system noise.

- In a linear dynamic system, the states update from one time tick to another.

  State ~~Update~~ propagation: $\underline{x}_{k+1} = \underline{F}_k \underline{x}_k + \underline{G}_k \underline{u}_k + \underline{\eta}_k$ system dynamics

  Measurement: $\underline{y}_k = \underline{H}_k \underline{x}_k + \underline{\xi}_k$

- $\underline{\eta}_k \sim \mathcal{N}(0, \underline{Q}_k)$ is the system noise

  $\underline{\xi}_k \sim \mathcal{N}(0, \underline{R}_k)$ is the measurement noise

- At every time tick there are 2 evidences which determine the current state.
  - Contribution (evidence) from the previous state, through state update
  - Evidence from the measurement made at the current time tick.

    (state update from observing the evidence)
- Notation

$\hat{\underline{x}}_{k|k-1}$ state estimate at time k given the measurements up to time $k-1$

$\hat{\underline{x}}_{k|k}$ state estimate after taking account the measurement up to time tick k

$\hat{\underline{y}}_k$ predicted observation at time tick k

1) Propagate step.

$\hat{x}_{k|k}$ $\hat{x}_{k+1|k}$

$k \longrightarrow k+1$

prior estimate

2) Update step

$\hat{x}_{k+1|k} \longrightarrow \hat{x}_{k+1|k+1}$

prior

posterior estimate

y_{k+1}

Update Step

- There are two sources that provide estimate to the state $\hat{\underline{x}}_k$
  1. Estimate from the state propagation $\hat{\underline{x}}_{k|k-1}$ prior estimate (after the propagation step)

state propagation

$$\hat{\underline{x}}_{k|k-1} = \underset{\text{fixed.}}{\overset{\text{true.}}{\underline{x}_k}} + \underline{e}_k$$

random noise

The estimate $\hat{\underline{x}}_{k|k-1}$ is considered to have a random deviation or error from the true state $\underline{x}_k$

The covariance $P_{k|k-1}$ of the error term summarises the uncertainty of prediction $\hat{\underline{x}}_{k|k-1}$ given the past history of measurements.

$$P_{k|k-1} = \mathbb{E}[\underline{e}_k \underline{e}_k^\top]$$

- There are two sources that provide estimate to the state $\hat{\underline{x}}_k$

   2. Evidence from the new measurement

$$\underline{y}_k = \underline{H}_k \underline{x}_k + \underline{\xi}_k$$

$$R_k = \mathbb{E}[\underline{\xi}_k \underline{\xi}_k^\top]$$

$\hat{x}_k$ accounted by the observation y_k.

$\hat{x}_{k|k-1}$ prior estimate

$\hat{x}_{k|k}$.

posterior

- Given the evidences about the unknown true state $\underline{x}_k$ our task is to compute an estimate $\hat{\underline{x}}_k$ of $\underline{x}_k$

- We collect the two evidences

$$\left[\begin{aligned} \hat{x}_{k|k-1} &= \underline{x}_k + \underline{e}_k \\ \underline{y}_k &= \underline{H}_k \underline{x}_k + \underline{\xi}_k \end{aligned}\right] \rightarrow \underline{\underline{\text{estimate}}} \text{ of the true value } x_k$$

$\hat{x}_{k|k}$

to form a system of equations

$$\underline{y} = \underline{H}\underline{x}_k + \underline{n}$$

from measurement

Update Step

from measurement

Update Step

- $y = Hx_k + n$ Combined equation

$$y = \begin{bmatrix} \hat{x}_{k|k-1} \\ y_k \end{bmatrix} \qquad H = \begin{bmatrix} I \\ H_k \end{bmatrix} \qquad n = \begin{bmatrix} e_k \\ n_k \end{bmatrix}$$

Given — Given — random noise.

gives the estimate.

We solve for the true x_k.

- $\hat{x}_{k|k-1} = Ix_k + e_k$

  $y_k = H_k x_k + \xi_k$

$\hat{x}_{k|k}$.

system noise affecting the state propagation

- The covariance matrix of the vector $\underline{n} = \begin{bmatrix} \underline{e}_k \\ \underline{\xi}_k \end{bmatrix}$ is formulated as

measurement ξ_k

defines the uncertainty of the noise

$$R = \begin{bmatrix} \underline{P}_{k|k-1} & 0 \\ 0 & \underline{R}_k \end{bmatrix}$$

Here we assume that the two noise vectors are independent.

- The formulation $\boxed{\underline{y} = \underline{H}\underline{x}_k + \underline{n}}$ is a classical estimation problem.

$\hat{x}_{k|k}$ posterior estimate

Linear system

Update Stage

$$\underline{y} = \begin{bmatrix} \hat{\underline{x}}_{k|k-1} \\ \underline{y}_k \end{bmatrix}$$

- The solution to the estimation problem $\underline{y} = \underline{H}_k \underline{x}_k + \underline{n}$ is

Covariance of the posterior estimate

$$\boxed{\underline{P}_{k|k} = (\underline{H}^\top \underline{R}^{-1} \underline{H})^{-1}}$$

$$\boxed{\hat{\underline{x}}_{k|k} = \underline{P}_{k|k} \underline{H}^\top \underline{R}^{-1} \underline{y}}$$

identical to the derivations done earlier

$$\underline{H} = \begin{bmatrix} \underline{I} \\ \underline{H}_k \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} \underline{P}_{k|k-1} & 0 \\ 0 & \underline{R}_k \end{bmatrix}$$

- Simplification

$$\underline{P}_{k|k}^{-1} = \underline{H}^\top \underline{R}^{-1} \underline{H}$$

$$= \begin{bmatrix} \underline{I} & \underline{H}_k^\top \end{bmatrix} \begin{bmatrix} \underline{P}_{k|k-1}^{-1} & 0 \\ 0 & \underline{R}_k^{-1} \end{bmatrix} \begin{bmatrix} \underline{I} \\ \underline{H}_k \end{bmatrix}$$

Posterior covar

$$= \underbrace{\underline{P}_{k|k-1}^{-1}}_{\text{prior covariance}} + \underline{H}_k^\top \underline{R}_k^{-1} \underline{H}_k$$

Revised estimates have been obtained using two evidences.

precision improves.

This is the update stage of the Kalman Filter

(Covariance of the prior estimate)

prior covariance (covariance of the ...

Simplifying the posterior estimate

- $$\boxed{\hat{x}_{k|k} = \underline{P}_{k|k}\underline{H}^{\top}\underline{R}^{-1}\underline{y}}$$

$$= \underline{P}_{k|k} \underbrace{\begin{bmatrix} I & \underline{H}_k^{\top} \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} \underline{P}_{k|k-1}^{-1} & 0 \\ 0 & \underline{R}_k^{-1} \end{bmatrix}}_{R.} \begin{bmatrix} \hat{x}_{k|k-1} \\ \underline{y}_k \end{bmatrix}$$

$$\underline{H} = \begin{bmatrix} \underline{I} \\ \underline{H}_k \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} \underline{P}_{k|k-1} & 0 \\ 0 & \underline{R}_k \end{bmatrix}$$

$$= \underline{P}_{k|k} \begin{bmatrix} \underline{P}_{k|k-1}^{-1} & \underline{H}_k^{\top}\underline{R}_k^{-1} \end{bmatrix} \begin{bmatrix} \hat{x}_{k|k-1} \\ \underline{y}_k \end{bmatrix}$$

$$= \underline{P}_{k|k}\left(\underline{P}_{k|k-1}^{-1}\hat{x}_{k|k-1} + \underline{H}_k^{\top}\underline{R}_k^{-1}\underline{y}_k\right)$$

prior covariance.

cov matrix of measurement error.

posterior prior

$$\underline{P}_{k|k}^{-1} = \underline{P}_{k|k-1}^{-1} + \underline{H}_k^{\top}\underline{R}_k^{-1}\underline{H}_k$$

y_k is not required

$$\underline{y} = \begin{bmatrix} \hat{\underline{x}}_{k|k-1} \\ \underline{y}_k \end{bmatrix} \qquad \underline{R} = \begin{bmatrix} \underline{P}_{k|k-1} & 0 \\ 0 & \underline{R}_k \end{bmatrix} \qquad \underline{H} = \begin{bmatrix} \underline{I} \\ \underline{H}_k \end{bmatrix}$$

- $\hat{\underline{x}}_{k|k} = \underline{P}_{k|k}\underline{H}^\top \underline{R}^{-1}\underline{y}$

$$= \underline{P}_{k|k}\left(\underbrace{\left(\underline{P}_{k|k}^{-1} - \underline{H}_k^\top \underline{R}_k^{-1}\underline{H}_k\right)}_{P_{k|k-1}^{-1}}\hat{\underline{x}}_{k|k-1} + \underline{H}_k^\top \underline{R}_k^{-1}\underline{y}_k\right)$$

$$= \left(\underline{I} - \underline{P}_{k|k}\underline{H}_k^\top \underline{R}_k^{-1}\underline{H}^k\right)\underbrace{\hat{\underline{x}}_{k|k-1}}_{\text{prior}} + \underline{P}_{k|k}\underline{H}_k^\top \underline{R}_k^{-1}\underline{y}_k$$

$$= \hat{\underline{x}}_{k|k-1} - \underline{P}_{k|k}\underline{H}_k^\top \underline{R}_k^{-1}\underline{H}^k\hat{\underline{x}}_{k|k-1} + \underline{P}_{k|k}\underline{H}_k^\top \underline{R}_k^{-1}\underline{y}_k$$

$$\underbrace{\text{posterior}}_{\hat{x}_{k|k}} = \underbrace{\hat{\underline{x}}_{k|k-1}}_{\text{prior}} + \underbrace{\left(\underline{P}_{k|k}\underline{H}_k^\top \underline{R}_k^{-1}\right)}_{\text{Matrix}}\underbrace{\left(\underline{y}_k - \underline{H}_k\hat{\underline{x}}_{k|k-1}\right)}_{\text{difference: residue}}$$

fusion of prior estimate & evidence from observation

$$x_k \xrightarrow{H_k} y_k$$

y_k : actual observation

$H_k\hat{x}_{k|k-1}$: predicted observation

observation.

- Introduce the Kalman Gain Matrix

$$\boxed{\underline{K}_k \triangleq \underline{P}_{k|k}\underline{H}_k^\top \underline{R}_k^{-1}}$$

cov of the posterior estimate

- Define the residue

$$\underline{r}_k \triangleq \underline{y}_k - \underline{H}_k \hat{\underline{x}}_{k|k-1}$$

The residue is the difference between the actual measurement and the predicted measurement.

update equation

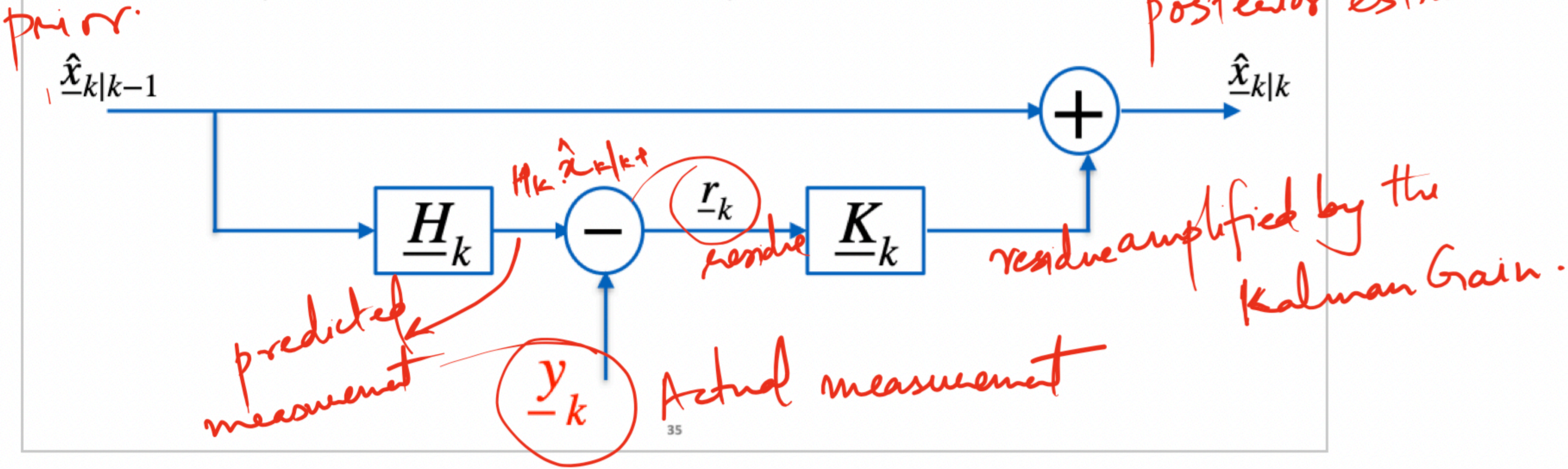
Update of the state estimate by accounting for the current observation.

$$\boxed{\hat{\underline{x}}_{k|k} = \hat{\underline{x}}_{k|k-1} + \underline{K}_k\left(\underline{y}_k - \underline{H}_k \hat{\underline{x}}_{k|k-1}\right)}$$

Taking the measurement into account (The Update Step)

- The Kalman gain matrix specifies the amount by which the residue must be multiplied (or amplified) to obtain the correction term to correct the old estimate $\hat{\underline{x}}_{k|k-1}$ and give the new estimate $\hat{\underline{x}}_{k|k}$

posterior estimate.



$\xrightarrow{\text{propagation}}$ prior estimate $\xrightarrow{\text{update}}$ posterior estimate $\xrightarrow{\text{propagation}}$

The Propagation Step

Propagation to the next time step

posterior of the previous step
is a random variable.

- Propagated estimate

prior estimate for the current time step.

$$\hat{\underline{x}}_{k+1|k} = \underline{F}_k \hat{\underline{x}}_{k|k} + \underline{G}_k \underline{u}_k$$

$+ \eta_k$.

- Note that we ignore the noise term in the propagation step because the estimate $\hat{\underline{x}}_{k+1|k}$ is computed using another random vector $\hat{\underline{x}}_{k|k}$

- Compare this with the state propagation equation

$$\boxed{\underline{x}_{k+1} = \underline{F}_k \underline{x}_k + \underline{G}_k \underline{u}_k + \eta_k}$$

true (not random)

- The covariance matrix of the propagated estimate is

$$\underline{P}_{k+1|k} = \mathbb{E}\left[(\underline{\hat{x}}_{k+1|k} - \underline{x}_{k+1})(\underline{\hat{x}}_{k+1|k} - \underline{x}_{k+1})^{\mathsf{T}}\right]$$

mean

$$= \mathbb{E}\left[\left(\underline{F}_k(\underline{\hat{x}}_{k|k} - \underline{x}_k) - \eta_k\right)\left(\underline{F}_k(\underline{\hat{x}}_{k|k} - \underline{x}_k) - \eta_k\right)^{\mathsf{T}}\right]$$

$$= \underline{F}_k \underbrace{\mathbb{E}\left[\left(\underline{\hat{x}}_{k|k} - \underline{x}_k\right)\left(\underline{\hat{x}}_{k|k} - \underline{x}_k\right)^{\mathsf{T}}\right]}_{P_{k|k}} \underline{F}_k^{\mathsf{T}} + \underbrace{\mathbb{E}\left[\eta_k \eta_k^{\mathsf{T}}\right]}_{Q_k} + \underbrace{\mathbb{E}\left[\eta_k(\underline{\hat{x}}_{k|k} - \underline{x}_k)\right]}_{} = 0$$

$$= \underline{F}_k \underline{P}_{k|k} \underline{F}_k^{\mathsf{T}} + \mathbb{E}\left[\eta_k \eta_k^{\mathsf{T}}\right]$$

$$= \underline{F}_k \underline{P}_{k|k} \underline{F}_k^{\mathsf{T}} + \underline{Q}_k$$

$$\boxed{\underline{P}_{k+1|k} = \underline{F}_k \underline{P}_{k|k} \underline{F}_k^{\mathsf{T}} + \underline{Q}_k}$$

$P_{k|k} \rightarrow P_{k+1|k}.$

For unbiased estimates, the true value is the mean of the random vector.

$E[\hat{n}\,\hat{x}] = \underbrace{E[\hat{n}]}_{\text{zero mean.}} E[\hat{x}]$

$\Rightarrow$ 1) noise is uncorrelated with the diff $(\hat{x}_{k|k} - x_k)$

2) noise is zero mean.

$\because$ the noise has covariance matrix Q_k

$\eta_k \sim N(0, Q_k)$

Kalman Filter Equations

No past observation

$\hat{x}_{k|k-1}$

$\hat{x}_{0|-1}$

- Initialisation

$$\underline{\hat{x}}_{0|-1} \triangleq \underline{\hat{x}}_0$$

$$\underline{P}_{0|-1} \triangleq \underline{P}_0$$

prior to posterior.

- Update

$$\underline{\hat{x}}_{k|k} = \underline{\hat{x}}_{k|k-1} + \underline{K}_k(\underline{y}_k - \underline{H}_k\underline{\hat{x}}_{k|k-1})$$

$$\underline{P}^{-1}_{k|k} = \underline{P}^{-1}_{k|k-1} + \underline{H}^\top_k \underline{R}^{-1}_k \underline{H}_k$$

$$\boxed{\underline{K}_k \triangleq \underline{P}_{k|k}\underline{H}^\top_k \underline{R}^{-1}_k}$$

posterior prior

$k \longrightarrow k+1$

- Propagation

$$\boxed{\underline{\hat{x}}_{k+1|k} = \underline{F}_k\underline{\hat{x}}_{k|k} + \underline{G}_k\underline{u}_k} \leftarrow \text{input}$$

$$\boxed{\underline{P}_{k+1|k} = \underline{F}_k\underline{P}_{k|k}\underline{F}^\top_k + \underline{Q}_k}$$

I. posterior covariance of current time step.

II. prior covariance of current time step. $P_{k|k-1}$

I. posterior covariance of current time step.

II. prior covariance of current time step. $P_{k|k-1}$

Alternate form of Kalman Gain Matrix