Knowledge Representation and Reasoning

Agent

- Perceive / Acquire
- Knowledge representation
- Reasoning
- Acting

Propositional Logic Concepts

- Logic is a study of principles used to
  - distinguish correct from incorrect reasoning.
- Formally it deals with
  - the notion of truth in an abstract sense and is concerned with the principles of valid inferencing.
- A proposition in logic is a declarative statements which are either true or false (but not both) in a given context. For example,
  - “Jack is a male”,
  - "Jack loves Mary" etc.

Cont..

- Given some propositions to be true in a given context,
  - logic helps in inferencing new proposition, which is also true in the same context.
- Suppose we are given a set of propositions such as
  - “It is hot today" and
  - “If it is hot it will rain", then
  - we can infer that
    - “It will rain today".

Well-formed formula

- Propositional Calculus (PC) is a language of propositions basically refers
  - to set of rules used to combine the propositions to form compound propositions using logical operators often called connectives such as Λ, V, ~, $\rightarrow$, $\leftrightarrow$
- Well-formed formula is defined as:
  - An atom is a well-formed formula.
  - If α is a well-formed formula, then ~α is a well-formed formula.
  - If α and β are well formed formulae, then (α Λ β), (α V β), ($\alpha \rightarrow \beta$), ($\alpha \leftrightarrow \beta$) are also well-formed formulae.
  - A propositional expression is a well-formed formula if and only if it can be obtained by using above conditions.

Truth Table

- Truth table gives us operational definitions of important logical operators.
  - By using truth table, the truth values of well-formed formulae are calculated.
- Truth table elaborates all possible truth values of a formula.
- The meanings of the logical operators are given by the following truth table.

P	Q	~P	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Equivalence Laws

Commutation

1.	$P \wedge Q$	$\cong$	$Q \wedge P$
2.	$P \vee Q$	$\cong$	$Q \vee P$

Association

1.	$P \wedge (Q \wedge R)$	$\cong$	$(P \wedge Q) \wedge R$
2.	$P \vee (Q \vee R)$	$\cong$	$(P \vee Q) \vee R$

Double Negation

$\sim (\sim P) \cong P$

Distributive Laws

1.	$P \wedge (Q \vee R)$	$\cong$	$(P \wedge Q) \vee (P \wedge R)$
2.	$P \vee (Q \wedge R)$	$\cong$	$(P \vee Q) \wedge (P \vee R)$

De Morgan's Laws

1.	$\sim (P \wedge Q)$	$\cong$	$\sim P \vee \sim Q$
2.	$\sim (P \vee Q)$	$\cong$	$\sim P \wedge \sim Q$

Law of Excluded Middle

$P \vee \sim P \cong$ T (true)

Law of Contradiction

$P \wedge \sim P \cong$ F (false)

P -> Q = ~ P V Q

Propositional Logic - PL

- PL deals with
  - the validity, satisfiability and unsatisfiability of a formula
  - derivation of a new formula using equivalence laws.
- Each row of a truth table for a given formula is called its **interpretation** under which a formula can be true or false.
- A formula α is called **tautology** if and only
  - if α is true for all interpretations.
- A formula α is also called **valid** if and only if
  - it is a **tautology.**

Cont..

- Let α be a formula and if there exist at least one interpretation for which α is true,
  - then α is said to be **consistent** (satisfiable) i.e., if $\exists$ a model for α, then α is said to be consistent .
- A formula α is said to be inconsistent (unsatisfiable), if and only if
  - α is always false under all interpretations.
- We can translate
  - simple declarative and
  - conditional (if .. then) natural language sentences into its corresponding propositional formulae.

Example

- Show that " It is humid today and if it is humid then it will rain so it will rain today" is a valid argument.
- **Solution:** Let us symbolize English sentences by propositional atoms as follows:

  A : It is humid

  B : It will rain

- Formula corresponding to a text:

  $\alpha : ((A \rightarrow B) \Lambda A) \rightarrow B$

- Using truth table approach, one can see that α is true under all four interpretations and hence is valid argument.

Cont..

Truth Table for $((A \rightarrow B) \wedge A) \rightarrow B$				
A	B	$A \rightarrow B = X$	$X \wedge A = Y$	$\mathbf{Y \rightarrow B}$
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	F	**T**
F	F	T	F	**T**

Cont…

- Truth table method for problem solving is
  - simple and straightforward and
  - very good at presenting a survey of all the truth possibilities in a given situation.
- It is an easy method to evaluate
  - a consistency, inconsistency or validity of a formula, but the size of truth table grows exponentially.
  - Truth table method is good for small values of n.
- For example, if a formula contains n atoms, then the truth table will contain 2^n entries.
  - A formula α : $(P \wedge Q \wedge R) \rightarrow (Q \vee S)$ is **valid** can be proved using truth table.
  - A table of 16 rows is constructed and the truth values of α are computed.
  - Since the truth value of α is true under all 16 interpretations, it is valid.

Cont..

- We notice that if P Λ Q Λ R is false, then α is true because of the definition of $\rightarrow$.
- Since P Λ Q Λ R is false for 14 entries out of 16, we are left only with two entries to be tested for which α is true.
  - So in order to prove the validity of a formula, all the entries in the truth table may not be relevant.
- Other methods which are concerned with proofs and deductions of logical formula are as follows:
  - Natural Deductive System
  - Axiomatic System
  - Semantic Tableaux Method
  - Resolution Refutation Method

Natural deduction method - ND

- ND is based on the set of few deductive inference rules.
- The name natural deductive system is given because it mimics the pattern of natural reasoning.
- It has about 10 deductive inference rules.

Conventions:

- E for Elimination.
- P, P_k , $(1 \leq k \leq n)$ are atoms.
- α_k, $(1 \leq k \leq n)$ and β are formulae.

ND Rules

Rule 1: I-Λ (Introducing Λ)

I-Λ : If $P_1, P_2, \ldots, P_n$ then $P_1 \Lambda P_2 \Lambda \ldots \Lambda P_n$

Interpretation: If we have hypothesized or proved $P_1, P_2, \ldots$ and P_n, then their conjunction $P_1 \Lambda P_2 \Lambda \ldots \Lambda P_n$ is also proved or derived.

Rule 2: E-Λ (Eliminating Λ)

E-Λ : If $P_1 \Lambda P_2 \Lambda \ldots \Lambda P_n$ then P_i ($1 \leq i \leq n$)

Interpretation: If we have proved $P_1 \Lambda P_2 \Lambda \ldots \Lambda P_n$, then any P_i is also proved or derived. This rule shows that Λ can be eliminated to yield one of its conjuncts.

Rule 3: I-V (Introducing V)

I-V : If P_i ($1 \leq i \leq n$) then $P_1 V P_2 V \ldots V P_n$

Interpretation: If any Pi ($1 \leq i \leq n$) is proved, then $P_1 V \ldots V P_n$ is also proved.

Rule 4: E-V (Eliminating V)

E-V : If $P_1 V \ldots V P_n$, $P_1 \rightarrow P, \ldots, P_n \rightarrow P$ then P

Interpretation: If $P_1 V \ldots V P_n$, $P_1 \rightarrow P, \ldots$, and $P_n \rightarrow P$ are proved, then P is proved.

Rules – cont..

Rule 5: I- $\rightarrow$ (Introducing $\rightarrow$)

I- $\rightarrow$: If **from** $\boldsymbol{\alpha_1}$**, …,** $\boldsymbol{\alpha_n}$ **infer** $\boldsymbol{\beta}$ **is proved** then $\boldsymbol{\alpha_1 \Lambda \ldots \Lambda\alpha_n \rightarrow \beta}$ is **proved**

Interpretation: If given α_1, α_2, …and α_n and from these we deduce β then $\alpha_1 \Lambda \alpha_2 \Lambda \ldots \Lambda\alpha_n \rightarrow \beta$ is also proved.

Rule 6: E- $\rightarrow$ (Eliminating $\rightarrow$) - *Modus Ponen*

E- $\rightarrow$: If $P_1 \rightarrow P$, P_1 then P

Rule 7: I- $\leftrightarrow$ (Introducing $\leftrightarrow$)

I- $\leftrightarrow$: If $P_1 \rightarrow P_2$, $P_2 \rightarrow P_1$ then $P_1 \leftrightarrow P_2$

Rule 8: E- $\leftrightarrow$ (Elimination $\leftrightarrow$)

E- $\leftrightarrow$: If $P_1 \leftrightarrow P_2$ then $P_1 \rightarrow P_2$, $P_2 \rightarrow P_1$

Examples

Example1: Prove that $P \wedge (Q \vee R)$ follows from $P \wedge Q$

Solution: This problem is restated in natural deductive system as "**from $P \wedge Q$ infer $P \wedge (Q \vee R)$**".

{Theorem}	**from $P \wedge Q$ infer $P \wedge (Q \vee R)$**	
{ premise}	$P \wedge Q$	(1)
{ E-$\wedge$, (1)}	P	(2)
{ E-$\wedge$, (1)}	Q	(3)
{ I-V , (3) }	$Q \vee R$	(4)
{ I-$\wedge$, (2, 4)}	**$P \wedge (Q \vee R)$**	**Conclusion**

Axiomatic System for PL

- It is based on the set of only three axioms and one rule of deduction.
  - It is minimal in structure but as powerful as the truth table and natural deduction approaches.
  - These methods basically require forward chaining strategy where we start with the given hypotheses and prove the goal.

Axiom1 (A1): $\alpha \rightarrow (\beta \rightarrow \alpha)$

Axiom2 (A2): $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$

Axiom3 (A3): $(\sim \alpha \rightarrow \sim \beta) \rightarrow (\beta \rightarrow \alpha)$

Modus Ponen (MP) defined as follows:

Hypotheses: $\alpha \rightarrow \beta$ and α ***Consequent:*** β

Examples

Examples: Establish the following:

1. {Q} |-(P→Q) i.e., P→Q is a deductive consequence of {Q}.

{Hypothesis}	Q	(1)
{Axiom A1}	$Q \to (P \to Q)$	(2)
{MP, (1,2)}	$\mathbf{P \to Q}$	**proved**

2. { P → Q, Q → R } |- (P → R) i.e., P → R is a deductive consequence of { P → Q, Q → R }.

{Hypothesis}	$P \to Q$	(1)
{Hypothesis}	$Q \to R$	(2)
{Axiom A1}	$(Q \to R) \to (P \to (Q \to R))$	(3)
{MP, (2, 3)}	$P \to (Q \to R)$	(4)
{Axiom A2}	$(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$	(5)
{MP , (4, 5)}	$(P \to Q) \to (P \to R)$	(6)
{MP, (1, 6)}	$\mathbf{P \to R}$	**proved**

Deduction Theorems in AS

Deduction Theorem:

If Σ is a set of hypotheses and α and β are well-formed formulae , then $\{\Sigma \cup \alpha\}$ |- β implies Σ |- $(\alpha \rightarrow \beta)$.

Converse of deduction theorem:

Given Σ |- $(\alpha \rightarrow \beta)$,

we can prove $\{\Sigma \cup \alpha\}$ |- β.

Useful Tips

1. Given α, we can easily prove $\beta \rightarrow \alpha$ for any well-formed formulae α and β.
2. **Useful tip**

   If $\alpha \rightarrow \beta$ is to be proved, then include α in the set of hypotheses $\sum$ and derive β from the set $\{\sum \cup \alpha\}$. Then using deduction theorem, we conclude $\alpha \rightarrow \beta$.

Semantic Tableaux System in PL

- Earlier approaches require
  - construction of proof of a formula from given set of formulae and are called direct methods.
- In **semantic tableaux**,
  - the set of rules are applied systematically on a formula or set of formulae to establish its consistency or inconsistency.
- *Semantic tableau*
  - binary tree constructed by using semantic rules with a formula as a root
- Assume α and β be any two formulae.

Semantic Tableaux Rules

Rule 1: A tableau for a formula $(\alpha \ \Lambda \ \beta)$ is constructed by adding both α and β to the same path (branch). This can be represented as follows:

$$\begin{array}{|l} \alpha \ \Lambda \ \beta \\ \alpha \\ \beta \end{array}$$

Rule 2: A tableau for a formula $\sim (\alpha \ \Lambda \ \beta)$ is constructed by adding two alternative paths one containing $\sim \alpha$ and other containing $\sim\beta$.

$$\begin{array}{ccc} & \sim (\alpha \ \Lambda \ \beta) & \\ \swarrow & & \searrow \\ \sim \alpha & & \sim \beta \end{array}$$

Rule 3: A tableau for a formula $(\alpha$ V $\beta)$ is constructed by adding two new paths one containing α and other containing β.

$$\begin{array}{ccc} & \alpha \ \mathrm{V} \ \beta & \\ \swarrow & & \searrow \\ \alpha & & \beta \end{array}$$

Rule 4: A tableau for a formula $\sim (\alpha$ V $\beta)$ is constructed by adding both $\sim \alpha$ and $\sim \beta$ to the same path. This can be expressed as follows:

$$\begin{array}{|l} \sim (\alpha \ \ \mathrm{V} \ \ \beta) \\ \sim \alpha \\ \sim \beta \end{array}$$

Rules - Cont..

Rule 5:

$\sim\ \sim\ \alpha$

α

Rule 6:

$\alpha \rightarrow \beta$

$\sim \alpha \qquad \beta$

Rule 7:

$\sim (\alpha \rightarrow \beta)$

α

$\sim \beta$

Rule 8: $\alpha \leftrightarrow \beta \cong (\alpha \wedge \beta) \vee (\sim \alpha \wedge \sim \beta)$

$\alpha \leftrightarrow \beta$

$\alpha \wedge \beta \qquad \sim \alpha \wedge \sim \beta$

Rule 9: $\sim (\alpha \leftrightarrow \beta) \cong (\alpha \wedge \sim \beta) \vee (\sim \alpha \wedge \beta)$

$\sim (\alpha \leftrightarrow \beta)$

$\alpha \wedge \sim \beta \qquad \sim \alpha \wedge \beta$

Consistency and Inconsistency

- If an atom P and ~ P appear on a same path of a semantic tableau,
  - then inconsistency is indicated and such path is said to be c**ontradictory** or **closed** (finished) path.
  - Even if one path remains **non contradictory** or **unclosed** (open), then the formula α at the root of a tableau is **consistent**.
- **Contradictory tableau** (or **finished tableau):**
  - It defined to be a tableau in which all the paths are contradictory or closed (finished).
- If a tableau for a formula α at the root is a contradictory tableau**,**
  - then a formula α is said to be inconsistent.

Example

- Show that α: $(Q \ \Lambda \sim R) \ \Lambda \ (R \rightarrow P)$ is ***consistent*** and find its model.

{Tableau root}	$(Q \ \Lambda \sim R) \Lambda (R \rightarrow P)$	(1)
{Apply rule 1 to 1}	$(Q \ \Lambda \ \sim R)$	(2)
	$(R \rightarrow P)$	(3)
{Apply rule 1 to 2}	Q	
{Apply rule 6 to 3}	$\sim R$	

$\sim R$ $\qquad\qquad$ P

open $\qquad\qquad$ **open**

- { Q = T, R = F } and { P = T , Q = T, R = F } are models of α.