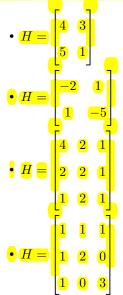
Practice problem 1:Optimization in ML

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- 1. Justify whether the following sets are convex or not (if yes proof, if not give $n \ge 2$) dimensional counter example):
 - $S = \{x \in \mathbb{R}^2 : x_1 + x_2 \le 10, x_1 + x_2 \ge 2, x_1, x_2 \ge 0\}.$
 - $S = \{x \in \mathbb{R}^2 : 2 \le x_1 \le 4, x_2 = 3\}.$
 - $S = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 4\}.$
 - $S = \{x \in \mathbb{R}^2 : \frac{x_1^2}{9} + \frac{x_2^2}{4} \le 1\}.$
 - $S = \{x \in \mathbb{R}^2 : |x_1| + |x_2| = 9\}.$
 - $S = \{x\mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2\}.$
 - $S = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \ge 2\}.$
- 2. Consider $S = \{(0,0), (4,1), (6,7), (2,5), (3,3)\}$. Plot S in 2-D graph and identify Conv(S).
- 3. Suppose $S = \{x^1, x^2, \dots, x^n\}$. Show that Conv(S) is a convex set.
- 4. Suppose S be a nonempty convex set in \mathbb{R}^n A be an $m \times n$ matrix and $\alpha > 0$ be a scalar. Show that the sets, $S_1 = \{y \in \mathbb{R}^m | y = Ax, x \in S \text{ and } S_2 = \{\alpha x : x \in S\}$ are convex sets.
- 5. Justify whether the following matrices are positive semi-definite/definite or not:



6. Justify whether the following functions are convex or not. If it is convex then find $\min_x |f(x)|$.

- $f(x) = x_1^2 + x_2^2$
- $f(x) = 4x_1^2 + x_2^2 2x_1x_2$
- $f(x) = -3x_1^2 + 4x_1x_2 3/2x_2^2$
- $f(x) = x_1 \log(x_1) + x_2 \log(x_2)$ for $x_1, x_2 > 0$. • f(x) = (4, 2, 3) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \frac{1}{2}(x_1, x_2, x_3)$ $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
- 7. Suppose f is a convex function and $x = (2,3)^T$. If f(x) = 12 and $\nabla f(x) = (-12,10)^T$ then find a lower bound of f(y) for $y = (0,0)^T$.

 Hint: If f is convex then $f(y) \ge f(x) + (y-x)^T \nabla f(x)$.
- 8. Justify if you can find a convex function f satisfying the following conditions:
 - $f((1,2,3)^T) = 5$, $f((4,5,7)^T) = 8$, $f((1.9,2.9,4.2)^T) = 6$.
 - $f((3,4)^T) = 10$, $f((4,3)^T = 8$, $f((3.5,3.5)^T = 5$.
 - $f((0,1)^T) = 12$, $\nabla f((0,1)^T) = (-2,-3)^T$, $f((2,2)^T) = 2.5$
 - $f((-1,-1)^T) = -5$, $\nabla f((-1,-1)^T) = (3,3)^T$, $f((0,0)^T) = 2.5$

Hint: Use either $f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y)$ or $f(y) \ge f(x) + \nabla f(x)^T (y - x)$.

- 9. Show that the function $f(x) = 4x_1^2 + 6x_2^2 9x_1x_2 + 3x_1 + 5x_2$ is convex. Then find $\min_{x \in \mathbb{R}^2} f(x)$.
- 10. Find the modulus of strong convexity of $f(x) = (x_1 3x_2)^2 + (2x_1 x_2)^2$. (Hint: $\sigma = eig_{min} \nabla^2 f(x)$)
- 11. Find the subdifferential of the following functions at x^* . Justify whether x^* is a minima of f or not.
 - (i) $f(x) = \max\{5x_1 + x_2, x_1^2 + x_2^2\}, x^* = (3,3)^T$.
 - (ii) $f(x) = \max\{x_1^2 + x_2^4, (2 x_1)^2 + (2 x_2)^2, 2e^{-x_1 + x_2}\}, X^* = (1, 1)^T$
 - (iii) $f(x) = \max\{(x_1 2)^2 + (x_2 + 2)^2, x_1^2 + 8x_2\}, x^* = (2, 0)^T.$
 - (iv) $f(x) = \max\{(x_1 2)^2 + (x_2 2)^2, x_1^2 + x_2^2\}, x^* = (1, 1)^T$.
 - (v) $f(x) = \max\{\sin(x_1) + \cos(x_2), \cos(x_1) + \sin(x_2)\}, \ 0 \le x_1, x_2 \le \pi/2, x^* = (\pi/4, \pi/4)^T.$