



Knowledge

Representation and

Reasoning



Agent

- Perceive / Acquire
- Knowledge representation
- Reasoning
- Acting

Propositional Logic Concepts

- Logic is a study of principles used to
 - distinguish correct from incorrect reasoning.
- Formally it deals with
 - the notion of truth in an abstract sense and is concerned with the principles of valid inferencing.
- A proposition in logic is a declarative statements which are either true or false (but not both) in a given context. For example,
 - “Jack is a male”,
 - "Jack loves Mary" etc.

Cont..

- Given some propositions to be true in a given context,
 - logic helps in inferencing new proposition, which is also true in the same context.
- Suppose we are given a set of propositions such as
 - "It is hot today" and
 - "If it is hot it will rain", then
 - we can infer that
 - "It will rain today".

Well-formed formula

- Propositional Calculus (PC) is a language of propositions basically refers
 - to set of rules used to combine the propositions to form compound propositions using logical operators often called connectives such as \wedge , \vee , \sim , \rightarrow , \leftrightarrow
- Well-formed formula is defined as:
 - An atom is a well-formed formula.
 - If α is a well-formed formula, then $\sim\alpha$ is a well-formed formula.
 - If α and β are well formed formulae, then $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$ are also well-formed formulae.
 - A propositional expression is a well-formed formula if and only if it can be obtained by using above conditions.

Truth Table

- Truth table gives us operational definitions of important logical operators.
 - By using truth table, the truth values of well-formed formulae are calculated.
- Truth table elaborates all possible truth values of a formula.
- The meanings of the logical operators are given by the following truth table.

P	Q	$\sim P$	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Equivalence Laws

Commutation

$$\begin{array}{lcl} 1. & P \wedge Q & \equiv \\ 2. & P \vee Q & \equiv \end{array}$$

$$\begin{array}{l} Q \wedge P \\ Q \vee P \end{array}$$

Association

$$\begin{array}{lcl} 1. & P \wedge (Q \wedge R) & \equiv \\ 2. & P \vee (Q \vee R) & \equiv \end{array}$$

$$\begin{array}{l} (P \wedge Q) \wedge R \\ (P \vee Q) \vee R \end{array}$$

Double Negation

$$\sim (\sim P) \equiv$$

$$P$$

Distributive Laws

$$\begin{array}{lcl} 1. & P \wedge (Q \vee R) & \equiv \\ 2. & P \vee (Q \wedge R) & \equiv \end{array}$$

$$\begin{array}{l} (P \wedge Q) \vee (P \wedge R) \\ (P \vee Q) \wedge (P \vee R) \end{array}$$

De Morgan's Laws

$$\begin{array}{lcl} 1. & \sim (P \wedge Q) & \equiv \\ 2. & \sim (P \vee Q) & \equiv \end{array}$$

$$\begin{array}{l} \sim P \vee \sim Q \\ \sim P \wedge \sim Q \end{array}$$

Law of Excluded Middle

$$P \vee \sim P \equiv$$

$$T \text{ (true)}$$

Law of Contradiction

$$P \wedge \sim P \equiv$$

$$F \text{ (false)}$$

$$P \rightarrow Q = \sim P \vee Q$$

Propositional Logic - PL

- PL deals with
 - the validity, satisfiability and unsatisfiability of a formula
 - derivation of a new formula using equivalence laws.
- Each row of a truth table for a given formula is called its **interpretation** under which a formula can be true or false.
- A formula α is called **tautology** if and only
 - if α is true for all interpretations.
- A formula α is also called **valid** if and only if
 - it is a **tautology**.

Cont..

- Let α be a formula and if there exist at least one interpretation for which α is true,
 - then α is said to be **consistent** (satisfiable) i.e., if \exists a model for α , then α is said to be consistent .
- A formula α is said to be inconsistent (unsatisfiable), if and only if
 - α is always false under all interpretations.
- We can translate
 - simple declarative and
 - conditional (if .. then) natural language sentences into its corresponding propositional formulae.

Example

- Show that "It is humid today and if it is humid then it will rain so it will rain today" is a valid argument.

- **Solution:** Let us symbolize English sentences by propositional atoms as follows:

A : It is humid

B : It will rain

- Formula corresponding to a text:

$\alpha : ((A \rightarrow B) \wedge A) \rightarrow B$

- Using truth table approach, one can see that α is true under all four interpretations and hence is valid argument.

Cont..

Truth Table for $((A \rightarrow B) \wedge A) \rightarrow B$

A	B	$A \rightarrow B = X$	$X \wedge A = Y$	$Y \rightarrow B$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Cont...

- Truth table method for problem solving is
 - simple and straightforward and
 - very good at presenting a survey of all the truth possibilities in a given situation.
- It is an easy method to evaluate
 - a consistency, inconsistency or validity of a formula, but the size of truth table grows exponentially.
 - Truth table method is good for small values of n .
- For example, if a formula contains n atoms, then the truth table will contain 2^n entries.
 - A formula $\alpha : (P \wedge Q \wedge R) \rightarrow (Q \vee S)$ is **valid** can be proved using truth table.
 - A table of 16 rows is constructed and the truth values of α are computed.
 - Since the truth value of α is true under all 16 interpretations, it is valid.

Cont..

- We notice that if $P \wedge Q \wedge R$ is false, then α is true because of the definition of \rightarrow .
- Since $P \wedge Q \wedge R$ is false for 14 entries out of 16, we are left only with two entries to be tested for which α is true.
 - So in order to prove the validity of a formula, all the entries in the truth table may not be relevant.
- Other methods which are concerned with proofs and deductions of logical formula are as follows:
 - Natural Deductive System
 - Axiomatic System
 - Semantic Tableaux Method
 - Resolution Refutation Method

Natural deduction method - ND

- ND is based on the set of few deductive inference rules.
- The name natural deductive system is given because it mimics the pattern of natural reasoning.
- It has about 10 deductive inference rules.

Conventions:

- E for Elimination.
- $P, P_k, (1 \leq k \leq n)$ are atoms.
- $\alpha_k, (1 \leq k \leq n)$ and β are formulae.

ND Rules

Rule 1: I- \wedge (Introducing \wedge)

I- \wedge : If P_1, P_2, \dots, P_n then $P_1 \wedge P_2 \wedge \dots \wedge P_n$

Interpretation: If we have hypothesized or proved P_1, P_2, \dots and P_n , then their conjunction $P_1 \wedge P_2 \wedge \dots \wedge P_n$ is also proved or derived.

Rule 2: E- \wedge (Eliminating \wedge)

E- \wedge : If $P_1 \wedge P_2 \wedge \dots \wedge P_n$ then P_i ($1 \leq i \leq n$)

Interpretation: If we have proved $P_1 \wedge P_2 \wedge \dots \wedge P_n$, then any P_i is also proved or derived. This rule shows that \wedge can be eliminated to yield one of its conjuncts.

Rule 3: I- \vee (Introducing \vee)

I- \vee : If P_i ($1 \leq i \leq n$) then $P_1 \vee P_2 \vee \dots \vee P_n$

Interpretation: If any P_i ($1 \leq i \leq n$) is proved, then $P_1 \vee \dots \vee P_n$ is also proved.

Rule 4: E- \vee (Eliminating \vee)

E- \vee : If $P_1 \vee \dots \vee P_n, P_1 \rightarrow P, \dots, P_n \rightarrow P$ then P

Interpretation: If $P_1 \vee \dots \vee P_n, P_1 \rightarrow P, \dots$, and $P_n \rightarrow P$ are proved, then P is proved.

Rules – cont..

Rule 5: I- \rightarrow (Introducing \rightarrow)

I- \rightarrow : If from $\alpha_1, \dots, \alpha_n$ infer β is proved then $\alpha_1 \wedge \dots \wedge \alpha_n \rightarrow \beta$ is proved

Interpretation: If given $\alpha_1, \alpha_2, \dots$ and α_n and from these we deduce β then $\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_n \rightarrow \beta$ is also proved.

Rule 6: E- \rightarrow (Eliminating \rightarrow) - *Modus Ponens*

E- \rightarrow : If $P_1 \rightarrow P, P_1$ then P

Rule 7: I- \leftrightarrow (Introducing \leftrightarrow)

I- \leftrightarrow : If $P_1 \rightarrow P_2, P_2 \rightarrow P_1$ then $P_1 \leftrightarrow P_2$

Rule 8: E- \leftrightarrow (Elimination \leftrightarrow)

E- \leftrightarrow : If $P_1 \leftrightarrow P_2$ then $P_1 \rightarrow P_2, P_2 \rightarrow P_1$

Examples

Example1: Prove that $P \wedge (Q \vee R)$ follows from $P \wedge Q$

Solution: This problem is restated in natural deductive system as "from $P \wedge Q$ infer $P \wedge (Q \vee R)$ ".

{Theorem}

from $P \wedge Q$ infer $P \wedge (Q \vee R)$

{ premise }	$P \wedge Q$	(1)
{ E- \wedge , (1) }	P	(2)
{ E- \wedge , (1) }	Q	(3)
{ I- \vee , (3) }	$Q \vee R$	(4)
{ I- \wedge , (2, 4) }	$P \wedge (Q \vee R)$	Conclusion

Axiomatic System for PL

- It is based on the set of only three axioms and one rule of deduction.
 - It is minimal in structure but as powerful as the truth table and natural deduction approaches.
 - These methods basically require forward chaining strategy where we start with the given hypotheses and prove the goal.

Axiom1 (A1): $\alpha \rightarrow (\beta \rightarrow \alpha)$

Axiom2 (A2): $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$

Axiom3 (A3): $(\sim \alpha \rightarrow \sim \beta) \rightarrow (\beta \rightarrow \alpha)$

Modus Ponens (MP) defined as follows:

Hypotheses: $\alpha \rightarrow \beta$ and α **Consequent:** β

Examples

Examples: Establish the following:

1. $\{Q\} \vdash \neg(P \rightarrow Q)$ i.e., $P \rightarrow Q$ is a deductive consequence of $\{Q\}$.

{Hypothesis}	Q	(1)
{Axiom A1}	$Q \rightarrow (P \rightarrow Q)$	(2)
{MP, (1,2)}	$P \rightarrow Q$	proved

2. $\{P \rightarrow Q, Q \rightarrow R\} \vdash (P \rightarrow R)$ i.e., $P \rightarrow R$ is a deductive consequence of $\{P \rightarrow Q, Q \rightarrow R\}$.

{Hypothesis}	$P \rightarrow Q$	(1)
{Hypothesis}	$Q \rightarrow R$	(2)
{Axiom A1}	$(Q \rightarrow R) \rightarrow (P \rightarrow (Q \rightarrow R))$	(3)
{MP, (2, 3)}	$P \rightarrow (Q \rightarrow R)$	(4)
{Axiom A2}	$(P \rightarrow (Q \rightarrow R)) \rightarrow$ $((P \rightarrow Q) \rightarrow (P \rightarrow R))$	(5)
{MP, (4, 5)}	$(P \rightarrow Q) \rightarrow (P \rightarrow R)$	(6)
{MP, (1, 6)}	$P \rightarrow R$	proved

Deduction Theorems in AS

Deduction Theorem:

If Σ is a set of hypotheses and α and β are well-formed formulae, then $\{\Sigma \cup \alpha\} \vdash \beta$ implies $\Sigma \vdash (\alpha \rightarrow \beta)$.

Converse of deduction theorem:

Given $\Sigma \vdash (\alpha \rightarrow \beta)$,
we can prove $\{\Sigma \cup \alpha\} \vdash \beta$.

Useful Tips

1. Given α , we can easily prove $\beta \rightarrow \alpha$ for any well-formed formulae α and β .

2. Useful tip

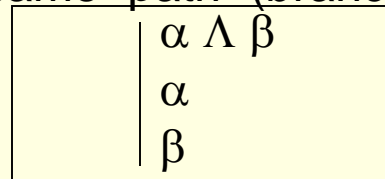
If $\alpha \rightarrow \beta$ is to be proved, then include α in the set of hypotheses Σ and derive β from the set $\{\Sigma \cup \alpha\}$. Then using deduction theorem, we conclude $\alpha \rightarrow \beta$.

Semantic Tableaux System in PL

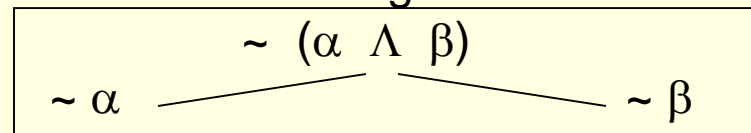
- Earlier approaches require
 - construction of proof of a formula from given set of formulae and are called direct methods.
- In **semantic tableaux**,
 - the set of rules are applied systematically on a formula or set of formulae to establish its consistency or inconsistency.
- *Semantic tableau*
 - binary tree constructed by using semantic rules with a formula as a root
- Assume α and β be any two formulae.

Semantic Tableau Rules

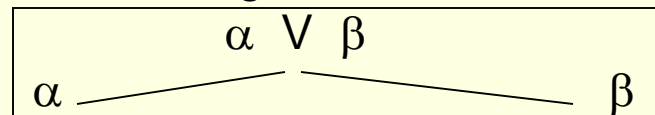
Rule 1: A tableau for a formula $(\alpha \wedge \beta)$ is constructed by adding both α and β to the same path (branch). This can be represented as follows:



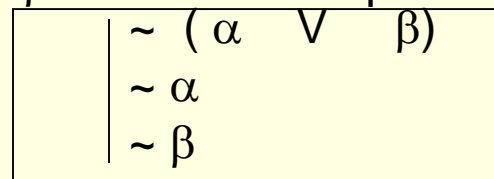
Rule 2: A tableau for a formula $\sim(\alpha \wedge \beta)$ is constructed by adding two alternative paths one containing $\sim \alpha$ and other containing $\sim \beta$.



Rule 3: A tableau for a formula $(\alpha \vee \beta)$ is constructed by adding two new paths one containing α and other containing β .

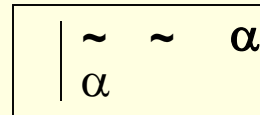


Rule 4: A tableau for a formula $\sim(\alpha \vee \beta)$ is constructed by adding both $\sim \alpha$ and $\sim \beta$ to the same path. This can be expressed as follows:

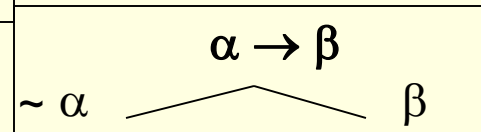


Rules - Cont..

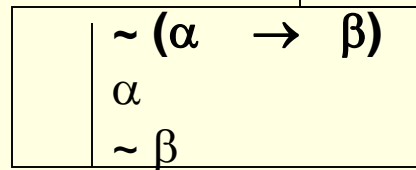
Rule 5:



Rule 6:

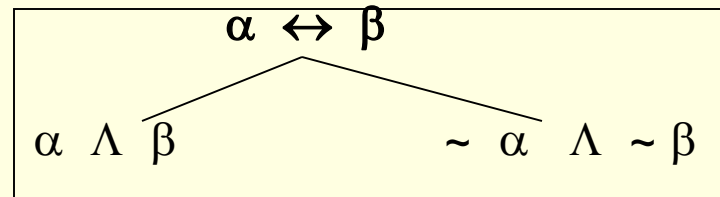


Rule 7:

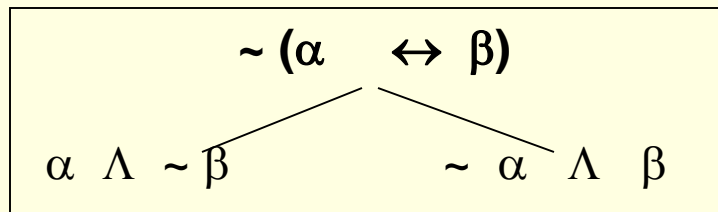


Rule 8:

$$\alpha \leftrightarrow \beta \equiv (\alpha \wedge \beta) \vee (\sim \alpha \wedge \sim \beta)$$



Rule 9: $\sim (\alpha \leftrightarrow \beta) \equiv (\alpha \wedge \sim \beta) \vee (\sim \alpha \wedge \beta)$



Consistency and Inconsistency

- If an atom P and $\sim P$ appear on a same path of a semantic tableau,
 - then inconsistency is indicated and such path is said to be **contradictory** or **closed** (finished) path.
 - Even if one path remains **non contradictory** or **unclosed** (open), then the formula α at the root of a tableau is **consistent**.
- **Contradictory tableau (or finished tableau):**
 - It defined to be a tableau in which all the paths are contradictory or closed (finished).
- If a tableau for a formula α at the root is a contradictory tableau,
 - then a formula α is said to be inconsistent.

Example

- Show that $\alpha: (Q \wedge \sim R) \wedge (R \rightarrow P)$ is **consistent** and find its model.

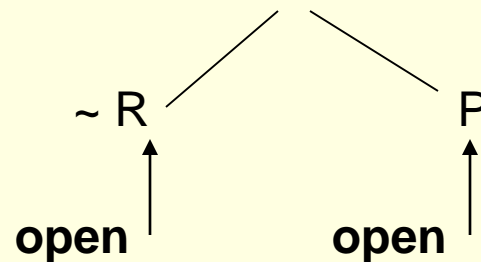
{Tableau root} $(Q \wedge \sim R) \wedge (R \rightarrow P)$ (1)

{Apply rule 1 to 1} $(Q \wedge \sim R)$ (2)

$(R \rightarrow P)$ (3)

{Apply rule 1 to 2} Q

{Apply rule 6 to 3} $\sim R$



- $\{Q = T, R = F\}$ and $\{P = T, Q = T, R = F\}$ are models of α .