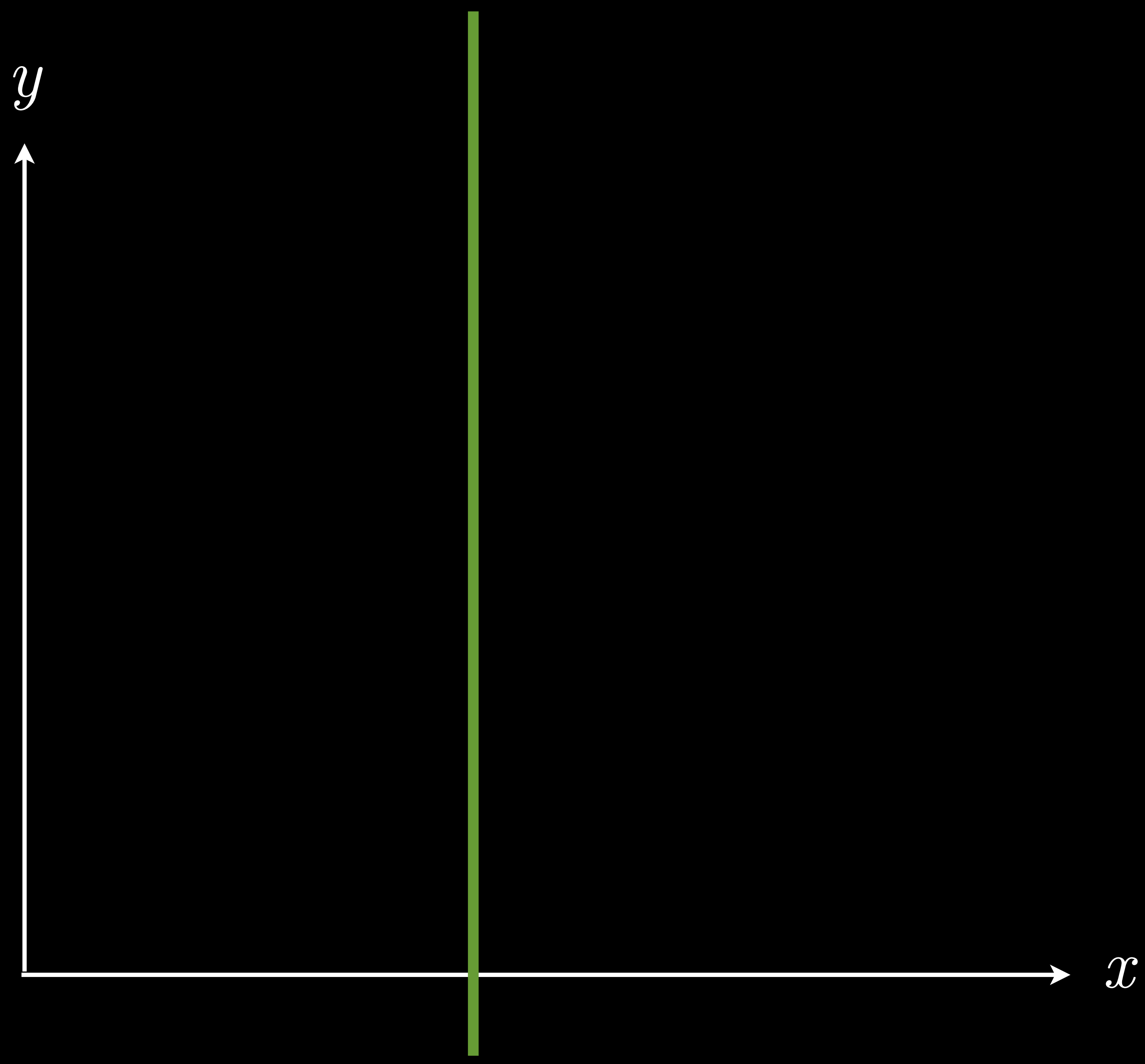
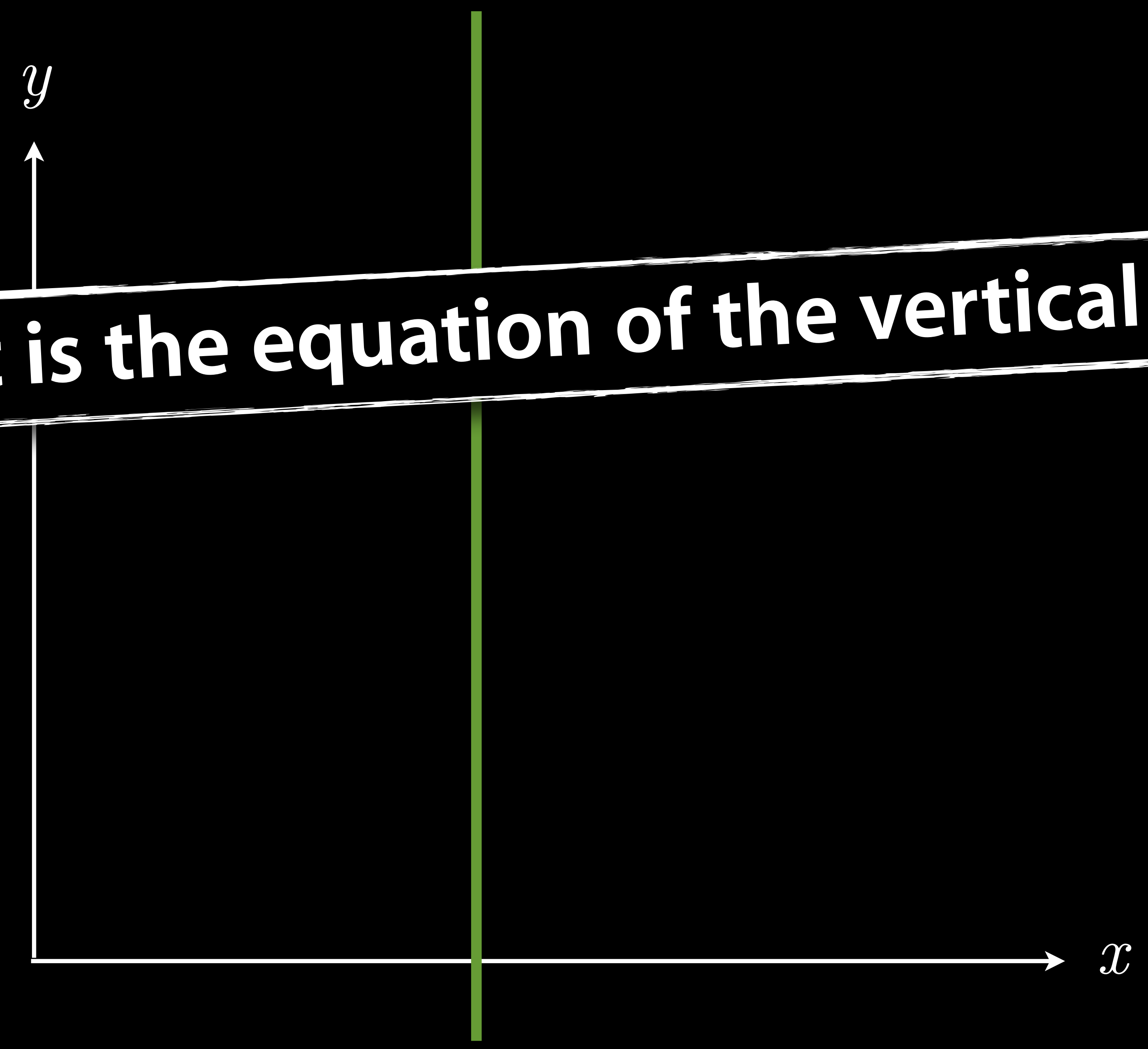


image space

Slope-Intercept

What is the equation of the vertical line?



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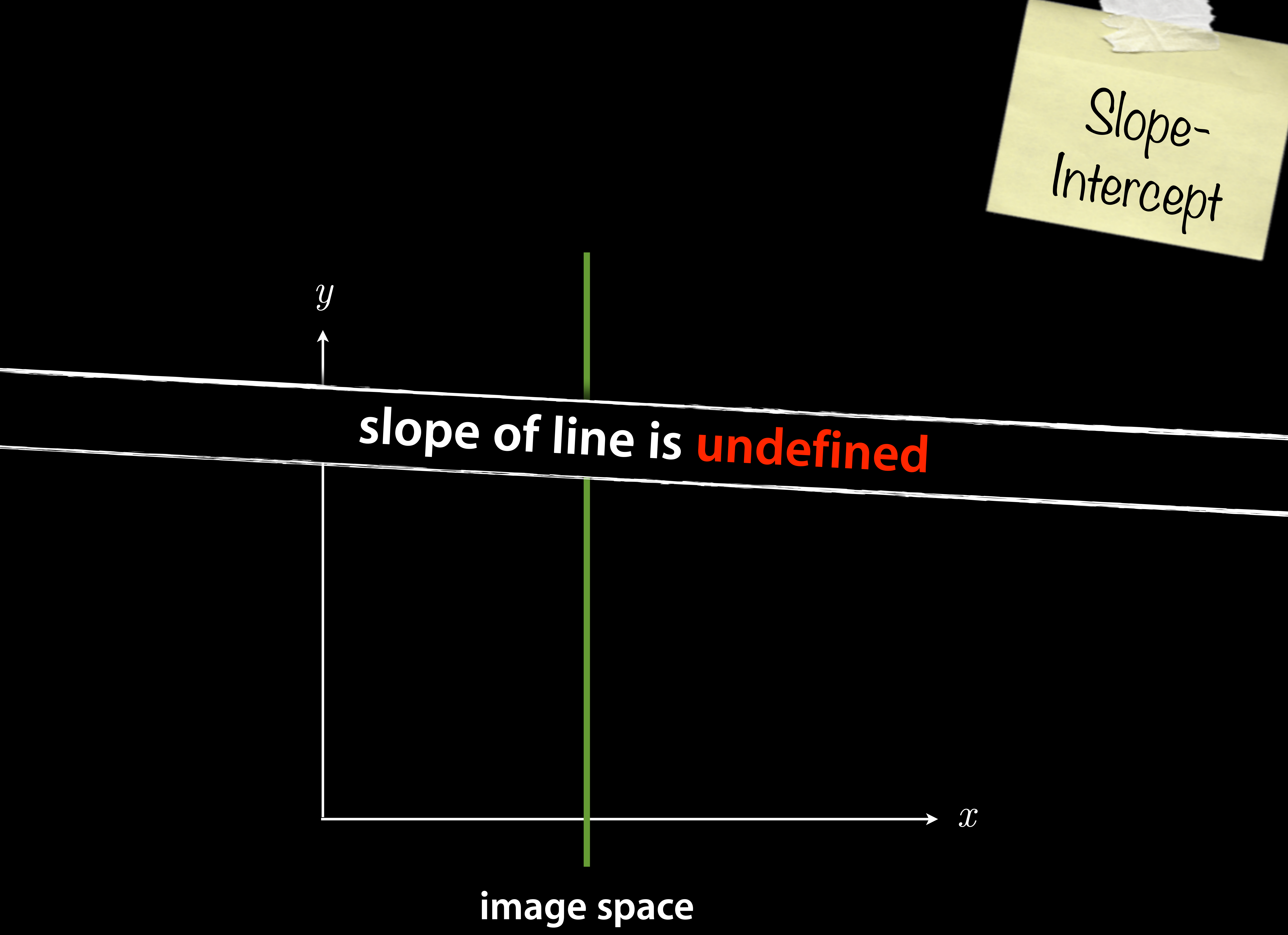

Slope-Intercept
y
slope of line is undefined
x
image space

Slope-Intercept

parameter space $(m,\ b)$ unbounded

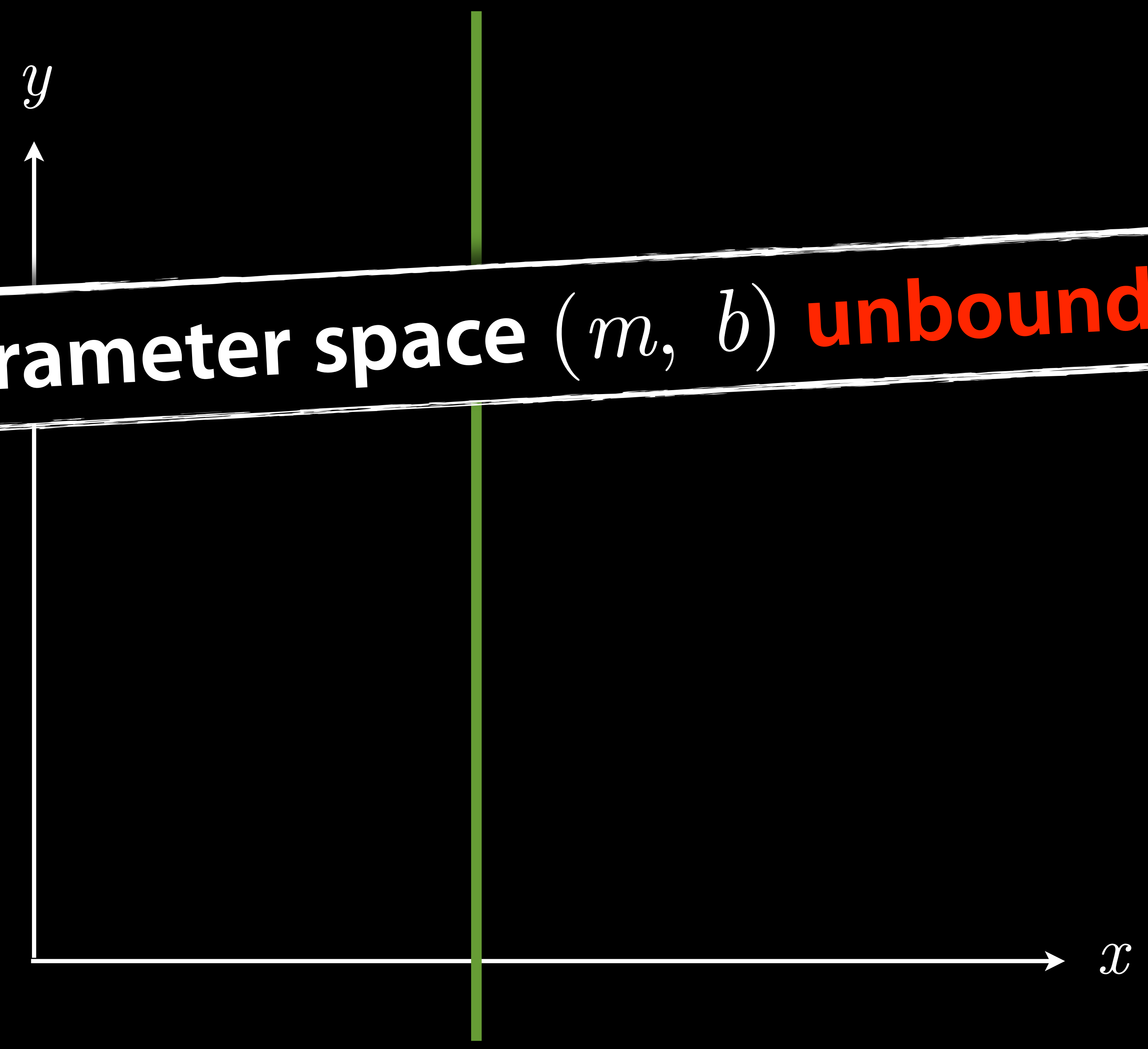
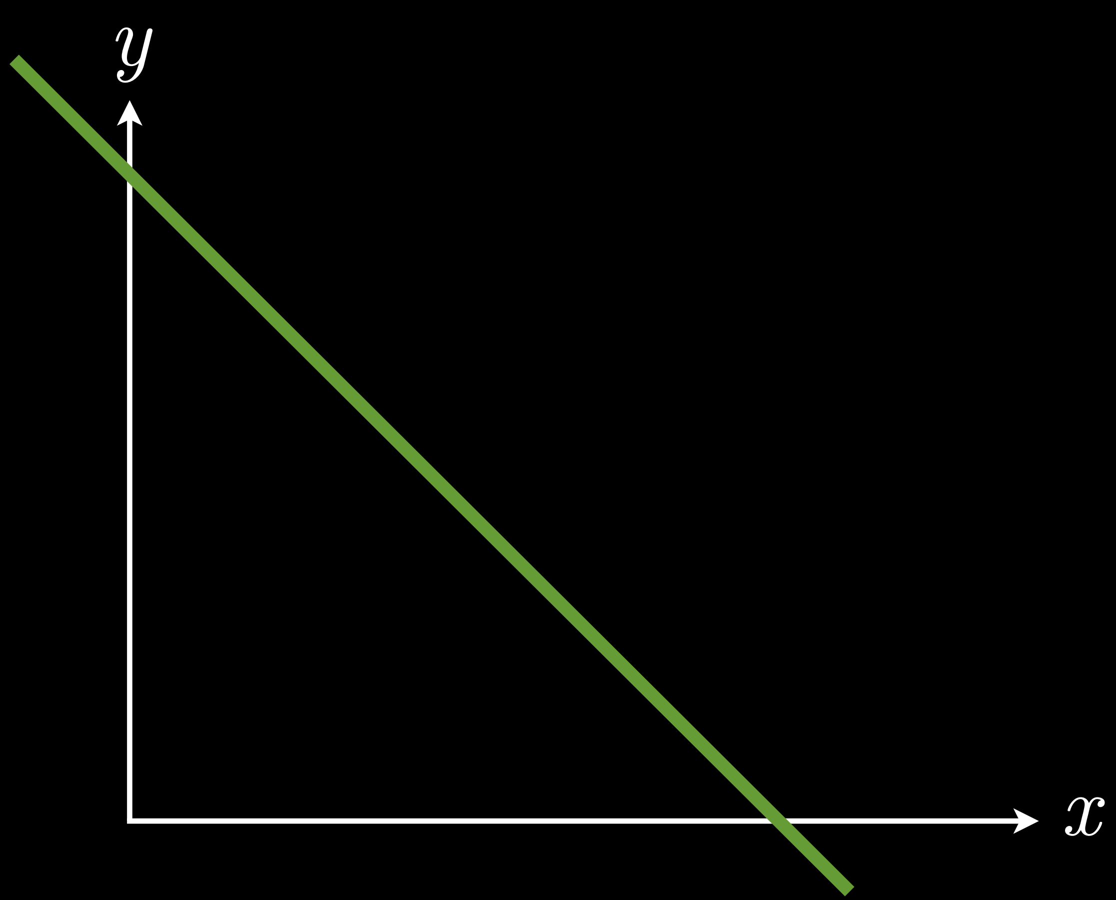


image space

Polar Representation



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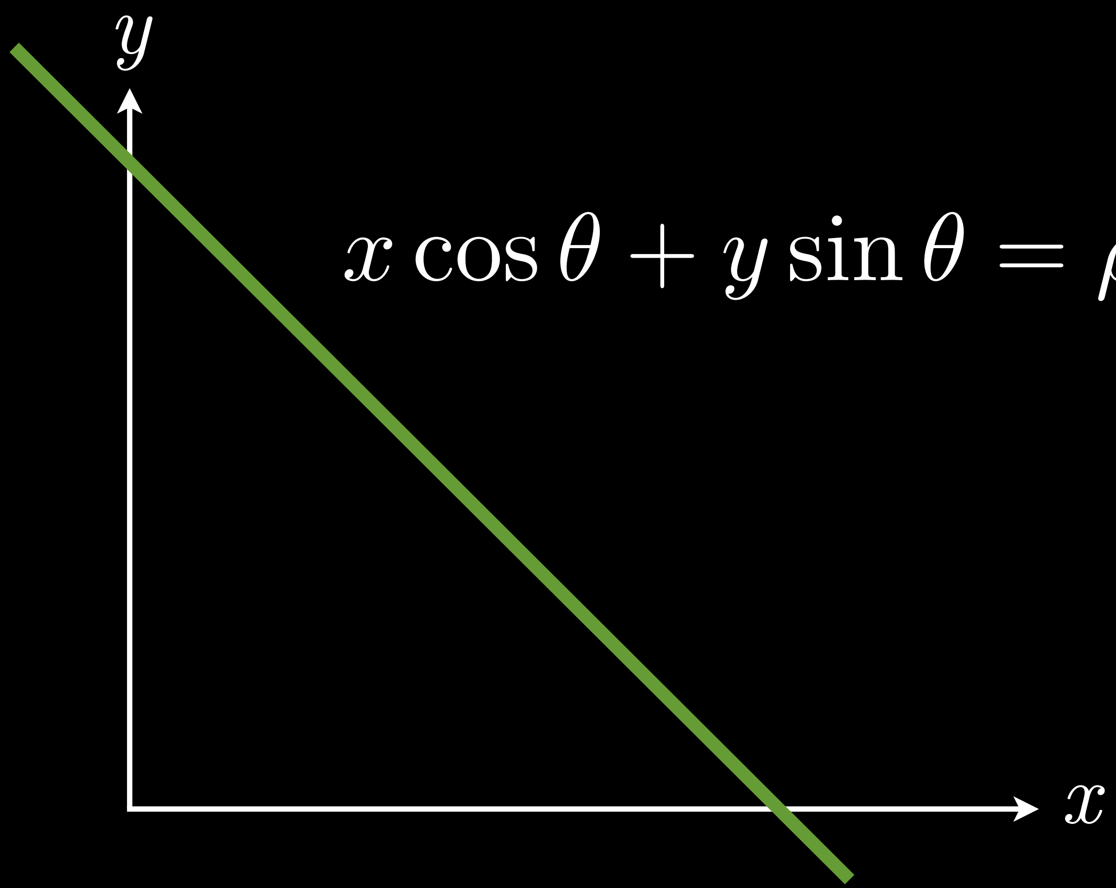

$$x\cos\theta + y\sin\theta = \rho$$

image space

Polar Representation

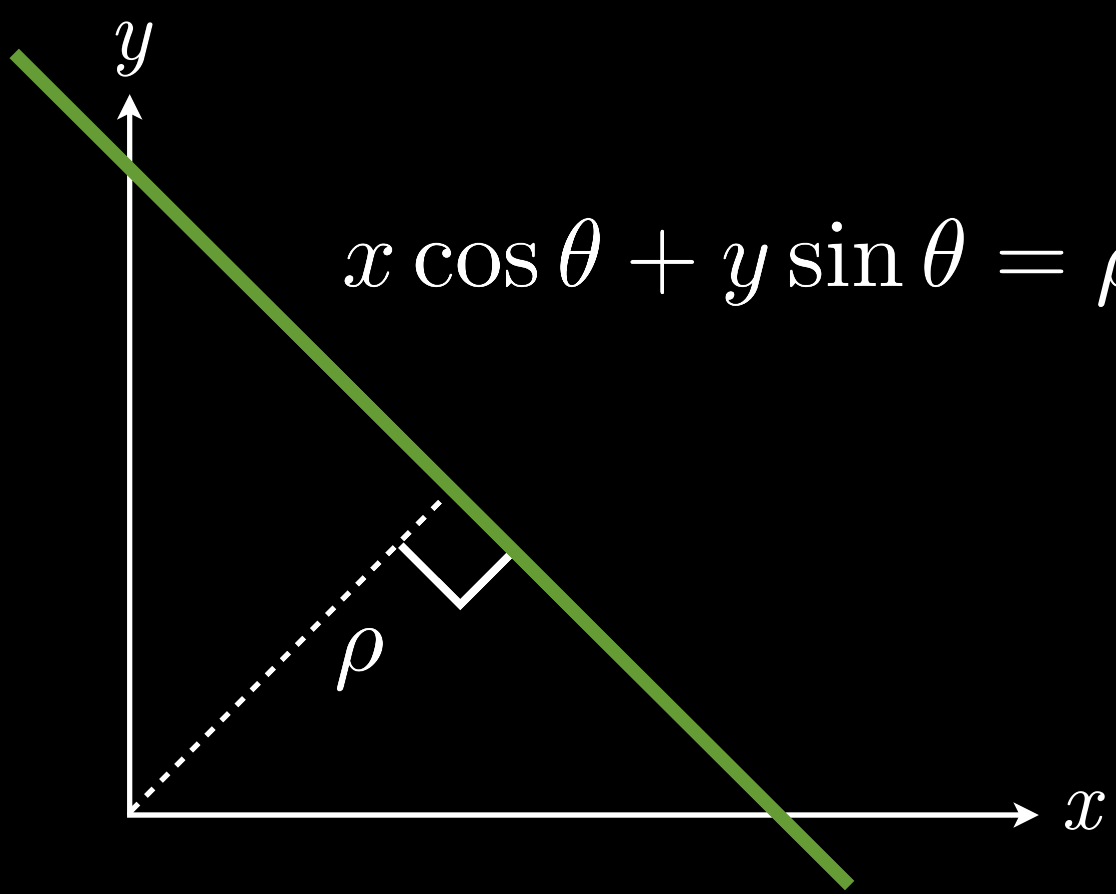

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image space

Polar Representation

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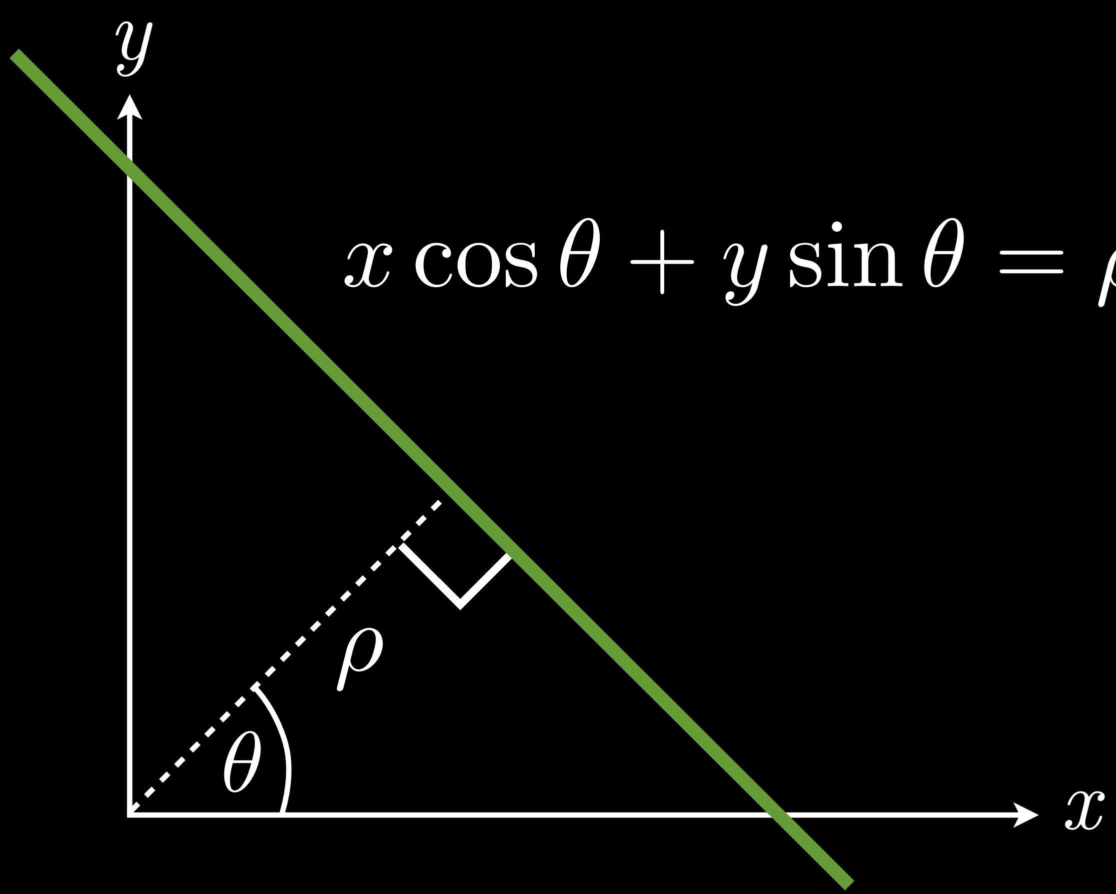

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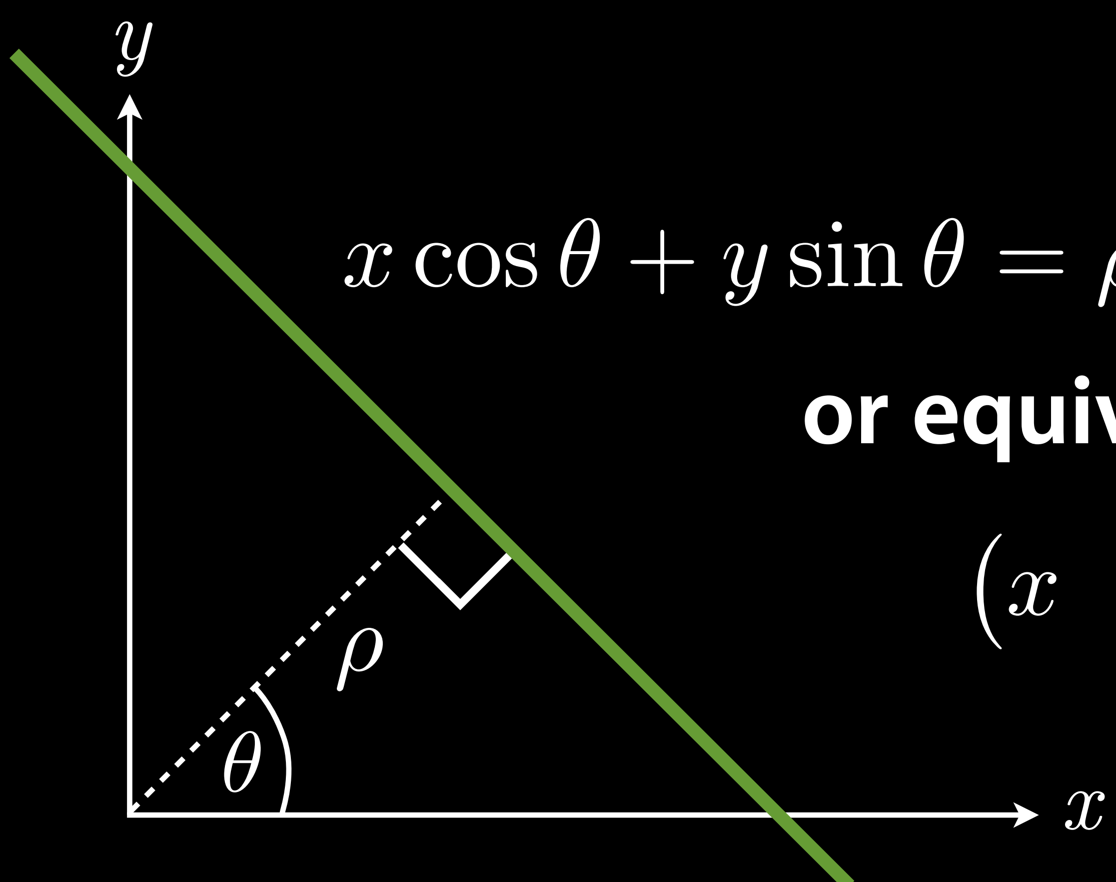
Polar Representation



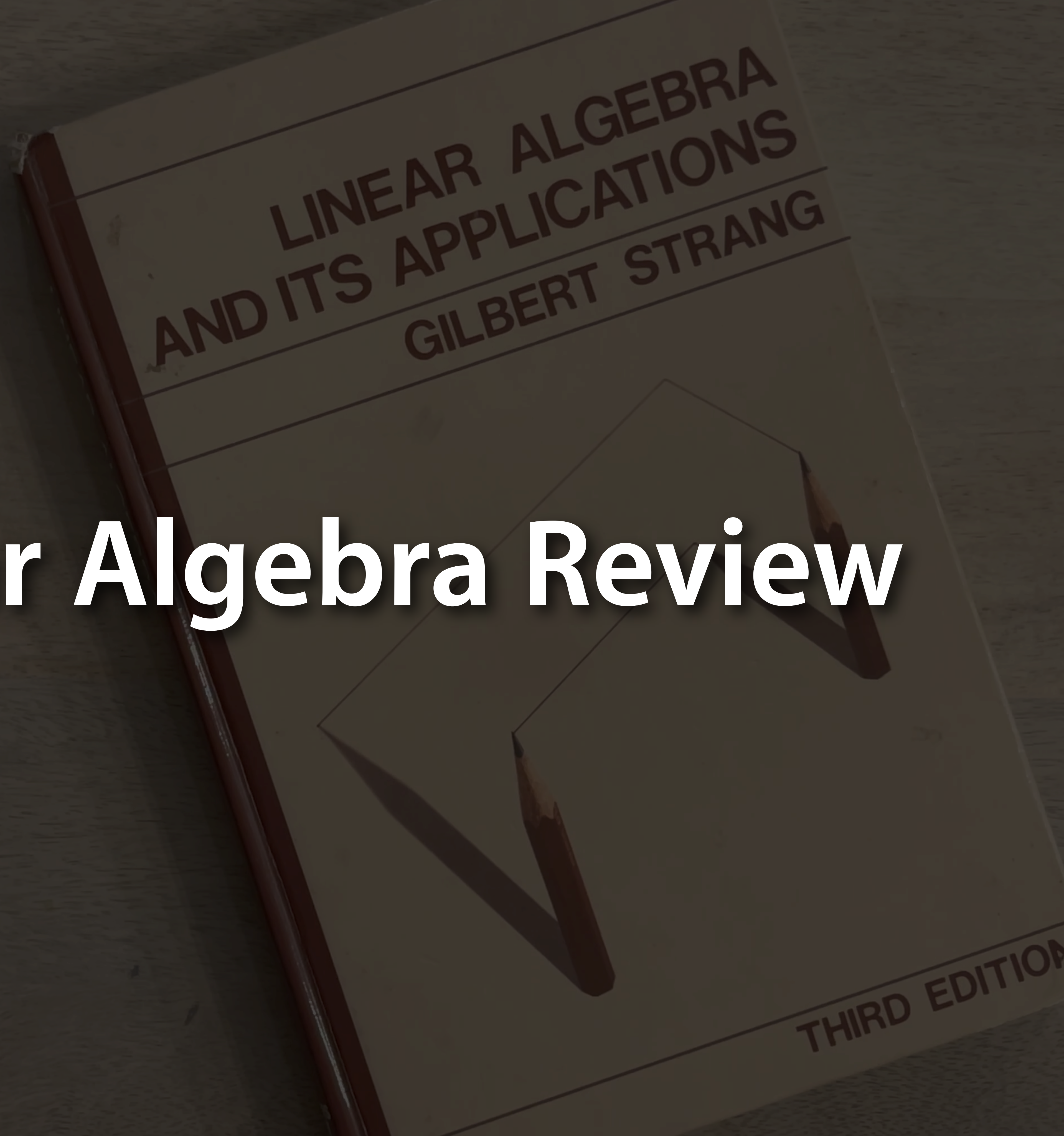
$$x \cos\theta + y \sin\theta = \rho$$

or equivalently

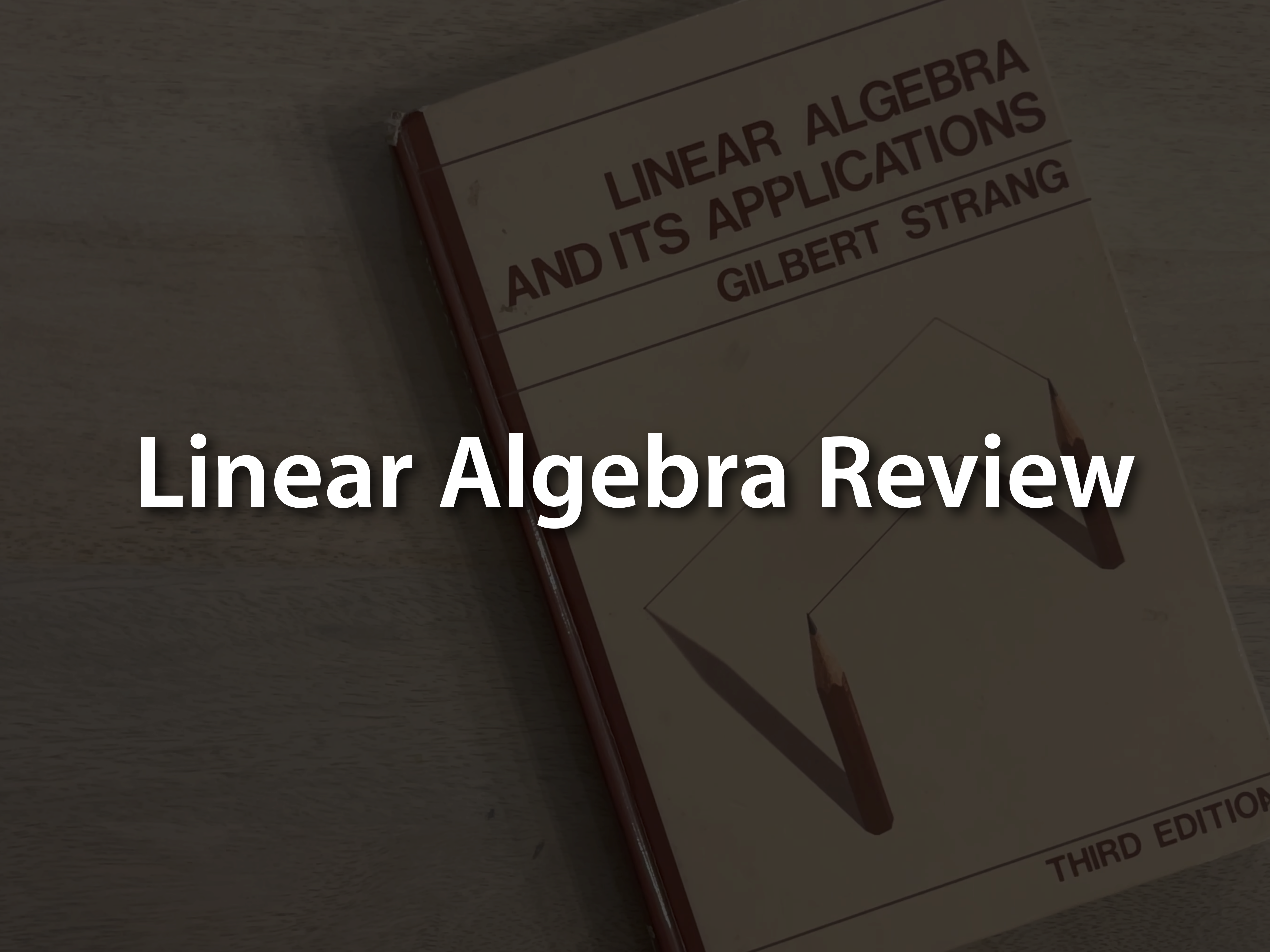
$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}^\top = \rho$$

image space

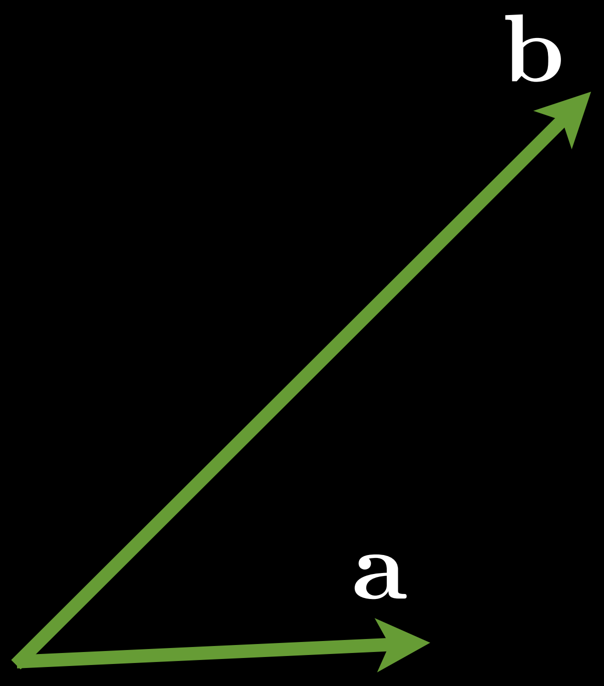
Linear Algebra Review

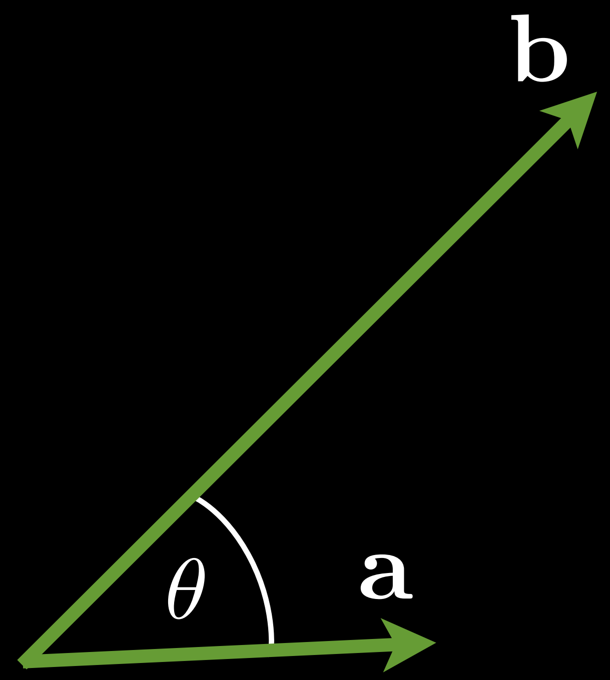


Linear Algebra Review

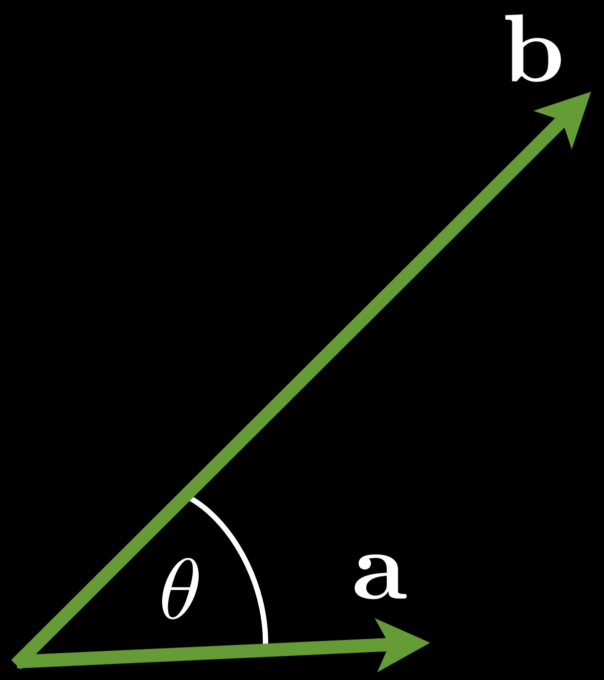



a

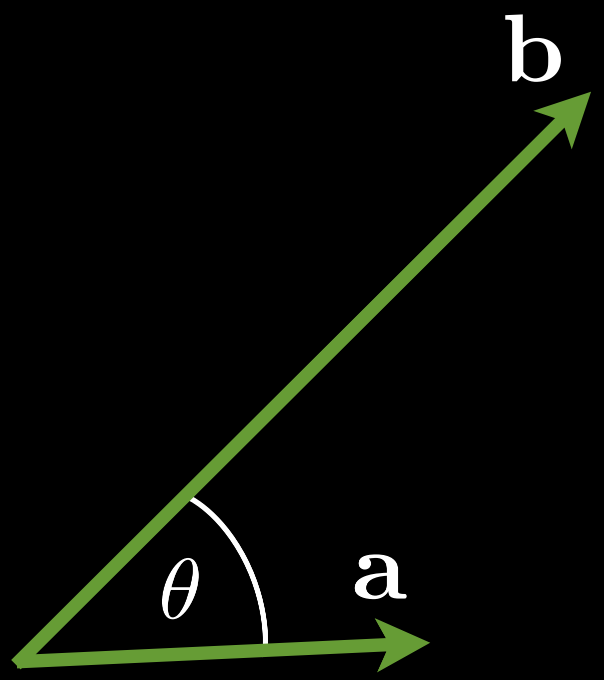

b
a



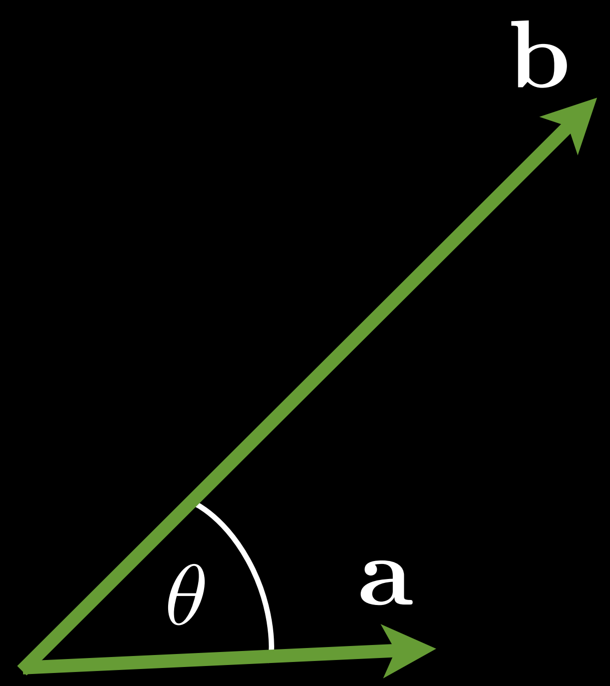
b
θ
a



$$\mathbf{a} \cdot \mathbf{b}$$

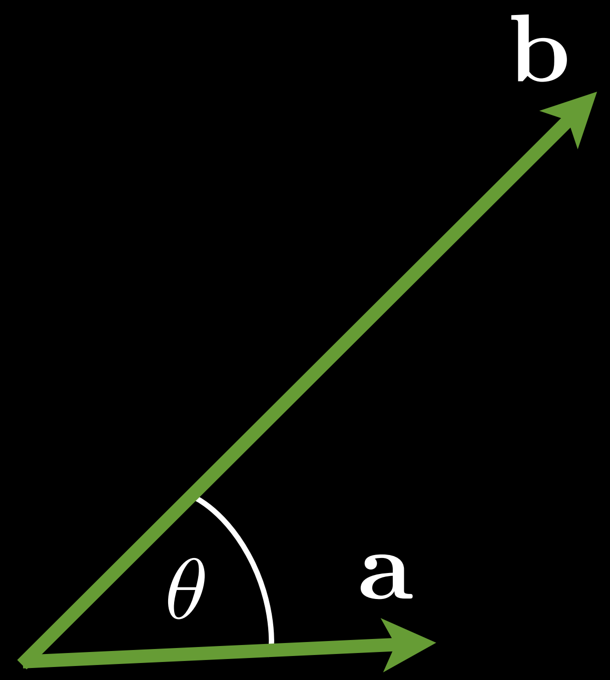


$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos \theta$$



$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

assume $\|\mathbf{a}\| = 1$



$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$$

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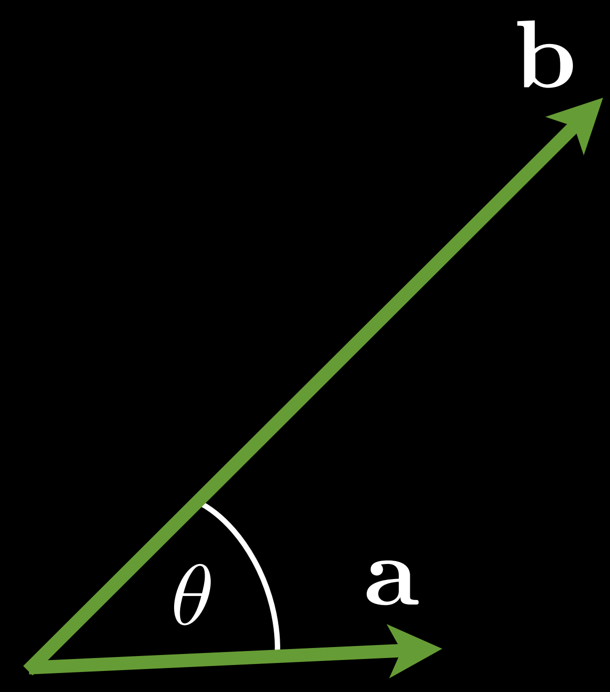
$$= \|\mathbf{b}\| \cos\theta$$

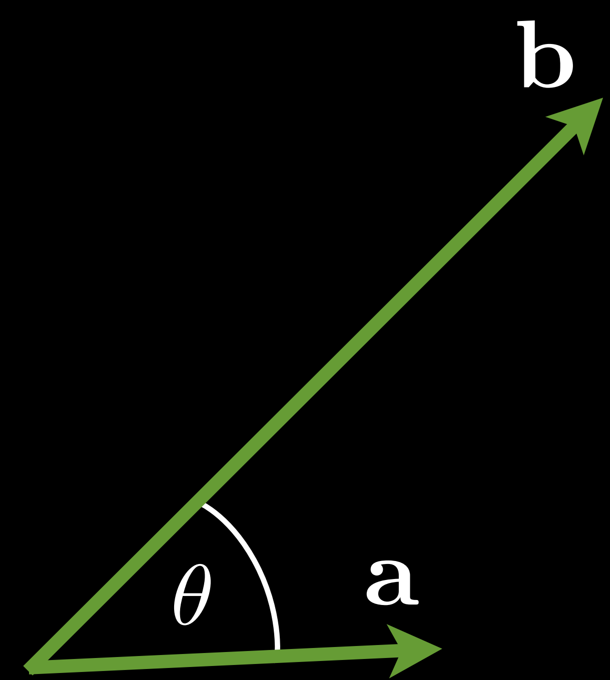


$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

assume $\|\mathbf{a}\| = 1$

$$= \|\mathbf{b}\| \cos \theta$$

What is the geometric interpretation?

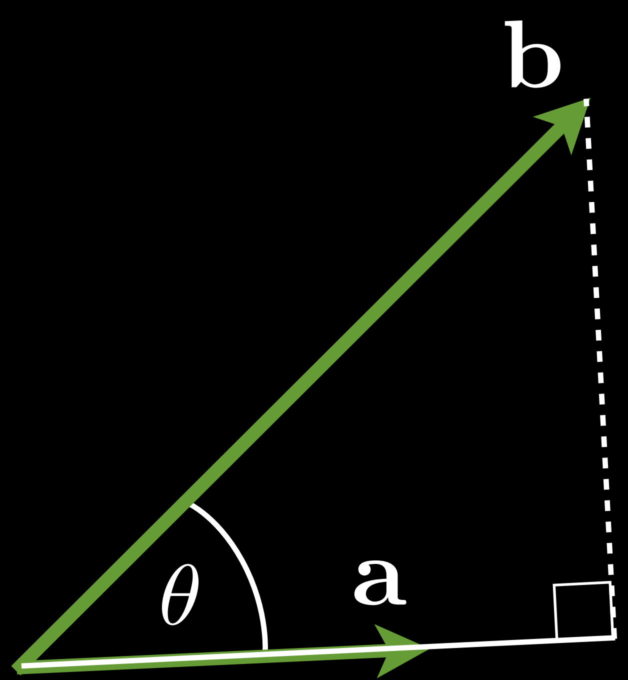


$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$$

assume $\|\mathbf{a}\| = 1$

$$= \|\mathbf{b}\| \cos\theta$$

Length of projection of b onto $\mathbf{a}$

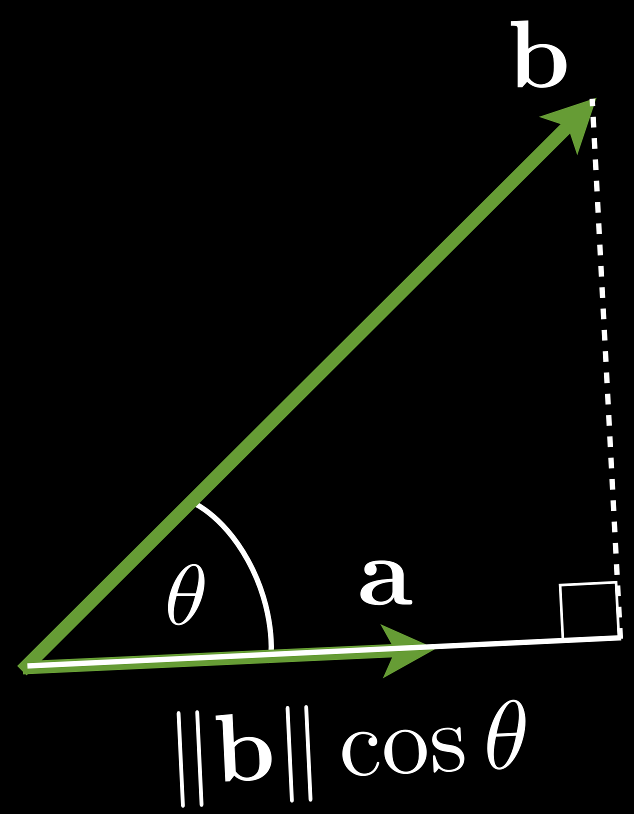


$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta$$

assume $\|\mathbf{a}\| = 1$

$$= \|\mathbf{b}\|\cos\theta$$

Length of projection of b onto a



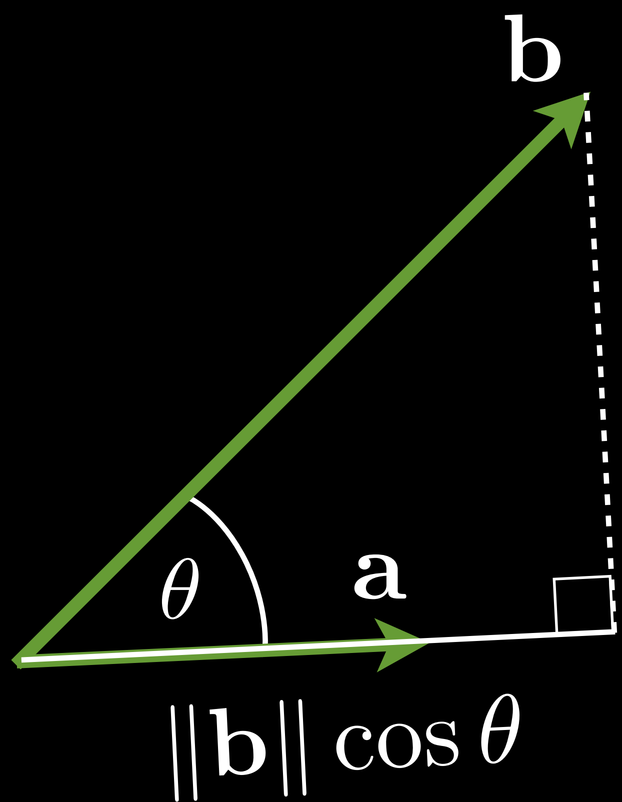
$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$

assume $\|\mathbf{a}\| = 1$

$$= \|\mathbf{b}\|\cos\theta$$

Length of projection of b onto a

Vector Projection

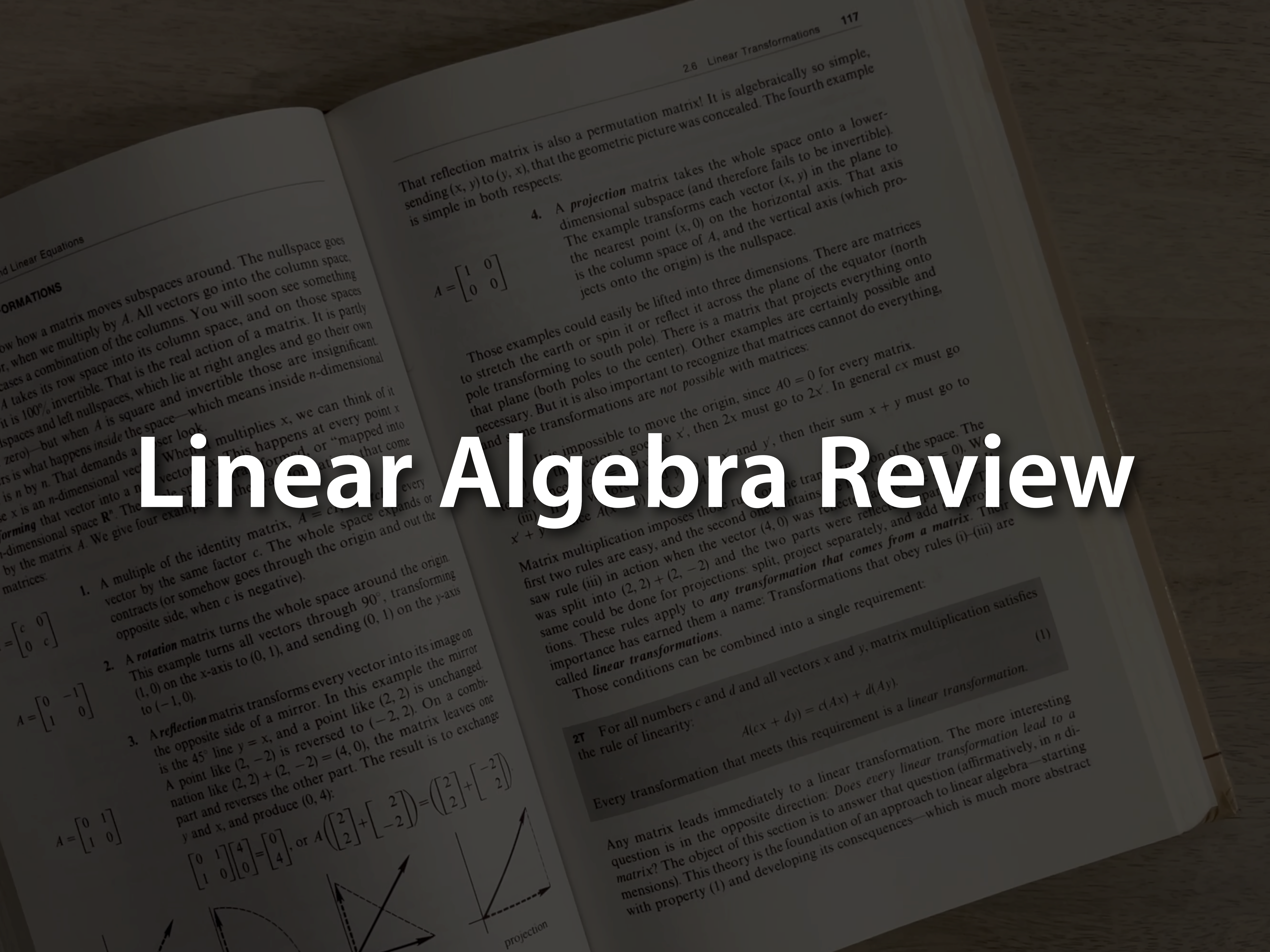


$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta$$

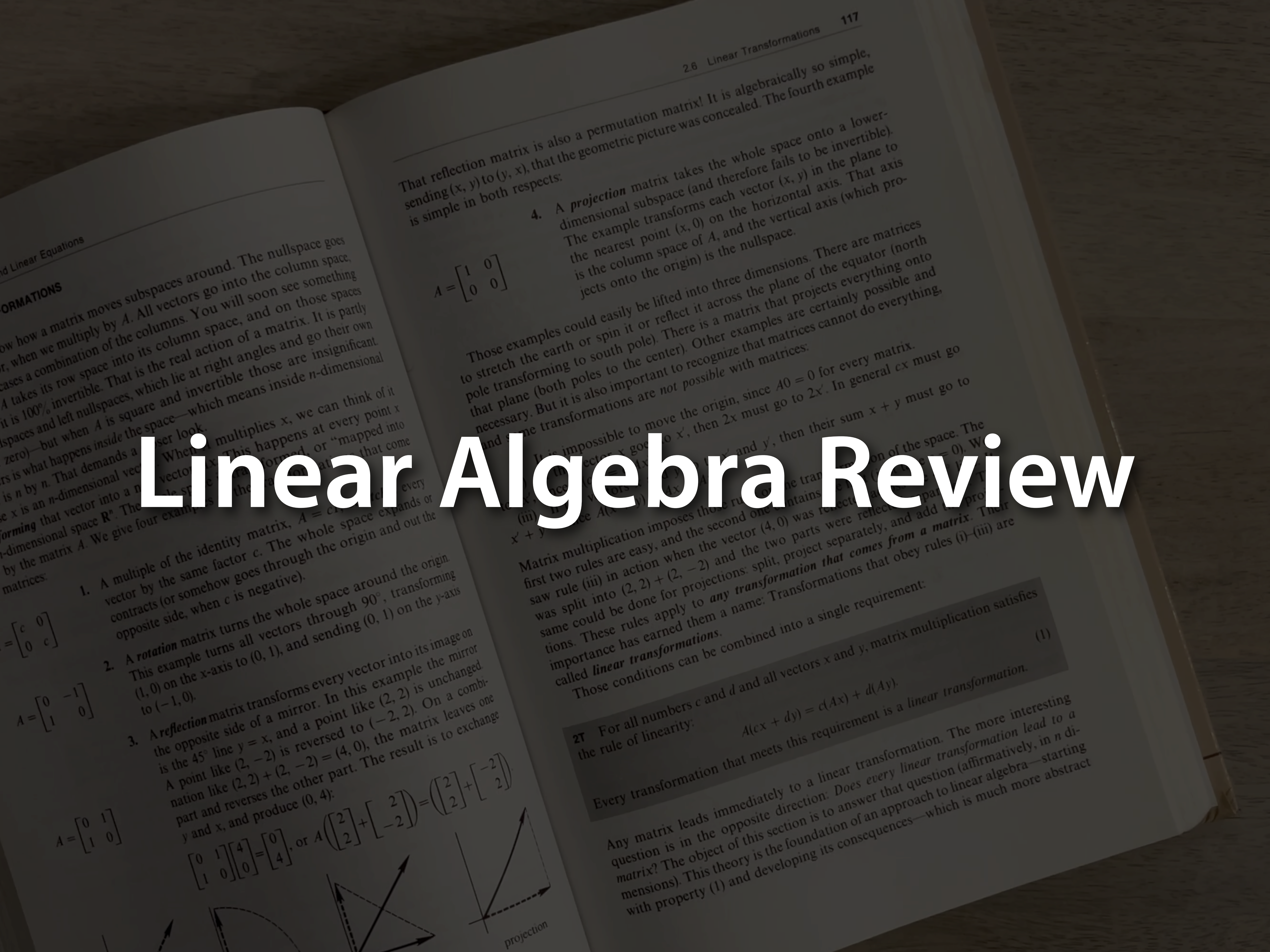
assume $\|\mathbf{a}\| = 1$

$$= \|\mathbf{b}\|\cos\theta$$

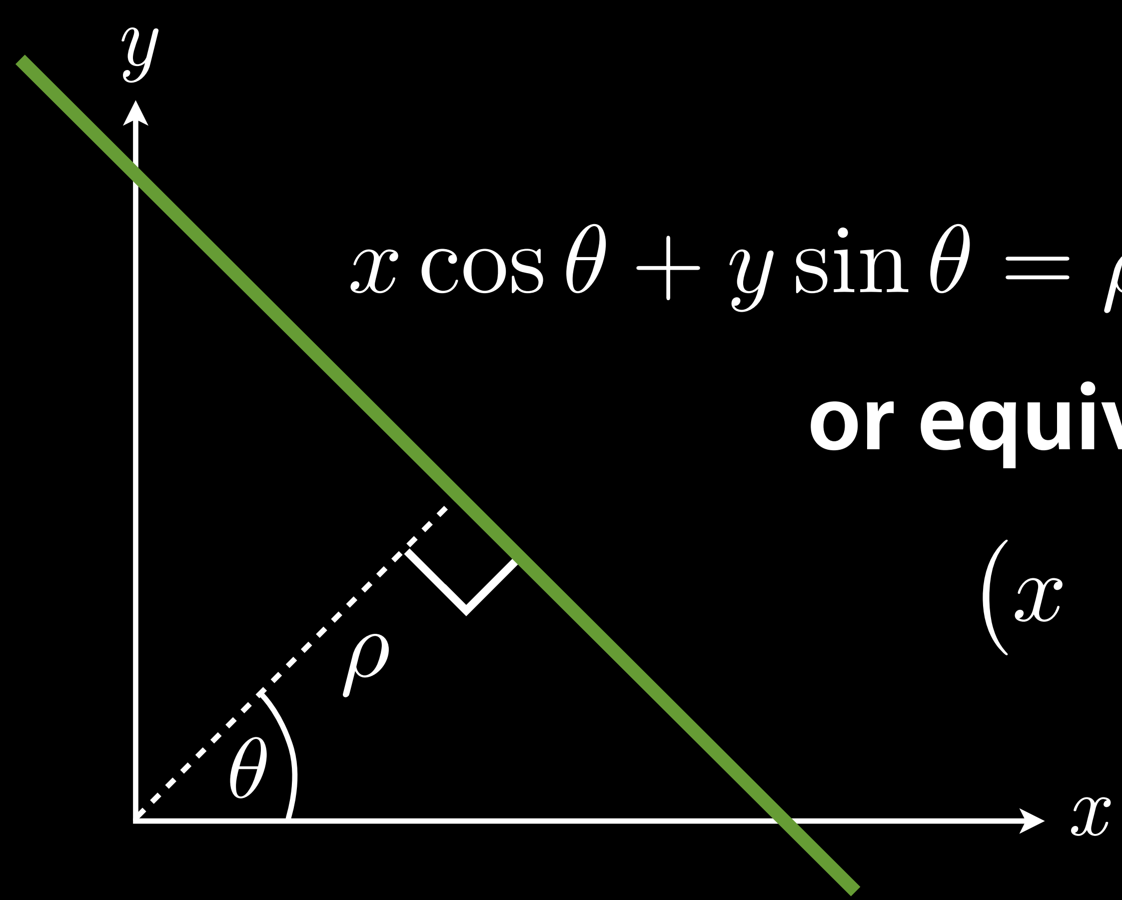
Length of projection of b onto a



Linear Algebra Review


Linear Algebra Review

Polar Representation



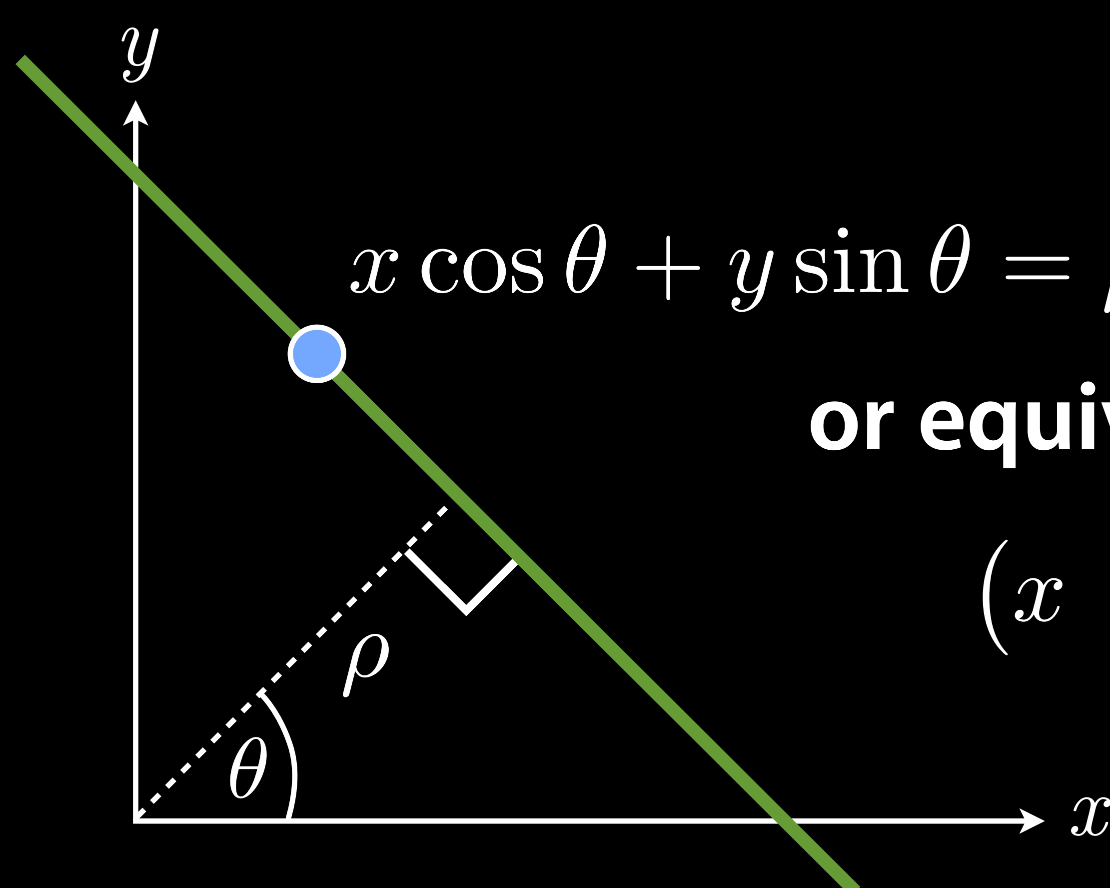
$$x \cos\theta + y \sin\theta = \rho$$

or equivalently

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}^\top = \rho$$

image space

Polar Representation



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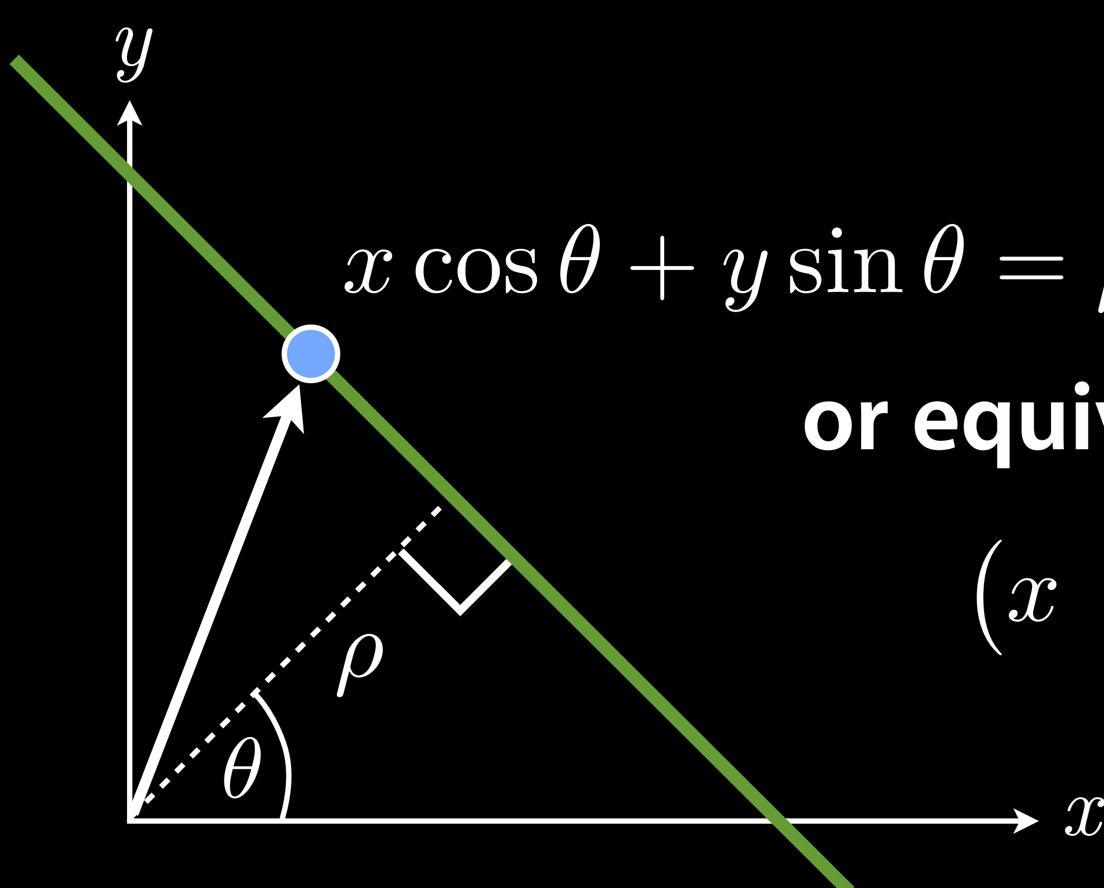

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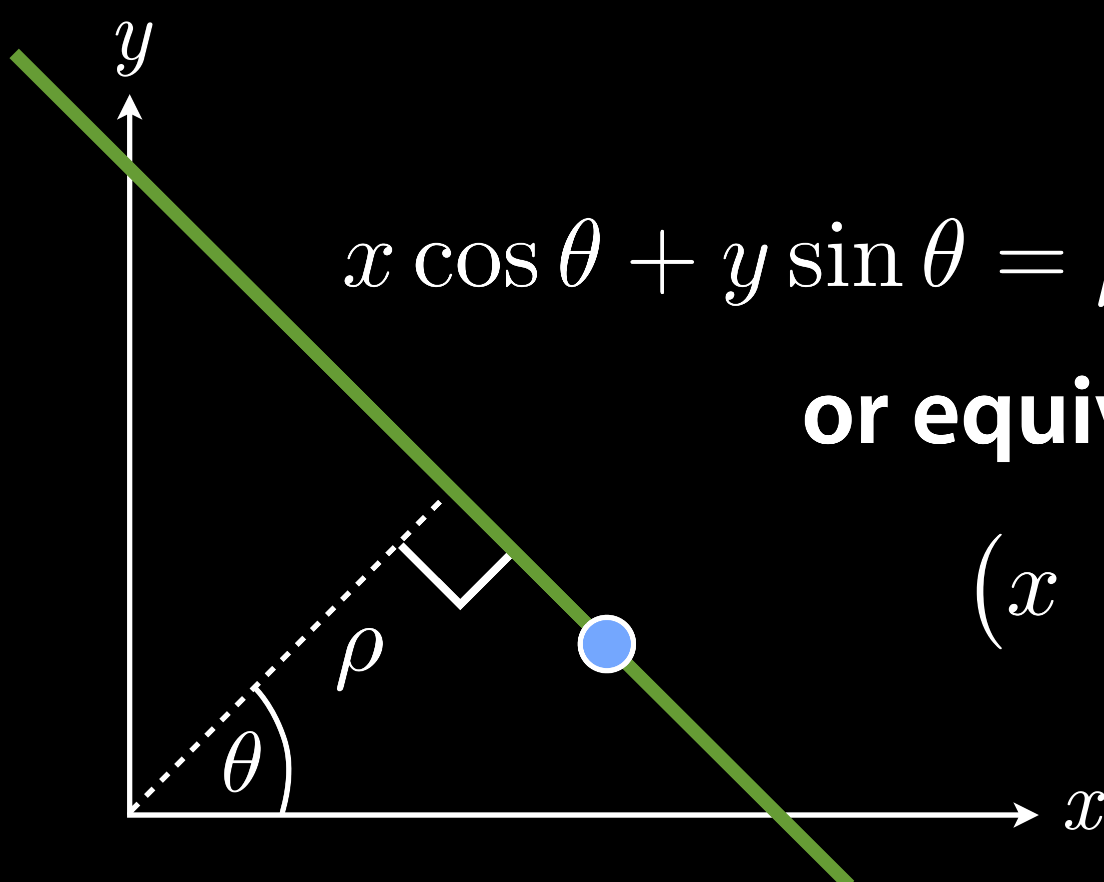

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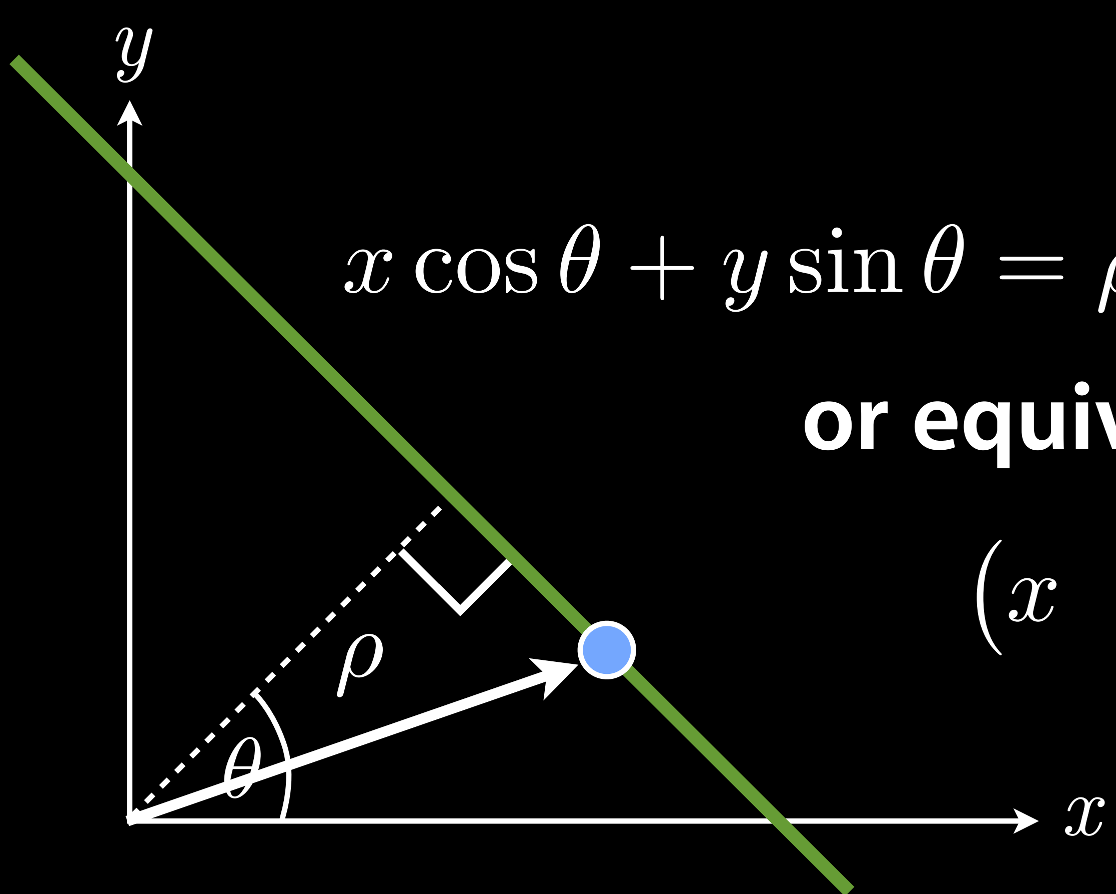

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image space

Polar Representation

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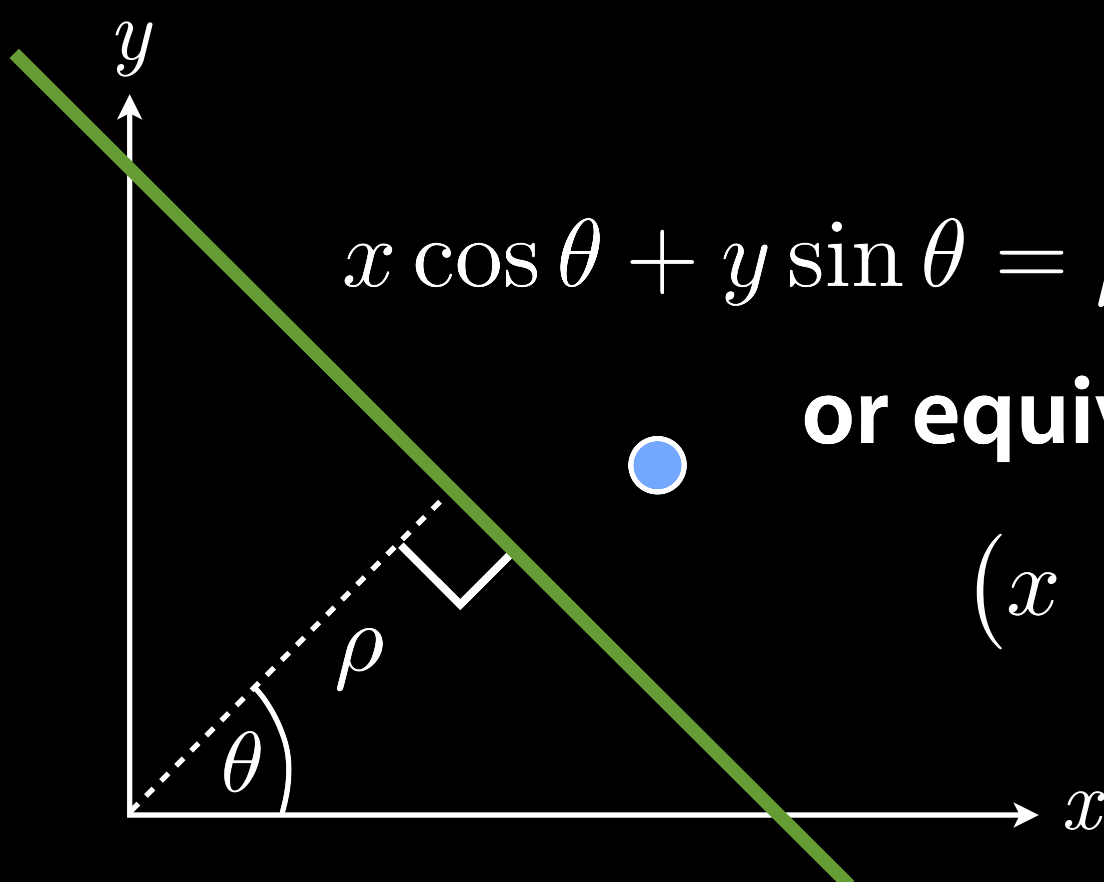

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Polar Representation

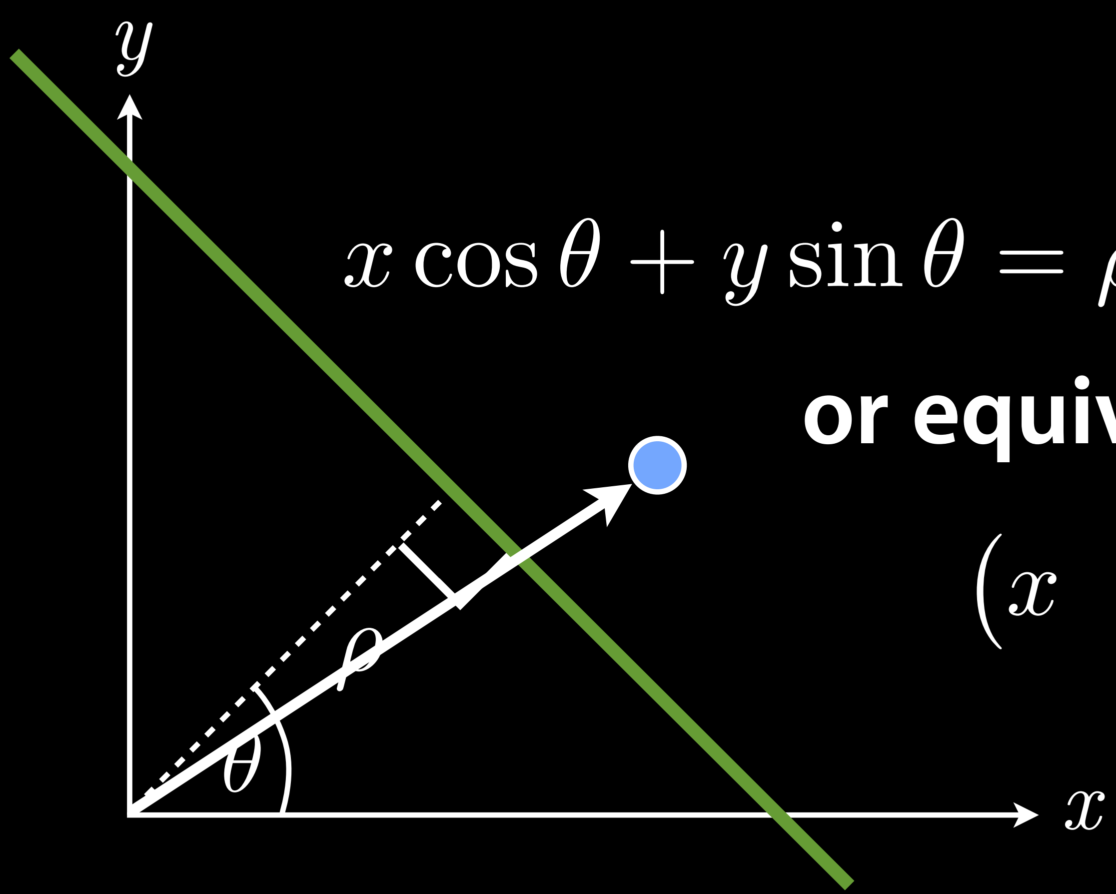

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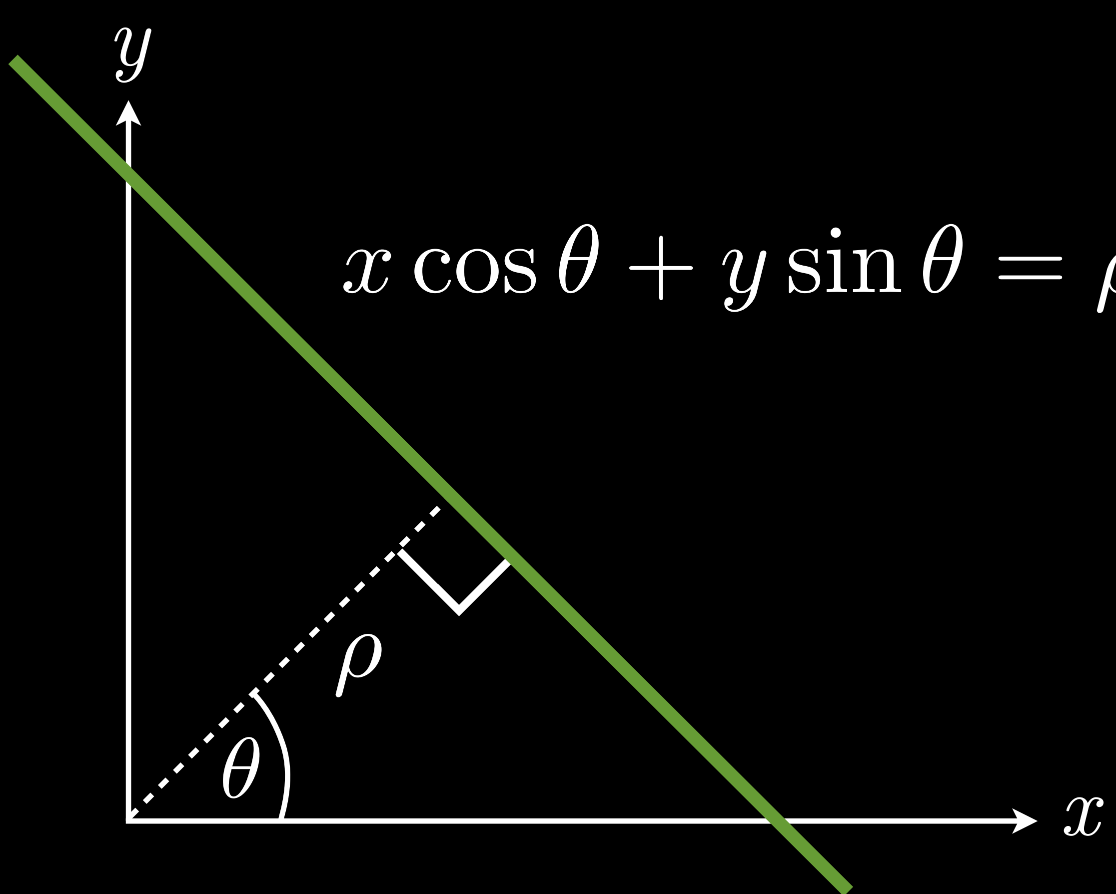
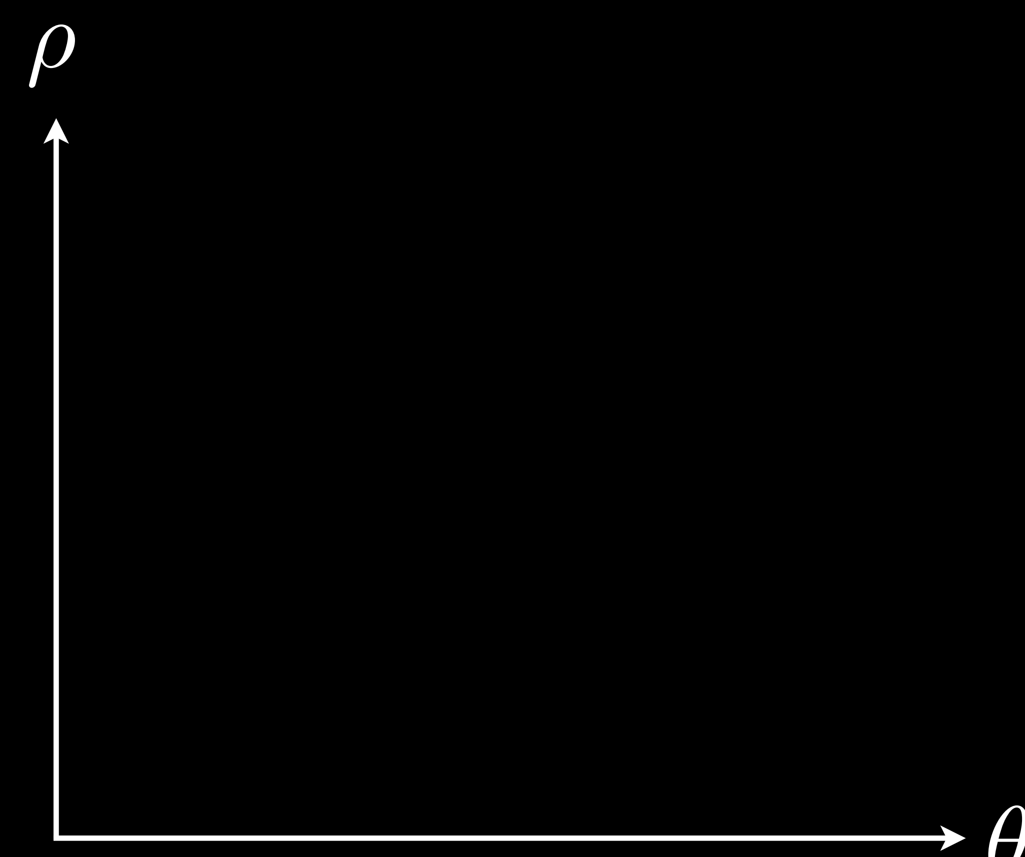
image space

Polar Representation



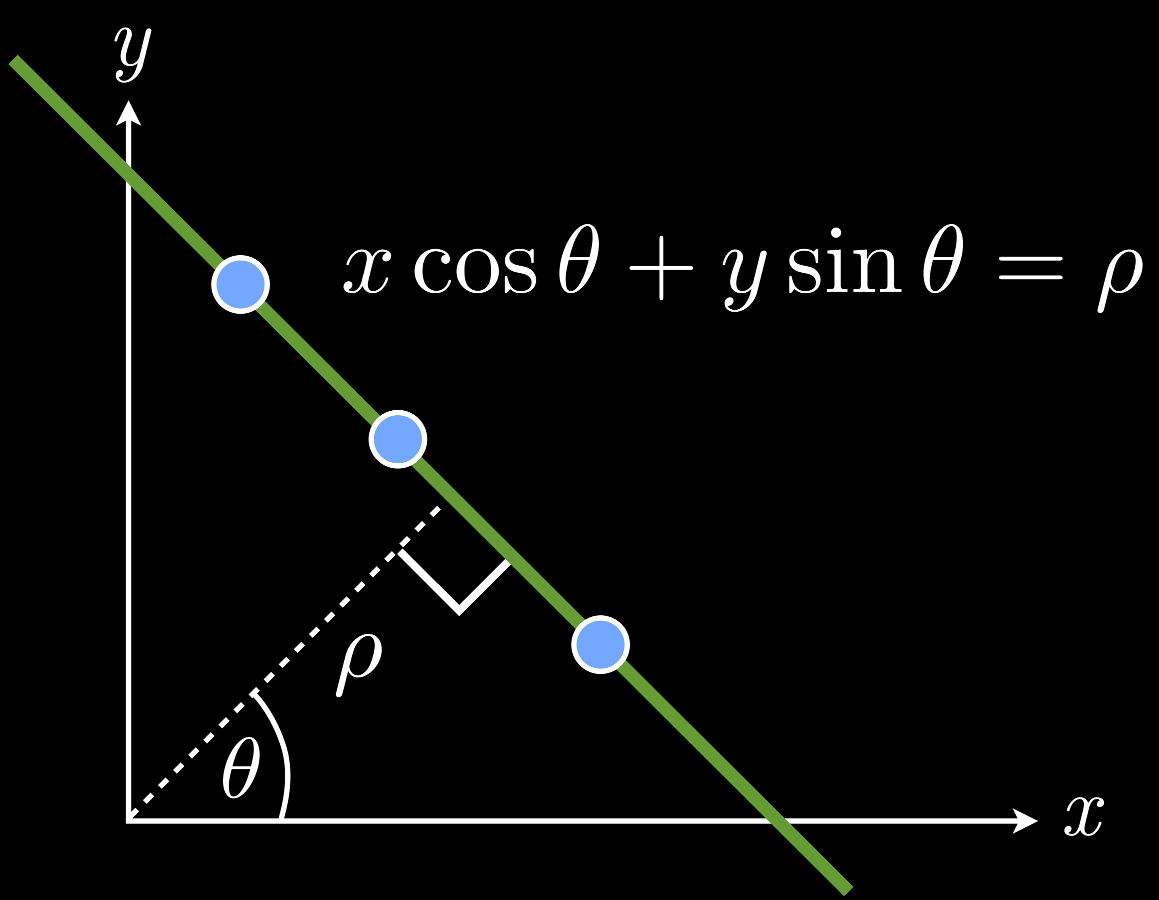
$$x \cos\theta + y \sin\theta = \rho$$

image space



Hough space

Polar Representation



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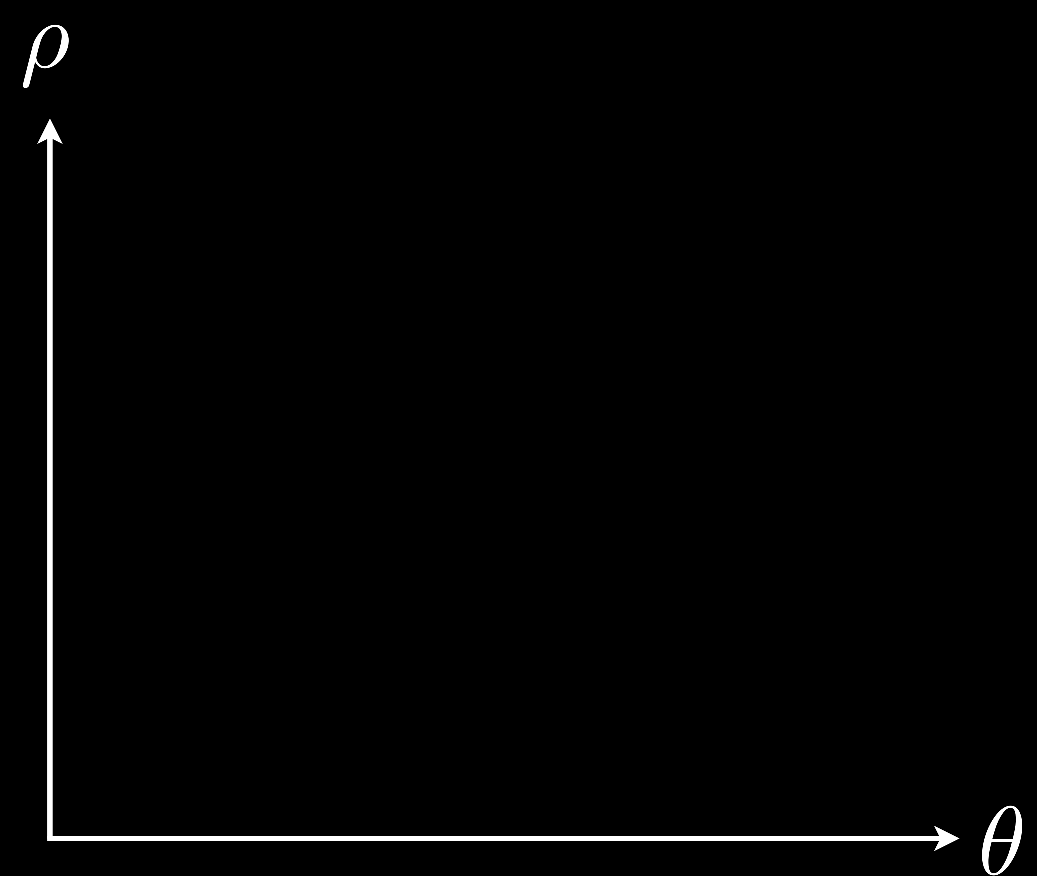


Hough space

Polar Representation

$$x\cos\theta + y\sin\theta = \rho$$

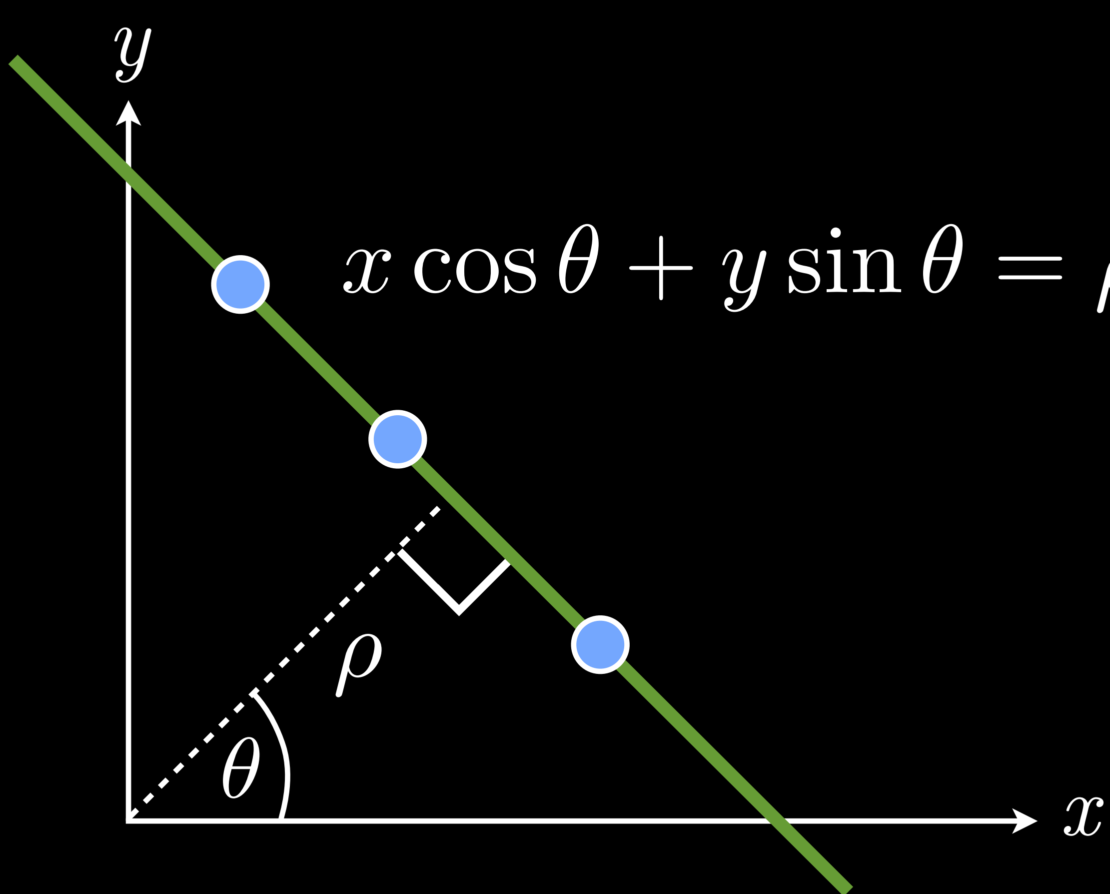
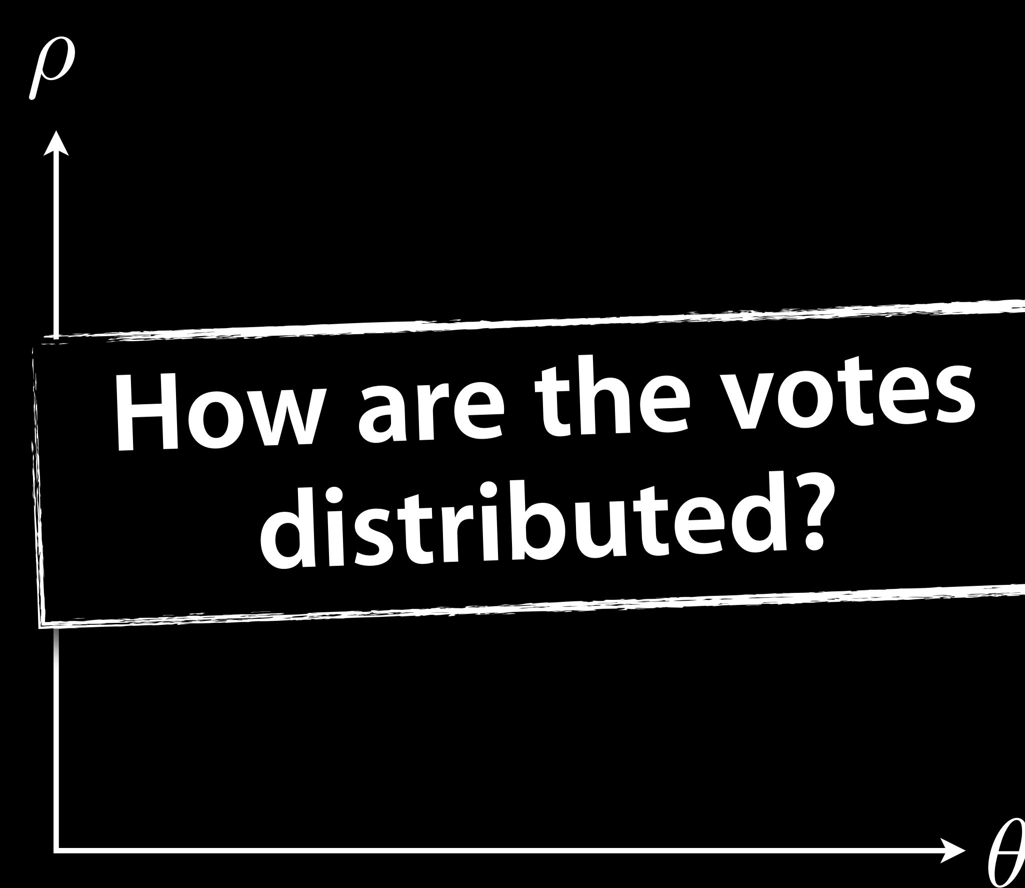


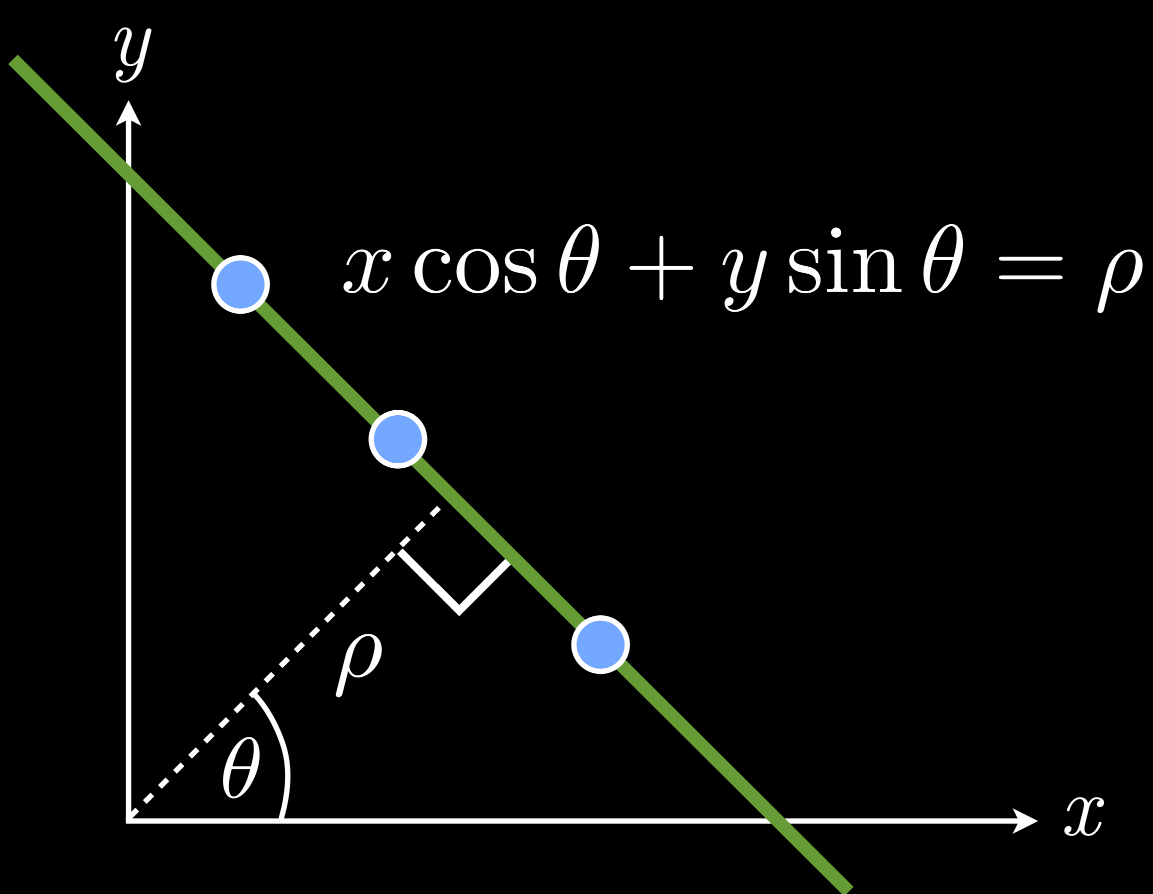
image space

How are the votes distributed?



Hough space

Polar Representation



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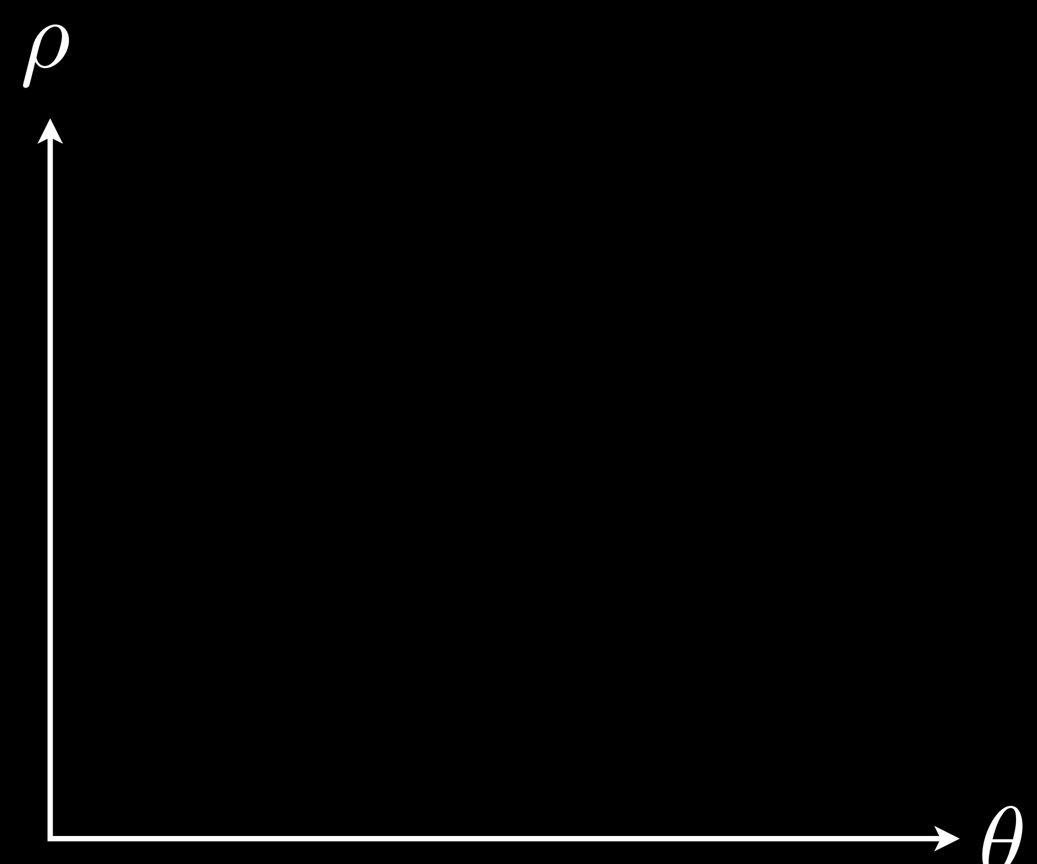

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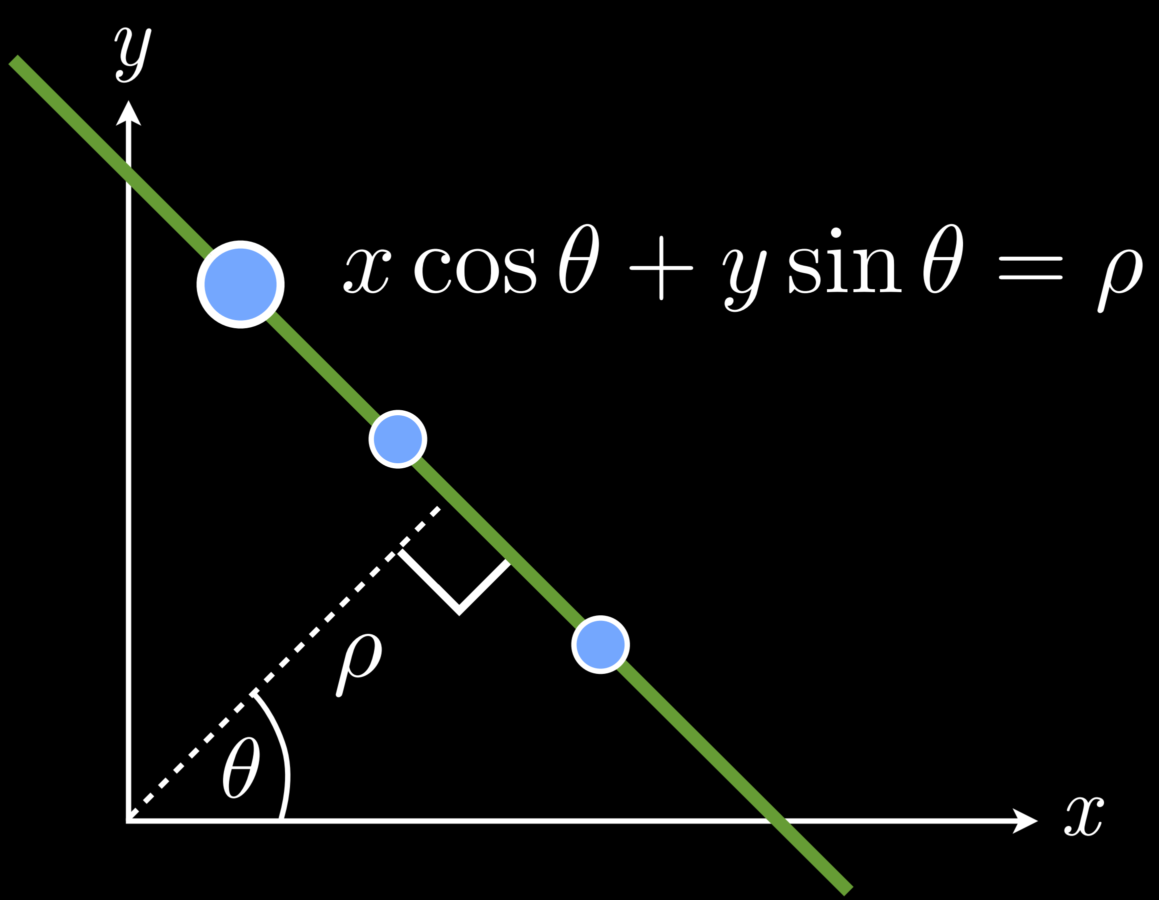
Polar Representation



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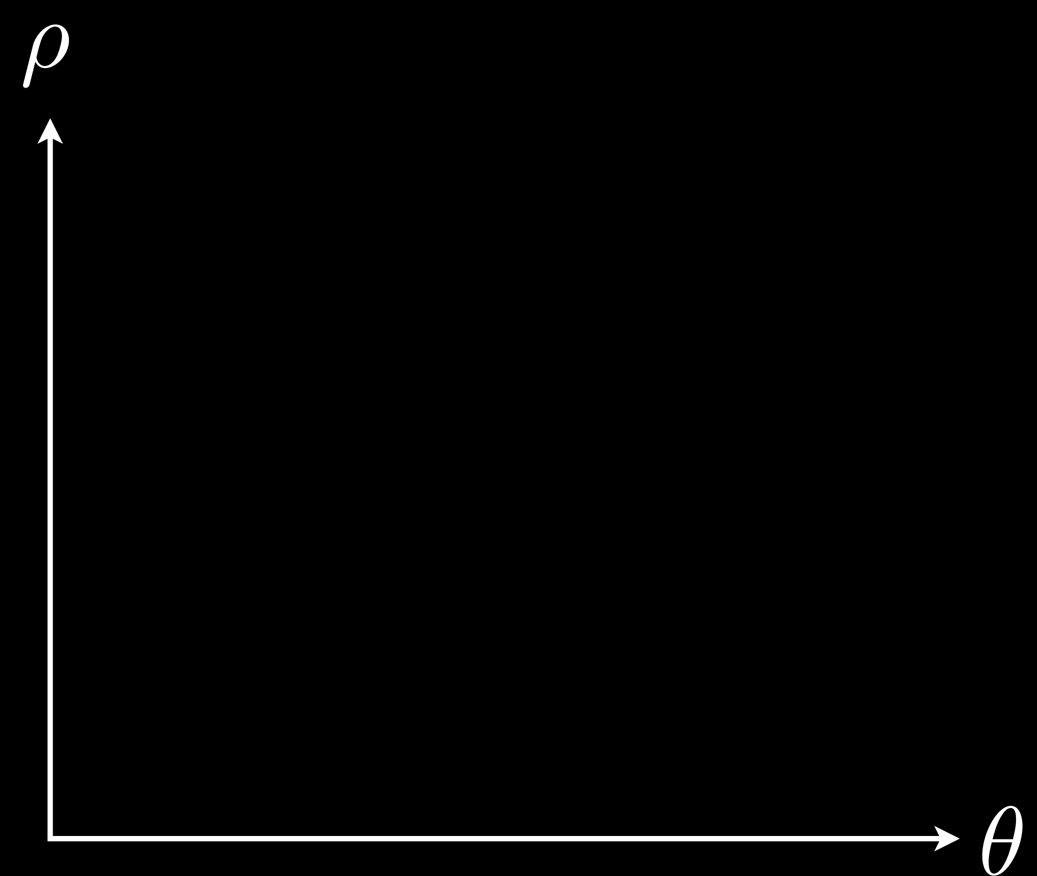

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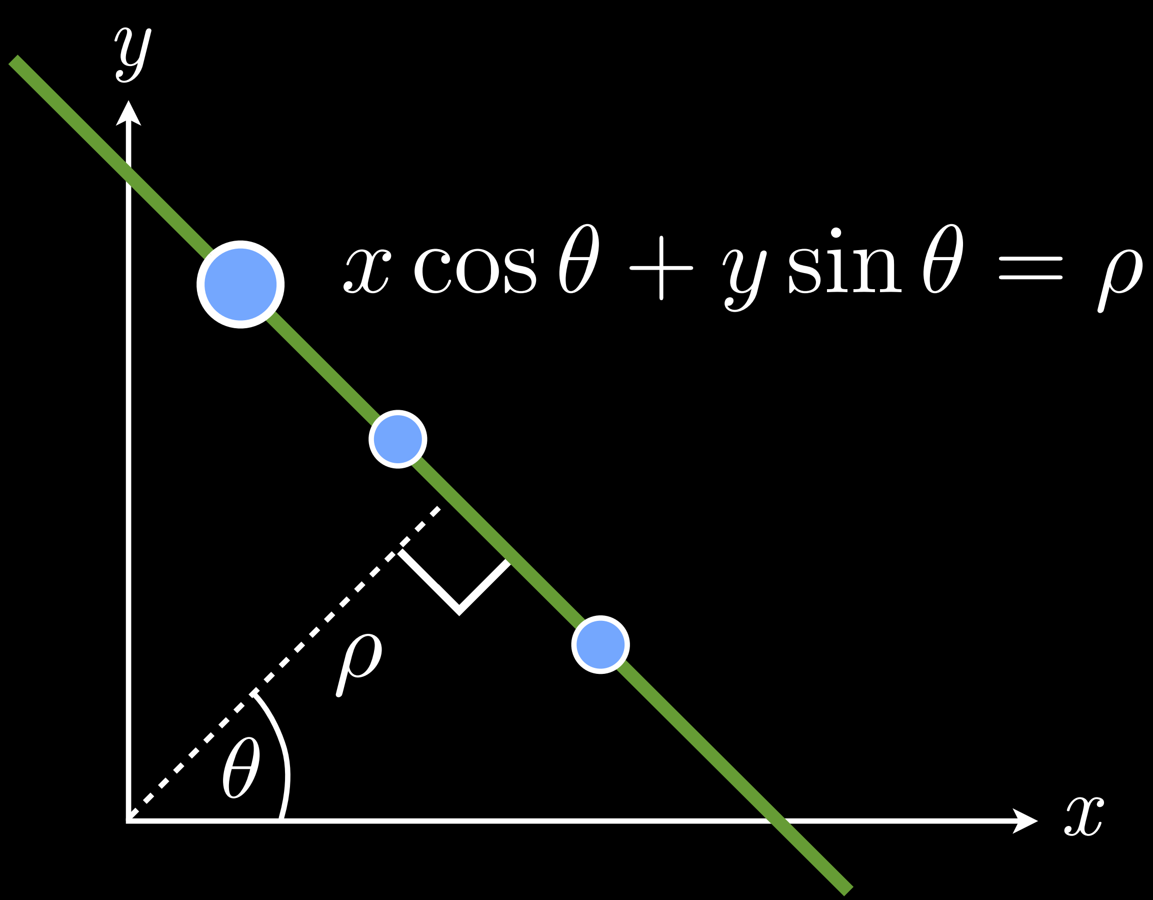

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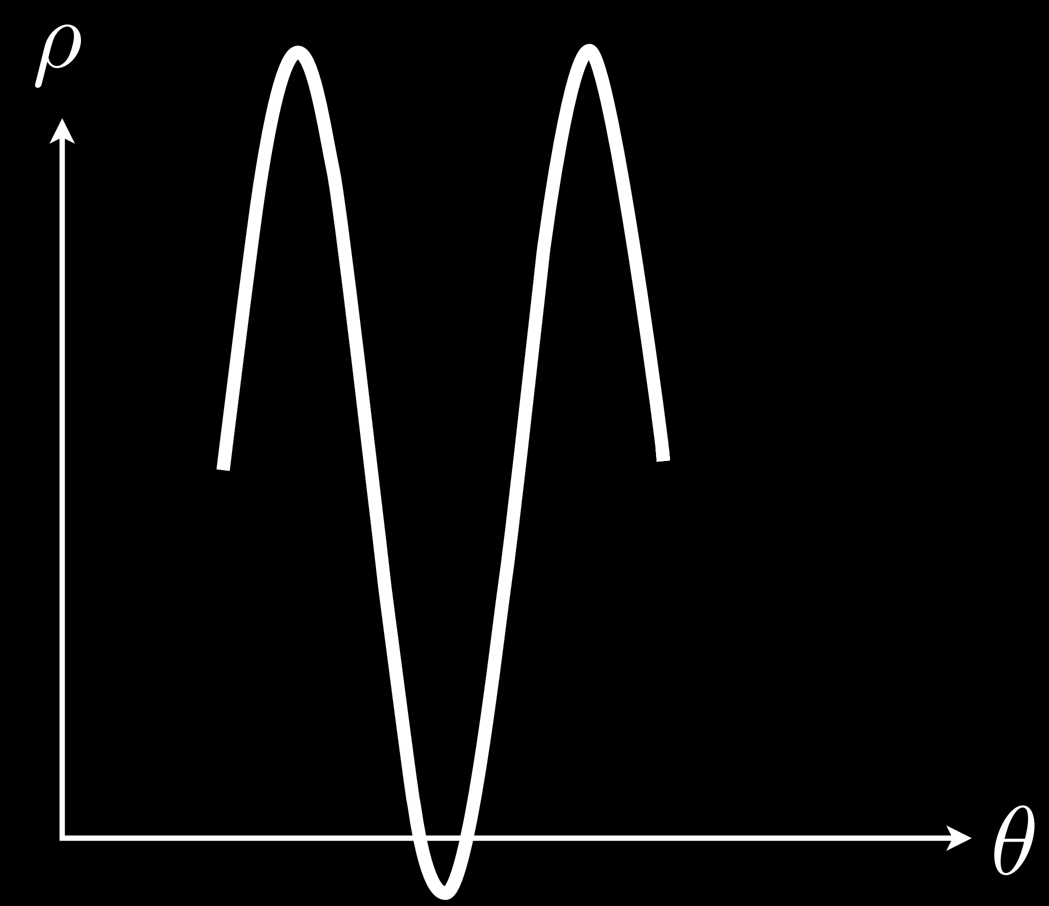

Hough space

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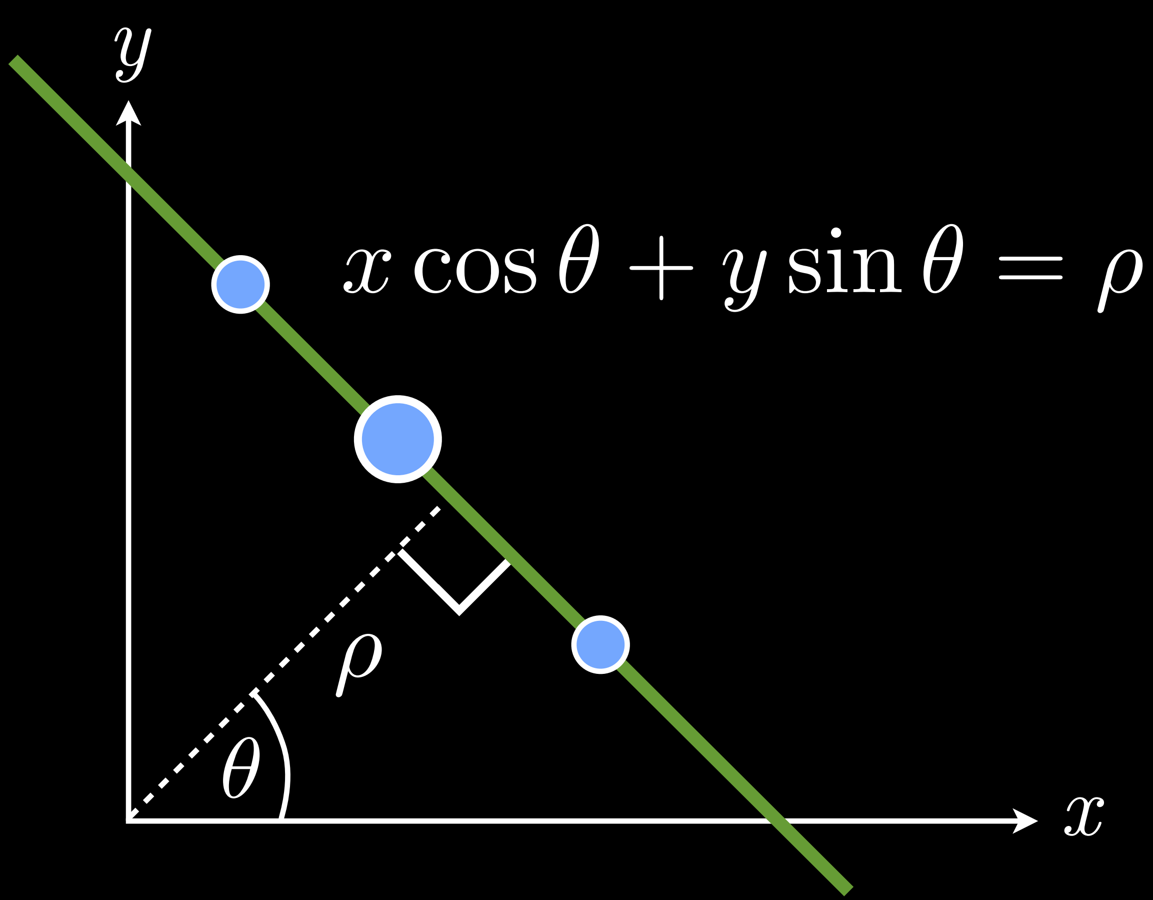

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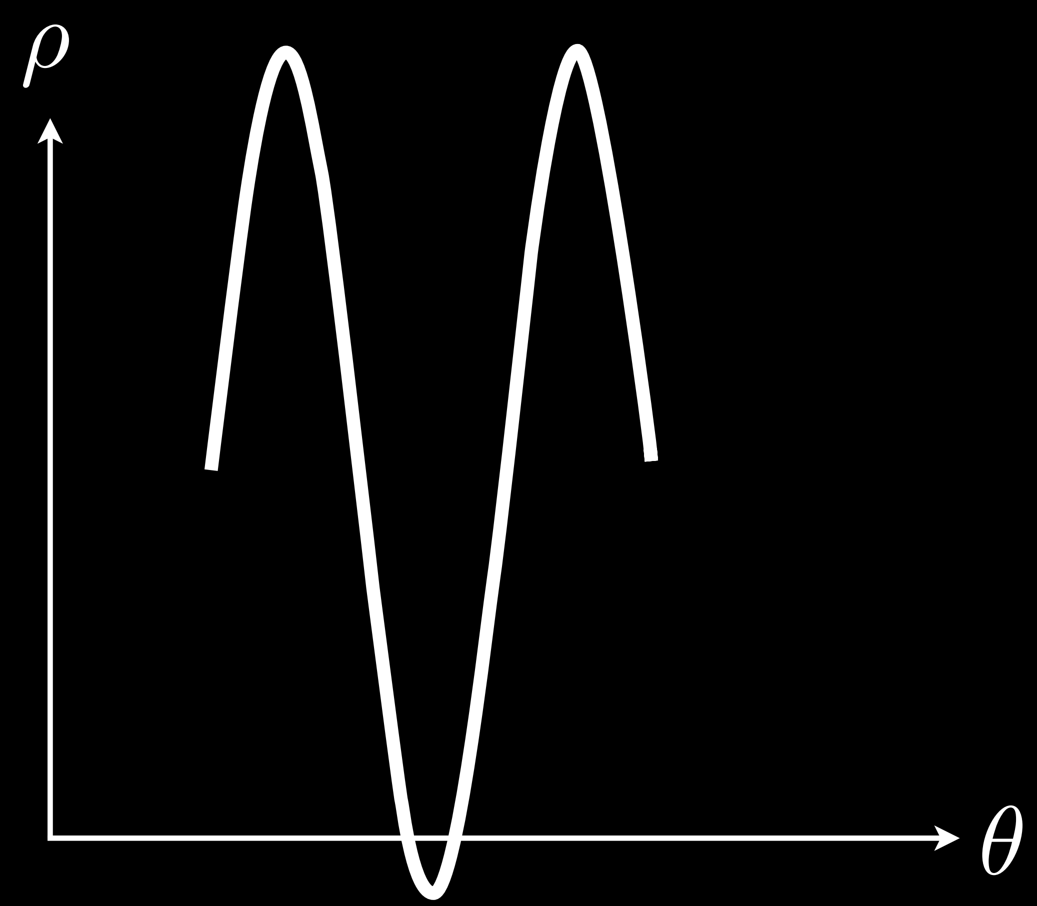
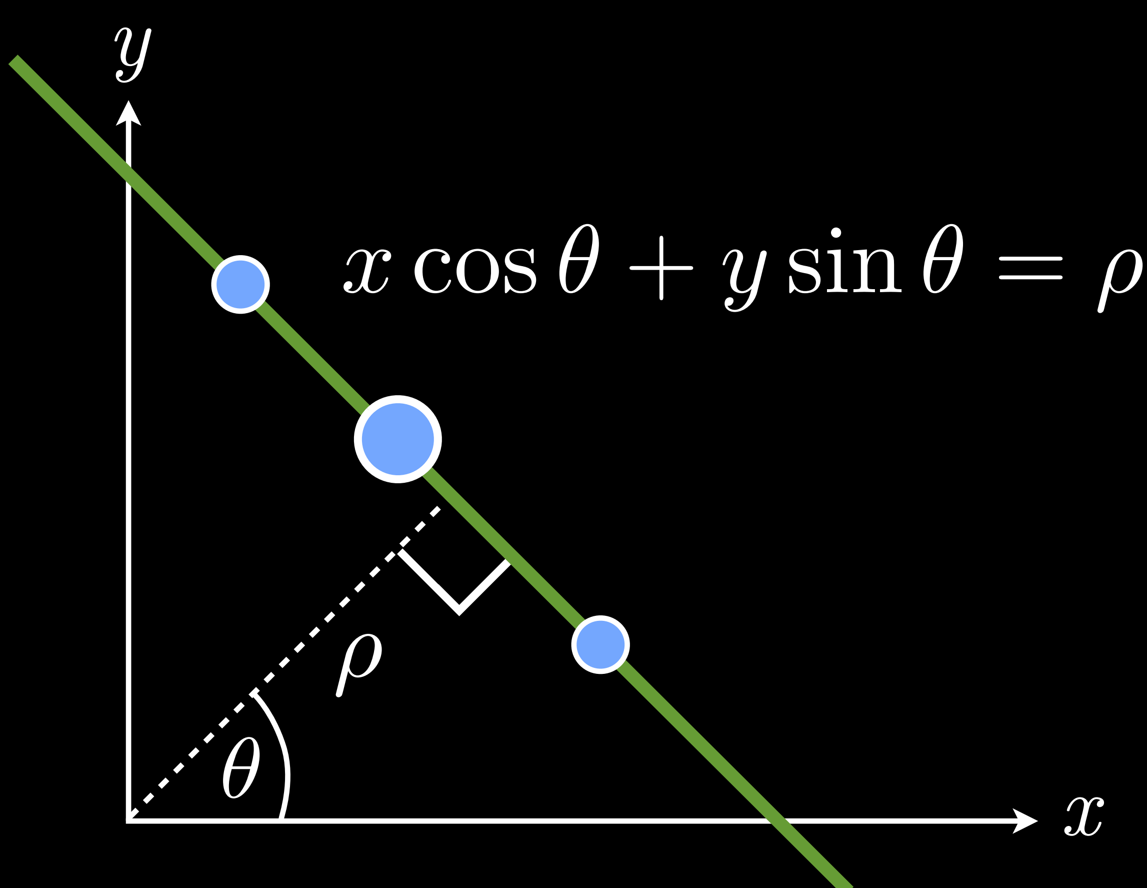


Hough space

Polar Representation



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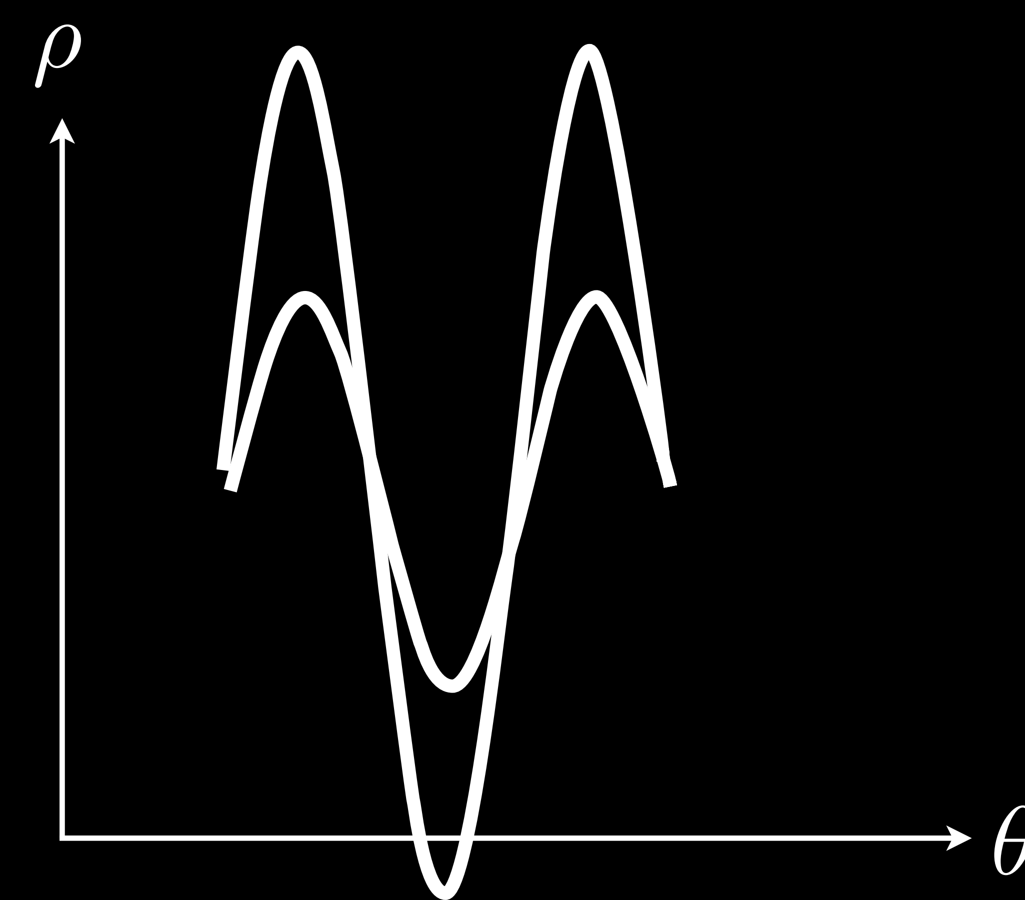

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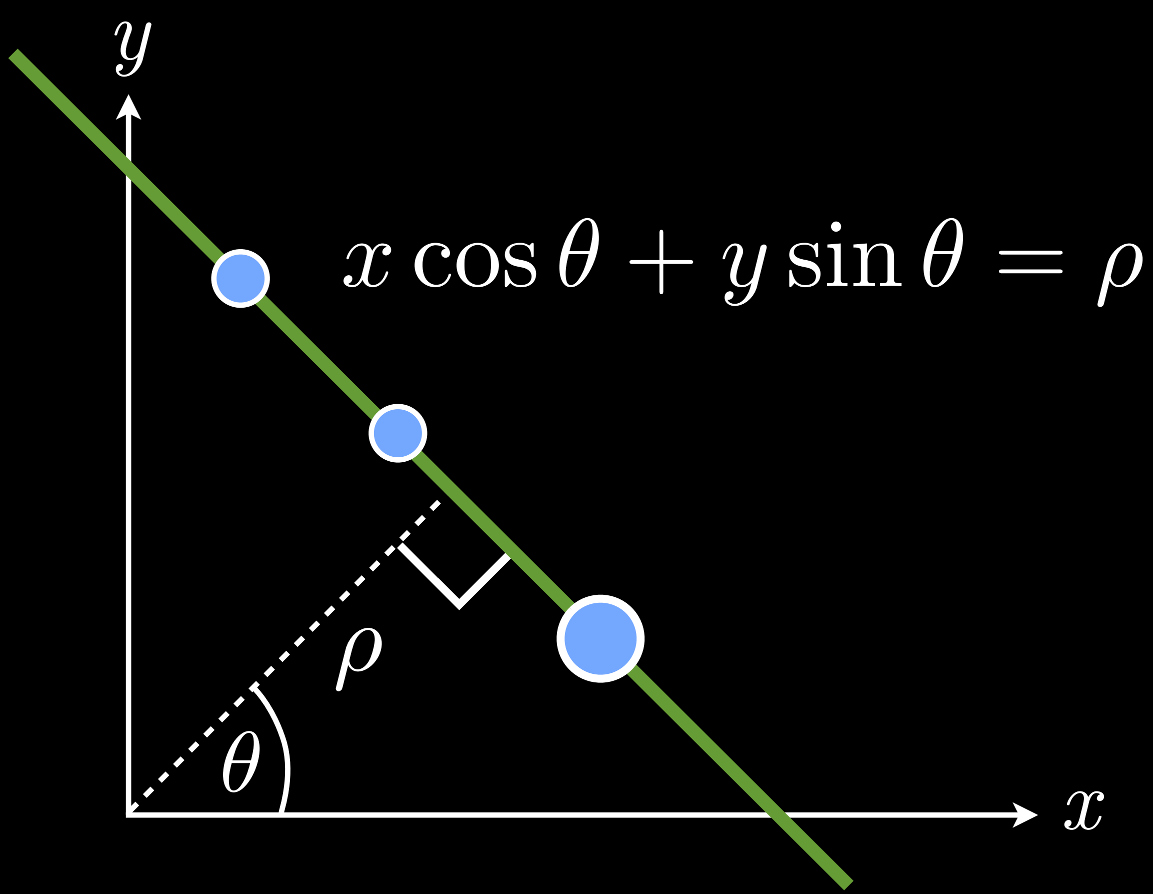

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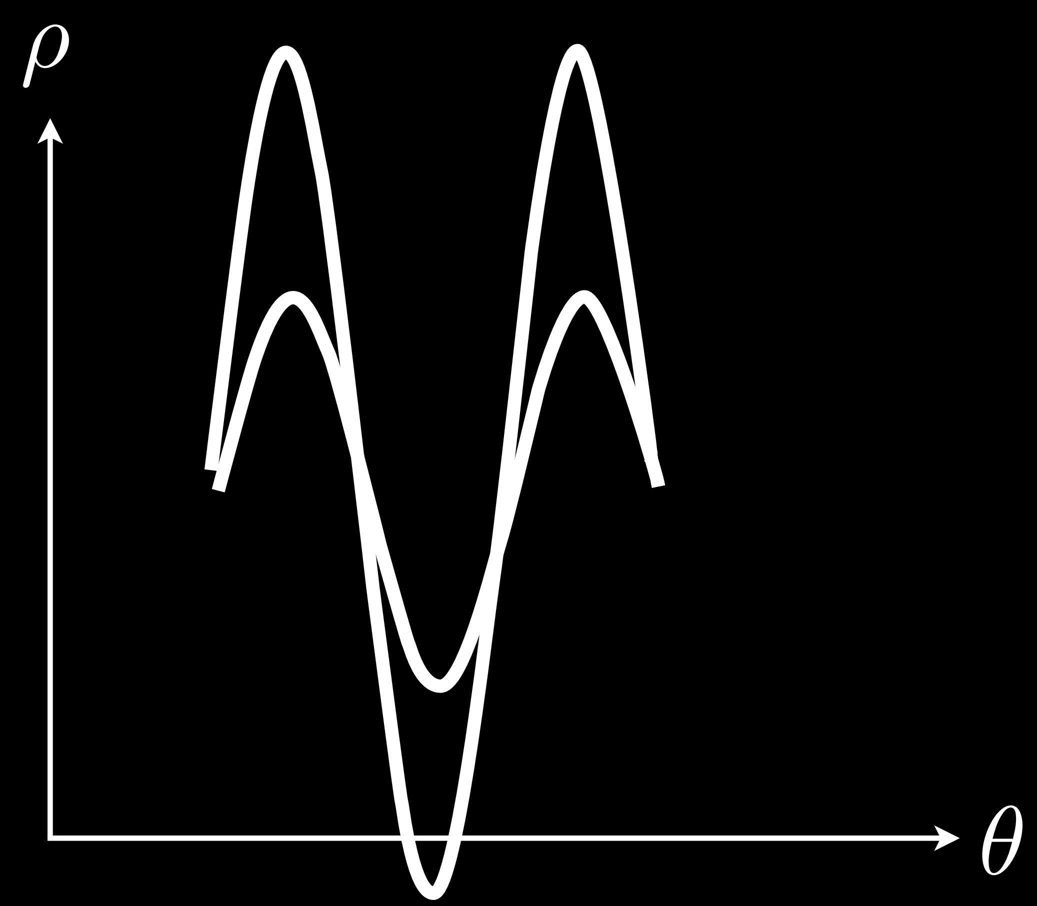

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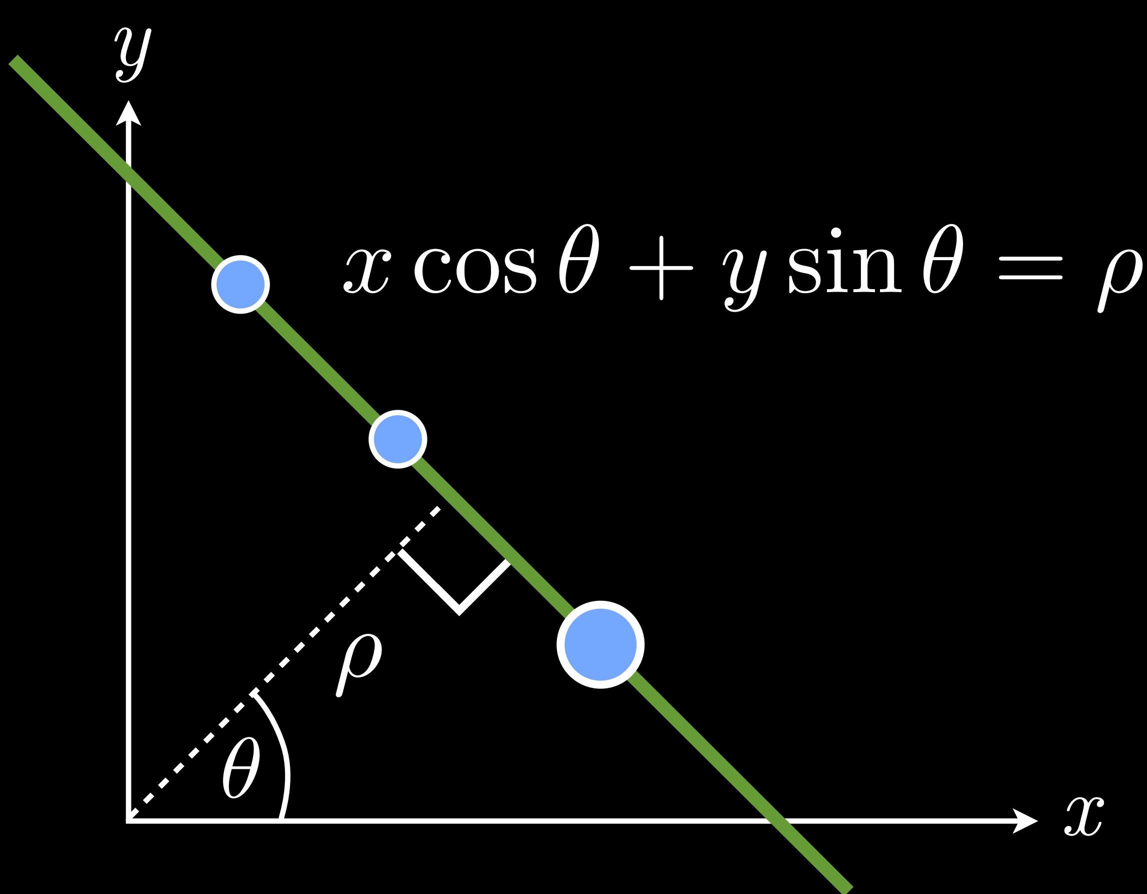

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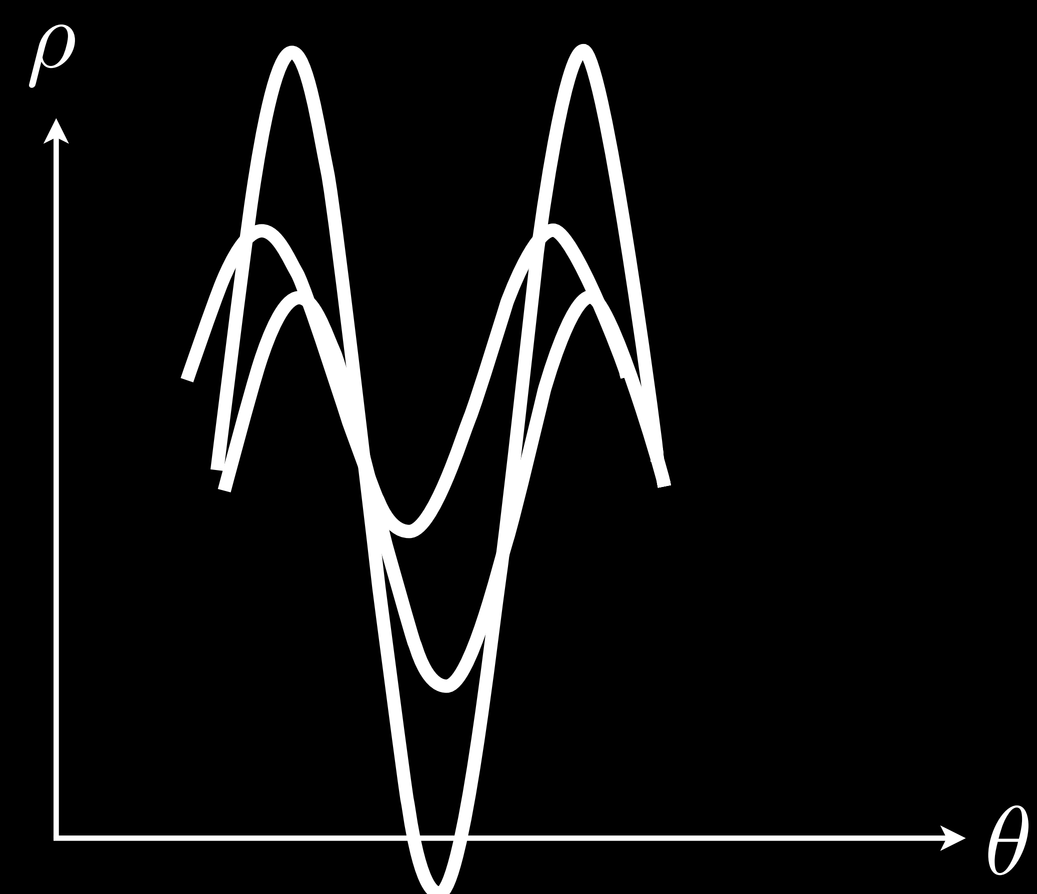

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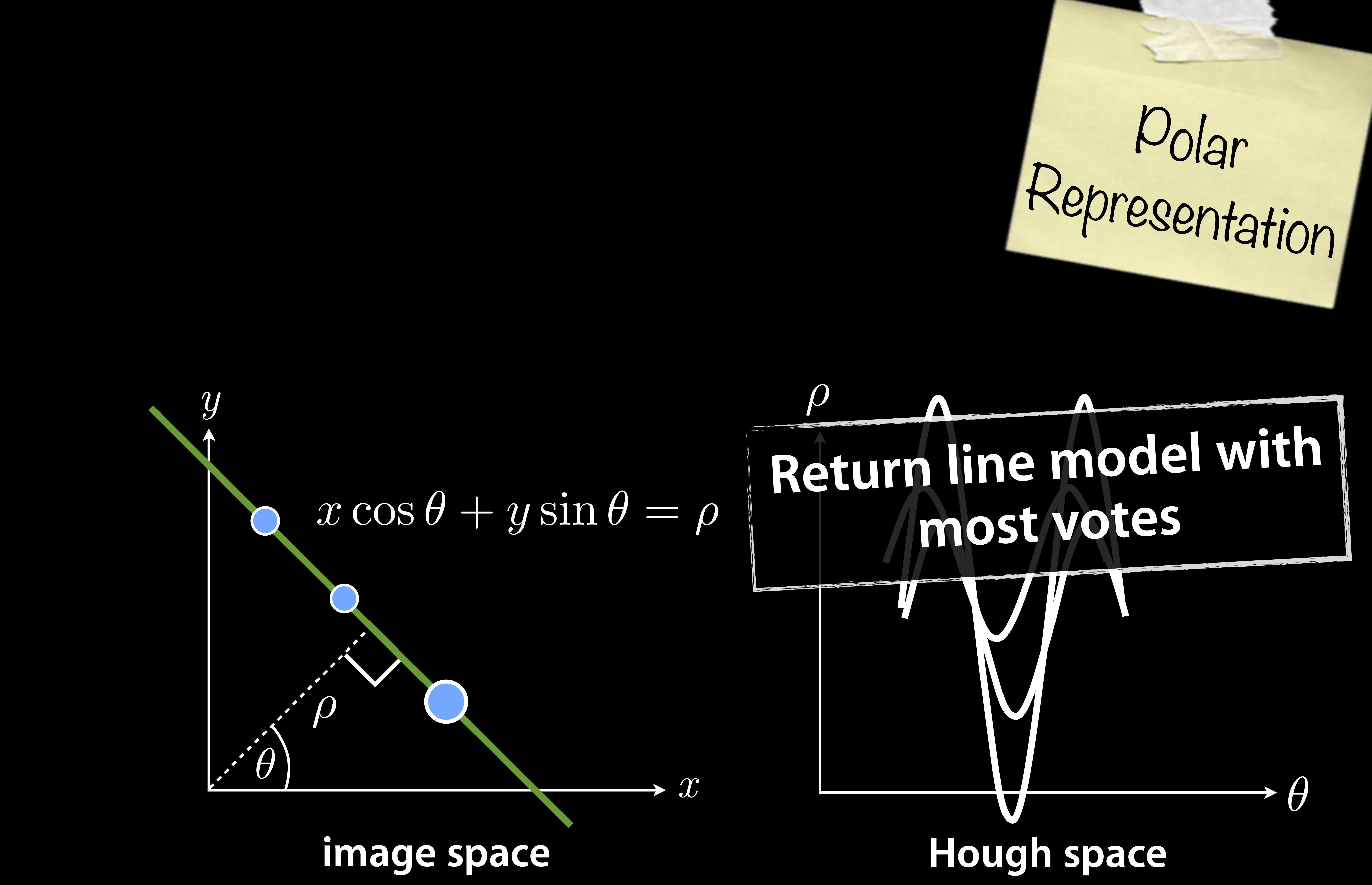
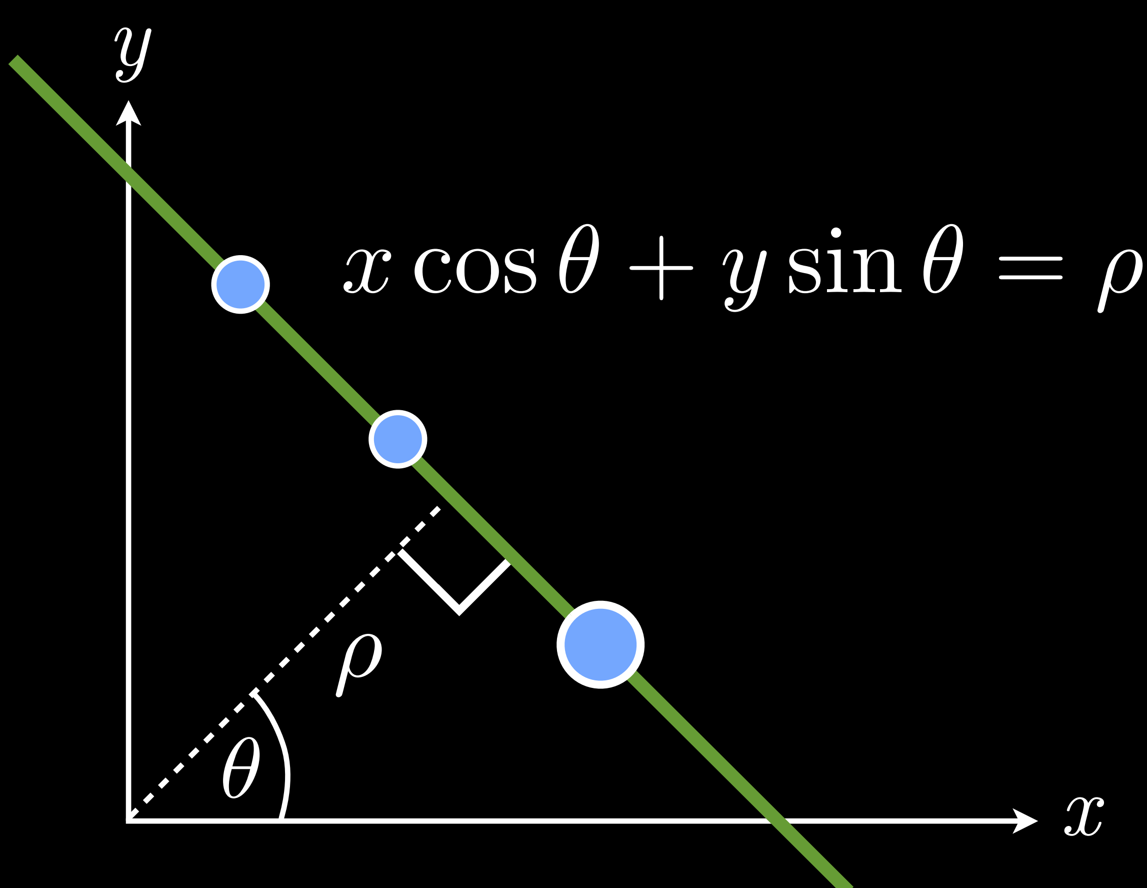


Polar Representation
$x \cos\theta + y \sin\theta = \rho$
y
x
ρ
θ
image space
ρ
θ
Return line model with most votes
Hough space

Polar Representation



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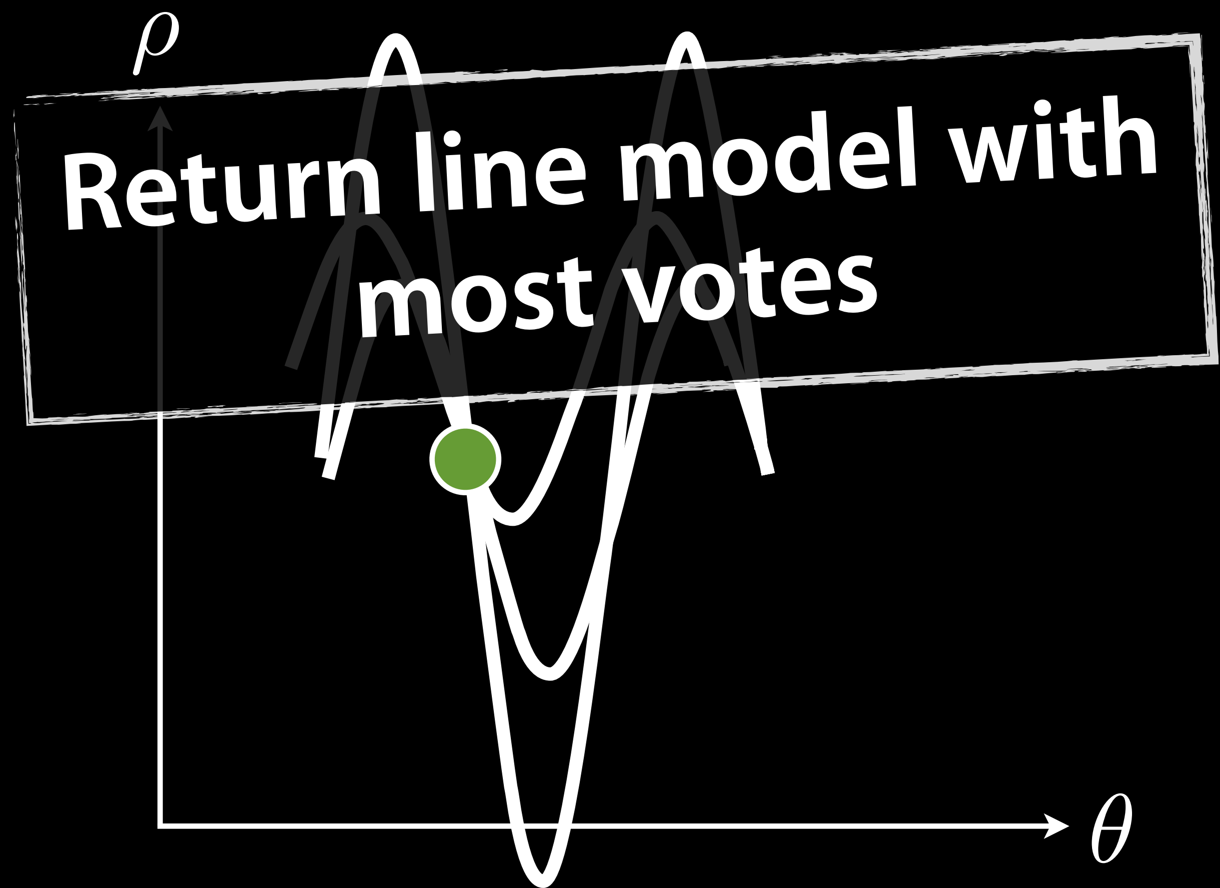

Hough space

Hough Lines

Initialize $H[\rho, \theta] = 0$

foreach edgel $e \in I[x, y]$ **do**

  foreach $\theta = \theta_{\min}$ to $\theta_{\max}$ **do**

    $\rho = x \cos\theta + y \sin\theta$

    $H[\rho, \theta] \mathrel{+}= 1$

  end

end

$\rho^*, \theta^* = \underset{\rho,\theta}{\operatorname{argmax}}\ H[\rho, \theta]$

Hough Lines

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Time complexity in terms of number of votes per point?

Hough Lines

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Hough Lines

extension #1

Initialize $H[\rho, \theta] = 0$

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Hough Lines

extension #1

Initialize $H[\rho, \theta] = 0$

foreach edgel $e \in I[x, y]$ **do**

$\quad \theta = \mathrm{angle}(\nabla(I[x, y]))$

$\quad \rho = x \cos\theta + y \sin\theta$

$\quad H[\rho, \theta] \mathrel{+}= 1$

end

$\rho^*, \theta^* = \underset{\rho, \theta}{\mathrm{argmax}}\ H[\rho, \theta]$

Hough Lines

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Initialize $H[\rho, \theta] = 0$

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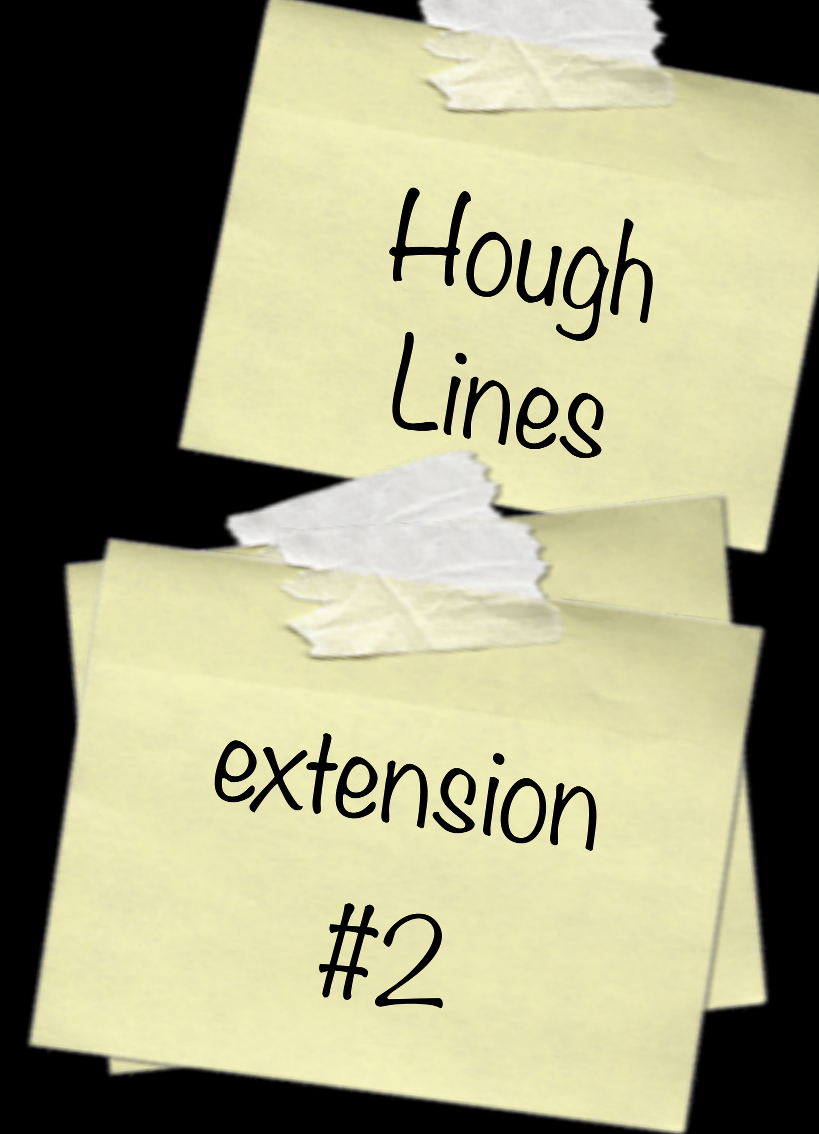
$$\rho^*, \theta^* = \underset{\rho, \theta}{\text{argmax}}\ H[\rho, \theta]$$



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foreach edgel $e \in I[x, y]$ **do**
    $\theta = \text{angle}(\nabla(I[x, y]))$
    $\rho = x \cos\theta + y \sin\theta$
    $H[\rho, \theta] \mathrel{+}= 1$
end
$\rho^*, \theta^* = \underset{\rho,\theta}{\text{argmax}}\ H[\rho, \theta]$

Hough Lines

extension #2

Initialize $H[\rho, \theta] = 0$
foreach edgel $e \in I[x, y]$ **do**
    $\theta = \text{angle}(\nabla(I[x, y]))$
    $\rho = x \cos\theta + y \sin\theta$
    $H[\rho, \theta] \mathrel{+}= 1$

end
$\rho^*, \theta^* = \underset{\rho, \theta}{\text{argmax}}\ H[\rho, \theta]$

Hough Lines

extension #2

Initialize $H[\rho, \theta] = 0$

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Hough Lines

extension #2

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Initialize $H[\rho, \theta] = 0$
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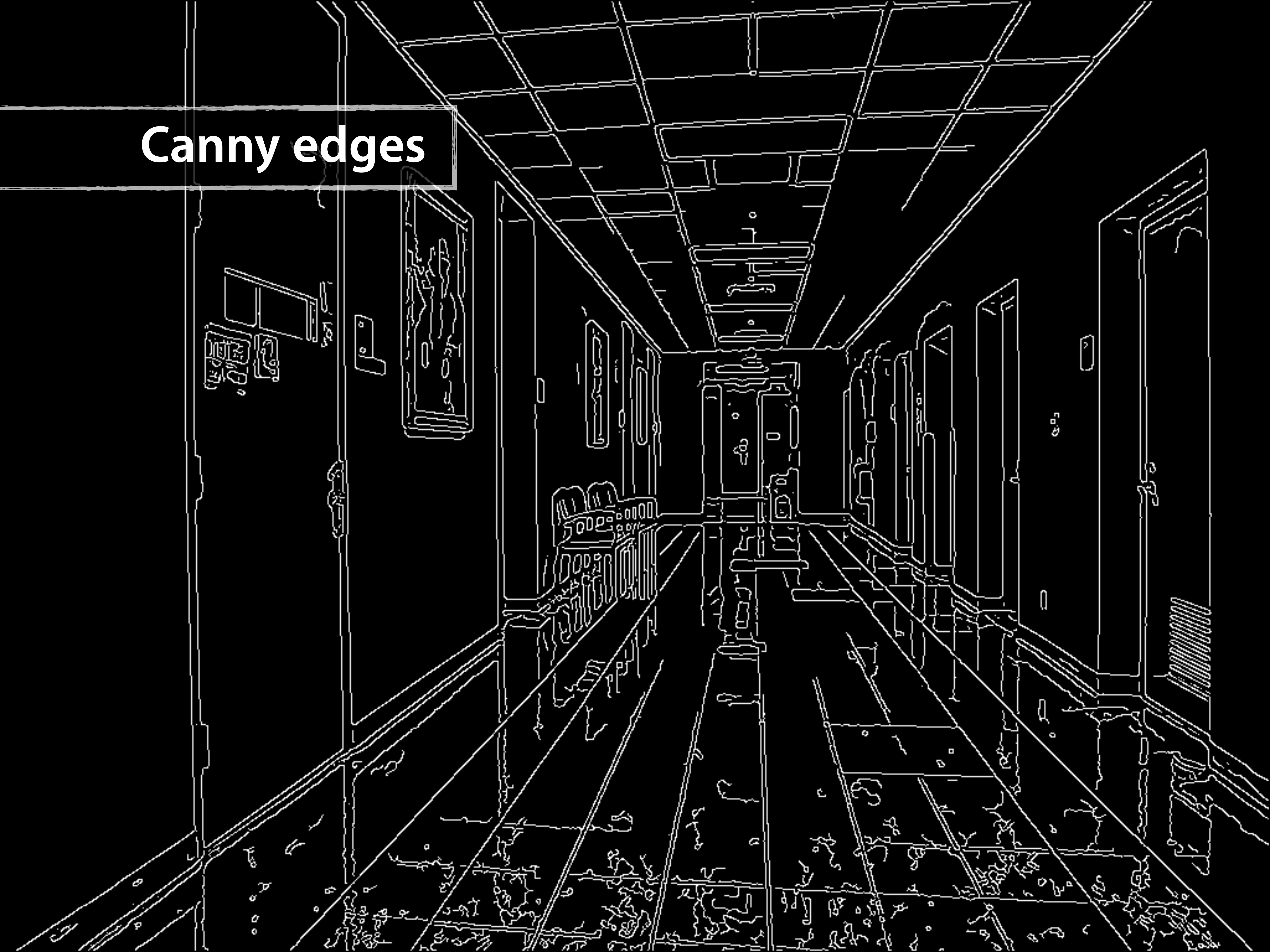
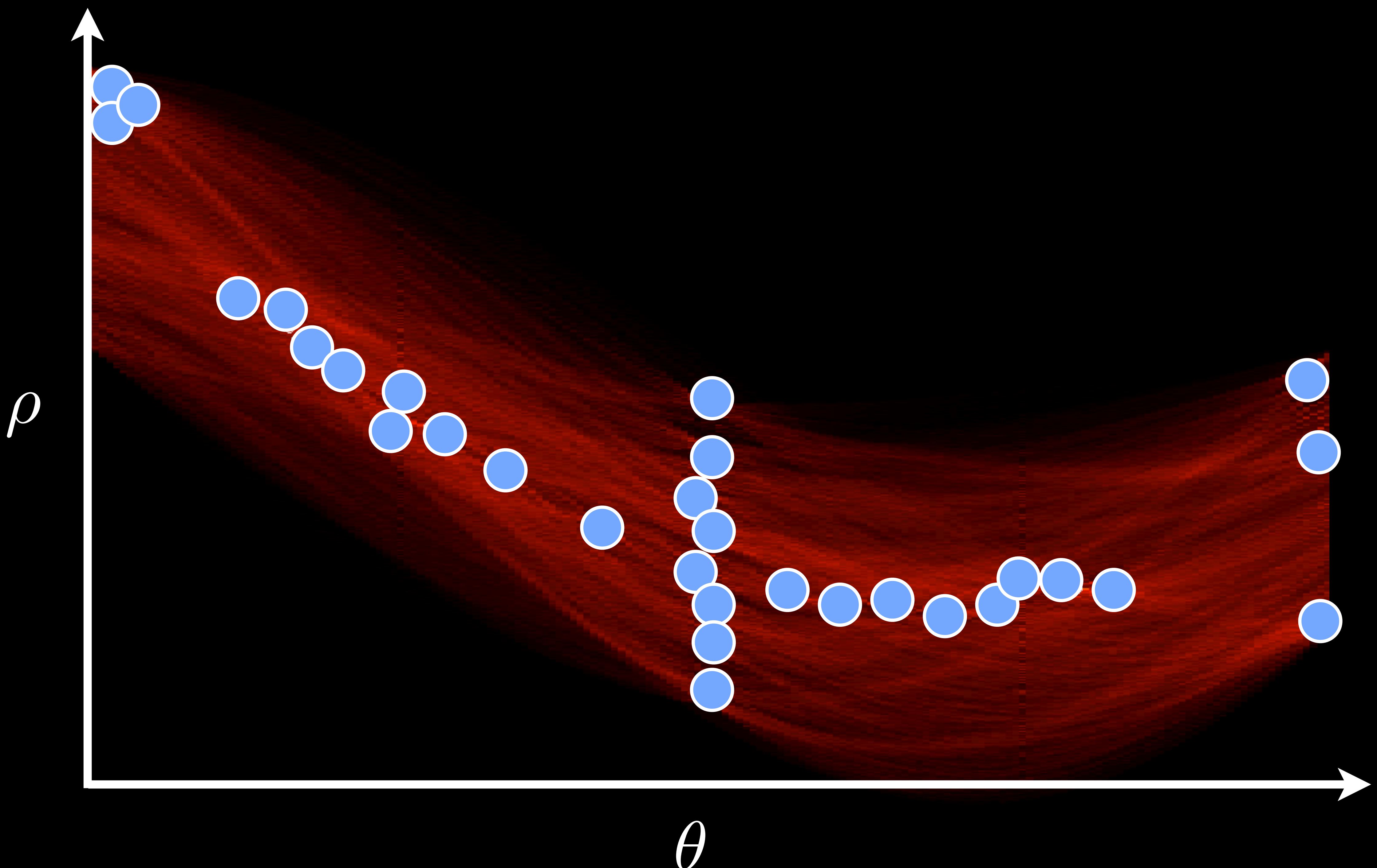
end
$\rho^*, \theta^* = \underset{\rho, \theta}{\mathrm{argmax}}\; H[\rho, \theta]$
```


input image


Canny edges


Hough lines

Hough space




Lane detection


Lane detection


Lane detection


Lane detection

Hough Circle Transform

Assume
Fixed Radius

$$(x_i - \alpha)^2 + (y_i - \beta)^2 = r^2$$

Assume Fixed Radius

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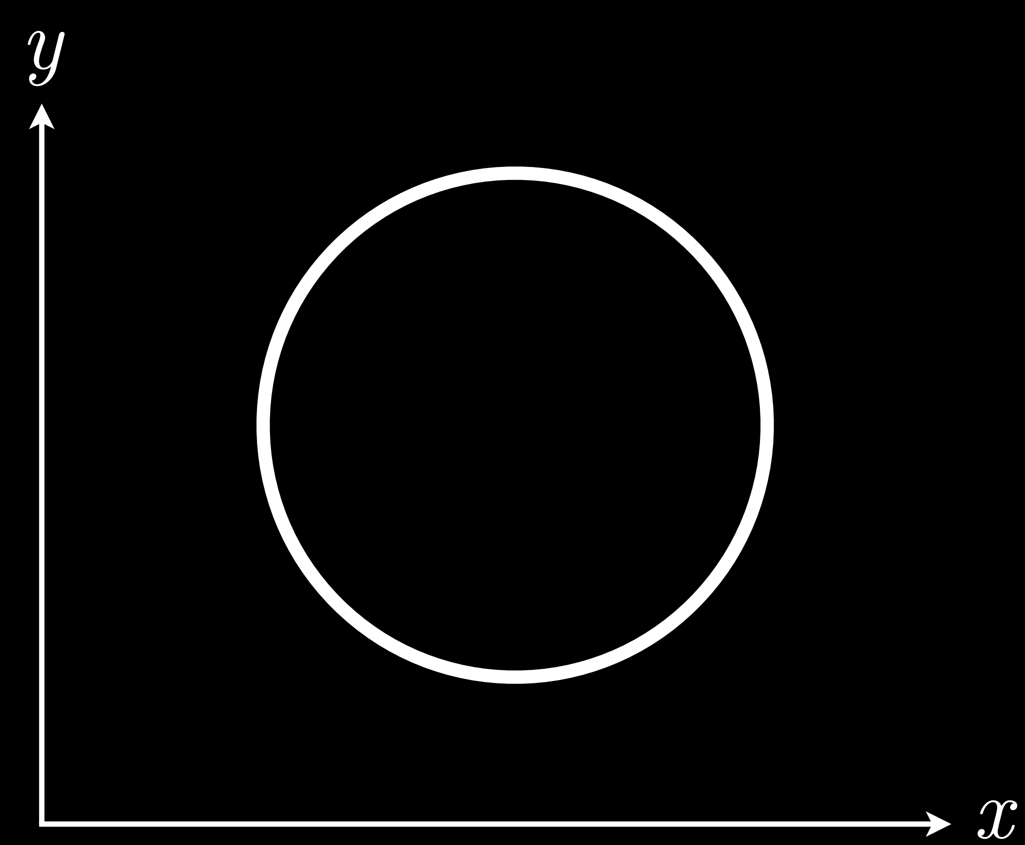
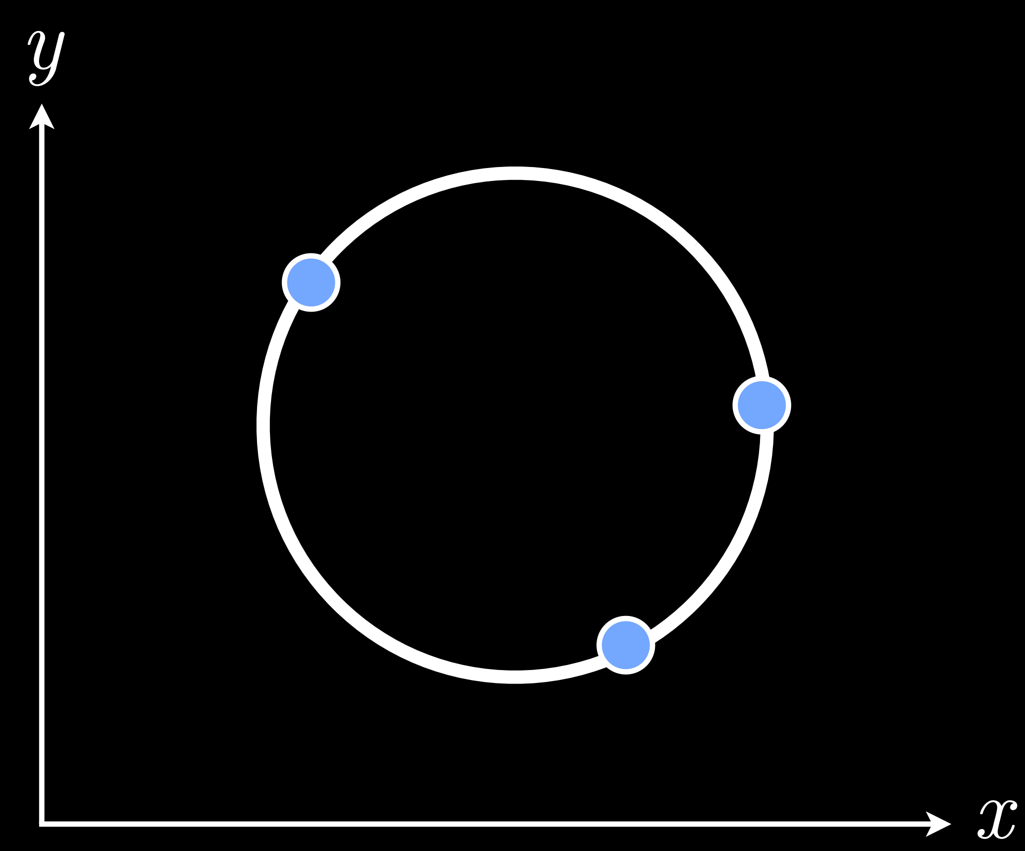


Assume Fixed Radius

$$(x_i - \alpha)^2 + (y_i - \beta)^2 = r^2$$



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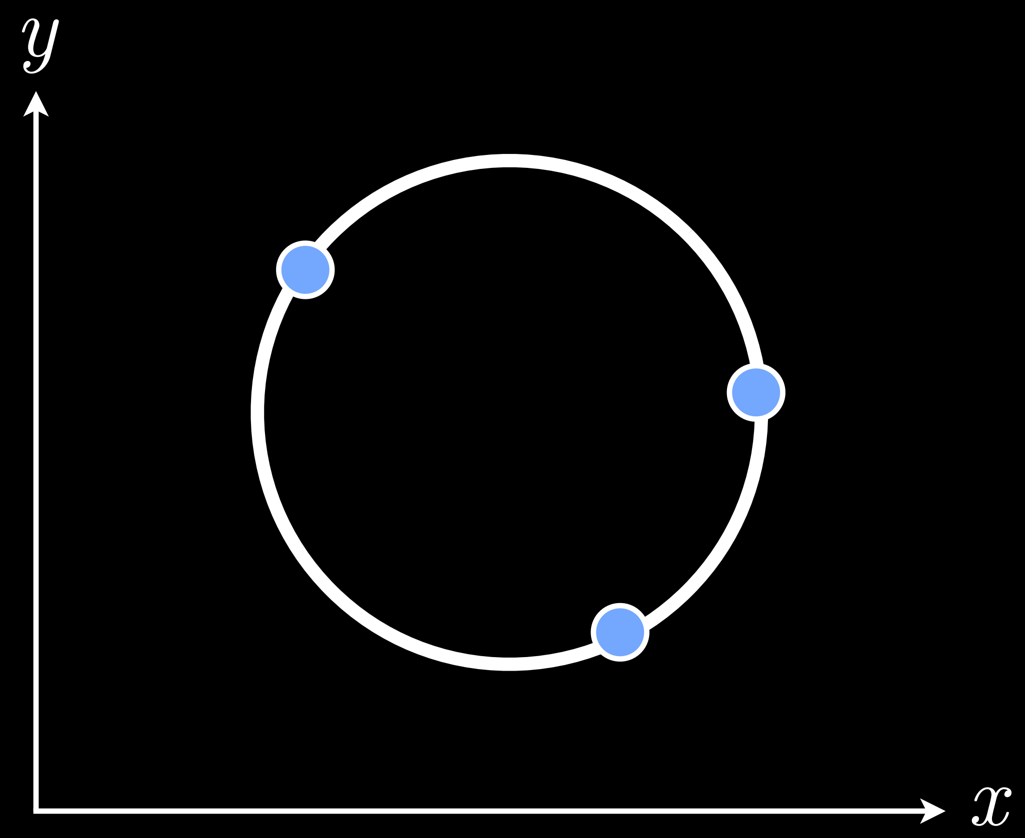
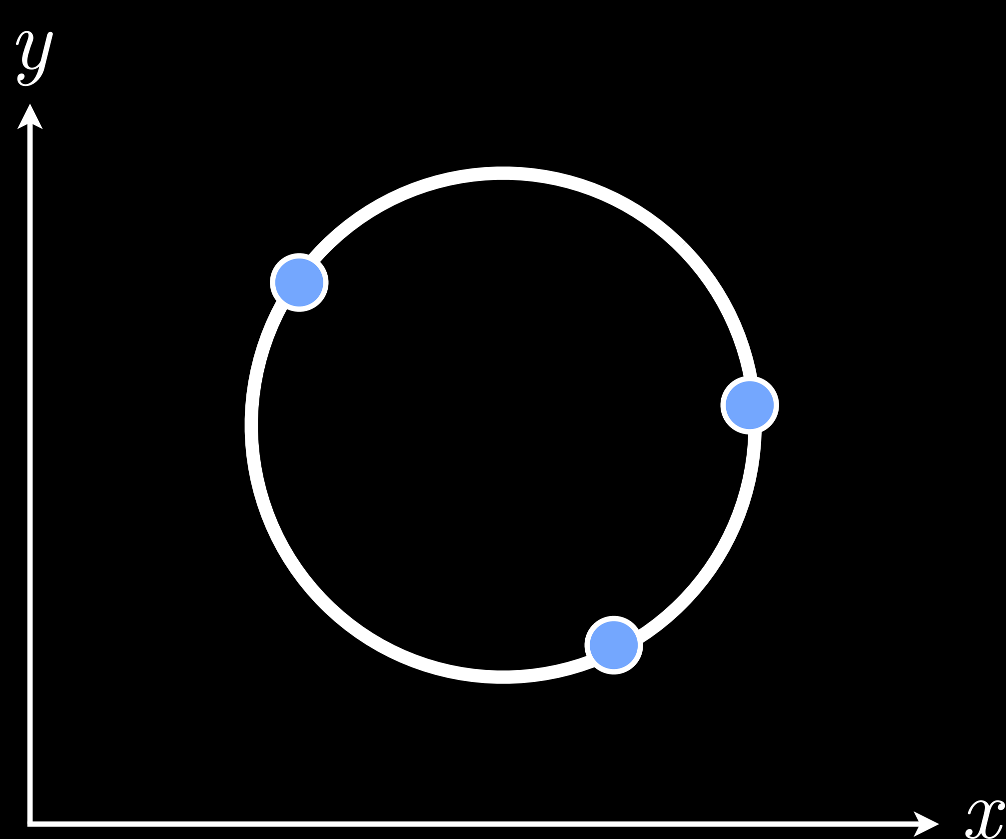
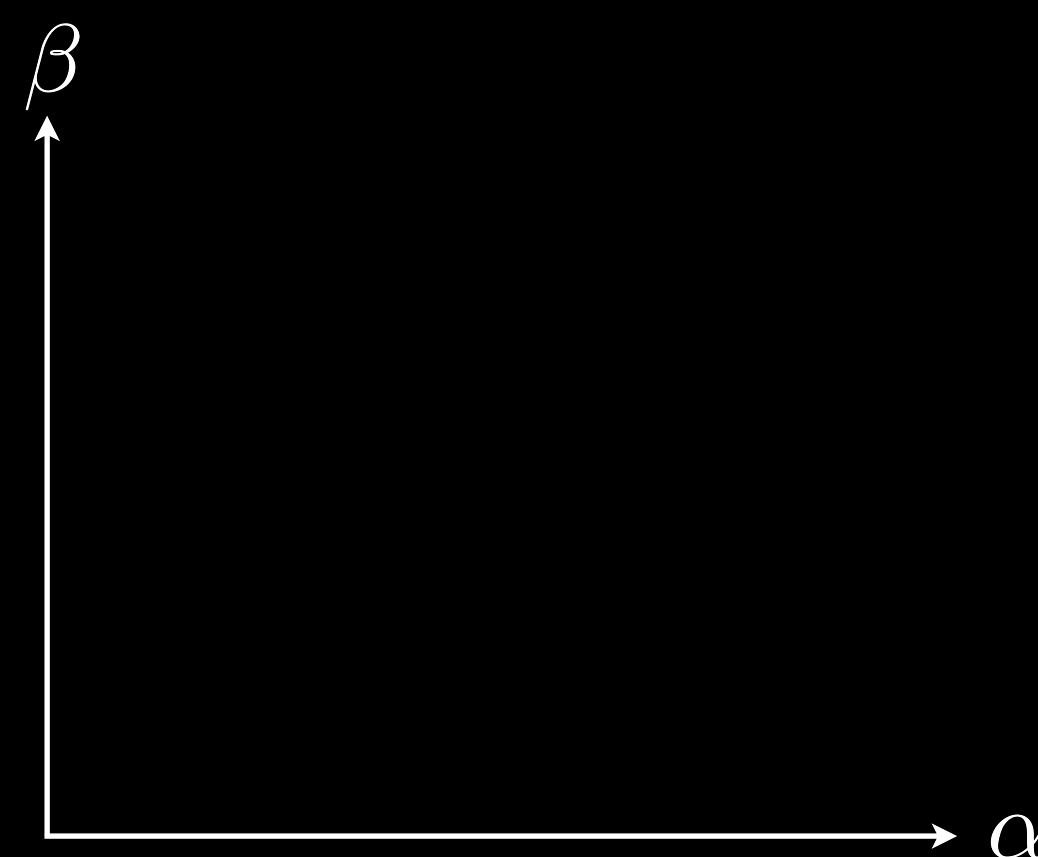


Parameter space?

$$(x_i - \alpha)^2 + (y_i - \beta)^2 = r^2$$





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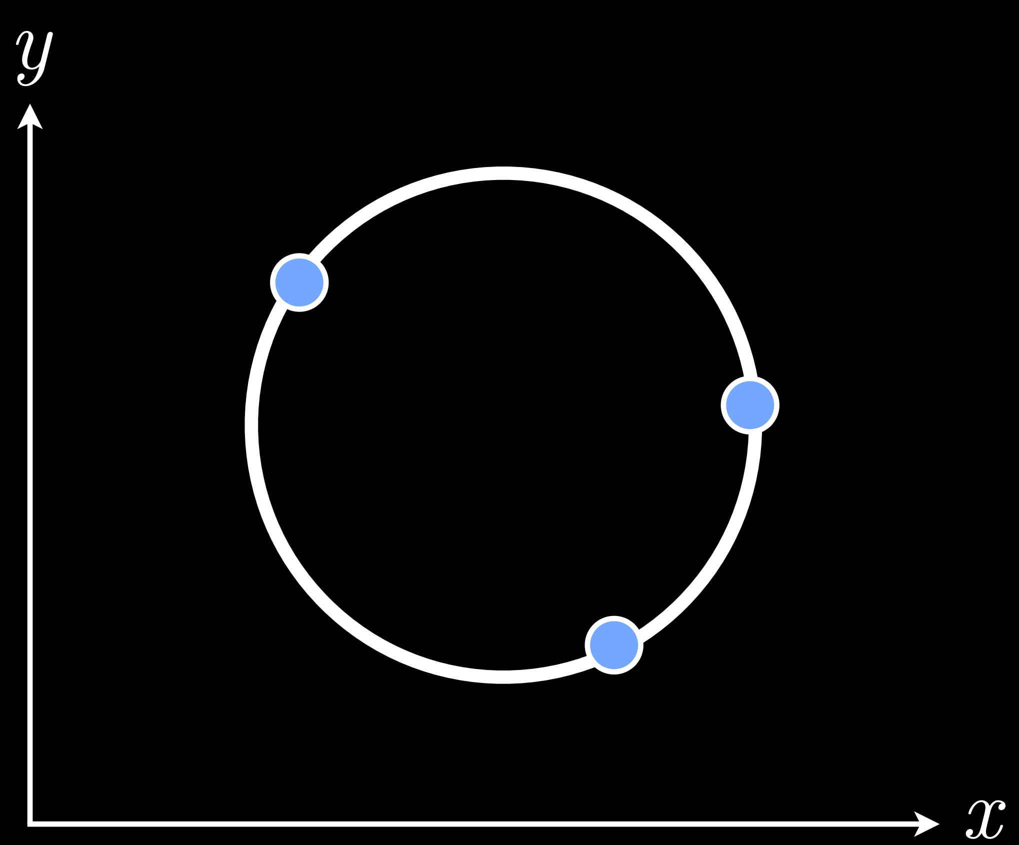
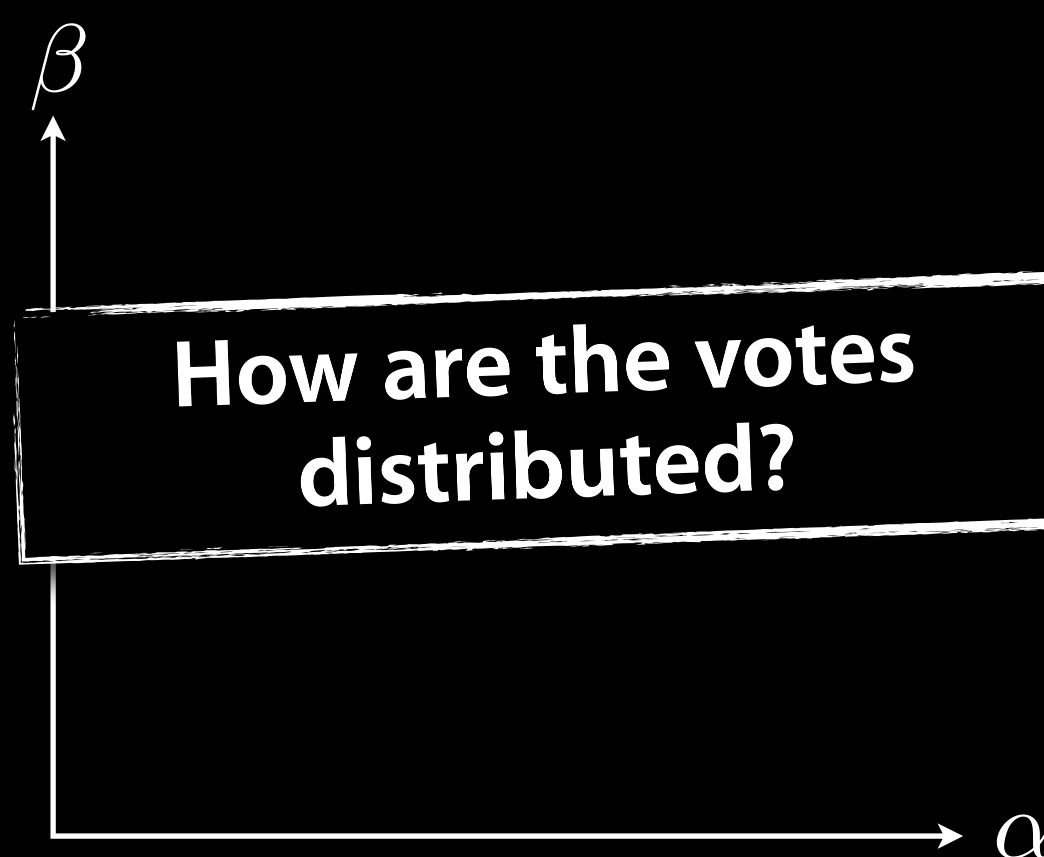
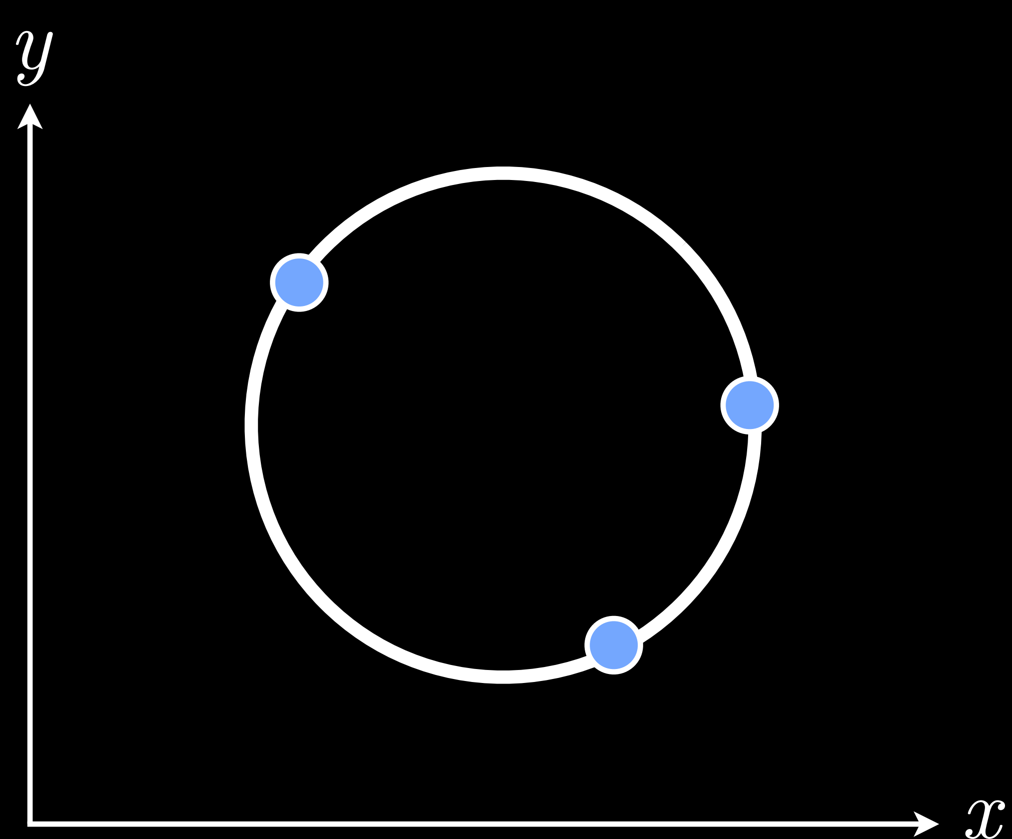


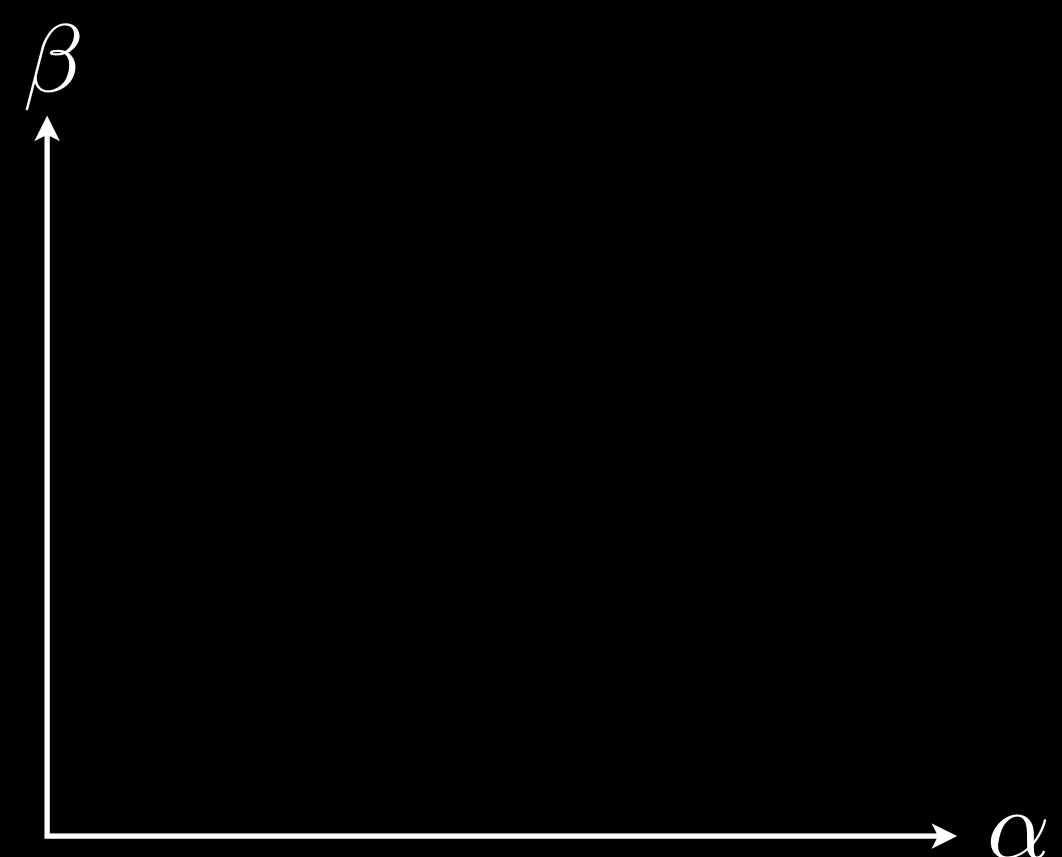


How are the votes distributed?

Assume Fixed Radius

$$(x_i - \alpha)^2 + (y_i - \beta)^2 = r^2$$





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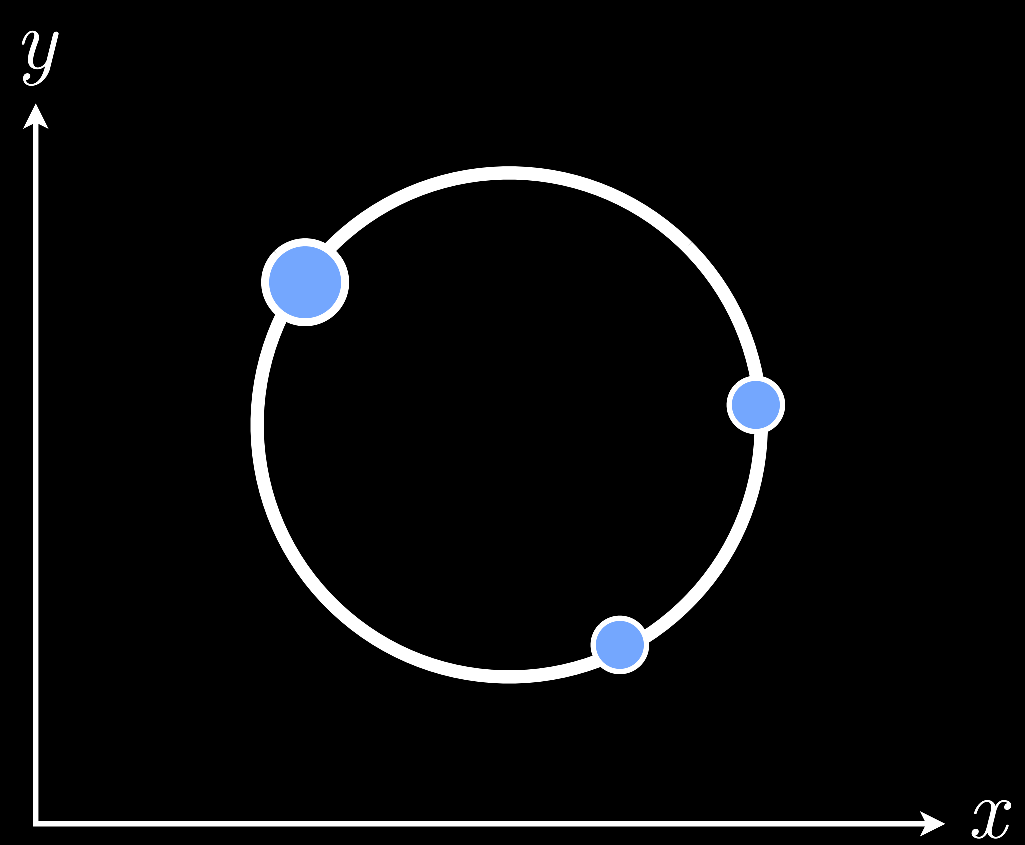

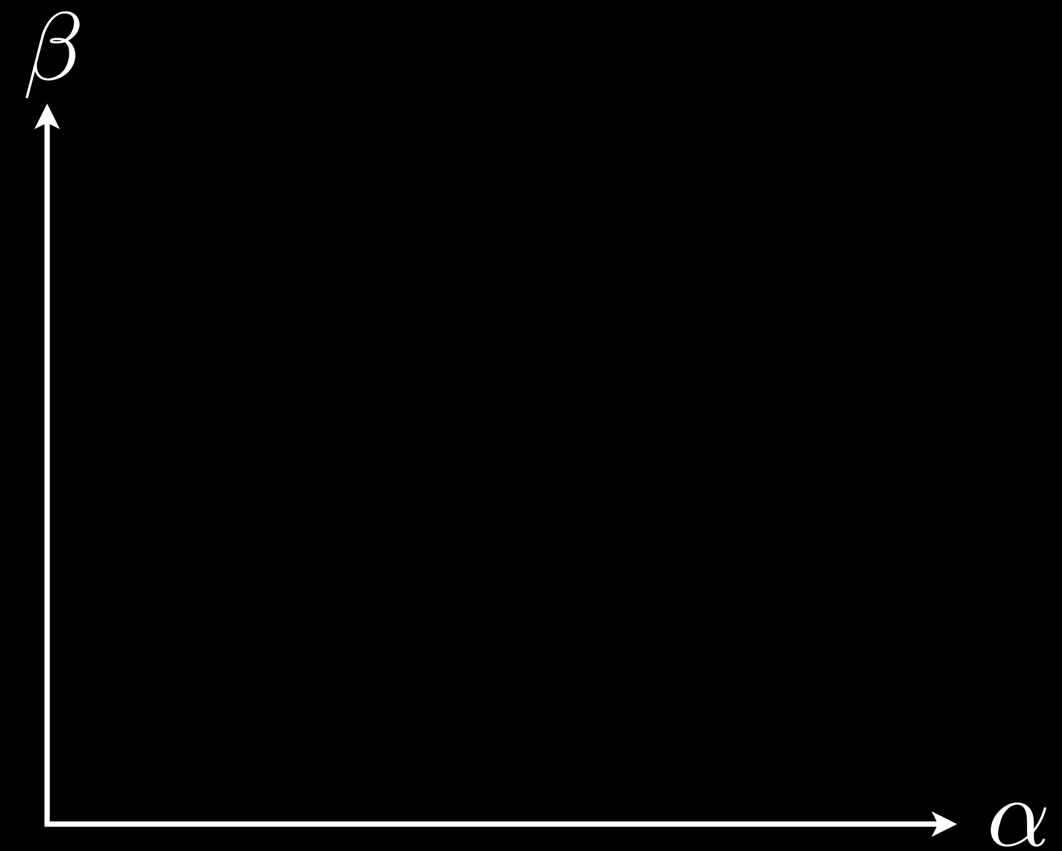

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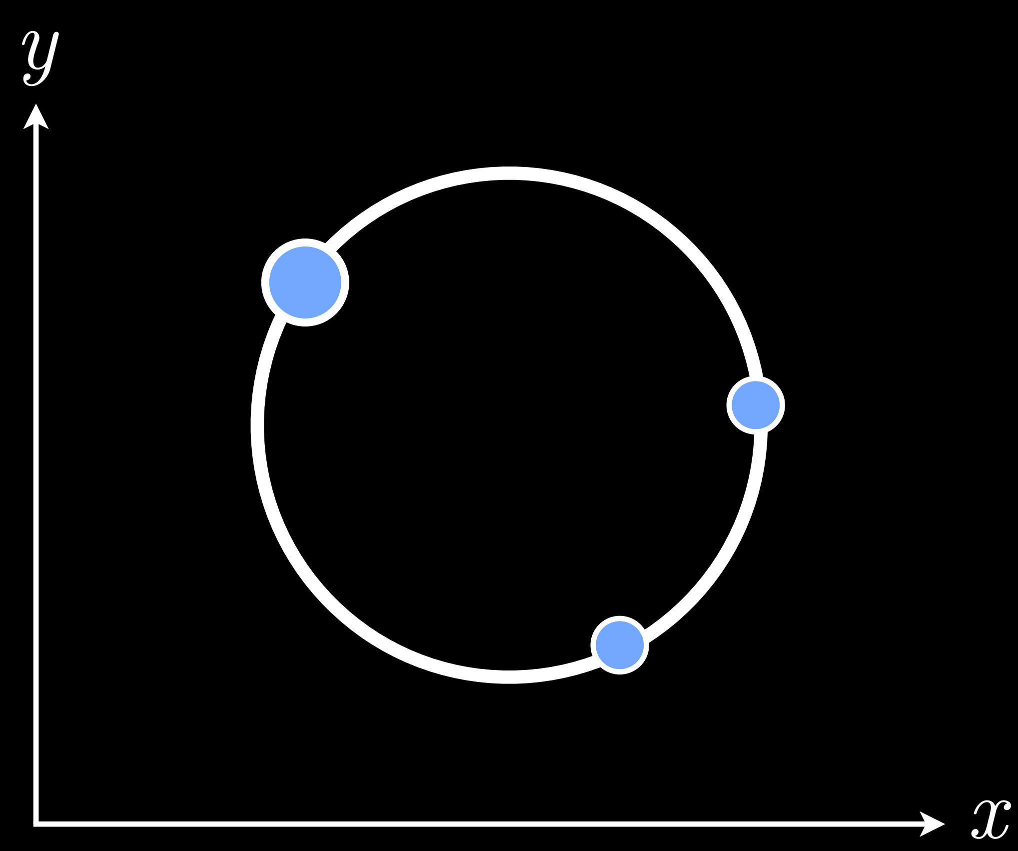

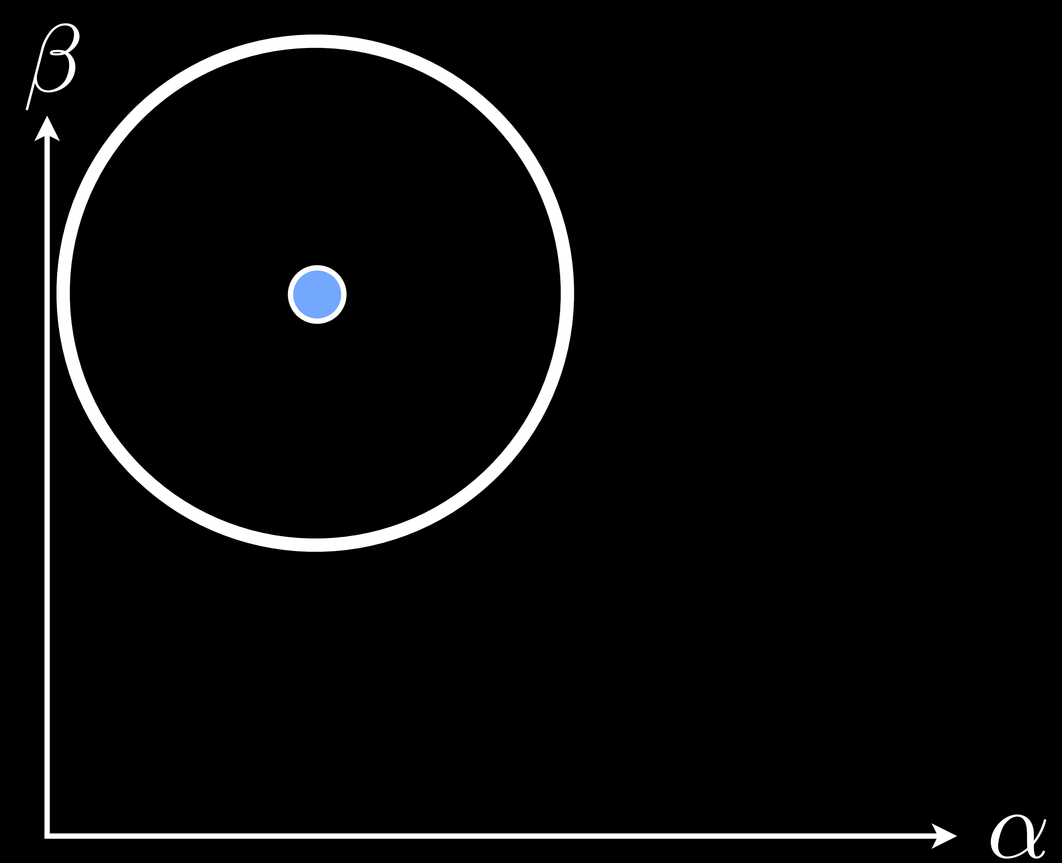

Assume Fixed Radius

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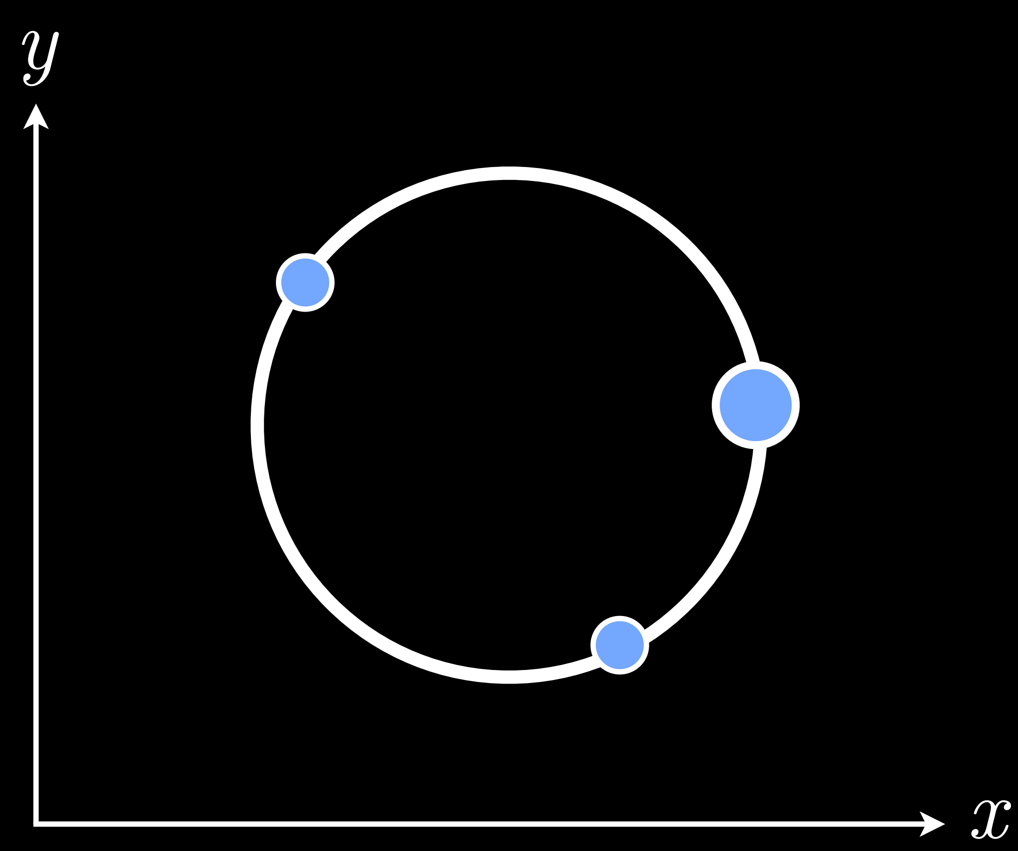
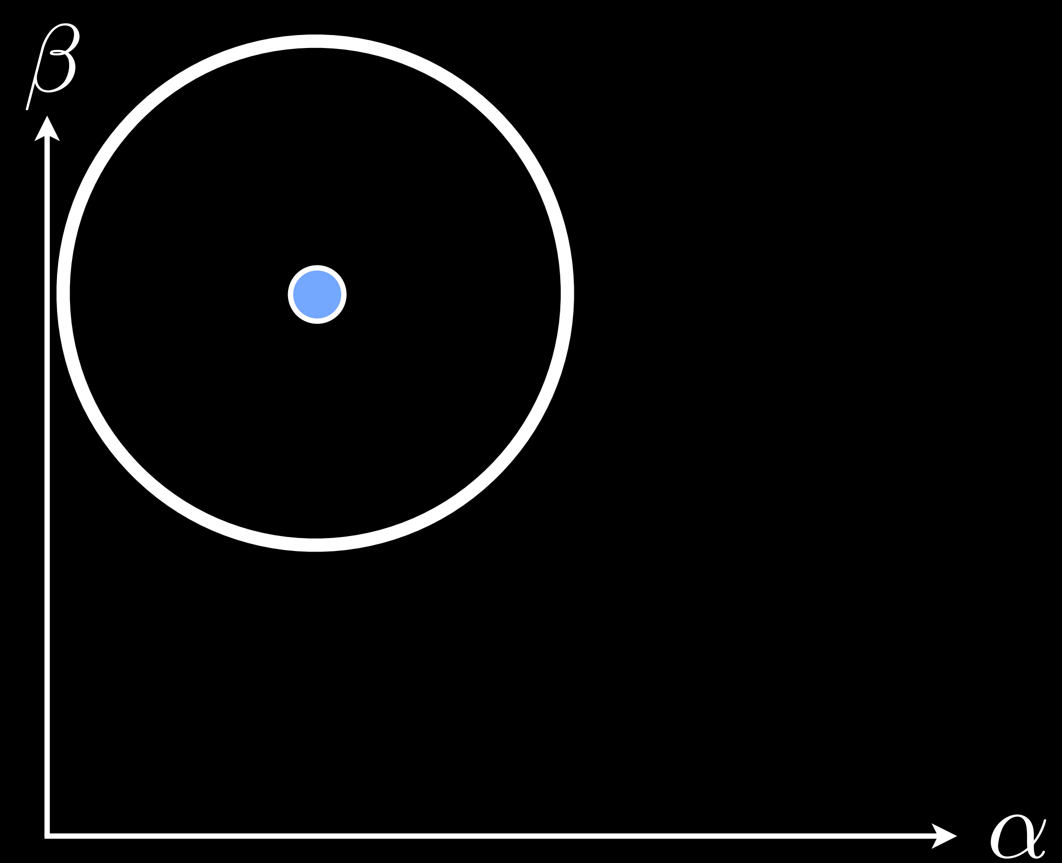
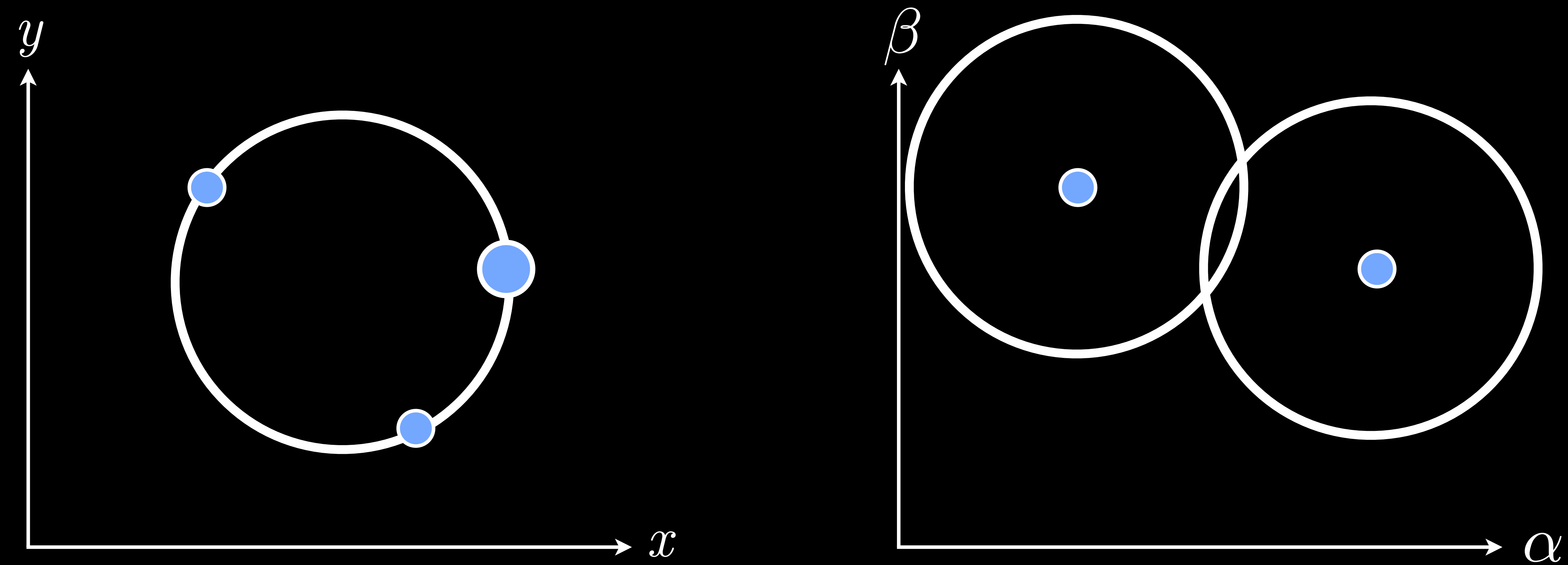




$$(x_i - \alpha)^2 + (y_i - \beta)^2 = r^2$$



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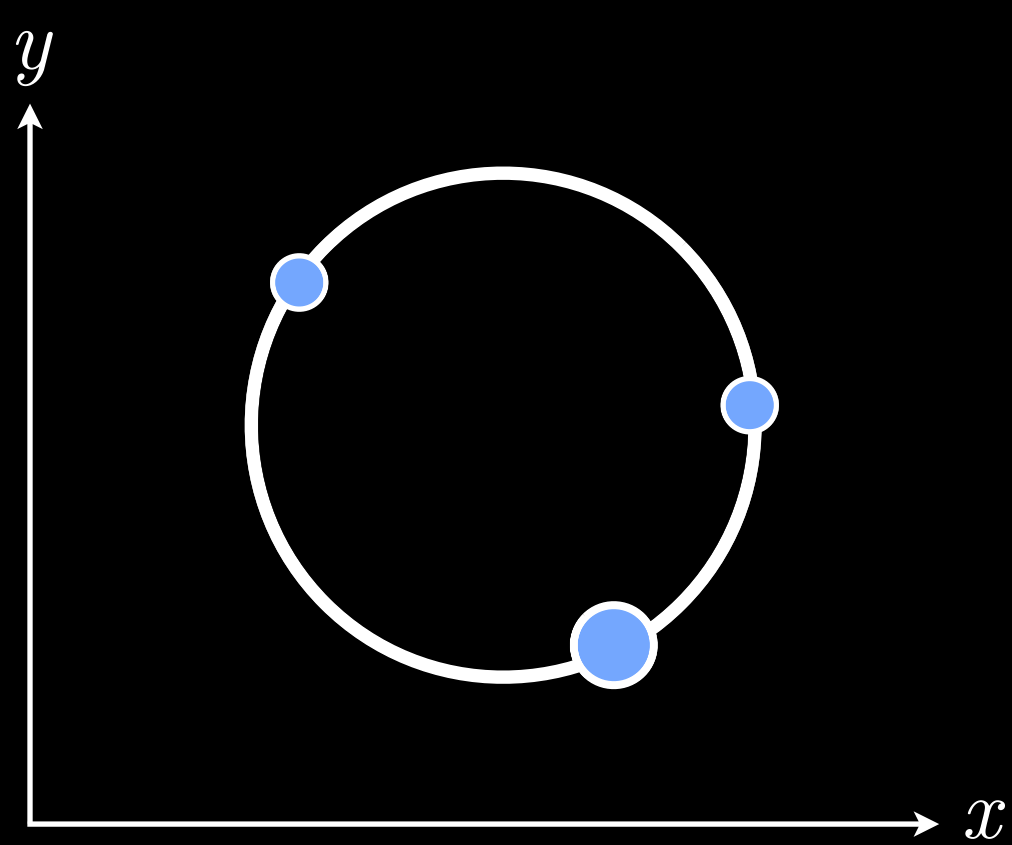

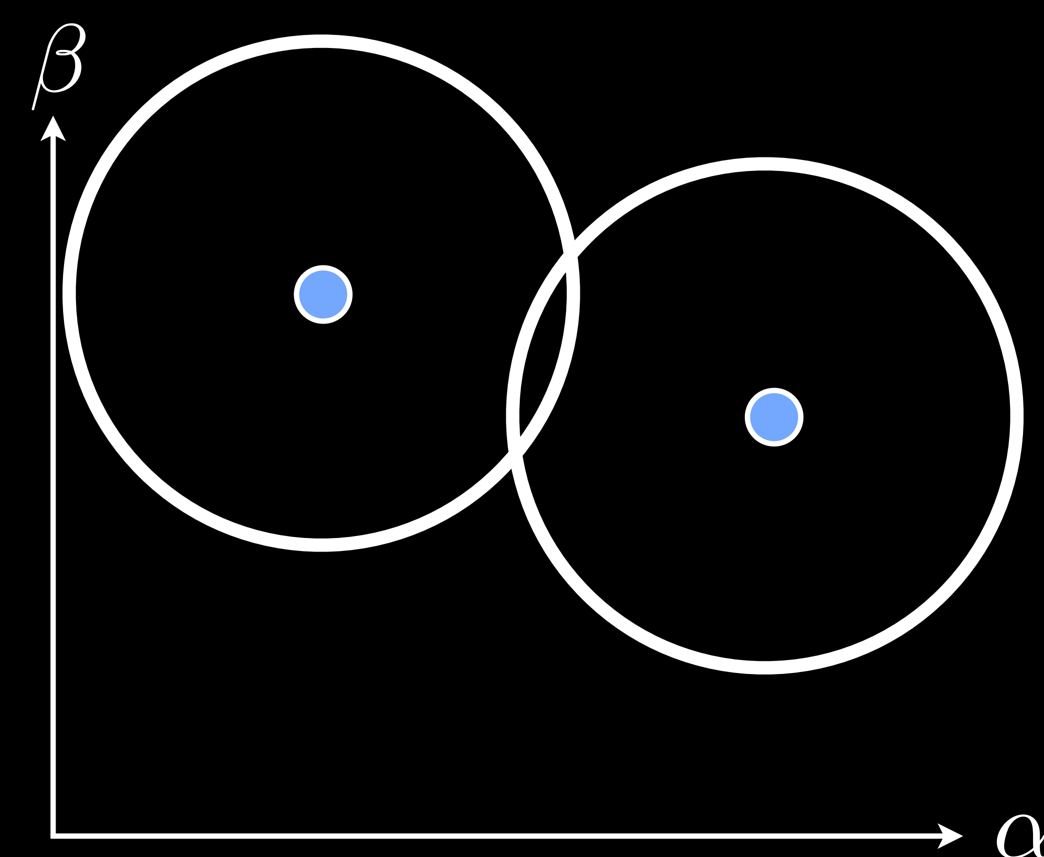

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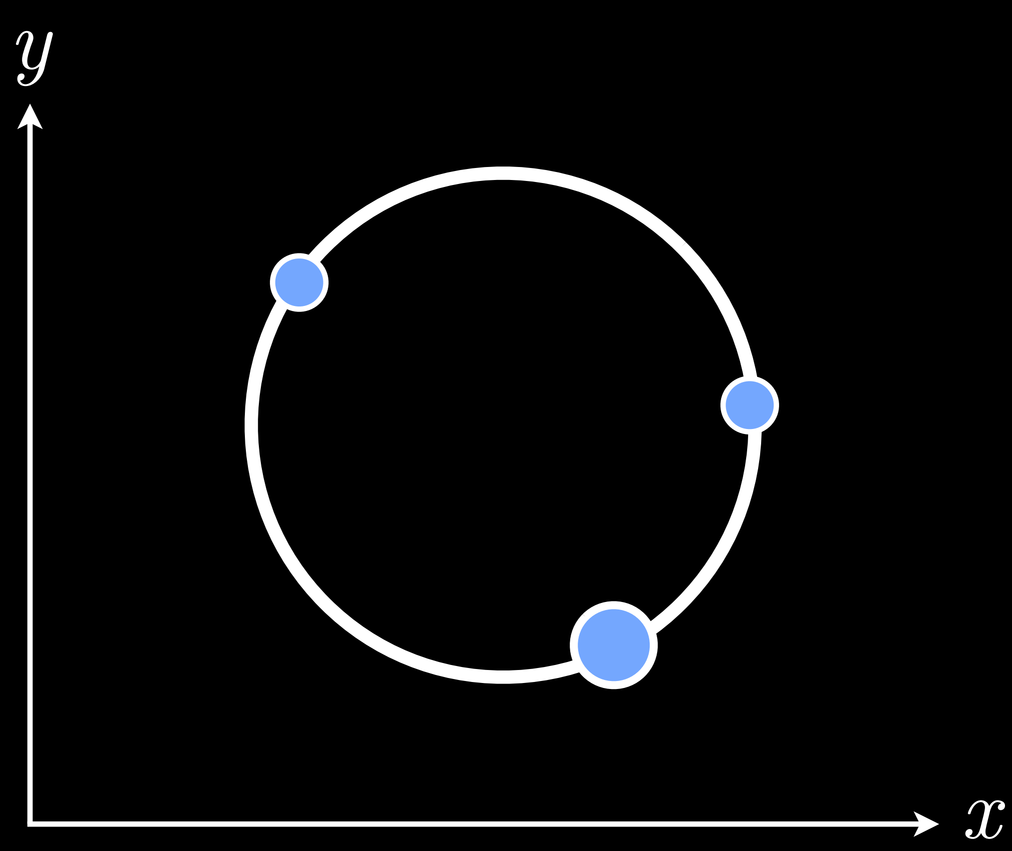
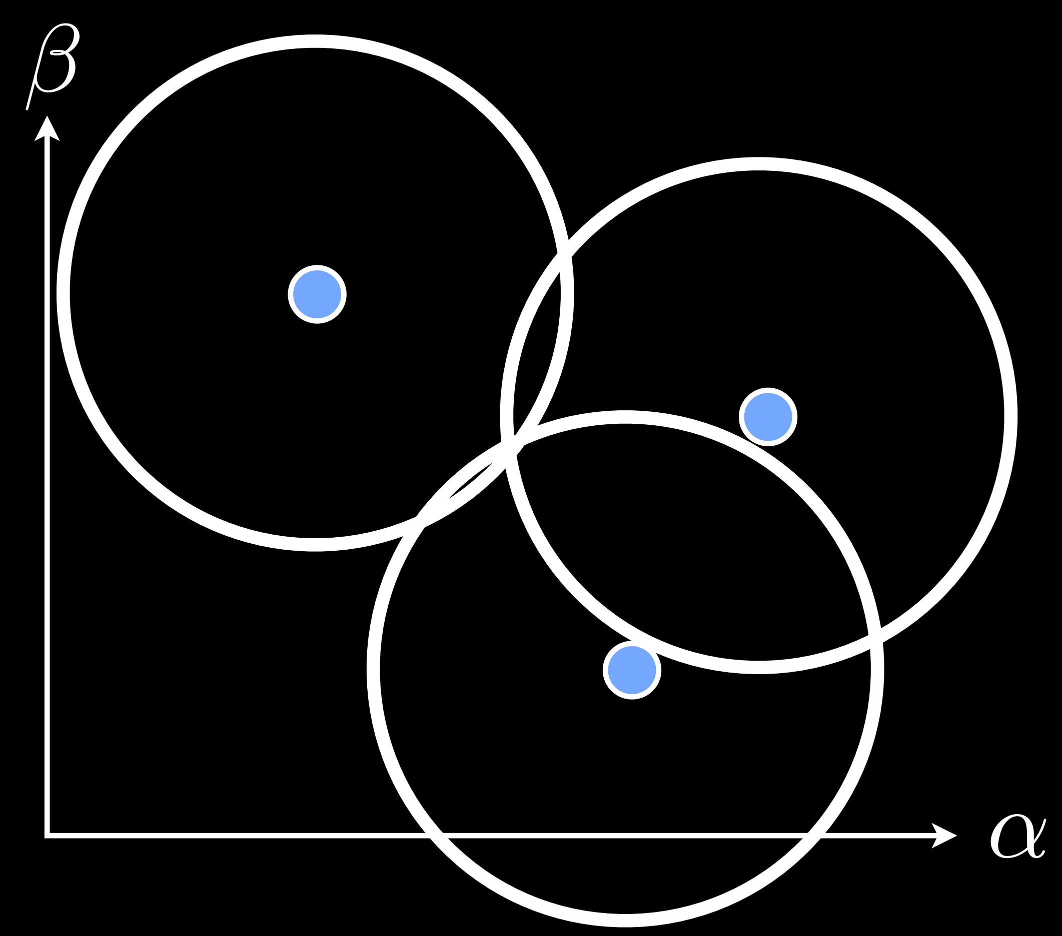
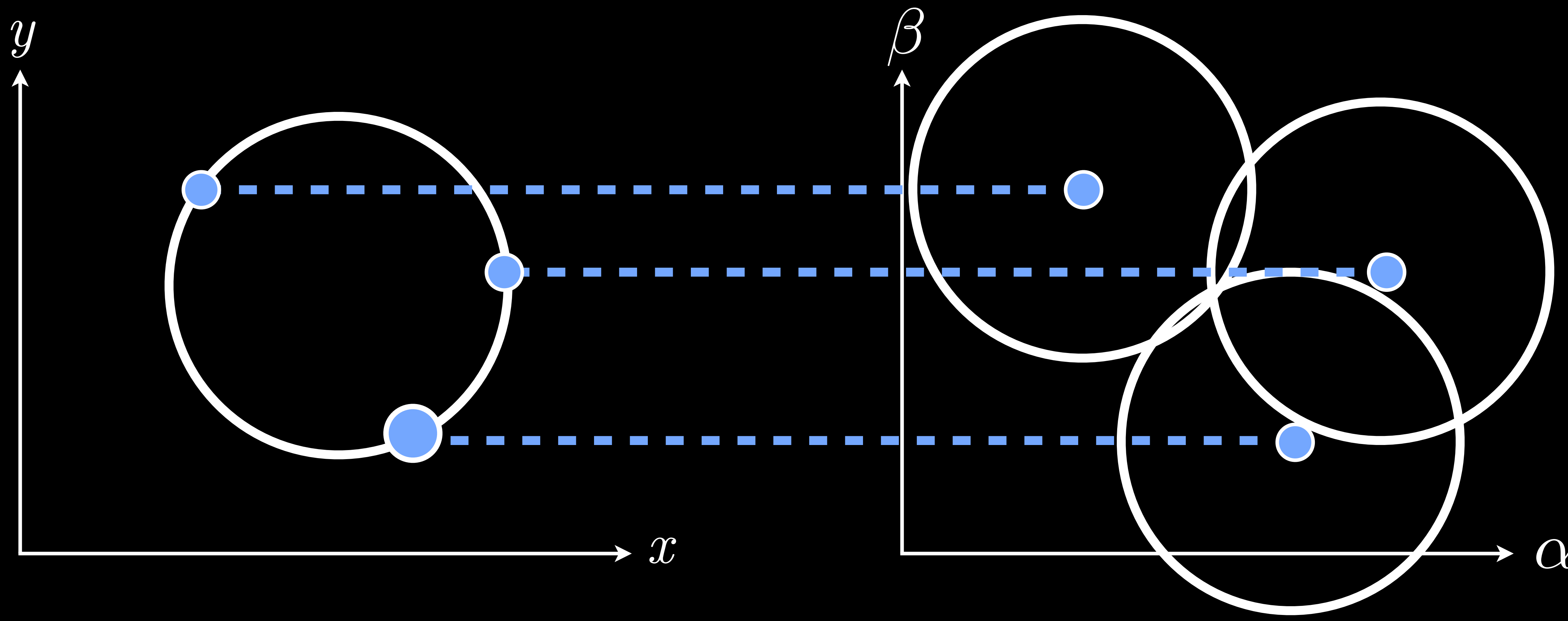




Assume Fixed Radius

$$(x_i - \alpha)^2 + (y_i - \beta)^2 = r^2$$



Assume Fixed Radius

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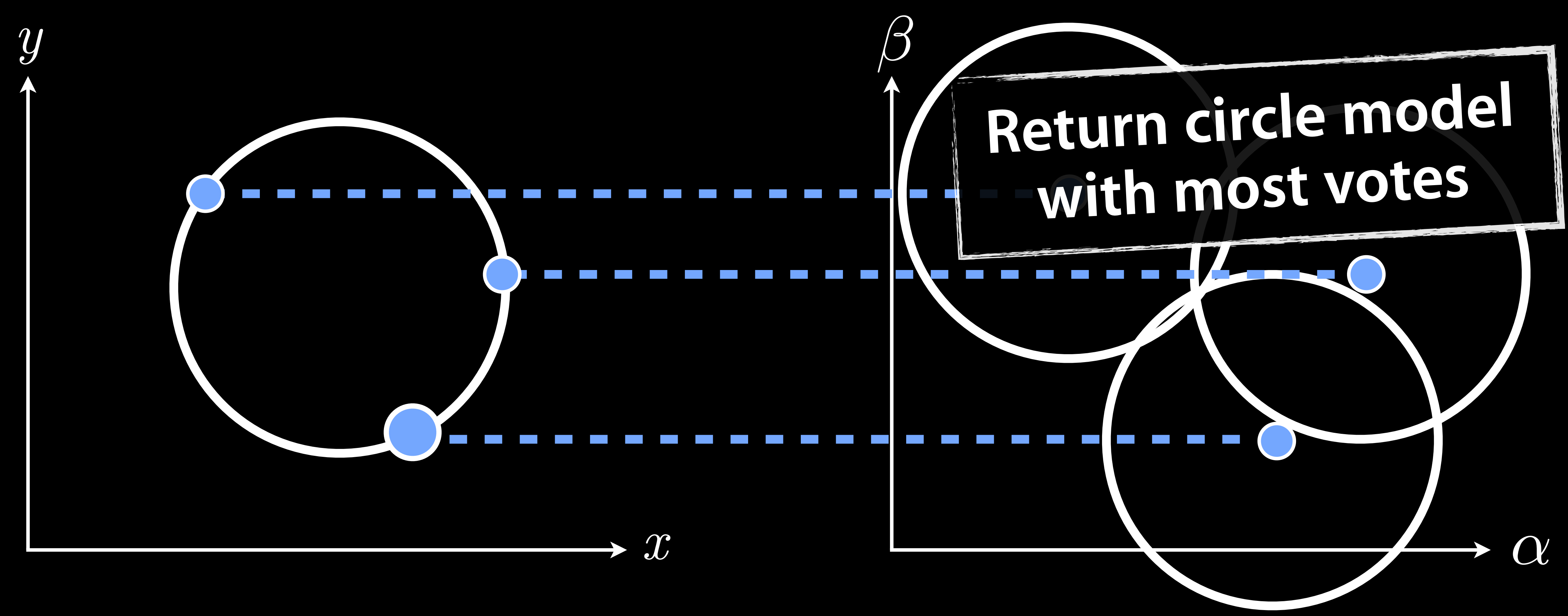

Assume Fixed Radius

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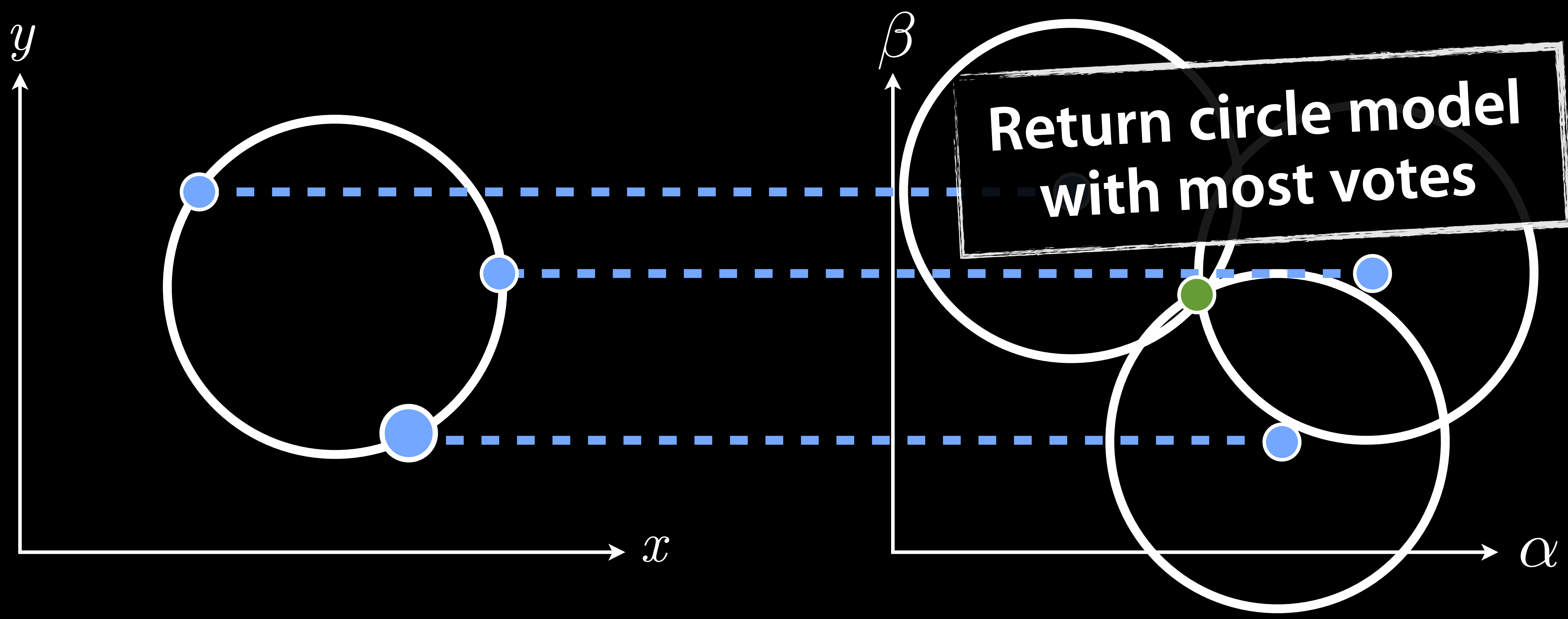

Assume Unknown Radius

$$(x_i - \alpha)^2 + (y_i - \beta)^2 = r^2$$

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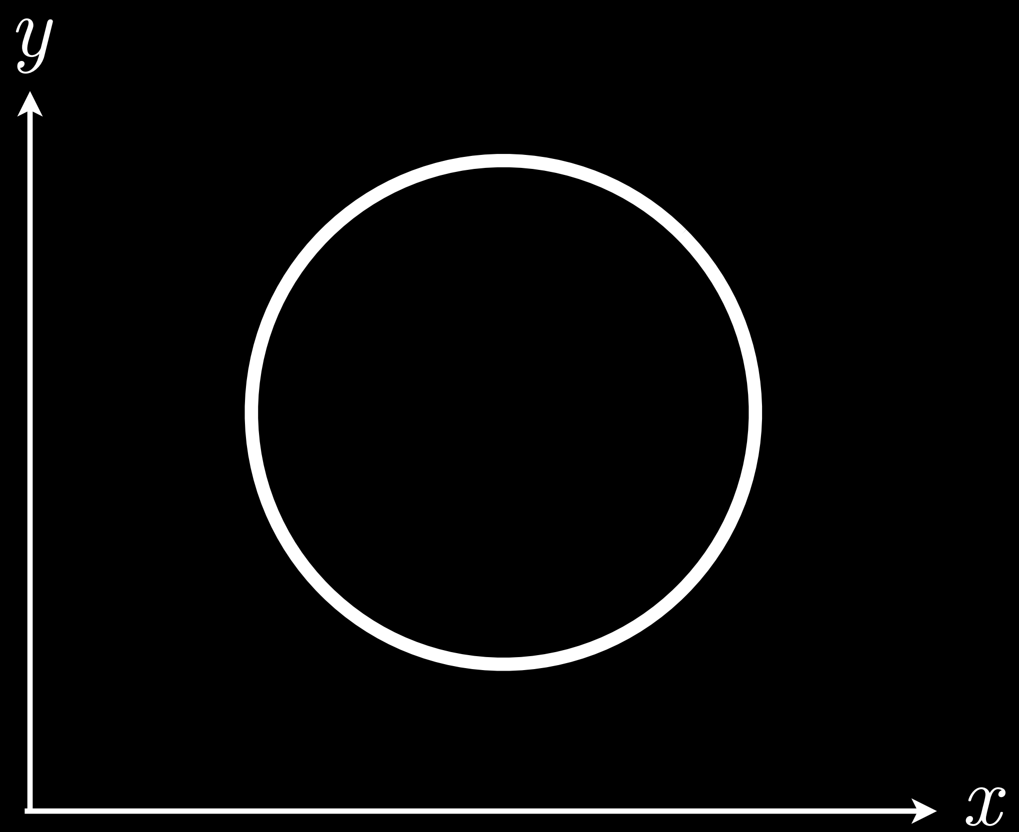

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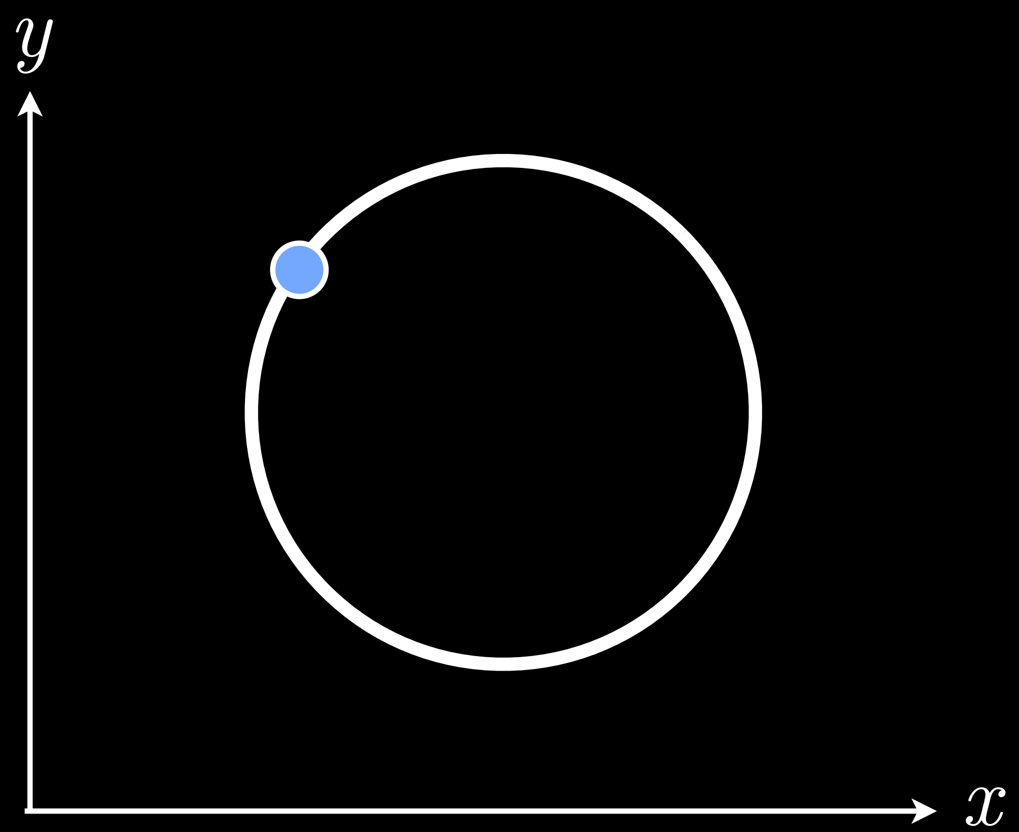

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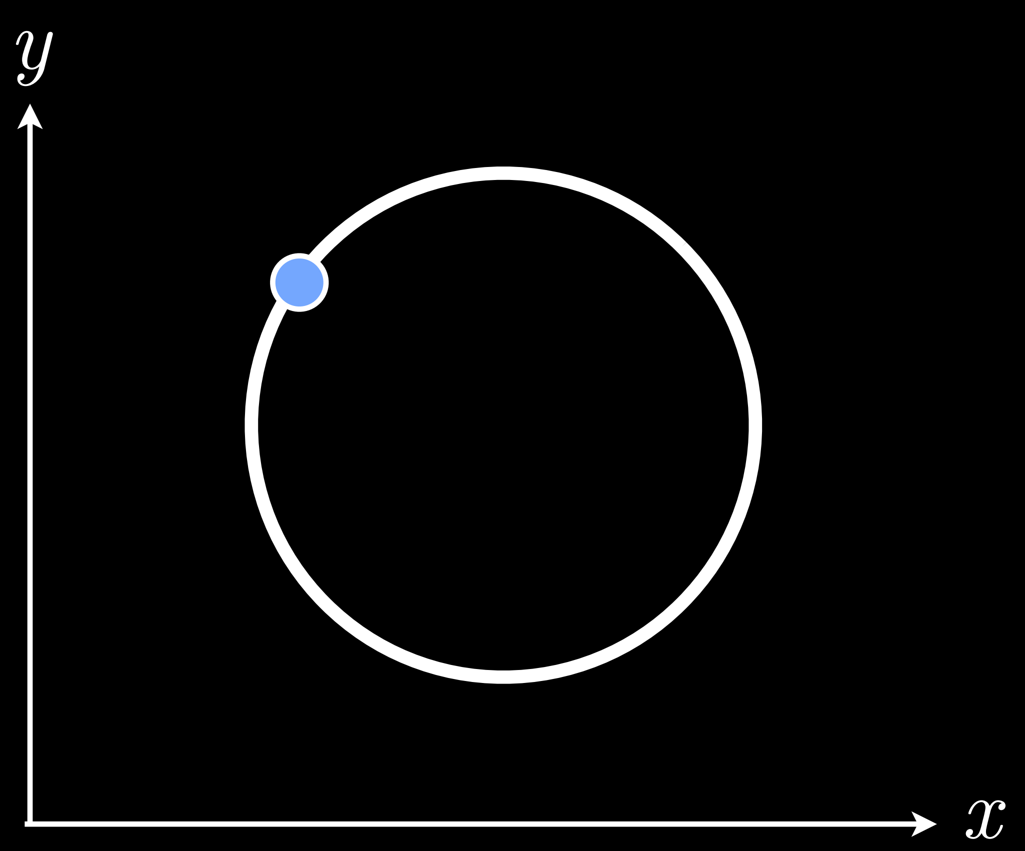
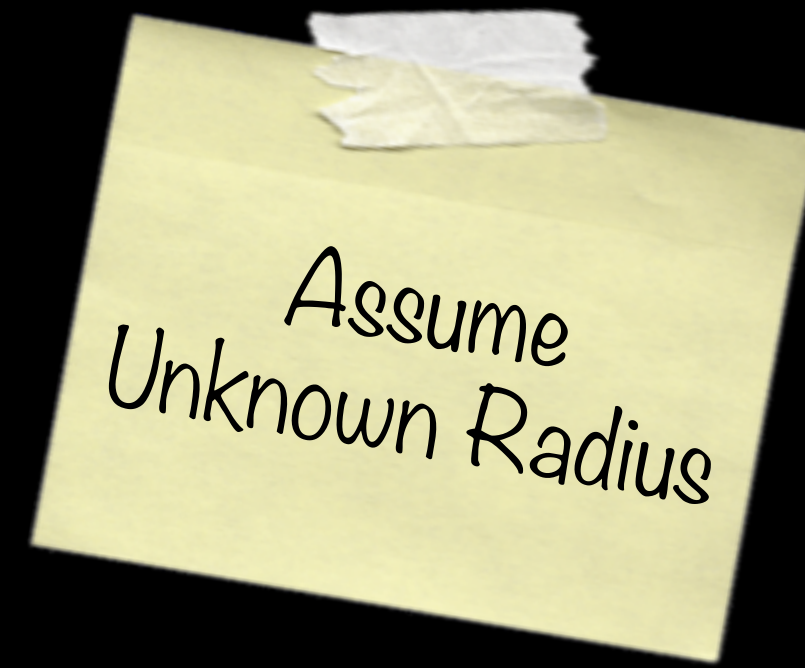
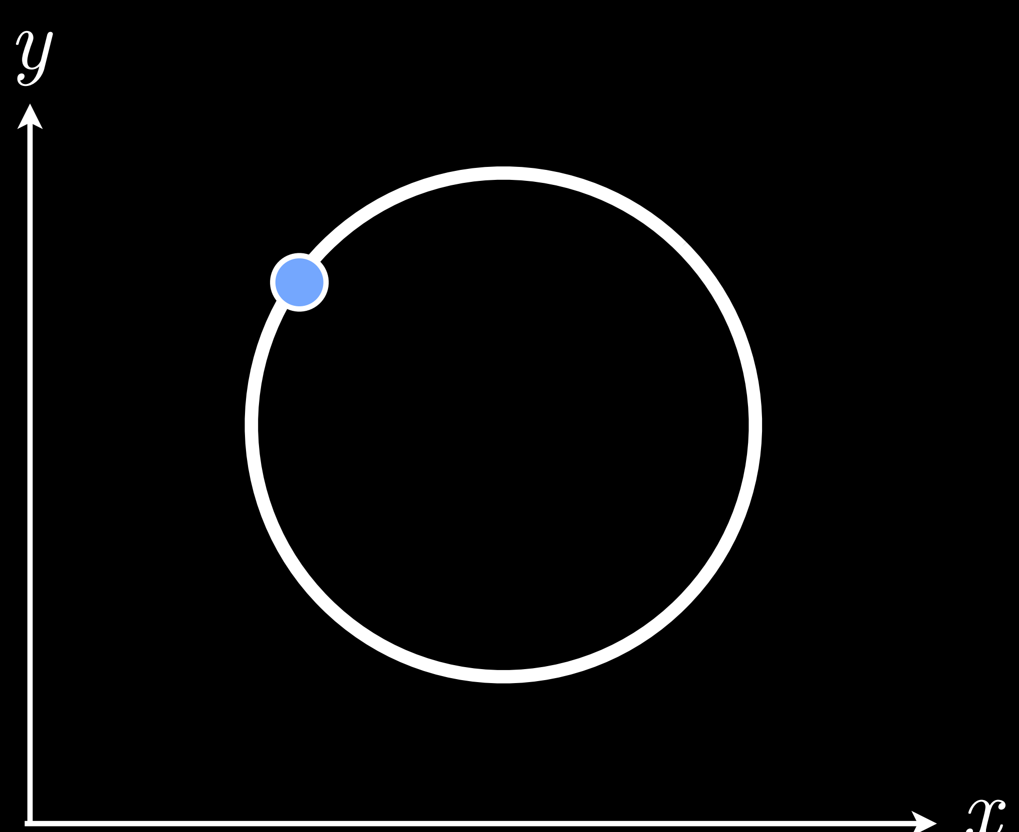
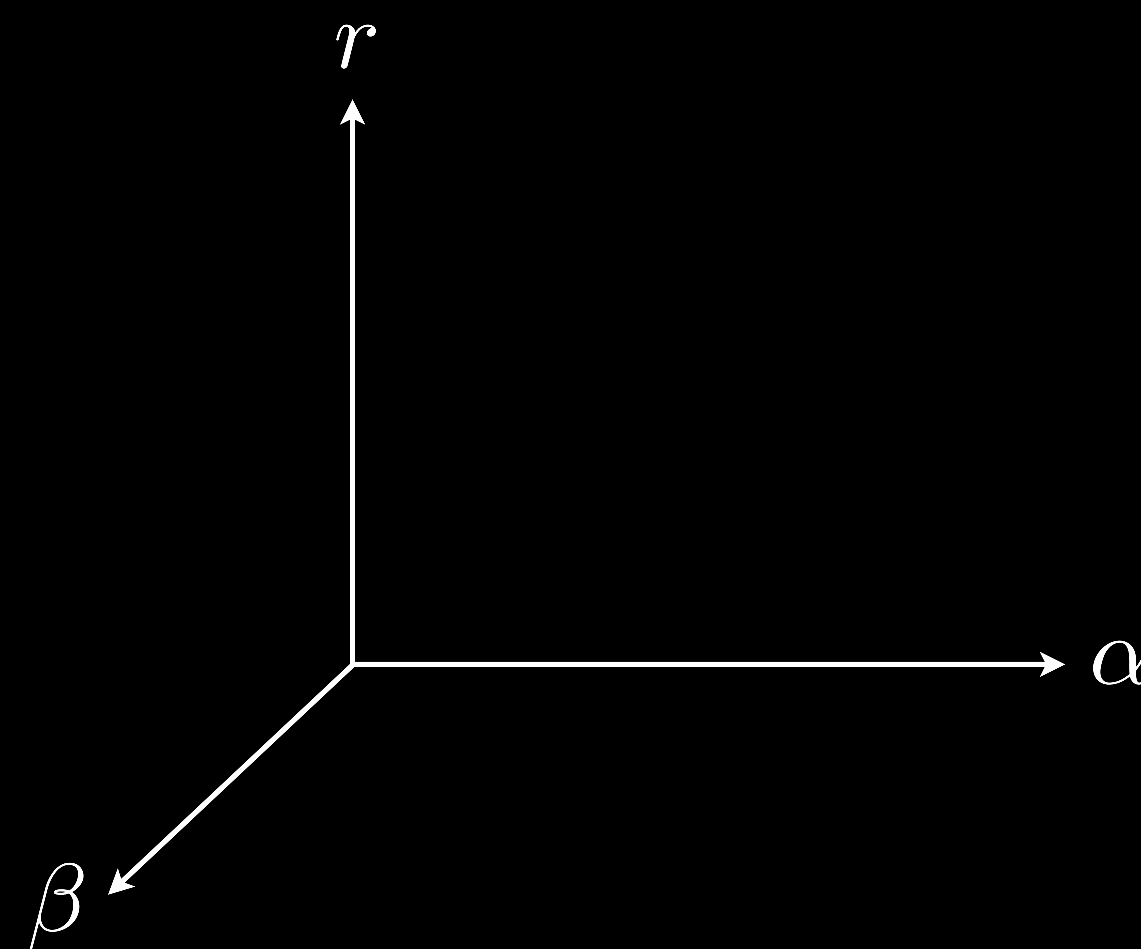


Parameter space?



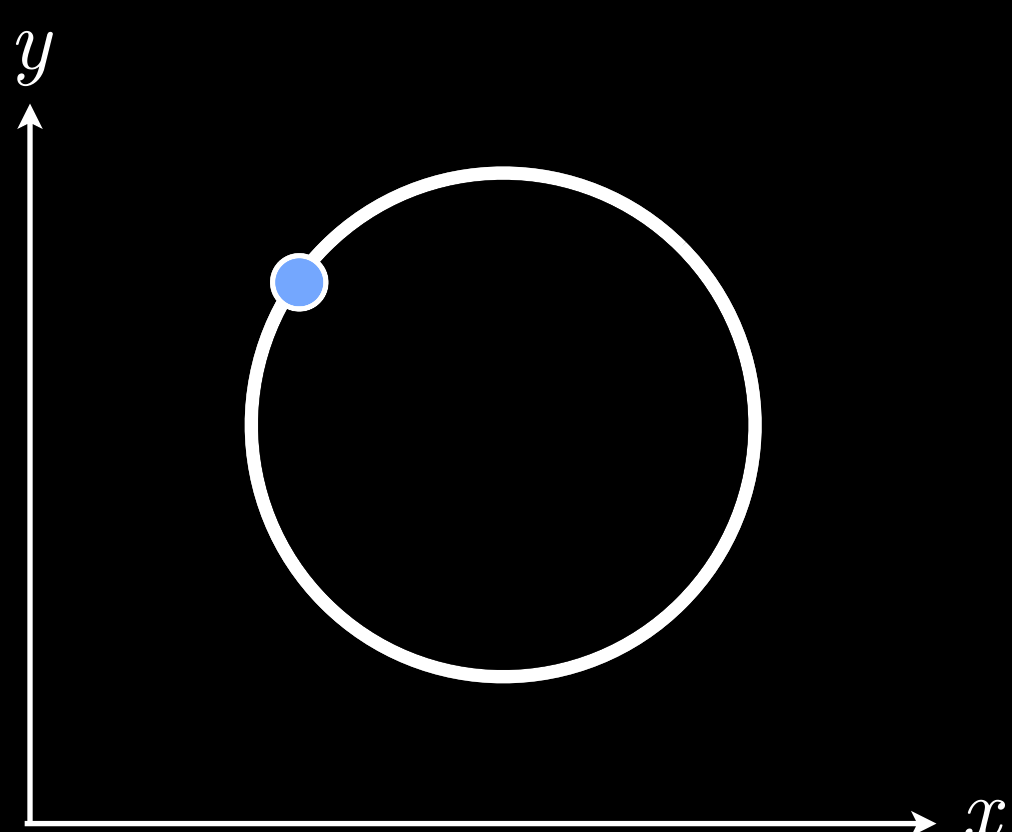
$$(x_i - \alpha)^2 + (y_i - \beta)^2 = r^2$$

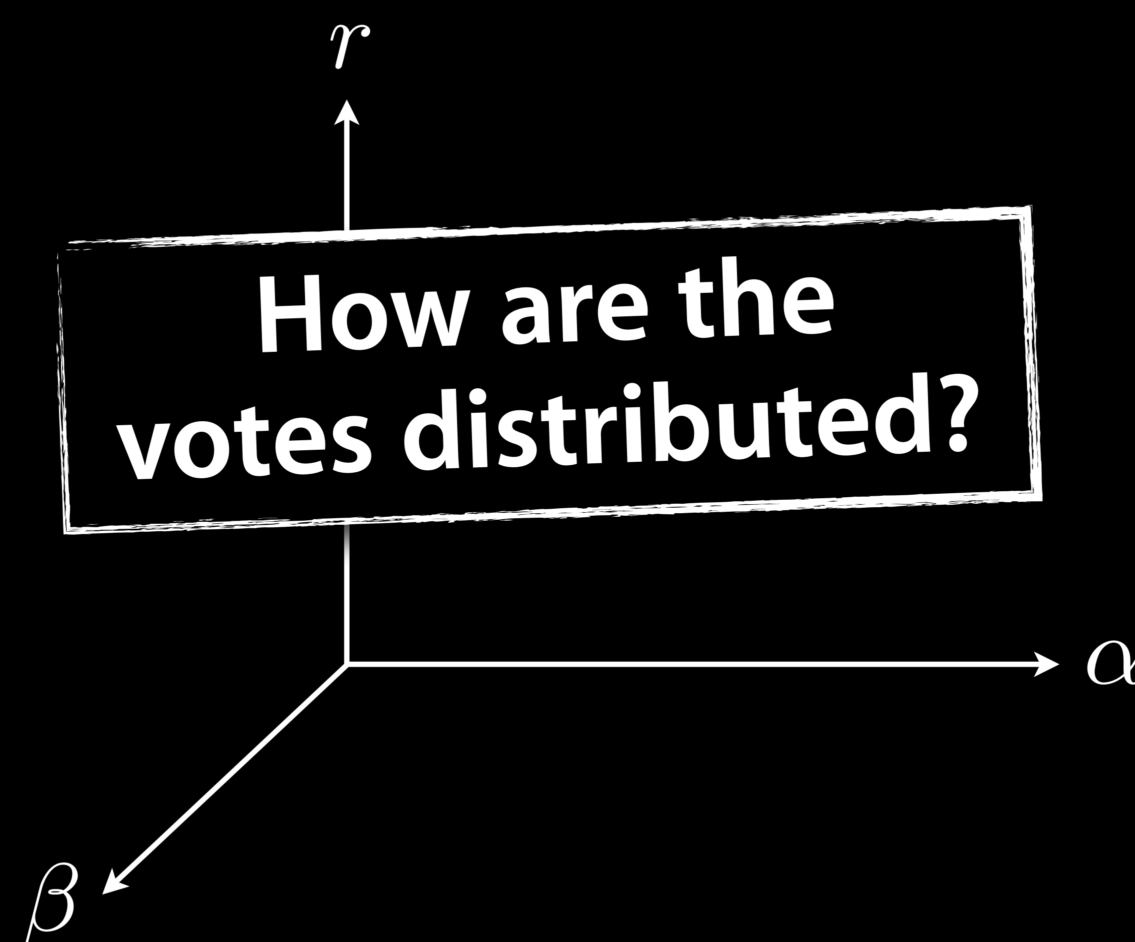




Assume
Unknown Radius

$$(x_i - \alpha)^2 + (y_i - \beta)^2 = r^2$$

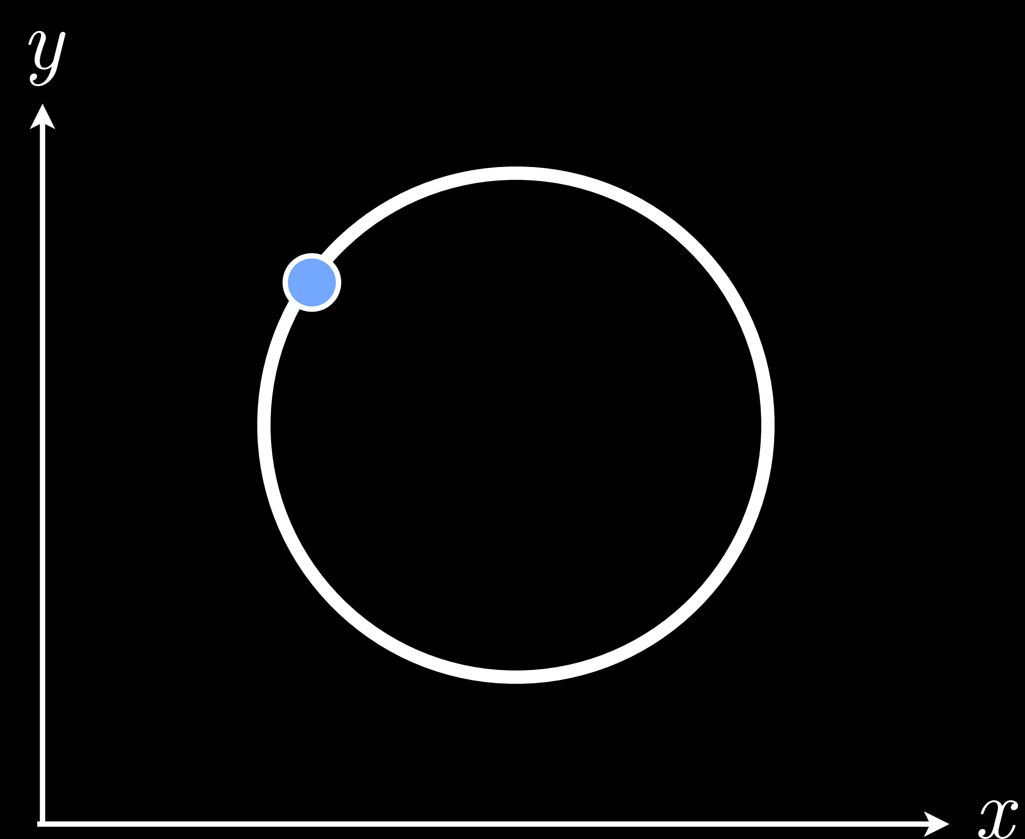


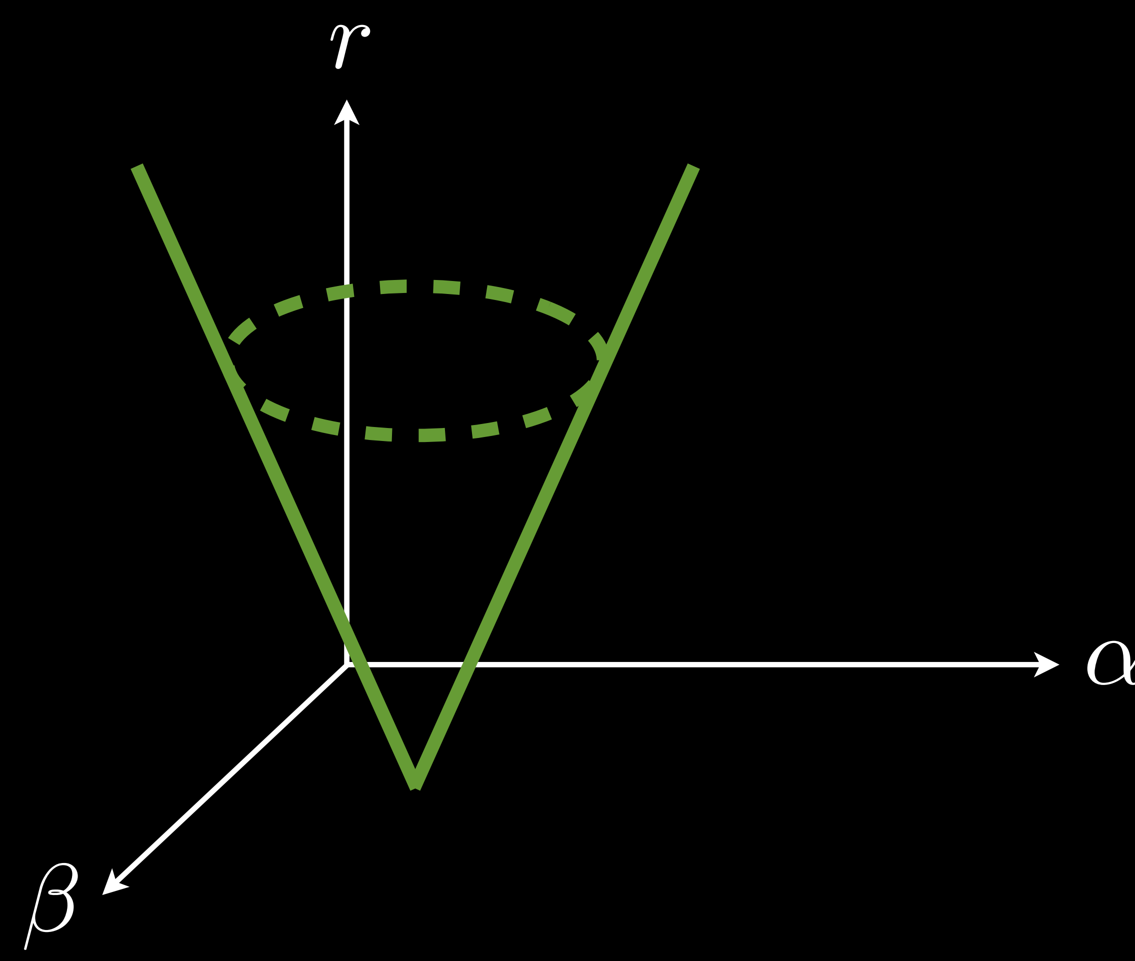


How are the votes distributed?

Assume Unknown Radius

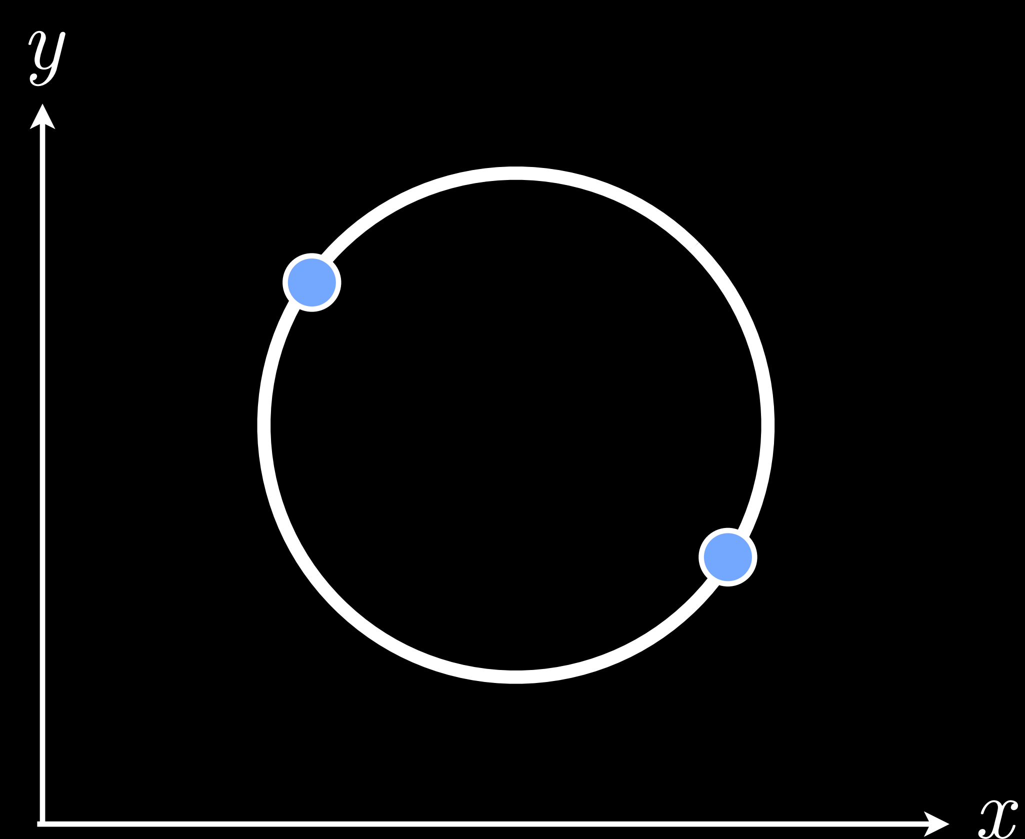
$$(x_i - \alpha)^2 + (y_i - \beta)^2 = r^2$$

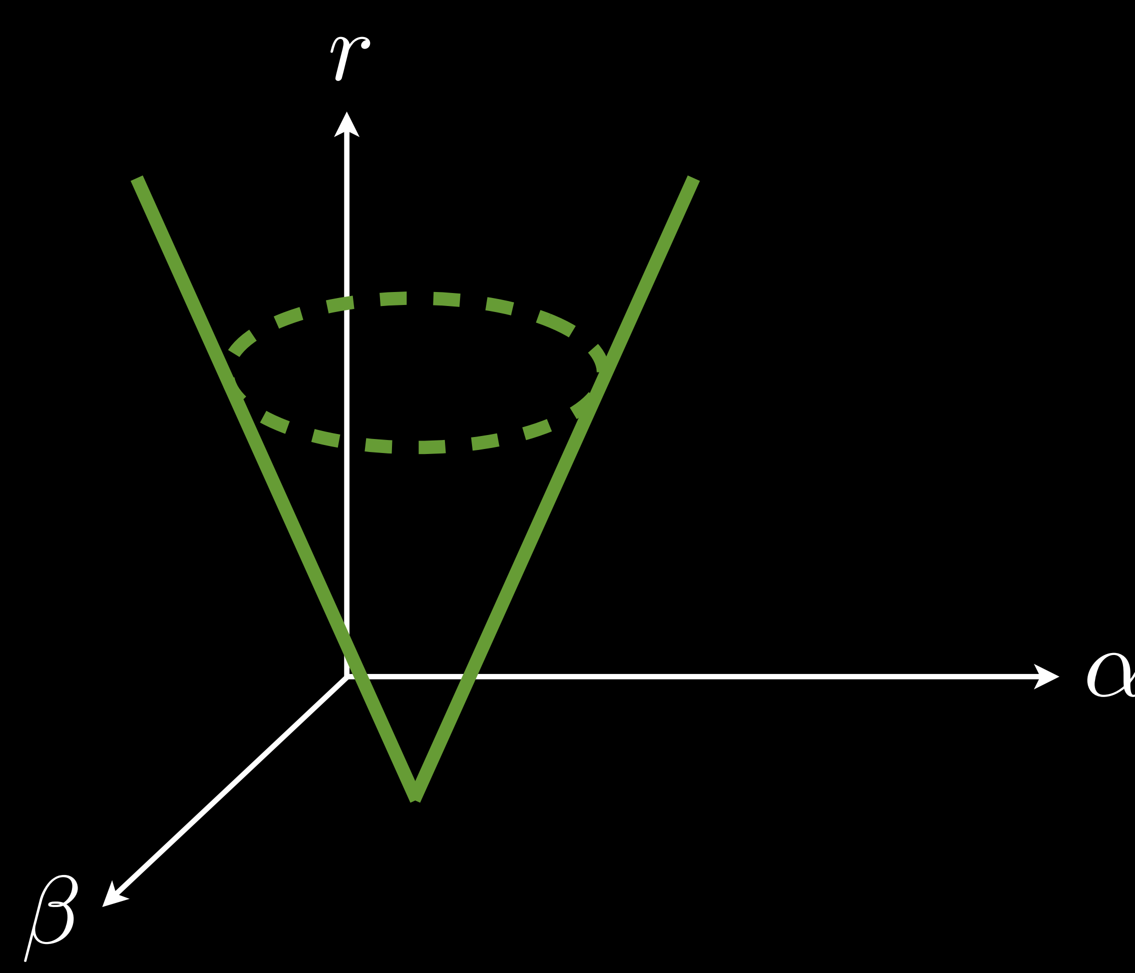




Assume
Unknown Radius

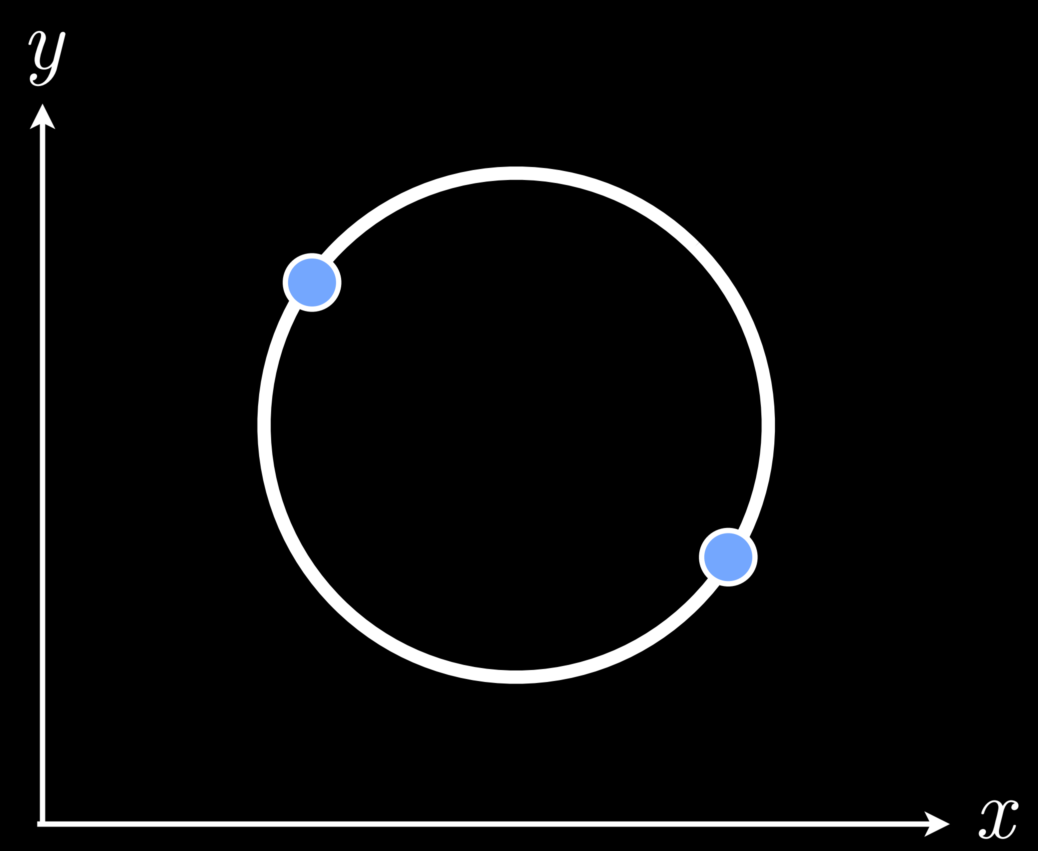
$$(x_i - \alpha)^2 + (y_i - \beta)^2 = r^2$$

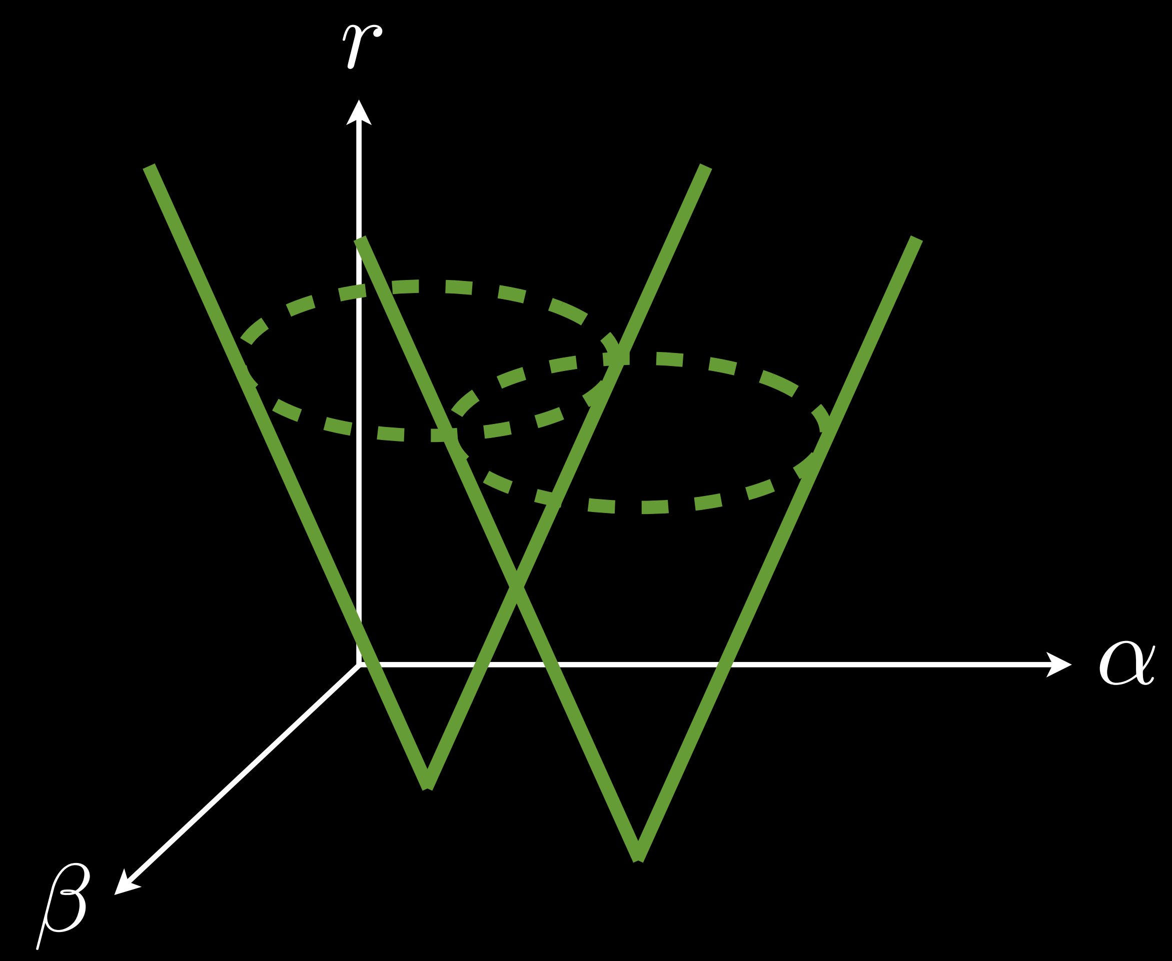




Assume Unknown Radius

$$(x_i - \alpha)^2 + (y_i - \beta)^2 = r^2$$



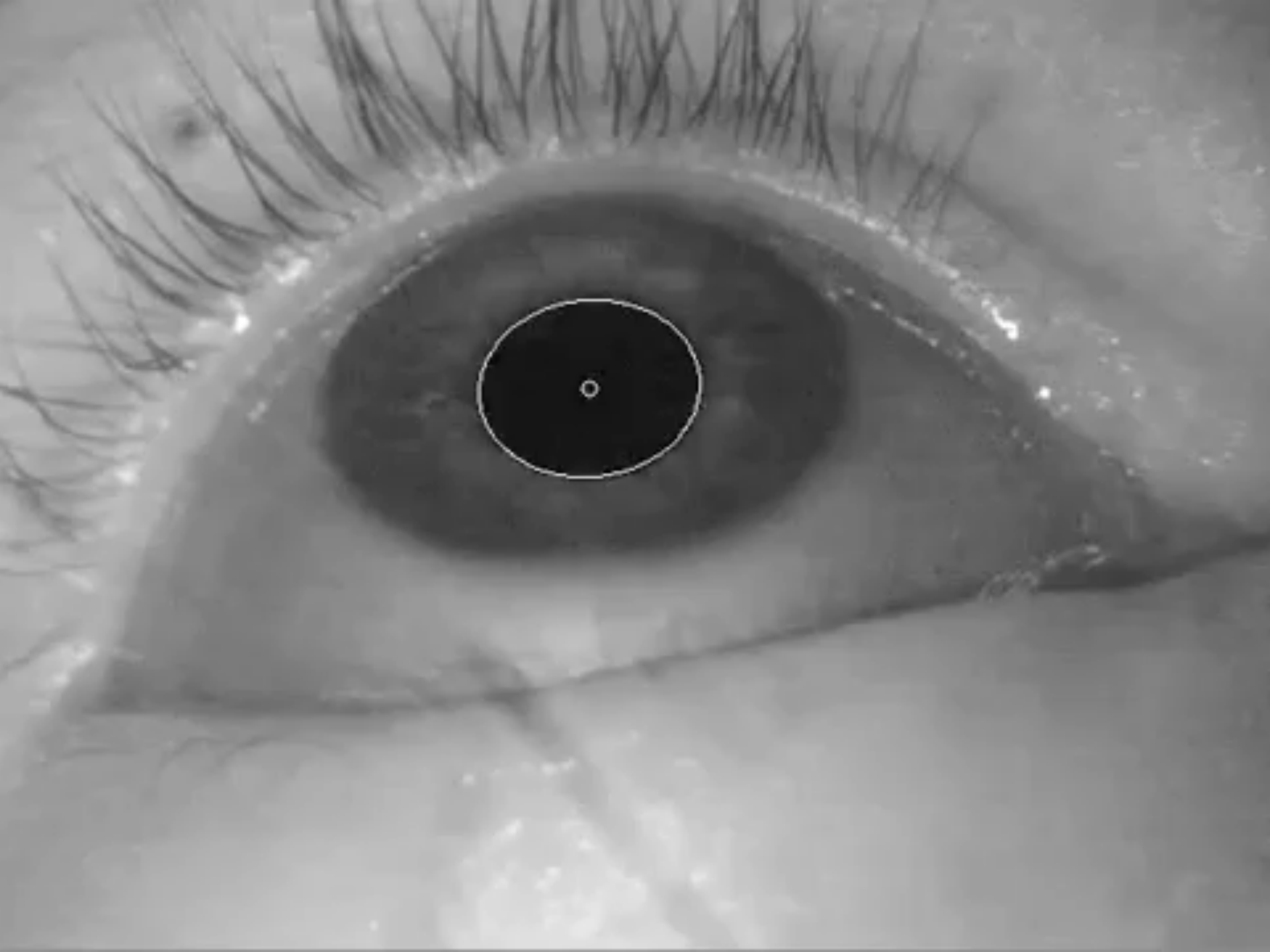


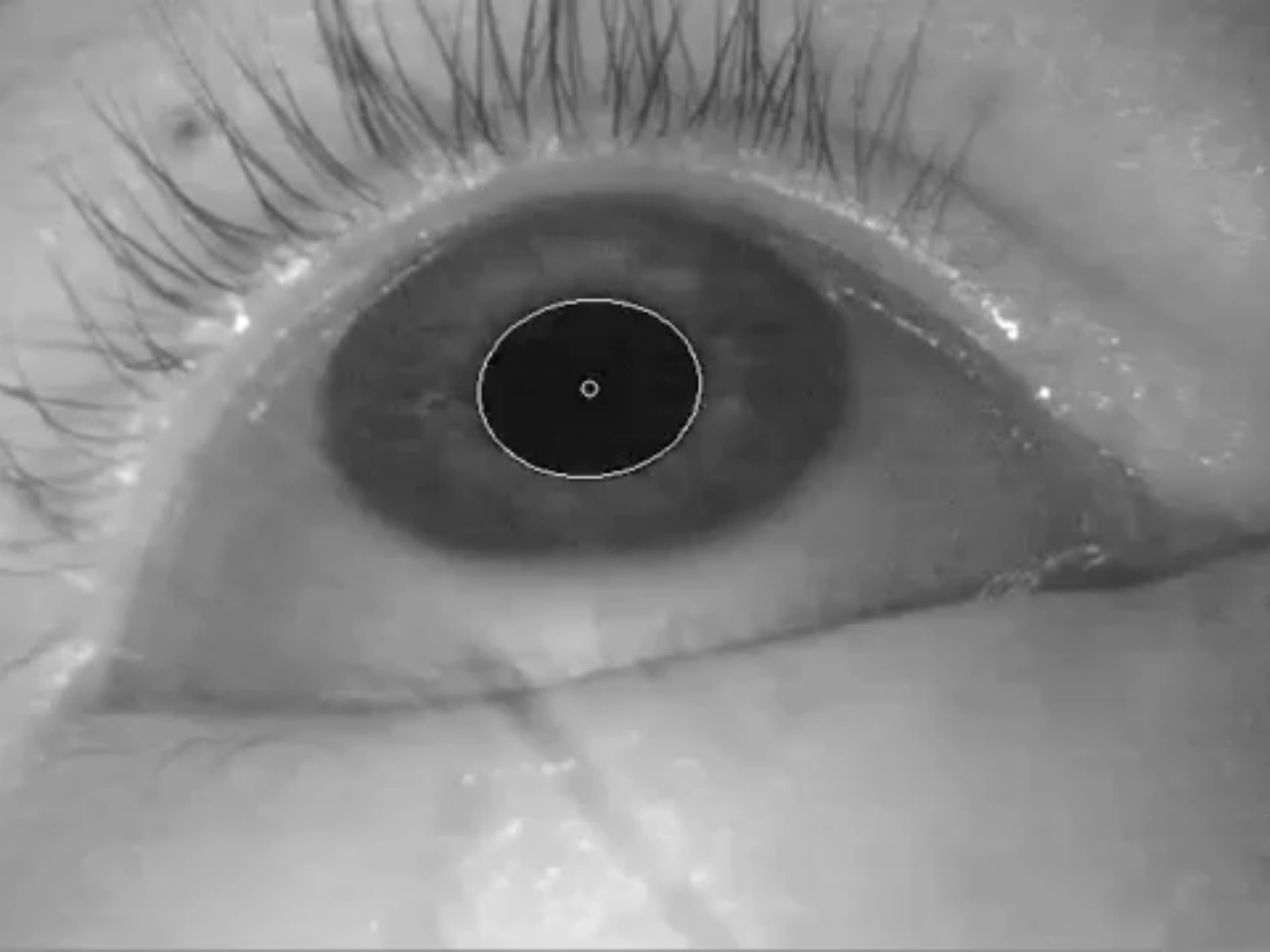
Unknown Radius
Known Gradient
Direction

$$(x_i - \alpha)^2 + (y_i - \beta)^2 = r^2$$


Yellow


Yellow





PROS

PROS

All points processed independently

PROS

All points processed independently

Robustness to OUTLIERS

PROS

All points processed independently

Robustness to OUTLIERS

Can detect multiple model instances in a single pass

CONS

Search time complexity increases exponentially with number of parameters

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Search time complexity increases exponentially with number of parameters

Non-target shapes can produce spurious peaks in parameter space

CONS

Search time complexity increases exponentially with number of parameters

Non-target shapes can produce spurious peaks in parameter space

Difficult to select an optimal quantization