Update Step

1) Propagate step.
2) Update step

$\hat{x}_{k|k}$ $\hat{x}_{k+1|k}$

$k \longrightarrow k+1$

prior estimate

$\hat{x}_{k+1|k} \longrightarrow \hat{x}_{k+1|k+1}$

prior

posterior estimate

y_{k+1}

- There are two sources that provide estimate to the state $\underline{\hat{x}}_k$

1. Estimate from the state propagation $\underline{\hat{x}}_{k|k-1}$ prior estimate (after the propagation step)

state propagation

true

$$\underline{\hat{x}}_{k|k-1} = \underline{x}_k + \underline{e}_k$$

fixed random noise

The estimate $\underline{\hat{x}}_{k|k-1}$ is considered to have a random deviation or error from the true state $\underline{x}_k$

The covariance $P_{k|k-1}$ of the error term summarises the uncertainty of prediction $\underline{\hat{x}}_{k|k-1}$ given the past history of measurements.

$$P_{k|k-1} = \mathbb{E}[\underline{e}_k \underline{e}_k^\top]$$

- There are two sources that provide estimate to the state $\underline{\hat{x}}_k$

  2. Evidence from the new measurement

$$\underline{y}_k = \underline{H}_k \underline{x}_k + \underline{\xi}_k$$

$$R_k = \mathbb{E}[\underline{\xi}_k \underline{\xi}_k^\top]$$

$\hat{x}_k$ accounted by the observation y_k.

$\hat{x}_{k|k-1}$ prior estimate

$\hat{x}_{k|k}$.

posterior

- Given the evidences about the unknown true state $\underline{x}_k$ our task is to compute an estimate $\hat{\underline{x}}_k$ of $\underline{x}_k$
- We collect the two evidences

$$\left[\begin{aligned} \hat{x}_{k|k-1} &= \underline{x}_k + \underline{e}_k \\ \underline{y}_k &= \underline{H}_k \underline{x}_k + \underline{\xi}_k \end{aligned}\right]$$

$\hat{x}_{k|k}$

estimate of the true value x_k

to form a system of equations

$$\underline{y} = \underline{H}\,\underline{x}_k + \underline{n}$$

from measurement

Update Step

Update Step

- $\underline{y} = \underline{H}\underline{x}_k + \underline{n}$ Combined equation

$$\underline{y} = \begin{bmatrix} \underline{\hat{x}}_{k|k-1} \\ \underline{y}_k \end{bmatrix} \qquad \underline{H} = \begin{bmatrix} \underline{I} \\ \underline{H}_k \end{bmatrix} \qquad \underline{n} = \begin{bmatrix} \underline{e}_k \\ \underline{n}_k \end{bmatrix}$$

Given Given noise. random

- $\underline{\hat{x}}_{k|k-1} = \underline{I}\underline{x}_k + \underline{e}_k$

$$\underline{y}_k = \underline{H}_k\underline{x}_k + \underline{\xi}_k$$

$\hat{x}_{k|k}$.

gives the estimate

We solve for the true x_k.

system noise affecting the state propagation.

- The covariance matrix of the vector $\underline{n} = \begin{bmatrix} \underline{e}_k \\ \underline{\xi}_k \end{bmatrix}$ is formulated as

measurement ξ_k.

defines the uncertainty of the noise

$$\underline{R} = \begin{bmatrix} \underline{P}_{k|k-1} & 0 \\ 0 & \underline{R}_k \end{bmatrix}$$

Here we assume that the two noise vectors are independent.

- The formulation $\boxed{\underline{y} = \underline{H}\underline{x}_k + \underline{n}}$ is a classical estimation problem.

Linear system

$\hat{x}_{k|k}$ posterior estimate

Update Stage

- The solution to the estimation problem $\underline{y} = \underline{H}_k \underline{x}_k + \underline{n}$ is

Covariance of the posterior estimate

$$\boxed{\underline{P}_{k|k} = (\underline{H}^\top \underline{R}^{-1} \underline{H})^{-1}}$$

$$\boxed{\hat{\underline{x}}_{k|k} = \underline{P}_{k|k} \underline{H}^\top \underline{R}^{-1} \underline{y}}$$

identical to the derivations done earlier

$$\underline{y} = \begin{bmatrix} \hat{\underline{x}}_{k|k-1} \\ \underline{y}_k \end{bmatrix}$$

$$\underline{H} = \begin{bmatrix} \underline{I} \\ \underline{H}_k \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} \underline{P}_{k|k-1} & 0 \\ 0 & \underline{R}_k \end{bmatrix}$$

- Simplification

$$\underline{P}_{k|k}^{-1} = \underline{H}^\top \underline{R}^{-1} \underline{H}$$

$$= \begin{bmatrix} \underline{I} & \underline{H}_k^\top \end{bmatrix} \begin{bmatrix} \underline{P}_{k|k-1}^{-1} & 0 \\ 0 & \underline{R}_k^{-1} \end{bmatrix} \begin{bmatrix} \underline{I} \\ \underline{H}_k \end{bmatrix}$$

Posterior covar

$$= \underbrace{\underline{P}_{k|k-1}^{-1}}_{\text{prior covariance}} + \underline{H}_k^\top \underline{R}_k^{-1} \underline{H}_k$$

(Covariance of the prior estimate)

Revised estimates have been obtained using two evidences.

precision improves.

This is the update stage of the Kalman Filter

Simplifying the posterior estimate

- $$\hat{x}_{k|k} = \underline{P}_{k|k}\underline{H}^\top \underline{R}^{-1}\underline{y}$$

$$= \underline{P}_{k|k} \underbrace{\begin{bmatrix} \underline{I} & \underline{H}_k^\top \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} \underline{P}_{k|k-1}^{-1} & 0 \\ 0 & \underline{R}_k^{-1} \end{bmatrix}}_{R.} \begin{bmatrix} \hat{x}_{k|k-1} \\ y_k \end{bmatrix}$$

$$= \underline{P}_{k|k} \begin{bmatrix} \underline{P}_{k|k-1}^{-1} & \underline{H}_k^\top \underline{R}_k^{-1} \end{bmatrix} \begin{bmatrix} \hat{x}_{k|k-1} \\ \underline{y}_k \end{bmatrix}$$

$$= \underline{P}_{k|k}\left(\underline{P}_{k|k-1}^{-1}\hat{x}_{k|k-1} + \underline{H}_k^\top \underline{R}_k^{-1} \underline{y}_k\right)$$

prior covariance.

$$\underline{H} = \begin{bmatrix} \underline{I} \\ \underline{H}_k \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} \underline{P}_{k|k-1} & 0 \\ 0 & \underline{R}_k \end{bmatrix}$$

cov matrix of measurement error.

posterior prior

$$\underline{P}_{k|k}^{-1} = \underline{P}_{k|k-1}^{-1} + \underline{H}_k^\top \underline{R}_k^{-1} \underline{H}_k$$

y_k is not required

$$y = \begin{bmatrix} \hat{x}_{k|k-1} \\ y_k \end{bmatrix} \qquad R = \begin{bmatrix} P_{k|k-1} & 0 \\ 0 & R_k \end{bmatrix} \qquad H = \begin{bmatrix} I \\ H_k \end{bmatrix}$$

- $$\hat{x}_{k|k} = P_{k|k} H^\top R^{-1} y$$

$$= P_{k|k}\Big(\underbrace{\big(P_{k|k}^{-1} - H_k^\top R_k^{-1} H_k\big)}_{P_{k|k-1}^{-1}} \hat{x}_{k|k-1} + H_k^\top R_k^{-1} y_k\Big)$$

$$= \big(I - P_{k|k} H_k^\top R_k^{-1} H^k\big) \underbrace{\hat{x}_{k|k-1}}_{\text{prior}} + P_{k|k} H_k^\top R_k^{-1} y_k$$

$$= \hat{x}_{k|k-1} - P_{k|k} H_k^\top R_k^{-1} H^k \hat{x}_{k|k-1} + P_{k|k} H_k^\top R_k^{-1} y_k$$

$$\underset{\hat{x}_{k|k}}{\text{posterior}} = \underbrace{\hat{x}_{k|k-1}}_{\text{prior}} + \underbrace{\big(P_{k|k} H_k^\top R_k^{-1}\big)}_{\text{Matrix}} \underbrace{\big(y_k - H_k \hat{x}_{k|k-1}\big)}_{\text{difference : residue}}$$

fusion of prior estimate & evidence from observation

$$x_k \xrightarrow{H_k} y_k$$

y_k : actual observation

$H_k \hat{x}_{k|k-1}$: predicted observation

- Introduce the Kalman Gain Matrix

$$\boxed{\underline{K}_k \triangleq \underline{P}_{k|k} \underline{H}_k^\top \underline{R}_k^{-1}}$$

cov of the posterior estimate

- Define the residue

$$\underline{r}_k \triangleq \underline{y}_k - \underline{H}_k \hat{\underline{x}}_{k|k-1}$$

The residue is the difference between the actual measurement and the predicted measurement.

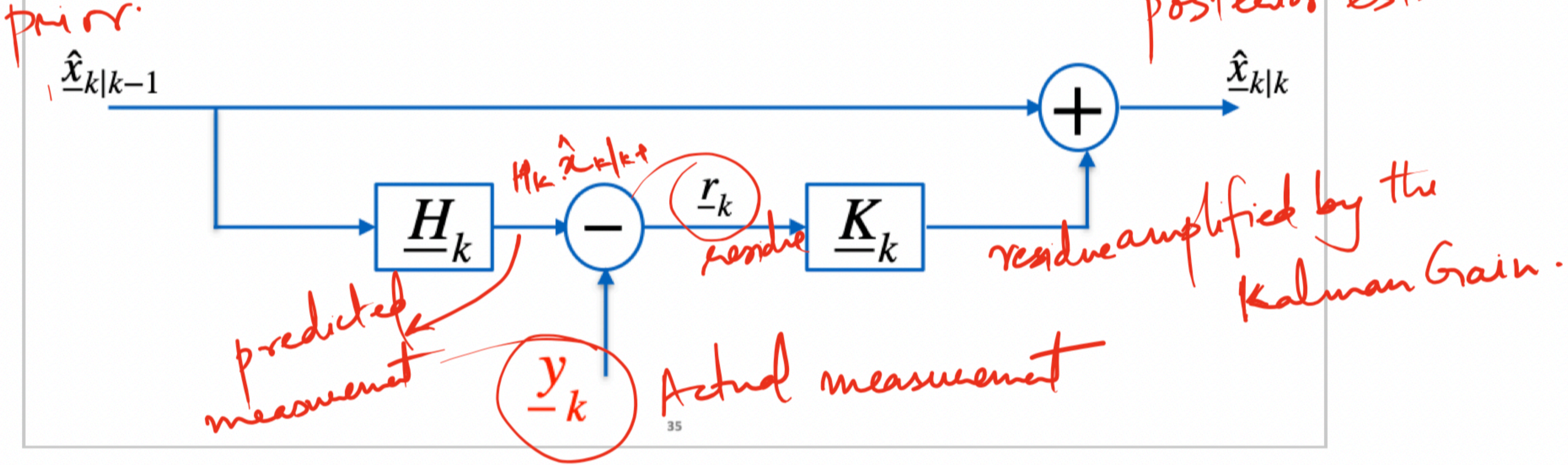
update equation

Update of the state estimate by accounting for the current observation.

$$\boxed{\hat{\underline{x}}_{k|k} = \hat{\underline{x}}_{k|k-1} + \underline{K}_k \left(\underline{y}_k - \underline{H}_k \hat{\underline{x}}_{k|k-1} \right)}$$

Taking the measurement into account (The Update Step)

- The Kalman gain matrix specifies the amount by which the residue must be multiplied (or amplified) to obtain the correction term to correct the old estimate $\underline{\hat{x}}_{k|k-1}$ and give the new estimate $\underline{\hat{x}}_{k|k}$



propagation → prior estimate —update→ posterior estimate —propagation→

The Propagation Step

Propagation to the next time step

posterior of the previous step

is a random variable.

- Propagated estimate

prior estimate for the current time step.

$$\underline{\hat{x}}_{k+1|k} = \underline{F}_k \underline{\hat{x}}_{k|k} + \underline{G}_k \underline{u}_k$$

$+ \eta_k$.

- Note that we ignore the noise term in the propagation step because the estimate $\underline{\hat{x}}_{k+1|k}$ is computed using another random vector $\underline{\hat{x}}_{k|k}$

- Compare this with the state propagation equation

$$\boxed{\underline{x}_{k+1} = \underline{F}_k \underline{x}_k + \underline{G}_k \underline{u}_k + \eta_k}$$

true (not random)

$$P_{k|k} \longrightarrow P_{k+1|k}.$$

For unbiased estimates, the true value is the mean of the random vector.

- The covariance matrix of the propagated estimate is

$$P_{k+1|k} = \mathbb{E}\left[(\underbrace{\hat{x}_{k+1|k}}_{\text{mean}} - x_{k+1})(\hat{x}_{k+1|k} - x_{k+1})^{\mathsf{T}}\right]$$

$$= \mathbb{E}\left[\left(F_k(\hat{x}_{k|k} - x_k) - \eta_k\right)\left(F_k(\hat{x}_{k|k} - x_k) - \eta_k\right)^{\mathsf{T}}\right]$$

$E[\hat{n}\hat{x}] = \underbrace{E[\hat{n}]}_{\text{zero mean.}} E[\hat{x}]$

$$= F_k \underbrace{\mathbb{E}\left[\left(\hat{x}_{k|k} - x_k\right)\left(\hat{x}_{k|k} - x_k\right)^{\mathsf{T}}\right]}_{P_{k|k}} F_k^{\mathsf{T}} + \underbrace{\mathbb{E}\left[\eta_k\eta_k^{\mathsf{T}}\right]}_{Q_k} + \underbrace{\mathbb{E}\left[\eta_k(\hat{x}_{k|k} - x_k)\right]}_{\Rightarrow} \rightarrow 0$$

2) noise is zero mean.

1) noise is uncorrelated with the diff $(\hat{x}_{k|k} - x_k)$

$$= F_k P_{k|k} F_k^{\mathsf{T}} + \mathbb{E}\left[\eta_k\eta_k^{\mathsf{T}}\right]$$

$\because$ the noise has covariance matrix Q_k

$$= F_k P_{k|k} F_k^{\mathsf{T}} + Q_k$$

$\eta_k \sim N(0, Q_k)$

$$\boxed{P_{k+1|k} = F_k P_{k|k} F_k^{\mathsf{T}} + Q_k}$$

Kalman Filter Equations

No past observation

$\hat{x}_{k|k-1}$

$\hat{x}_{0|-1}$

- Initialisation

$$\underline{\hat{x}}_{0|-1} \triangleq \underline{\hat{x}}_0$$

$$\underline{P}_{0|-1} \triangleq \underline{P}_0$$

prior to posterior go.

- Update

$$\underline{\hat{x}}_{k|k} = \underline{\hat{x}}_{k|k-1} + \underline{K}_k(\underline{y}_k - \underline{H}_k\underline{\hat{x}}_{k|k-1})$$

$$\underline{P}^{-1}_{k|k} = \underline{P}^{-1}_{k|k-1} + \underline{H}^{\top}_k \underline{R}^{-1}_k \underline{H}_k$$

$$\boxed{\underline{K}_k \triangleq \underline{P}_{k|k}\underline{H}^{\top}_k\underline{R}^{-1}_k}$$

posterior prior

$k \longrightarrow k+1$

- Propagation

$$\boxed{\underline{\hat{x}}_{k+1|k} = \underline{F}_k\underline{\hat{x}}_{k|k} + \underline{G}_k\underline{u}_k} \leftarrow \text{input}$$

$$\boxed{\underline{P}_{k+1|k} = \underline{F}_k\underline{P}_{k|k}\underline{F}^{\top}_k + \underline{Q}_k}$$

I. posterior covariance of current time step.

II. prior covariance of current time step. $P_{k|k-1}$

II. prior covariance of current time step. $P_{k|k-1}$

Alternate form of Kalman Gain Matrix

II. K_k ~ prior covariance.

post est. prior residue

- $\underbrace{\hat{\underline{x}}_{k|k}}_{\text{post est.}} = \underbrace{\hat{\underline{x}}_{k|k-1}}_{\text{prior}} + \underline{K}_k \underbrace{(\underline{y}_k - \underline{H}_k \hat{\underline{x}}_{k|k-1})}_{\text{residue}}$

$= \hat{\underline{x}}_{k|k-1} + \underline{K}_k(\underline{H}_k \underline{x}_k + \xi_k - \underline{H}_k \hat{\underline{x}}_{k|k-1})$

unknown → To solve

$y_k = H_k x_k + \xi_k.$

$\hat{x}_{k|k}$ should be as close as possible to the true value.

- Posterior estimates

$$\underline{P}_{k|k} = \mathbb{E}\left[\left(\underline{x}_k - \hat{\underline{x}}_{k|k}\right)\left(\underline{x}_k - \hat{\underline{x}}_{k|k}\right)^{\mathsf{T}}\right]$$

rewrite

- $\underline{x}_k - \hat{\underline{x}}_{k|k} = \underline{x}_k - \hat{\underline{x}}_{k|k-1}(\underline{I} - \underline{K}_k\underline{H}_k) - \underline{x}_k\underline{K}_k\underline{H}_k - \underline{K}_k\xi_k$

$\mathbb{E}\left(x_k - \hat{x}_{k|k}\right)^2$

sum of the diagonal elements of $P_{k|k}$.

$$P_{k|k} = \mathbb{E}\left[\left(\left(I - K_k H_k\right)\left(x_k - \hat{x}_{k|k-1}\right) - K_k \xi_k\right)\left(\left(I - K_k H_k\right)\left(x_k - \hat{x}_{k|k-1}\right) - K_k \xi_k\right)^\top\right]$$

$$= \left(I - K_k H_k\right)\mathbb{E}\left[\left(x_k - \hat{x}_{k|k-1}\right)\left(x_k - \hat{x}_{k|k-1}\right)^\top\right]\left(I - K_k H_k\right) + K_k \underbrace{\mathbb{E}\left[\xi_k \xi_k^\top\right]}_{R_k.} K_k^\top$$

$$= \left(I - K_k H_k\right) P_{k|k-1} \left(I - K_k H_k\right)^\top + K_k R_k K_k^\top$$

rearrange.

$$= P_{k|k-1} - K_k H_k P_{k|k-1} - P_{k|k-1} H_k^\top K_k^\top + K_k \left(H_k P_{k|k-1} H_k^\top + R_k\right) K_k^\top$$

- The mean squared error between $\underline{x}_k$ and $\hat{\underline{x}}_{k|k}$ can be minimised by minimising the trace of the covariance matrix $\underline{P}_{k|k}$

- $$\operatorname{Tr}\left[\underline{P}_{k|k}\right] = \underbrace{\operatorname{Tr}\left[\underline{P}_{k|k-1}\right]}_{} - 2\operatorname{Tr}\left[\underline{K}_k\underline{H}_k\underline{P}_{k|k-1}\right] + \operatorname{Tr}\left[\underline{K}_k\left(\underline{H}_k(\underline{H}_k\underline{P}_{k|k-1}\underline{H}_k^\top + \underline{R}_k\right)\underline{K}_k^\top\right]$$

Minimize.

Trace.

- We want to find the Kalman Gain matrix $\underline{K}_k$ which can minimise $\operatorname{Tr}\left[\underline{P}_{k|k}\right]$

- Taking derivative of $\mathrm{Tr}\left[\underline{P}_{k|k}\right]$ with respect to $\underline{K}_k$ and equating it to 0

$$\left(\frac{dT_r[P_{k|k}]}{dK_k}\right) = -2\left(\underline{H}_k\underline{P}_{k|k-1}\right)^{\top} + 2\underline{K}_k\left(\underline{H}_k\underline{P}_{k|k-1}\underline{H}_k^{\top} + \underline{R}_k\right)$$

$$= 0$$

- This gives $\left(\underline{H}_k\underline{P}_{k|k-1}\right)^{\top} = \underline{K}_k\left(\underline{H}_k\underline{P}_{k|k-1}\underline{H}_k^{\top} + \underline{R}_k\right)$

$$\boxed{\underline{K}_k = \underline{P}_{k|k-1}\underline{H}_k^{\top}\left(\underline{H}_k\underline{P}_{k|k-1}\underline{H}_k^{\top} + \underline{R}_k\right)^{-1}}$$

II.

- Some results that were used in the previous slide

Let $\underline{A}_{n\times m}$ and $\underline{X}_{n\times m}$

$$\frac{d}{d\underline{X}}\underbrace{\mathrm{Tr}\left[\underline{A}\,\underline{X}\right]}_{\text{scalar}} = \frac{d}{d\underline{X}}\mathrm{Tr}\left[\underline{X}\,\underline{A}\right] = \underline{A}^{\top}$$

$$\frac{d}{d\underline{X}}\mathrm{Tr}\left[\underline{A}\,\underline{X}^{\top}\right] = \frac{d}{d\underline{X}}\mathrm{Tr}\left[\underline{X}^{\top}\underline{A}\right] = \underline{A}$$

$$\frac{d}{d\underline{X}}\mathrm{Tr}\left[\underline{X}^{\top}\underline{A}\,\underline{X}\right] = \left(\underline{A}+\underline{A}^{\top}\right)\underline{X}$$

$$\frac{d}{d\underline{X}}\mathrm{Tr}\left[\underline{A}\,\underline{X}\,\underline{B}\right] = \left(\underline{B}\,\underline{A}\right)^{\top}$$

substitute

- The update step of the covariance matrix can now be derived as

$$\underline{P}_{k|k} = \underline{P}_{k|k} - \underline{K}_k \underline{H}_k \underline{P}_{k|k-1} - \underline{P}_{k|k-1} \underline{H}_k^\top \underline{K}_k^\top + \underline{K}_k \left(\underline{H}_k \underline{P}_{k|k-1} \underline{H}_k^\top + \underline{R}_k \right) \underline{K}_k^\top$$

- Substituting $\underline{K}_k = \underline{P}_{k|k-1} \underline{H}_k^\top \left(\underline{H}_k \underline{P}_{k|k-1} \underline{H}_k^\top + \underline{R}_k \right)^{-1}$

$$\underline{P}_{k|k} = \underline{P}_{k|k-1} - \underline{P}_{k|k-1} \underline{H}_k^\top \left(\underline{H}_k \underline{P}_{k|k-1} \underline{H}_k^\top + \underline{R}_k \right)^{-1} \underline{H}_k \underline{P}_{k|k-1}$$

$$\cancel{- \underline{P}_{k|k-1} \underline{H}_k^\top \left(\underline{H}_k \underline{P}_{k|k-1} \underline{H}_k^\top + \underline{R}_k \right)^{-1} \underline{H}_k \underline{P}_{k|k-1}}$$

$$\cancel{+ \underline{P}_{k|k-1} \underline{H}_k^\top \underbrace{\left(\underline{H}_k \underline{P}_{k|k-1} \underline{H}_k^\top + \underline{R}_k \right)^{-1} \left(\underline{H}_k \underline{P}_{k|k-1} \underline{H}_k^\top + \underline{R}_k \right)}_{\text{Identity.}} \left(\underline{H}_k \underline{P}_{k|k-1} \underline{H}_k^\top + \underline{R}_k \right)^{-1} \underline{H}_k \underline{P}_{k|k-1}}$$

- $$\underline{P}_{k|k} = \underline{P}_{k|k-1} - \underline{P}_{k|k-1}\underline{H}_k^\top\left(\underline{H}_k\underline{P}_{k|k-1}\underline{H}_k^\top + \underline{R}_k\right)^{-1}\underline{H}_k\underline{P}_{k|k-1}$$

$$= \underline{P}_{k|k-1} - \underline{K}_k\underline{H}_k\underline{P}_{k|k-1}$$

$$= \left(\underline{I} - \underline{K}_k\underline{H}_k\right)\underline{P}_{k|k-1}$$

$$\boxed{\underline{P}_{k|k} = \left(\underline{I} - \underline{K}_k\underline{H}_k\right)\underline{P}_{k|k-1}}$$

x_k

F_k.

G_k

Initial Estimates $\quad \hat{\underline{x}}_0, \underline{P}_0$

Kalman Gain $\quad \underline{K}_k = \underline{P}_{k|k-1}\underline{H}_k^{\top}\left(\underline{H}_k\underline{P}_{k|k-1}\underline{H}_k^{\top} + \underline{R}_k\right)^{-1}$

Update Estimates

(accounts for the current observation)

$$\hat{\underline{x}}_{k|k} = \hat{\underline{x}}_{k|k-1} + \underline{K}_k(\underline{y}_k - \underline{H}_k\hat{\underline{x}}_{k|k-1})$$

$$\underline{P}_{k|k} = \left(\underline{I} - \underline{K}_k\underline{H}_k\right)\underline{P}_{k|k-1}$$

Propagate Estimates to the next time step $k+1$

$$\hat{\underline{x}}_{k+1|k} = \underline{F}_k\hat{\underline{x}}_{k|k} + \underline{G}_k\underline{u}_k$$

$$\underline{P}_{k+1|k} = \underline{F}_k\underline{P}_{k|k}\underline{F}_k^{\top} + \underline{Q}_k$$

prior estimates

Linear dynamics.

Kalman filter is smoothening the state estimates

stability (in the presence of system & observation).