Practice problem 1:Optimization in ML

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1. Justify whether the following sets are convex or not (if yes proof, if not give $n(\geq 2)$ dimensional counter example):
   - $S = \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 10, x_1 + x_2 \geq 2, x_1, x_2 \geq 0\}$.
   - $S = \{x \in \mathbb{R}^2 : 2 \leq x_1 \leq 4, x_2 = 3\}$.
   - $S = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\}$.
   - $S = \{x \in \mathbb{R}^2 : \frac{x_1^2}{9} + \frac{x_2^2}{4} \leq 1\}$.
   - $S = \{x \in \mathbb{R}^2 : |x_1| + |x_2| = 9\}$.
   - $S = \{x\mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2\}$.
   - $S = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \geq 2\}$.
2. Consider $S = \{(0,0), (4,1), (6,7), (2,5), (3,3)\}$. Plot S in 2-D graph and identify $Conv(S)$.
3. Suppose $S = \{x^1, x^2, \ldots, x^n\}$. Show that $Conv(S)$ is a convex set.
4. Suppose S be a nonempty convex set in $\mathbb{R}^n$ A be an $m \times n$ matrix and $\alpha > 0$ be a scalar. Show that the sets, $S_1 = \{y \in \mathbb{R}^m | y = Ax, x \in S$ and $S_2 = \{\alpha x : x \in S\}$ are convex sets.
5. Justify whether the following matrices are positive semi-definite/definite or not:
   - $H = \begin{bmatrix} 4 & 3 \\ 5 & 1 \end{bmatrix}$
   - $H = \begin{bmatrix} -2 & 1 \\ 1 & -5 \end{bmatrix}$
   - $H = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$
   - $H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

6. Justify whether the following functions are convex or not. If it is convex then find $\min_x \; f(x)$.

- $f(x) = x_1^2 + x_2^2$
- $f(x) = 4x_1^2 + x_2^2 - 2x_1x_2$
- $f(x) = -3x_1^2 + 4x_1x_2 - 3/2x_2^2$
- $f(x) = x_1 \log(x_1) + x_2 \log(x_2)$ for $x_1, x_2 > 0$.
- $f(x) = (4, 2, 3)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \frac{1}{2}(x_1, x_2, x_3)\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

7. Suppose f is a convex function and $x = (2, 3)^T$. If $f(x) = 12$ and $\nabla f(x) = (-12, 10)^T$ then find a lower bound of $f(y)$ for $y = (0, 0)^T$.

   Hint: If f is convex then $f(y) \geq f(x) + (y - x)^T \nabla f(x)$.

8. Justify if you can find a convex function f satisfying the following conditions:

- $f((1, 2, 3)^T) = 5$, $f((4, 5, 7)^T) = 8$, $f((1.9, 2.9, 4.2)^T) = 6$.
- $f((3, 4)^T) = 10$, $f((4, 3)^T = 8$, $f((3.5, 3.5)^T = 5$.
- $f((0, 1)^T) = 12$, $\nabla f((0, 1)^T) = (-2, -3)^T$, $f((2, 2)^T) = 2.5$
- $f((-1, -1)^T) = -5$, $\nabla f((-1, -1)^T) = (3, 3)^T$, $f((0, 0)^T) = 2.5$

   Hint: Use either $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ or $f(y) \geq f(x) + \nabla f(x)^T(y - x)$.

9. Show that the function $f(x) = 4x_1^2 + 6x_2^2 - 9x_1x_2 + 3x_1 + 5x_2$ is convex. Then find $\min_{x \in \mathbb{R}^2} f(x)$.

10. Find the modulus of strong convexity of $f(x) = (x_1 - 3x_2)^2 + (2x_1 - x_2)^2$. (Hint: $\sigma = eig_{min} \nabla^2 f(x)$)

11. Find the subdifferential of the following functions at x^*. Justify whether x^* is a minima of f or not.

   (i) $f(x) = \max\{5x_1 + x_2, x_1^2 + x_2^2\}$, $x^* = (3, 3)^T$.

   (ii) $f(x) = \max\{x_1^2 + x_2^4, (2 - x_1)^2 + (2 - x_2)^2, 2e^{-x_1 + x_2}\}$, $X^* = (1, 1)^T$

   (iii) $f(x) = \max\{(x_1 - 2)^2 + (x_2 + 2)^2, x_1^2 + 8x_2\}$, $x^* = (2, 0)^T$.

   (iv) $f(x) = \max\{(x_1 - 2)^2 + (x_2 - 2)^2, x_1^2 + x_2^2\}$, $x^* = (1, 1)^T$.

   (v) $f(x) = \max\{\sin(x_1) + \cos(x_2), \cos(x_1) + \sin(x_2)\}$, $0 \leq x_1, x_2 \leq \pi/2$, $x^* = (\pi/4, \pi/4)^T$.