Implementation of Chattering-Free Digital Sliding-Mode Control With State Observer and Disturbance Rejection

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***Abstract*—This report details the practical implementation of the novel discrete-time sliding-mode control system as outlined in the paper by Vincent Acary, Bernard Brogliato, and Yury V. Orlov. It provides a comprehensive account of how the theoretical constructs in the paper are translated into a functioning control system. Central to this implementation is the utilization of the multivalued dynamics on the sliding surface, ensuring smooth stabilization without the chattering effect, a common issue in time-discretized systems. The report delves into how disturbances are effectively attenuated, demonstrating the robustness of the controller even under large sampling periods. The implementation leverages the implicit Euler method, with a focus on practicality and ease of use, incorporating projections onto the interval [1, 1] or employing quadratic programming. Additionally, the report explores the application of the zero-order-hold (ZOH) method in this context. The implementation is tested on first- and second-order perturbed systems, with conditions matching the theoretical framework, and includes a comparative analysis of classical and twisting sliding-mode controllers. This report not only validates the theoretical findings but also serves as a guide for practical application in real-world control systems.**

***Index Terms*—Backward Euler method, discrete-time sliding mode, disturbance compensation, sliding-mode control, twisting controller, and the zero-order-hold method.**

I. Introduction

In our implementation, sliding-mode control is acknowledged as a significant sector within feedback control, encompassing a breadth of applications. The digital definition and execution of sliding mode systems, despite being a focal point of numerous studies since the advent of seminal works, presents ongoing challenges, particularly in fully grasping the methodologies and in circumventing issues such as numerical chattering. The aim of our implementation is manifold: firstly, to demonstrate the numerical incorporation of the multivalued aspect of discontinuous sliding-mode controllers via an implicit Euler approach, effectively eliminating numerical chattering observed in explicit implementations; secondly, to adapt this approach to scenarios where only a subset of the state is observable; and thirdly, to illustrate the robustness of this strategy against disturbances, ensuring chattering suppression and disturbance rejection even when the system is influenced by external perturbations.

In our implementation, both the numerical chattering and the disturbance chattering are suppressed, with the latter being attenuated by the sampling time h, where $h > 0$. The disturbance chattering, which is a result of high frequency disturbances acting on the system, is reduced by a factor of h in the digital implementation compared to the ideal (analytical) continuous-time system.

The major features of the implicit causal discrete-time input are that it does not alter the continuous-time system sliding surface, which has a codimension larger than one, ensuring the integrity of the sliding motion in the discrete-time system. This is achieved by implementing an implicit Euler controller, which avoids the numerical chattering present in explicit implementations.

Definition 1: Let $h = t_{k+1} - t_k > 0$ be the sampling period for $k \geq 0$. An m-discrete-time sliding surface Σ_d is a codimension m subspace of the state space, such that the discrete state vector x_k approximates $x(t_k)$ and satisfies $x_k \in \Sigma_d$ for all $k_{\min} \leq k \leq k_{\max}$, $1 \leq k_{\min}$, and all $h > 0$.

The controllers designed in this paper are aimed at stabilizing an unperturbed nominal plant, and are robust against disturbances, preserving the exact sliding mode in the discrete case.

Although the systematic design of controllers to guarantee sliding-mode control is not new, the specific design that ensures robustness against disturbances while maintaining system performance is considered novel. The organization of the paper is as follows: Section II analyzes a simple first-order system, both with and without disturbance compensation, followed by an extension to higher-order systems using Euler and the Zero-Order Hold (ZOH) methods in Section III. Section IV discusses several types of controllers and compares their performance. The continuous-time system is introduced, followed by its time-discretization and simulation results. The paper concludes with a summary of findings in the final section.

Notation: The sign function $\mathrm{sgn}(x)$ used throughout is defined as a multivalued function:

$$\mathrm{sgn}(x) = \begin{cases} \{+1\}, & \text{if } x > 0, \\ \{-1\}, & \text{if } x < 0, \\ [-1, 1], & \text{if } x = 0. \end{cases} \tag{1}$$

Let $K \subset \mathbb{R}^n$ be a closed non-empty convex set. The normal cone to K at $x \in K$, denoted $N_K(x)$, and the projection of y onto K in the metric defined by a positive definite matrix M, $\mathrm{proj}_M(K; y)$, are defined as follows:

$$N_K(x) = \{z \in \mathbb{R}^n \mid z^T(y - x) \leq 0 \text{ for all } y \in K\}, \tag{2}$$

$$\mathrm{proj}_M(K; y) = \mathrm{argmin}_{x \in K} \frac{1}{2}(x - y)^T M (x - y). \tag{3}$$

For any $x, y \in \mathbb{R}$, one has $x \in \mathrm{sgn}(y)$ if and only if $y \in N_{[-1,1]}(x)$.

II. First-Order System

We analyze in this section the simplest case to illustrate how the method works. Two cases are treated: without and with disturbance compensation (in the continuous-time system). The basic ideas are illustrated on a simple first-order system.

A. The Case Without Disturbance Compensation

Let us start by considering the following basic sliding mode system:

$$\begin{cases} \dot{x}(t) = -a\tau(t) + \phi(t) \\ \tau(t) \in \mathrm{sgn}(\dot{x}(t)) \end{cases} \tag{4}$$

where $\phi(\cdot)$ is the Lebesgue measurable perturbation such that $\|\phi\|_\infty < \rho < a$. The control input is here $u(t) = \tau(t)$. It may be seen, in the language of differential inclusions theory, as a Lebesgue measurable *selection* of the set-valued right-hand side of the system [33]. Choosing correctly this selection is the object of the following discretization. The system (3) has $x = 0$ as its unique equilibrium point, which is globally asymptotically stable and is reached in finite time (this may be shown with the Lyapunov function $V(\dot{x}) = \dot{x}^2$). The discrete-time sliding mode system is implemented as follows:

$$\begin{aligned} \dot{x}_{k+1} &= x_k - ah\tau_{k+1} \\ \tau_{k+1} &\in \mathrm{sgn}(\dot{x}_{k+1}) \\ x_{k+1} &= x_k - ah\tau_{k+1} + h\phi_{k+1} \end{aligned} \tag{5}$$

The first two lines of (4) may be considered as the *nominal unperturbed plant*, from which one computes the input at time t_k. The input is said implicit since it involves $\dot{x}_{k+1}$ in the sign multifunction. It is however a causal input as shown next, and $\dot{x}_{k+1}$ is just an intermediate variable which does not explicitly enter into the controller. The third line is the Euler approximation of the plant, on which the disturbance is acting. One has $u(t) = \tau_{k+1}$ on the time-interval $[t_k, t_{k+1}]$.

Proposition 1: Let x_0 be the given initial state. Then after a finite number of steps k_0 one obtains that $\dot{x}_k = 0$ and $x_k = h\phi_k$ for all $k > k_0$. In other words, the disturbance is attenuated by a factor h. Moreover the approximated derivative of the state satisfies $(x_{k+1} - x_k)/h = \phi_{k+1} - \phi_k$ for all $k > k_0 + 1$ whereas $(\dot{x}_{k+1} - \dot{x}_k)/h = 0$ for all $k > k_0$. The control input takes values inside the sign multifunction multivalued part on the sliding surface for all $k > k_0$.

Derivation: We commence our analysis with the initial condition that the absolute value of x_0 is greater than ah, where both are positive. We examine the following reformulated system:

$$\dot{x}_{k+1} = x_k - ah\tau_{k+1} \quad \text{and} \quad \tau_{k+1} \in \mathrm{sgn}(\dot{x}_{k+1}), \tag{6}$$

which, when applying the previously established relations (1) and (2), simplifies to the inclusion $\tau_{k+1} + \frac{x_k}{ah} \in -N_{[-1,1]}(\tau_{k+1})$, leading us to deduce that $\tau_{k+1} = \mathrm{proj}_{[-1,1]}\left(\frac{x_k}{ah}\right)$. Consequently, we derive the following scenarios:

- If x_k exceeds ah, then x_{k+1} is reduced by ah, and the sign function yields 1.
- If x_k is less than $-ah$, then x_{k+1} is increased by ah, with the sign function yielding -1.
- If x_k is within the bounds of $\pm ah$, then $\dot{x}_{k+1}$ falls within the range of $(-ah, ah)$, and the sign function spans the interval $[-1, 1]$.

This leads us to infer:

- For x_k greater than ah, x_{k+1} is formulated as $x_k + h(\phi_{k+1} - a)$, which will inevitably decrease since $\rho - a$ is negative.
- For x_k less than $-ah$, x_{k+1} is expressed as $x_k + h(\phi_{k+1} + a)$, which will increase due to $-\rho + a$ being positive.

It follows that if the initial value $|x_0|$ surpasses ah, it will take $k_0 = \left\lfloor \frac{|x_0|}{h(|a-\rho|)} \right\rfloor$ iterations for $\dot{x}_k$ to stabilize at zero, where $\lfloor \cdot \rfloor$ denotes the floor function. Upon reaching k_0, x_k falls within $(-ah, ah)$, and the only viable solution for $\dot{x}_k$ becomes zero. This conclusion further implies that the magnitude of x_{k_0} is less than ah. Should the initial value $|x_0|$ be equal to or less than ah, then k_0 is promptly determined to be 1.

To ascertain the subsequent value of $\dot{x}_k$, the ensuing generalized system must be resolved:

$$\begin{aligned} \dot{x}_{k+1} &= x_{k_0} - ah\tau_{k+1} \\ \tau_{k+1} &\in \mathrm{sgn}(\dot{x}_{k+1}) \end{aligned} \tag{7}$$

Inspection reveals that the unique solution to our system at k_0+1 is $\dot{x}_{k_0+1} = 0.1$. By extending this logic, we deduce that $\dot{x}_k = 0$ for every k greater than or equal to k_0. Consequently, the difference $(\dot{x}_{k+1} - \dot{x}_k)/h$ is zero for all k exceeding k_0. We now proceed under the assumption that for all k beyond k_0, the following holds true:

$$\dot{x}_{k+1} = x_k - ah\tau_{k+1} = 0, \quad k \geq k_0, \tag{8}$$

which leads us to express τ_{k+1} as:

$$\tau_{k+1} = \frac{x_k}{ha}. \tag{9}$$

With this relation, the next state x_{k+1} is determined by:

$$x_{k+1} = h\phi_{k+1}, \tag{10}$$

and thus:

$$x_k = h\phi_k, \quad \tau_{k+1} = \frac{\phi_k}{a}, \quad \text{for all } k \geq k_0 + 1. \tag{11}$$

This implies that the incremental change in x_k over h equates to $\phi_{k+1} - \phi_k$ for each step past k_0.

It is noteworthy that the backward Euler discretization of the unperturbed system aligns with the zero-order holder (ZOH) discretization when applied to the system (3). For the perturbed plant, the primary distinction between (4) and the ZOH discretization is the substitution of $h\phi_{k+1}$ in (4) with the integral of $\phi(t)$ over $[t_k, t_{k+1}]$ in the ZOH method. The expression for $h\phi_{k+1}$ in equations (8) and (9) is replaced by the integral of $\phi(t)$ over $[t_k, t_{k+1}]$ which is confined below ah. The ZOH method preserves the disturbance attenuation property. In essence, the state $x(\cdot)$ of the plant satisfies $x(t) \leq \int_{t_0}^{t} \phi(s)ds \leq h\phi$. To ensure the disturbance attenuation remains effective, the discretization of both the controller and the disturbance signal must adhere to the same method (implicit Euler or ZOH). The Lyapunov function for the system, denoted by $V_k = |\dot{x}_k|$, provides a measure of stability for the nominal system.

B. The Case With Disturbance Compensation

Consideration is given to the scenario where disturbance compensation is applied. The aim here is to mitigate the impact of disturbances on the foundational system. For this, we introduce a compensatory variable $\hat{x}$ within the dynamic equation $\dot{\hat{x}}(t) = -a\tau_1(t)$, with $\tau_1(t)$ aligned with the signum function of $x(t)$. Let $e = x - \hat{x}$, and adopt a control law $u = -a\text{sgn}(x(t)) - a\text{sgn}(e(t))$, with the constraints that $a > 0, a > \bar{a}$ and $a < \alpha$. This gives rise to the following closed-loop system:

$$\begin{aligned}
\dot{\hat{x}}(t) &= -a\tau_1(t) - \bar{a}\tau_2(t) + \phi(t) \\
\dot{e}(t) &= -\bar{a}\tau_2(t) + \phi(t) \\
\tau_1(t) &\in \text{sgn}(x(t)) \\
\tau_2(t) &\in \text{sgn}(e(t)),
\end{aligned} \tag{12}$$

where $\phi(\cdot)$ denotes a disturbance with the property that $\|\phi\|_\infty < \rho$ and $\rho < \min(a, \alpha)$. The system's fixed point $(x, e) = (0, 0)$ can be demonstrated, through a method considered standard in literature [34], to be globally and robustly stable by employing the nonsmooth Lyapunov function $V(x, e) = |x| + |e|$. Furthermore, the system is designed to reach the sliding surface $e = 0$ in finite time, where it will operate under the sliding dynamics $\dot{x}(t) = -a\tau_1(t) + \phi(t)$. The stipulation that $a < \alpha$ ensures the stability of the control system.

The system does not directly converge to the origin but initially exhibits sliding behavior on the surface $e = 0$. It is clear from the system's dynamics that on this surface, x behaves as if no disturbances are present. The discrete sliding mode system's implementation is delineated as follows:

$$\begin{aligned}
\dot{x}_{k+1} &= x_k - ah\tau_{1,k+1} - \bar{a}h\tau_{2,k+1} \\
\dot{e}_{k+1} &= e_k - \bar{a}h\tau_{2,k+1} \\
\tau_{1,k+1} &\in \text{sgn}(\dot{x}_{k+1}) \\
\tau_{2,k+1} &\in \text{sgn}(\dot{e}_{k+1})
\end{aligned} \tag{13}$$

and the process for updating the dynamics of the plant is given by:

$$\begin{aligned}
x_{k+1} &= x_k - ah\tau_{1,k+1} - \bar{a}h\tau_{2,k+1} + h\phi_{k+1} \\
e_{k+1} &= e_k - \bar{a}h\tau_{2,k+1} + h\phi_{k+1}.
\end{aligned} \tag{14}$$

Proposition 2: Assuming x_0, e_0 as the system's initial states, one can show that after a finite sequence of steps k_0, the error e_k vanishes for all k greater than k_0, and there exists a k_1 such that $\dot{x}_k = 0$ for all k greater than $k_0 + k_1$, while x_k aligns with $h\phi_k$ for these steps. The detailed proof is below. As a result, the discrete-time controller ensures the state of the nominal system converges to the origin within a finite time frame, while the plant's state matches the disturbance scaled by h. The discrete-time closed-loop system can thus be summarized by the following equations:

$$\begin{aligned}
x_{k+1} &= x_k - ah\tau_{1,k+1} - \bar{a}h\tau_{2,k+1} + h\phi_{k+1} \\
e_{k+1} &= e_k - \bar{a}h\tau_{2,k+1} + h\phi_{k+1} \\
\tau_{1,k+1} &= \text{proj}\left([-1, 1]; \frac{x_k - \bar{a}h\tau_{2,k+1}}{ah}\right) \\
\tau_{2,k+1} &= \text{proj}\left([-1, 1]; \frac{e_k}{\bar{a}h}\right).
\end{aligned} \tag{15}$$

This system is notable for its straightforward implementation using nested projections.

Proof: Referring to equation (11), it is established that:

$$\begin{cases} \dot{e}_{k+1} = e_k - \alpha h\tau_{2,k+1}, \\ \tau_{2,k+1} \in \text{sgn}(\dot{e}_{k+1}). \end{cases} \tag{16}$$

This pair of equations aligns with the initial statements in (4). As a result, the inferences made there are applicable here, with the adjustment of replacing a with α. Consequently, the dynamic behavior of e_k is expressed as:

$$e_{k+1} = e_k - \alpha h \text{proj}\left([1, 1]; \frac{e_k}{\alpha}\right) + h\phi_{k+1}. \tag{17}$$

Beyond the point k_0, the discrete trajectory adheres to the sliding surface where $\dot{e}_k = 0$, with $\tau_{2,k+1} = \frac{e_k}{\alpha}$ and $e_k = h\phi_{k+1}$. By invoking equation (1), the following is deduced:

$$\begin{cases} \dot{x}_{k+1} = x_k - h\phi_{k+1} - \alpha h\tau_{1,k+1}, \\ \tau_{1,k+1} \in \text{sgn}(\dot{x}_{k+1}), \\ x_{k+1} = x_k - \alpha h\tau_{1,k+1}. \end{cases} \tag{18}$$

Subsequently, this leads to the implication:

$$x_{k+1} = x_k - \alpha h \text{proj}\left([1, 1]; \frac{x_k - h\phi_{k+1}}{\alpha h}\right). \tag{19}$$

Adopting a similar analytical approach as in the proof of Proposition 1 and modifying x_k by $x_k - h\phi_{k+1}$ in the opening line of (4), and $x_k + h\phi_{k+1}$ by x_k in the latter part, one can deduce after a limited series of iterations that $\dot{x}_k = 0$, $\tau_{1,k+1} = (\dot{x}_k - h\phi_{k+1})/\alpha h$, yielding:

$$x_{k+1} = x_k - e_k - (x_k - h\phi_{k+1}) + h\phi_{k+1} = h\phi_{k+1}. \quad (20)$$

Assuming now that:

$$\begin{cases} \dot{x}_{k+1} = x_k - \alpha h\tau_{1,k+1} - \alpha h\tau_{2,k+1} = 0, \\ \dot{e}_{k+1} = e_k - \alpha h\tau_{2,k+1} = 0, \end{cases} \quad (21)$$

it follows that:

$$\begin{cases} \tau_{1,k+1} = \frac{x_k - e_k}{\alpha h}, \\ \tau_{2,k+1} = \frac{e_k}{\alpha h}. \end{cases} \quad (22)$$

Upon executing the update process delineated in (12), we arrive at:

$$\begin{cases} \dot{x}_{k+1} = h\phi_{k+1}, \\ \dot{e}_{k+1} = h\phi_{k+1}. \end{cases} \quad (23)$$

It can be concluded that upon achieving the sliding mode condition for $\dot{x}$ and $\dot{e}$:

$$\begin{cases} \dot{x}_{k+1} = h\phi_{k+1}, \\ \dot{e}_{k+1} = h\phi_{k+1}, \end{cases} \quad (24)$$

holds true for all $k \geq k_0$.

Furthermore:

$$\begin{cases} \tau_{1,k+1} = 0, \\ \tau_{2,k+1} = \frac{\phi_k}{\alpha}, \end{cases} \quad (25)$$

is valid for all $k \geq k_0 + 1$

C. Extension to Higher Order Systems

The methodology discussed extends seamlessly to n-th order systems with the equivalent-control-based sliding-mode-controller (ECB-SMC) and also to linear time-invariant systems with disturbance. The system is characterized by $\dot{x}(t) = Ax(t) + Bu(t) + D\phi(t)$ with $\|\phi(t)\| \leq \bar{\phi}_{\max}$ for all t, and where D is a matrix of appropriate dimensions. Let us select a sliding surface $\Sigma = \{x \in \mathbb{R}^n | Cx = 0, C \in \mathbb{R}^{m\times n}\}$, where m is the number of inputs $u(t)$. The ECB-SMC takes the form $u = -(CB)^{-1}(CAx - \alpha(CB)^{-1}\text{sgn}(Cx))$, assuming CB is invertible. The reduced closed-loop dynamics are $\dot{z}(t) = -\alpha\tau + CD\phi(t)$, with τ following the signum function of z, and is globally stable. The discretized system is:

$$x_{k+1} = x_k + hAx_k + hBu_{k+1} + hD\phi_{k+1}. \quad (26)$$

For the nominal system, the evolution is described by the simple equation $\dot{x}_{k+1} = (I + hA)x_k + hBu_{k+1}$. The implicit Euler controller is specified as:

$$\begin{aligned} &u_{k+1} = -(CB)^{-1}CAx_k - \alpha(CB)^{-1}\tau_{k+1} \\ &\tau_{k+1} \in \text{Sgn}(C\dot{x}_{k+1}). \end{aligned} \quad (27)$$

As such, τ_{k+1} is computed by:

$$\tau_{k+1} \in \text{Sgn}(CAx_k - \alpha h\tau_{k+1}) \Leftrightarrow \tau_{k+1} = \text{proj}\left([-1,1]^m; \frac{1}{\alpha h}CAx_k\right), \quad (28)$$

where $[-1,1]^m = [-1,1] \times \ldots \times [-1,1]$ repeated m times. Consequently, the controller to be applied at time t_k is:

$$u_{k+1} = -(CB)^{-1}CAx_k - \alpha(CB)^{-1}\text{proj}\left([-1,1]^m; \frac{1}{\alpha h}CAx_k\right). \quad (29)$$

This results in the following system representation, with $z_k = CAx_k$ and $\dot{z}_k = C\dot{x}_k$:

$$\begin{aligned} &\dot{z}_{k+1} = z_k - \alpha h\tau_{k+1} \\ &\tau_{k+1} \in \text{Sgn}(\dot{z}_{k+1}) \\ &z_{k+1} = z_k - \alpha h\tau_{k+1} + hCD\phi_{k+1}, \end{aligned} \quad (30)$$

which parallels the system (4). Therefore, the discrete-time system achieves the sliding surface $C\dot{x}_k = 0$ after a finite number of steps regardless of the bounded initial conditions, and the system progresses smoothly on this surface while the disturbance influence on CAx_k is reduced by a factor of h.

Remark 1: The discrete-time input, as derived from [35, Equation (9.36)] and referenced in [5], [20], and [12], when applied to equation (14) results in a linear expression. This contrasts with equation (17), which includes a projection onto the set $[-1,1]^m$ that introduces a nonlinearity characteristic of the implicit Euler method. Despite both controllers being conceptualized to drive the sliding surface to zero, their methodologies are distinct. In practice, the controllers discussed in this work can be solved using a suitable complementarity problem solver [2]. To ensure disturbance attenuation, the plant and the controller must be discretized using the same scheme (either backward Euler or ZOH). The ZOH discretization of the ECB-SMC controller on the interval $[t_k, t_{k+1}]$ is formulated as:

$$x_{k+1} = A^*(h)x_k + \alpha B^*(h)\tau_{k+1} + \phi^*(h), \quad (31)$$

with $A^*(h)$, $B^*(h)$, and $\phi^*(h)$ defined as integral expressions based on e^{Ah}, $\int_0^h e^{A(t-s)}dt$, and $D\phi(t)$, respectively.

The nominal system evolves according to $\dot{x}_{k+1} = (I + hA)x_k + hBu_{k+1}$. The implicit Euler controller is determined as follows:

$$\begin{aligned} &u_{k+1} = -(CB)^{-1}CAx_k - \alpha(CB)^{-1}\tau_{k+1} \\ &\tau_{k+1} \in \text{Sgn}(C\dot{x}_{k+1}). \end{aligned} \quad (32)$$

Thus, τ_{k+1} can be computed as:

$$\tau_{k+1} \in \text{Sgn}(CAx_k - \alpha h\tau_{k+1}) \Leftrightarrow \tau_{k+1} = \text{proj}\left([-1,1]^m; \frac{1}{\alpha h}CAx_k\right), \quad (33)$$

where $[-1,1]^m$ signifies a Cartesian product of the interval $[-1,1]$ taken m times. The controller that should be applied at time t_k is then:

$$u_{k+1} = -(CB)^{-1}CAx_k - \alpha(CB)^{-1}\text{proj}\left([-1,1]^m; \frac{1}{\alpha h}CAx_k\right). \quad (34)$$

From this, we derive that, with $z_k = CAx_k$ and $\dot{z}_k = C\dot{x}_k$:

$$\begin{aligned} &\dot{z}_{k+1} = z_k - \alpha h\tau_{k+1} \\ &\tau_{k+1} \in \text{Sgn}(\dot{z}_{k+1}) \\ &z_{k+1} = z_k - \alpha h\tau_{k+1} + hCD\phi_{k+1}, \end{aligned} \quad (35)$$

which is analogous to equation (4). Therefore, similar to Proposition 1, we can conclude for this discrete-time system that provided $\alpha > \|\phi\|\|CD\|\phi_{\max}$, the sliding surface $C\dot{x}_k = 0$ is reached after a finite number of steps for any bounded initial condition, and the system operates smoothly on this surface with the disturbance impact on CAx_k being diminished by h.

As $h \to 0$, it holds that $A^*(h)$ approaches $I + Ah - hB(CB)^{-1}CA$ and $B^*(h)$ approximates $hB(CB)^{-1}$, leading to the conclusion that both implicit Euler and ZOH methods converge to the same discrete-time system for small sampling periods. Additionally, it can be shown that $\|\phi^*(h)\|$ is bounded by $h\|\phi\|\|D\|\phi_{\max}$ plus higher-order terms. This brings us to the generalized equation:

$$\dot{x}_{k+1} = A^*(h)x_k - \alpha B^*(h)\tau_{k+1} + \phi^*(h) \in \mathrm{Sgn}(C\dot{x}_{k+1}) \tag{36a}$$

$$x_{k+1} = A^*(h)x_k - \alpha B^*(h)\tau_{k+1} + C\phi^*(h) \tag{36b}$$

$$C\dot{x}_{k+1} = CA^*(h)x_k - \alpha CB^*(h)\tau_{k+1} \in \mathrm{Sgn}(C\dot{x}_{k+1}) \tag{36c}$$

$$C\dot{x}_{k+1} = CA^*(h)x_k - \alpha CB^*(h)\tau_{k+1} + C\phi^*(h). \tag{36d}$$

Assuming $CB^*(h)$ is symmetric positive definite, which is true for sufficiently small h if CB is invertible, the solution for τ_{k+1} from the above system can be expressed as a projection:

$$\tau_{k+1} = \mathrm{proj}_{CB^*(h)}\left([-1,1]^m; \frac{CA^*(h)x_k}{\alpha}\right) \tag{37}$$

$$= \mathrm{argmin}_{\zeta\in[-1,1]^m} \frac{1}{2}(\zeta - \frac{CA^*(h)x_k}{\alpha})^T X \tag{38}$$

$$CB^*(h)(\zeta - \frac{CA^*(h)x_k}{\alpha}). \tag{39}$$

With the controller computed as a solution to a quadratic program, it remains bounded even as the sampling time diminishes. The system's behavior under the ZOH method maintains the disturbance attenuation on the nominal discrete-time system's sliding surface. Lemma 1 states that if $C\dot{x}_k = 0$ for some $k \geq 0$, then $\|C\dot{x}_{k+1}\|$ is constrained by $h\|\phi\|\|D\|\phi_{\max}$. Thus, the system achieves the sliding surface in a finite number of steps, and the sliding mode control philosophy is preserved.

Proposition 3: Given $h > 0$, if the solution to (20a) maintains $\|x_k\| \leq M$ for all $k \geq 0$ and some $M < +\infty$, and $CB^*(h)$ is symmetric positive definite with $CB^*(h) \geq \gamma I_m$ for some $\gamma > 0$, then there exists a solution to (20a) that converges to the sliding surface in finite time.

It is established that there exists a constant $\delta(h^2, M)$ ensuring that if $\alpha > (m/\gamma)\|\phi\|\|D\|\phi_{\max} + \delta(h^2, M)$, then $C\dot{x}_{k+1} = 0$ for a certain $k \geq 0$ implies $C\dot{x}_{k+n} = 0$ for all $n \geq 2$.

Proof: Based on Lemma 1, the first line of equation (20b) can be expressed at step $k+2$ as:

$$C\dot{x}_{k+2} = (C + O(h^2))x_{k+1} - \alpha CB^*(h)\tau_{k+2} \tag{40}$$

$$= C\phi^*(h) + O(h^2)x_{k+1} - \alpha CB^*(h)\tau_{k+2}. \tag{41}$$

The equations (23) and $\tau_{k+2} \in \mathrm{Sgn}(C\dot{x}_{k+2})$ collectively form a generalized equation with a unique solution, given $\alpha CB^*(h)$ is positive definite. This can be reformulated as:

$$0 \in \tfrac{1}{\alpha}(CB^*(h))^{-1}C\dot{x}_{k+2} - \tfrac{1}{\alpha}(CB^*(h))^{-1} \times (C\phi^*(h) + O(h^2)x_{k+1}) + \mathrm{Sgn}(C\dot{x}_{k+2}). \tag{42}$$

Hence, if the vector $\frac{1}{\alpha}(CB^*(h))^{-1}(C\phi^*(h) + O(h^2)x_{k+1})$ is within $[-1,1]^m$, then $C\dot{x}_{k+2} = 0$ is the unique solution. Under the proposition's assumptions, one has:

$$\left\|(CB^*(h))^{-1}(C\phi^*(h) + O(h^2)x_{k+1})\right\|_1 \leq \frac{m}{\gamma}\|\phi\|\|D\|\phi_{\max} + \delta(h^2, M), \tag{43}$$

where $\delta(h^2, M)$ is an upper bound for the $O(h^2)$ term. This upper bound is solely dependent on M, the matrices of the system, and h, indicating uniformity with respect to the step number k.

Applying Lemma 1 demonstrates the disturbance attenuation on the nominal system's discrete-time sliding surface.

D. Numerical Simulations

Numerical simulations were performed using MATLAB, which is commonly employed for computing solutions to dynamical systems. Although the original paper utilized the SICONOS software package, renowned for its non-smooth dynamical system simulations, our implementation aims to replicate the continuous-time behavior of the plant with MATLAB's precise integration capabilities. The controller sampling time was set significantly larger than the machine precision to mimic a zero-order hold (ZOH) method: $h = 10^{-1}$ s. The disturbance was modeled as $\phi(t) = \phi\sin(\omega t)$, allowing for the simulation of the system described in equation (10).

The theoretical framework outlined in the paper is demonstrated through figures where the parameters are set as follows: in Fig. 1, $a = 1$, $\alpha = 2$, $\omega = 5$, and $\phi = 0.1$; in Figs. 2, $a = 1$, $\alpha = 2$, $\omega = 100$, and $\phi = 0.1$. These simulations clearly exhibit the effectiveness of the disturbance attenuation in the MATLAB environment, consistent with the theoretical predictions.

III. Second-Order Systems

We now turn our attention to a broader category of systems, specifically second-order systems. Adopting a similar approach as with the first-order systems, we commence with a recapitulation of the continuous-time case, followed by the discretization process. The numerical simulations, carried out using MATLAB to validate the theoretical constructs, will be presented subsequent to the theoretical discourse.

A. First-Order Sliding-Mode Stabilization With Disturbance Compensation

1) The Continuous-Time System: The dynamics of the plant are described by the following equation:

$$\dot{x}(t) = u(t) + \phi(x(t), t) \tag{44}$$

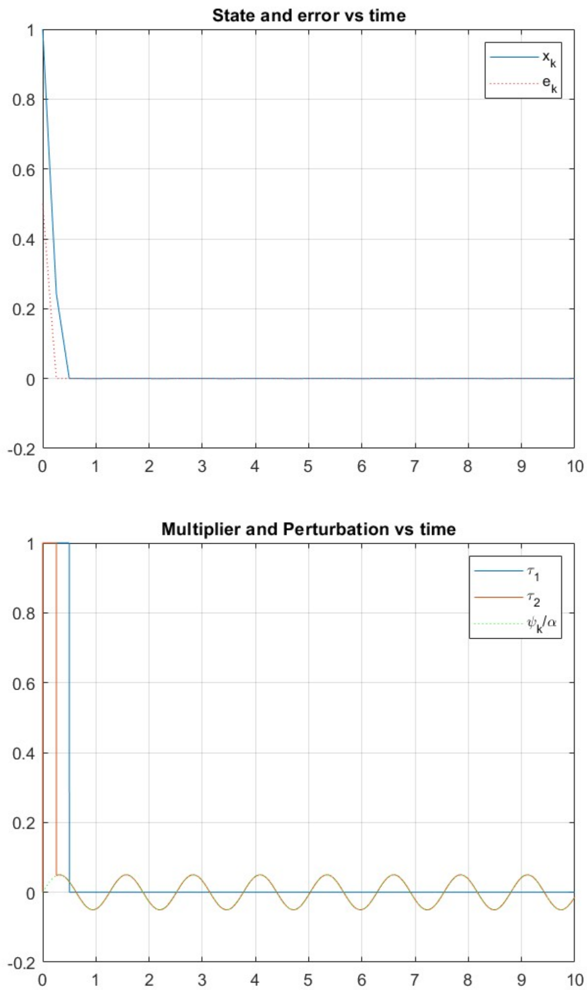

Fig. 1. For system (12), $\omega = 5$

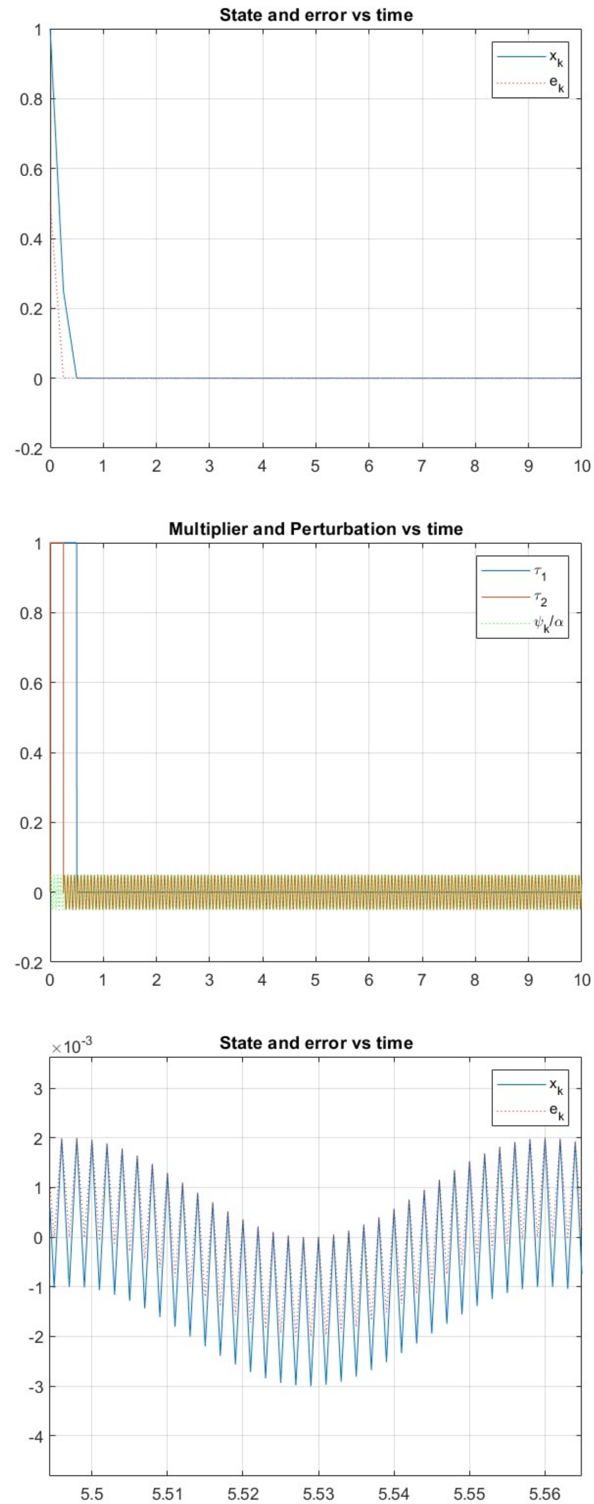

Fig. 2. For system (12), $\omega = 100$, the zoomed plot of state and error vs time shows the effectiveness of the controller

where $x(t) \in \mathbb{R}$ represents the state vector, $u(t) \in \mathbb{R}$ is the control input, and $\phi(x,t) \in \mathbb{R}$ denotes the disturbance affecting the system. This disturbance is assumed to be an unknown function with an established upper bound $\phi_{\max} > 0$, satisfying:

$$|\phi(x,t)| < \phi_{\max} \tag{45}$$

for almost all $x, t \in \mathbb{R}$. The plant model, which mirrors the structure of the actual system, is given by:

$$\dot{\hat{x}}(t) = u(t) + v(t) \tag{46}$$

where $v(t) \in \mathbb{R}$ stands as the model input. The dynamics of the error are then given by:

$$\dot{e}(t) = -v(t) + \phi(x(t), t) \tag{47}$$

where $e = x - \hat{x}$ is the deviation of the model state from the actual state. The error dynamics, incorporating the sliding-mode control input, are expressed as:

$$v(t) = k_e e(t) + k_s s_e(t) + M_v sgn(s_e(t)) \tag{48}$$

Here, k_e and k_s are control gains, and M_v is a gain parameter for the discontinuous component of the control.

The system is guaranteed to be globally asymptotically stable given that $M_v > \phi_{\text{max}}$ and $s_e(t) = \dot{e}(t) + k_e e(t)$ where k_e and k_s are positive constants. To replicate this outcome, it is sufficient to recast the state equation for s_e, yielding:

$$\dot{s}_e(t) \in -k_s s_e(t) - M_v \text{sgn}(s_e(t)) + \phi(x(t), t) \tag{49}$$

which posits $s_e^* = 0$ as the sole fixed point, assured to be globally finite-time stable. Hence, the equivalent control method concludes that:

$$v_{eq}(t) = \phi(x(t), t) \tag{50}$$

holds true on the surface $s_e = 0$. Consequently, the anticipated control law is given by:

$$u(t) = -v(t) - M_v \text{sgn}(s_x(t)) - k_x \dot{x}(t) \tag{51}$$

With $s_x(t) = \dot{x}(t) + k_x x(t)$, the system compensates for the disturbance $\phi(x, t)$ asymptotically. Indeed, when the sliding mode is active on the surface $s_e = 0$, the plant's dynamics can be represented as a disturbance-free system:

$$\begin{aligned} \dot{s}_x(t) &= e - M_v \text{sgn}(s_x(t)) - k_e \dot{e}(t) \\ \dot{e}(t) &= -k_e e(t) \end{aligned} \tag{52}$$

On this sliding surface, we have $M_v \text{sgn}(s_e(t)) = \phi(x(t), t)$. As the dynamics in (34) has a globally asymptotically stable fixed point at $s_x^* = 0$, it results in the desired disturbance compensation.

Summarizing the above, we obtain the following global asymptotic stability of the closed-loop system, represented by the state vector $z = [e \ s_e \ s_x]^T$. The dynamics of the coupled plant/error system in the closed-loop are described by:

$$\dot{z}(t) = \begin{bmatrix} -k_e & 1 & 0 & 0 \\ 0 & -k_s & 0 & 0 \\ 0 & 0 & -k_x & 1 \\ -k_e & -k_s & 0 & 0 \end{bmatrix} z(t) - \begin{bmatrix} 0 & 0 \\ M_v & 0 \\ 0 & 0 \\ M_v & M_J \end{bmatrix} \begin{bmatrix} \tau(t) \\ \phi(x(t), t) \end{bmatrix} \tag{53}$$

where $\tau(t)$ is within the signum set defined by the system's dynamics. It is important to note that the (e, s_e) subdynamics is distinct from the (x, s_x) subdynamics, which are not affected by the perturbations.

Proposition 4: Consider the closed-loop system with positive gains k_e, k_s, M_x, M_v, and an external disturbance $\phi(x, t)$ such that it satisfies the condition for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$ with $M_v > \phi_{\text{max}}$. Then, the system evolves on the sliding surfaces $s_e = 0$ and $s_x = 0$, and the dynamics is governed by asymptotically stable, disturbance-free equations.

The proof of Proposition 4 is standard and thus omitted. The parameter k_v ensures faster convergence of the error dynamics compared to the state variables of the plant, and the controller magnitude M_x is only required to be positive. The higher M_x is, the more conservative the rate.

The Backward Euler Time-Discretization: The discretization process is similar to that of the first-order case. Considering the error dynamics and discretizing it as:

$$\begin{aligned} \dot{s}_{e,k+1} &= s_{e,k} - hk_s s_{e,k} - hM_v \tau_{1,k+1} \\ \dot{e}_{k+1} &= e_k + h\dot{e}_{k+1} \end{aligned} \tag{54}$$

with $\tau_{1,k+1} \in \text{sgn}(\dot{s}_{e,k+1})$. This system leads to a unique solution due to the properties of the sign multifunction.

Lemma 2: Assuming the sliding surface is reached at a certain step k_0 and remains active, the evolution of e_k on the sliding surface is given by a recursive equation which leads to the expression of e_{k+1} as a function of previous errors and the disturbance.

Implementing $e_{k+1} = e_k + h\dot{e}_k$ yields $e_k = (1 - hk_e)e_{k-1} + h^2 \phi_k$ and, by similar computations considering small h, the discrete-time error is a sum of an asymptotically vanishing term and a term dependent on the disturbance, attenuated by h^2. Hence, the discretized second part of the error dynamics is:

$$\begin{aligned} &\dot{s}_{e,k+1} = s_{e,k} - hk_e \dot{e}_k - hk_s s_{e,k} - hM_v \tau_{1,k+1} - hM_x \tau_{2,k+1} \\ &\tau_{2,k+1} \in \text{sgn}(\dot{s}_{x,k+1}) \\ &s_{x,k+1} = s_{x,k} - hk_e \dot{e}_k - hk_s s_{e,k} - hM_v \tau_{1,k+1} \\ &\qquad - hM_x \tau_{2,k+1} + h\phi(x_{k+1}, t_{k+1}) \\ &x_{k+1} = x_k + h\dot{x}_{k+1} \end{aligned} \tag{55}$$

For $k \geq k_0$, $(1 - hk_s)s_{e,k} = hM_v \tau_{1,k+1}$ and $s_{e,k} = h\phi(x_{k+1}, t_{k+1})$, the system evolves on $s_{e,k} = 0$ leading to:

$$\begin{aligned} &s_{x,k+1} = s_{x,k} - h\phi(x_{k+1}, t_{k+1}) - hM_x \tau_{2,k+1} \\ &\tau_{2,k+1} \in \text{sgn}(\dot{s}_{x,k+1}) \\ &s_{x,k+1} = s_{x,k} - hk_e \dot{e}_k - hM_x \tau_{2,k+1} - h^2 \alpha_k - hM_x \tau_{2,k+1} \end{aligned} \tag{56}$$

Notably, this form is analogous to previous results except for the term $h^2 \alpha_k$, showing that disturbance effects are still attenuated by a factor h.

Proposition 5: For the discrete-time system representing the system's dynamics on the sliding surface and assuming $M_x > \phi_{\text{max}}$, there exists $k_1 < +\infty$, $k_1 > k_0$, such that for all $k \geq k_1$, $s_{x,k} = 0$ holds. Then:

$$|s_{x,k+1}| \leq h\phi_{\text{max}} + |e_k| + h^2 \alpha \phi_{\text{max}} \tag{57}$$

The proof is omitted as it follows similar arguments as previous propositions regarding finite-time convergence.

Proposition 6: Suppose that for $k \geq 0$ the system evolves on the sliding surface $s_{x,k} = 0$, implying that $|s_{x,k+1}| \leq h\phi_{\text{max}} + |e_k|$. Then, the state x_k can be expressed as:

$$x_k = (1 + hk_x)^{-1} x_0 - h(1 + hk_x)^{-1} \sum_{i=0}^{k-1} \xi_{i,k}, \tag{58}$$

where $\xi_{i,k} = (1 + hk_x)^{-i} (\epsilon_{k-1-i} + h^2 \alpha_{k-1-i} + h\phi_{k-i})$. The proof follows from the assumption of $s_{x,k+1} = 0$.

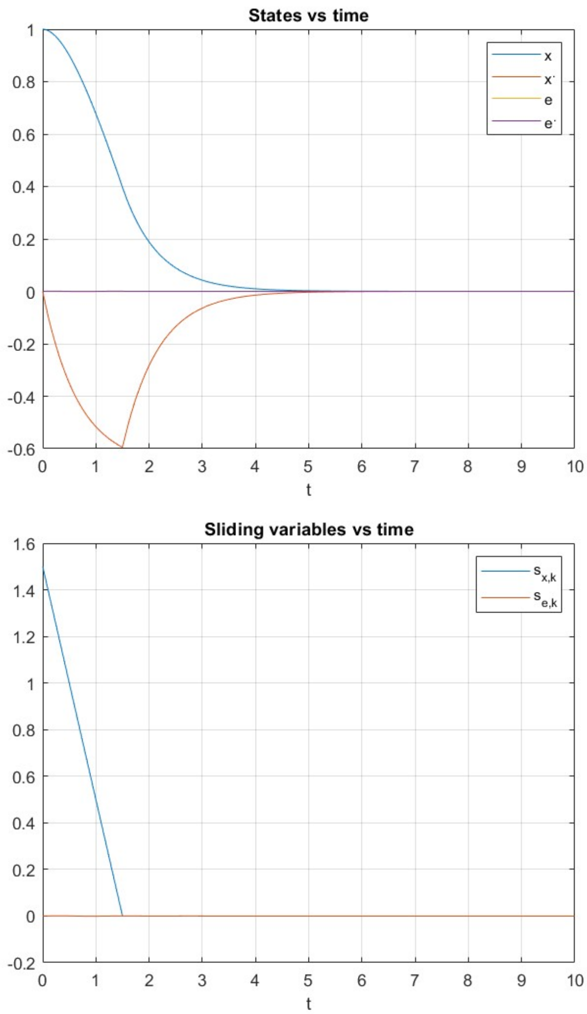

Fig. 3. For System(59)

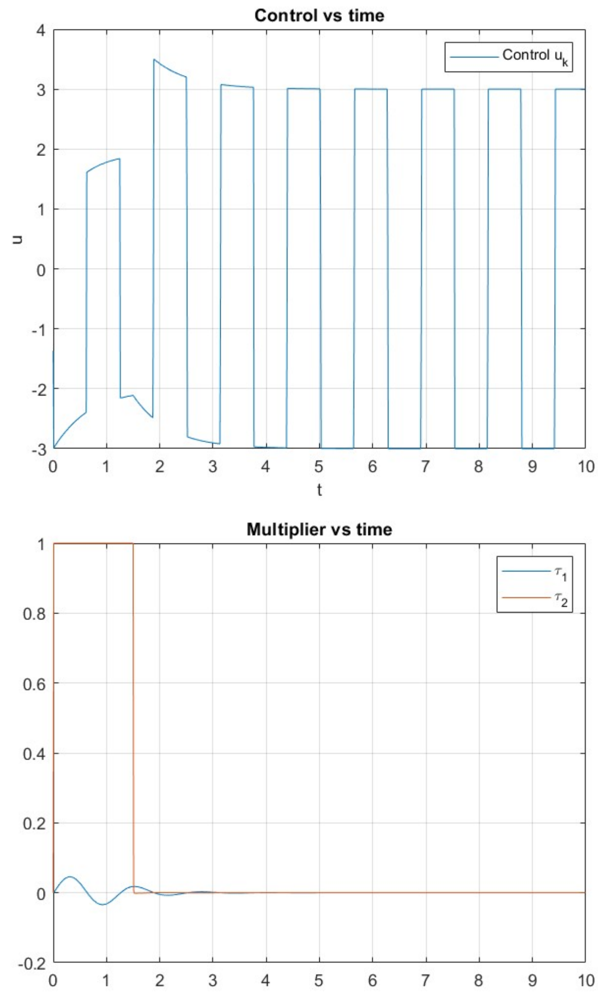

Fig. 4. For System(59)

The Backward Euler Time-Discretization: The discrete-time system can be written as:

$$\begin{aligned}
s_{x,k+1} &= s_{x,k} - hk_e\dot{e}_k - hk_s s_{e,k} - hM_v\tau_{1,k+1} \\
&\quad - hM_x\tau_{2,k+1} + h\phi(x_{k+1}, t_{k+1}) \\
s_{e,k+1} &= s_{e,k} - hk_s s_{e,k} - hM_v\tau_{1,k+1} + h\phi(x_{k+1}, t_{k+1}) \\
x_{k+1} &= x_k + h\dot{x}_{k+1} \\
e_{k+1} &= e_k + h\dot{e}_k \\
\tau_{1,k+1} &= \text{proj}_{[-1,1]}\left(\frac{s_{e,k} - hk_s s_{e,k}}{hM_v}\right) \\
\tau_{2,k+1} &= \text{proj}_{[-1,1]}\left(\frac{s_{x,k} - hk_e\dot{e}_k - hk_s s_{e,k} - hM_x\tau_{1,k+1}}{hM_x}\right).
\end{aligned} \tag{59}$$

The error dynamics reveal that $hk_e\dot{e}_k$ is equivalent to $e_k + h^2\alpha_k$, with α_k being bounded by a constant α for all k, and e_k exhibiting exponential decay.

B. Position Feedback Stabilization of a Double Integrator

Transitioning to other forms of sliding-mode control, such as twisting and super-twisting algorithms, these strategies offer advantages including finite-time stability and improved disturbance attenuation.

I) Finite-Time Stabilizing State Feedback Synthesis: We introduce a static feedback controller that stabilizes the double integrator system:

$$\begin{aligned}
\dot{x}(t) &= y(t), \\
\dot{y}(t) &= u(t),
\end{aligned} \tag{60}$$

with a feedback law $u(x, y) = -\nu\text{sgn}(y) - \mu\text{sgn}(x)$ proposed to ensure finite-time stabilization of the system.

Theorem 1: Consider the dynamics of the closed-loop system as in equations (45) and (46). This system has a unique fixed point $(x, y) = (0, 0)$ which is globally finite-time stable, provided the controller parameters satisfy $\nu > \mu > 0$. The proof is detailed in Example 3.2 and Section 4.6 of [27]. Now, for the disturbance-corrupted version:

$$\begin{aligned}
\dot{x}(t) &= y(t), \\
\dot{y}(t) &= u(t) + \phi(x(t), y(t), t),
\end{aligned} \tag{61}$$

we examine the robustness of the closed-loop system (46), (47) against external disturbances $\phi(x, y, t)$, which are locally integrable functions. According to [27, Theorem 4.2], the system remains finite-time stable under any bounded disturbance $\phi(\cdot)$ with:

$$\text{ess sup}_{t\geq 0}|\phi(x(t), y(t), t)| \leq \phi_{\max}, \tag{62}$$

and this robustness is attributed to the high frequency switching in the sliding mode controller.

Theorem 2: Given μ and $\nu > 0$, the system (46), (47) has a globally asymptotically finite-time stable fixed point, despite disturbances satisfying the previous conditions.

We propose an implicit Euler time-discretization where $\phi_{k+1} = \phi(x_{k+1}, y_{k+1}, t_{k+1})$:

$$\begin{aligned} \dot{\tilde{x}}_{k+1} &= x_k + h\dot{y}_{k+1}, \\ \dot{\tilde{y}}_{k+1} &= y_k - h\nu\tau_{1,k+1} - h\mu\tau_{2,k+1} + h\phi_{k+1}, \\ \tau_{1,k+1} &\in \text{sgn}(\dot{\tilde{x}}_{k+1}), \\ \tau_{2,k+1} &\in \text{sgn}(\dot{\tilde{y}}_{k+1}), \\ x_{k+1} &= x_k + h\dot{y}_{k+1}, \\ y_{k+1} &= y_k - h\nu\tau_{1,k+1} - h\mu\tau_{2,k+1} + h\phi_{k+1}, \end{aligned} \tag{63}$$

and we find that:

$$\tau_{2,k+1} = \text{proj}_{[-1,1]}\left(\frac{y_k - h\nu\tau_{1,k+1}}{h\mu}\right). \tag{64}$$

The discrete-time system is constructed as before, calculating the input from a nominal unperturbed system and then integrating the computed input into the dynamics.

Lemma 3: The controller $(\tau_{1,k+1}, \tau_{2,k+1})^T$ is a causal input at time $t = t_k$ and does not introduce singularities as h tends to zero.

Proof: It is established that:

$$\dot{y}_{k+1} = y_k - h\upsilon_{1,k+1} + h\mu\text{proj}\left([1, 1]; \frac{(\dot{y}_k - h\upsilon_{1,k+1})}{h\mu}\right). \tag{65}$$

Consequently, we can express $\dot{y}_{k+1}$ as:

$$\dot{y}_{k+1} = y_k - h\upsilon_{1,k+1} - \dots \tag{66}$$

The projection term is given by:

$$\text{proj}([h\mu, h\mu]; y_k - h\upsilon_{1,k+1}) = y_k - h\upsilon_{1,k+1}. \tag{67}$$

Hence, there are three "modes" possible:

- If $y_k - h\upsilon_{1,k+1} > h\mu$, it follows that

  $$\begin{aligned} y_{k+1} &= y_k - h\upsilon_{1,k+1} - h\mu, \\ y_{k+1} &= y_k - h\upsilon_{1,k+1} - h\mu + h\phi_{k+1}, \end{aligned}$$

  and similarly for $\dot{x}_{k+1}$. Moreover,

  $$\begin{aligned} x_{k+1} &= x_k + h(y_k - h\upsilon_{1,k+1}) - h^2\mu, \\ \upsilon_{1,k+1} &= \frac{1}{(h^2\mu)}\text{proj}([h^2\mu, h^2\mu]; x_k + hy_k - h^2\mu), \\ \tau_{2,k+1} &= 1. \end{aligned}$$

  There are sub-modes within this mode:

  (i-1) If $x_k + hy_k - h^2\mu > h^2\nu$, then $\tau_{1,k+1} = 1$, and so forth.
  (i-2) If $x_k + hy_k - h^2\mu < h^2\nu$, then $\tau_{1,k+1} = -1$, and so forth.
  (i-3) If $|x_k + hy_k - h^2\mu| \leq h^2\nu$, then $\tau_{1,k+1} = \frac{(x_k+hy_k-h^2\mu)}{h^2\nu}$, and so forth.

- If $y_k - h\upsilon_{1,k+1} \leq -h\mu$, then $y_{k+1} = y_k - h\upsilon_{1,k+1} + h\mu$, and similarly for x_{k+1}. In this case, $\tau_{1,k+1} = -1$ and $\tau_{2,k+1} = -1$.

  (ii-1) If $x_k + hy_k + h^2\mu > h^2\nu$, then $\tau_{1,k+1} = 1$, and so forth.
  (ii-2) If $x_k + hy_k + h^2\mu \leq -h^2\nu$, then $\tau_{1,k+1} = -1$, and so forth.
  (ii-3) If $|x_k + hy_k + h^2\mu| \leq h^2\nu$, then $\tau_{1,k+1} = -\frac{(x_k+hy_k+h^2\mu)}{h^2\nu}$, and so forth.

For modes (i) and (ii), the value for $\tau_{1,k+1}$ is derived from the generalized equation and in all cases, $\tau_{2,k+1}$ is determined from a separate condition.

Lemma 4: If $\phi_k = 0$, and given $|y_{k_0} - h\nu\tau_{1,k_0+1}| \leq h\mu$ and $x_{k_0} = 0$ for some $k_0 \geq 0$, then $|y_k - h\nu\tau_{1,k+1}| \leq h\mu$ for all $k \geq k_0$, hence $x_{k+1} = y_{k+1} = 0$ for all $k \geq k_0$.

Proof:

(a) From the previous condition (iii), it is inferred that $|y_{k_0} - h\upsilon_{1,k_0+1}| \leq h\mu$ and $x_{k_0} = 0$ imply $y_{k_0+1} = x_{k_0+1} = 0$. Therefore, we have:

$$|y_{k_0+1} - h\upsilon_{1,k_0+2}| = |h\upsilon_{1,k_0+2}|.$$

Assuming $h\upsilon_{1,k_0+2} > h\mu$, from conditions (i) and (ii), we deduce:

$$\begin{aligned} \upsilon_{1,k_0+2} &= \frac{1}{(h^2\nu)}\text{proj}\left([h^2\nu, h^2\nu]; x_{k_0+1} + hy_{k_0+1} \pm h^2\mu\right) \\ &= \frac{1}{(h^2\nu)}\text{proj}\left([h^2\nu, h^2\nu]; \pm h^2\mu\right) = \pm\frac{\mu}{\nu}, \end{aligned}$$

since $\mu < \nu$. Thus, $|y_{k_0+1} - h\upsilon_{1,k_0+2}| = h\mu$ is not possible, leading to a contradiction. Consequently, we have $|h\upsilon_{1,k_0+2}| \leq h\mu$ and so $|y_{k_0+1} - h\upsilon_{1,k_0+2}| \leq h\mu$ and $x_{k_0+2} = y_{k_0+2} = 0$. Hence, $x_{k_0+2} = x_{k_0+1} = 0$. This argument can be extended to the next step, proving (a).

(b) Proceeding from $x_k = y_k = 0$, we deduce that $x_{k+1} = hy_{k+1}$ and $y_{k+1} = -h\upsilon_{1,k+1} - h\upsilon_{2,k+1}$. Furthermore, $\upsilon_{1,k+1}$ lies within the signum function of the negative sum $(-h\upsilon_{1,k+1} - h\upsilon_{2,k+1})$, from which we conclude that:

$$\upsilon_{1,k+1} = \text{proj}\left([-1, 1]; -\frac{\upsilon_{2,k+1}}{\nu}\right),$$

completing the proof for (b).

We have that

$$\tau_{2,k+1} = \text{proj}\left([-1, 1]; -\frac{\nu}{\mu}\tau_{1,k+1}\right). \tag{68}$$

Given that $\mu < \nu$, it follows that

$$\tau_{1,k+1} = -\frac{\nu}{\mu}\tau_{2,k+1}. \tag{69}$$

It is also apparent that

$$\left|\frac{\nu}{\mu}\tau_{1,k+1}\right| \leq \frac{\nu}{\mu} < 1. \tag{70}$$

Consequently, since $\tau_{1,k+1} \in \text{sgn}(x_{k+1})$ and $x_{k+1} = 0$, we deduce that

$$\tau_{1,k+1} = 0. \tag{71}$$

From the fact that $x_{k+1} = hy_{k+1}$, it follows that $y_{k+1} = 0$. This reasoning is iterative and can be applied to subsequent steps. Moreover, it can be readily shown that

$$|\tau_{1,k+1}| \leq \frac{\nu}{\mu}, \tag{72}$$

which concludes the proof of part (b).

Proposition 7: Assuming $\dot{\hat{x}}_{k+1} = 0$ and $\dot{y}_{k+1} = 0$, it follows that $y_{k+1} = h\phi_{k+1}$ and thus $|x_{k+1}| \leq h^2\phi_{\max}$.

Proposition 8: If $\dot{\tilde{x}}_{k+1} = 0$ for some $k \geq 0$, then $x_{k+1} = h^2\phi_{k+1}$. Moreover, if $\dot{x}_{k+1} = 0$ for all $k \geq 0$, then $y_{k+1} = h(\phi_{k+1} - \phi_k)$ and the sum $\sum_{k=1}^{n} y_k = h(\phi_n - \phi_0)$, while $y_{k+1} = -h\phi_k$ for all $k \geq 0$.

The differential inclusions for the system can be written as:

$$\begin{aligned} \dot{z}(t) &= Az(t) - B\tau(t) + \Phi(z(t), t), \\ \tau(t) &\in \text{Sgn}(Cz(t)), \end{aligned} \tag{73}$$

where A, B, C, and $\Phi(\cdot)$ are defined accordingly, and the system is subject to the non-sliding condition along both $\dot{\hat{x}}$ and $\dot{y}$.

The focus of the current study is on the stability analysis of the velocity observer of the supertwisting observer:

$$\begin{aligned} \dot{\hat{x}}(t) &= \dot{y}(t) + k_1 |x(t) - \hat{x}(t)|^{0.5} \,\text{sgn}\,(x(t) - \hat{x}(t)), \\ \dot{\hat{y}}(t) &= u(t) + k_3 \text{sgn}\,(x(t) - \hat{x}(t)) + k_4\,(x(t) - \hat{x}(t)), \end{aligned} \tag{74}$$

which was first proposed in [11] with $k_2, k_4 = 0$ and is now augmented with nontrivial linear gains $k_2, k_4 > 0$. The observer error dynamics are governed by the following second-order system:

$$\begin{aligned} \dot{e}_1(t) &= e_2(t) - k_1 |e_1(t)|\, \text{sgn}\,(e_1(t)) - k_2 e_1(t), \\ \dot{e}_2(t) &= -k_3 \text{sgn}\,(e_1(t)) - k_4 e_1(t). \end{aligned} \tag{75}$$

Theorem 3: Given $k_1, k_3 > 0$ and $k_2, k_4 \geq 0$, the system is globally finite-time stable.

Theorem 4: If the system is affected by a uniformly bounded disturbance ϕ satisfying certain conditions, then the system is globally finite-time stable.

For finite-time stabilizing position feedback synthesis, we consider the double integrator system with the velocity estimate $\hat{y}$ and augment the control law with a term compensating the disturbance.

The stabilizing position feedback laws are given by:

$$u = -\mu \text{sgn}(\dot{y}) - \nu \text{sgn}(x) \tag{76}$$

or with the disturbance compensating term, by:

$$u = -\mu \text{sgn}(\dot{y}) - \nu \text{sgn}(x) - k_3 \text{sgn}(e_1) \tag{77}$$

Theorem 5: Let the system (47) be affected by a uniformly bounded disturbance (48) and driven by the feedback (53), (58) [respectively, (59)] with gains μ, ν satisfying (49), and observer parameters k_i, the system is globally finite-time stable.

The discretized system is given by:

$$\begin{aligned} &\tilde{x}_{k+1} = x_k + h\tilde{y}_{k+1}, \\ &\tilde{y}_{k+1} = y_k - h\mu\tau_{2,k+1} - h\nu\tau_{1,k+1}, \\ &\hat{x}_{k+1} \in \hat{x}_k + hk_1 \left[x_k - \hat{x_k}\right]^{\frac{1}{2}} \text{sgn}(\tilde{x}_{k+1} - \hat{x}_{k+1}) \\ &\qquad + hk_2(\tilde{x}_{k+1} - \hat{x}_{k+1}) + h\hat{y}_{k+1} \\ &\hat{y}_{k+1} \in \hat{y}_k - h\mu\tau_{2,k+1} - h\nu\tau_{1,k+1} + hk_3 \text{sgn}(\tilde{x}_{k+1} - \hat{x}_{k+1}) \\ &\qquad + hk_4(\tilde{x}_{k+1} - \hat{x}_{k+1}) \\ &x_{k+1} = x_k + hy_{k+1} \\ &y_{k+1} = y_k - h\mu\tau_{2,k+1} - h\nu\tau_{1,k+1} + h\phi_{k+1} \\ &\tau_{1,k+1} \in \text{sgn}(\tilde{x}_{k+1}), \quad \tau_{2,k+1} \in \text{sgn}(\tilde{y}_{k+1}) \end{aligned} \tag{78}$$

Lemma 5: The unperturbed discrete-time multivalued system (60) has a unique equilibrium point $(\dot{x}^*, \dot{y}^*, x^*, y^*)^T = (0, 0, 0, 0, 0)^T$.

Proof: Recall that in the unperturbed case we may consider $\dot{\hat{x}} = x_k$ and $\dot{\hat{y}} = y_k$ for all k. From the first line of (60) it follows that $x^* = x^* + hy^* \Rightarrow y^* = 0$. From the second line one has $y^* = y^* - h\mu\tau_2^* - h\nu\tau_1^*$, so that $\mu\tau_2^* + \nu\tau_1^* \geq 0 \Rightarrow 0 \in \text{sgn}(x^*) + (\nu/\mu)\text{sgn}(y^*)$ which is satisfied if and only if $x^* = y^* = 0$ because $\nu > \mu$ from (49). From the third line $\dot{\hat{x}}^* = \ldots$.

The proof shows that in the unperturbed case, the equilibrium is reached if and only if the observer gains μ, ν are selected appropriately according to the conditions given.

$$\dot{x}^* + hk_1 \left|\dot{x}^* - \dot{\hat{x}}^*\right| \text{sgn}(\dot{x}^* - \dot{\hat{x}}^*) + hk_2(\dot{x}^* - \dot{\hat{x}}^*) + h\dot{y}^* = \dot{x}^*,$$

which is equivalent to $0 \in k_1 \left|\dot{x}^* - \dot{\hat{x}}^*\right| \text{sgn}(\dot{x}^* - \dot{\hat{x}}^*) + k_2(\dot{x}^* - \dot{\hat{x}}^*)$. The unique solution of this generalized equation is $\dot{x}^* = \dot{\hat{x}}^*$.

Notice from the fourth and seventh lines of (78) that

$$\tau_{2,k+1} = \text{proj}\left([-1, 1]; \zeta_k\right)$$

where (1) has been used and

$$\zeta_k = \frac{\dot{y}_k - h\mu\tau_{1,k+1} + hk_3 \text{sgn}(\dot{x}_{k+1} - \dot{\hat{x}}_{k+1}) + hk_4(\dot{x}_{k+1} - \dot{\hat{x}}_{k+1})}{\mu}$$

Similarly to the twisting algorithm we may determine three main modes for (78):

- if $\frac{\dot{y}_k}{h} - \mu\tau_{1,k+1} + k_3 \text{sgn}(\dot{x}_{k+1} - \dot{\hat{x}}_{k+1}) + k_4(\dot{x}_{k+1} - \dot{\hat{x}}_{k+1}) \geq \mu$: $\tau_{2,k+1} = 1$,
- if $\frac{\dot{y}_k}{h} - \mu\tau_{1,k+1} + k_3 \text{sgn}(\dot{x}_{k+1} - \dot{\hat{x}}_{k+1}) + k_4(\dot{x}_{k+1} - \dot{\hat{x}}_{k+1}) \leq -\mu$: $\tau_{2,k+1} = -1$,
- if $\frac{\dot{y}_k}{h} - \mu\tau_{1,k+1} + k_3 \text{sgn}(\dot{x}_{k+1} - \dot{\hat{x}}_{k+1}) + k_4(\dot{x}_{k+1} - \dot{\hat{x}}_{k+1})$ is between $-\mu$ and μ: $\tau_{2,k+1}$ is determined by the projection.

If we consider the sub-mode (i-1) where the system's output $\hat{y}_k$ exceeds the sum of the disturbance compensation term $h\mu$ and the scaled state feedback $(\hat{x}_k/h)$ by $h\nu$, then the control input $\tau_{1,k+1}$ is activated to 1. The condition for this activation is $(\hat{y}_k/h) - \nu + k_3\text{sgn}(\hat{x}_{k+1} - \hat{x}_k) + k_4(\hat{x}_{k+1} - \hat{x}_k) \geq \mu$. This translates to the control law ensuring that the addition of the state feedback and the controller's gains are greater than the disturbance and the state feedback terms. Hence, the updated state $\hat{x}_{k+1}$ becomes equal to the previous state plus the effect of the controller's action.

To calculate $\hat{x}_{k+1}$ in the (i-1) mode, we have:

$$\hat{x}_{k+1} = \hat{x}_k + h(\hat{y}_k - h\mu + (\hat{x}_k/h)) \quad \text{given} \quad \tau_{1,k+1} = 1 \tag{79}$$

$$\hat{y}_{k+1} = \hat{y}_k - h\mu\tau_{1,k+1} + h\mu\tau_{2,k+1} + h\phi_{k+1} \tag{80}$$

The proof of the disturbance attenuation on the nominal system sliding mode is established by showing that for a large enough k, the system reaches a state where the disturbances are attenuated by a factor proportional to the time step squared h^2.

Remark 3 discusses the practical implementation of twisting controllers and supertwisting controllers, which are computed from the state values at each discrete time step.

The discrete-time system's dynamics can be written as:

$$\hat{x}_{k+1} = F\hat{x}_k + G\hat{x}_k - H\text{sgn}(C\hat{x}_k) \tag{81}$$

$$\hat{x}_{k+1} = (I_4 - hF)^{-1}(G\hat{z}_k + hH(\hat{z}_k)\text{sgn}(C\hat{z}_k)) \tag{82}$$

where F, G, H, C are matrices defined from the system dynamics, and the control law is derived by applying a projection operator to ensure that the control input stays within the admissible range. The control input is then used to update the state for the next time step, ensuring the system's evolution towards the desired state.

Consequently, the system can be easily executed in an online setting. This is applicable for both systems discussed previously, including system (50).

B. Numerical Simulations

Simulations of the system described in (59) are carried out with the disturbance $\Phi = \phi(t)$, employing the same conditions delineated in Section II-D, and utilizing the designated software package.

The simulations are conducted using MATLAB. When $\phi(t) \equiv 0$, the results are illustrated in Fig. 3. In the case where $\phi(t) = \phi\sin(\omega t)$ with $\phi = 0.1$ and $\omega = 5$, the findings are presented in Fig. 3. The chosen parameters are $k_e = k_s = 5$, $k_v = 1.5$, $M_x = 1$, and $M_u = 2$. Observations from Fig 3 show that since $T_{1,k}$ and $T_{2,k}$ are within the interval $(-1, 1)$.

The simulation of the discrete-time system described by equation (78) was conducted in MATLAB using the same setup as in Section II-D, except the sampling interval was selected to be $h = 10^{-2}$. The system was initialized with $x_0 = 2$, $y_0 = \dot{x}_0 = 1$, $ildex_0 = 1$, and $ildey_0 = 1$. The control parameters were set to $\mu = 1$, $u = 2$, and all k_i terms (where i ranges from 1 to 4) were equal to 5. The disturbance input $\phi(t) = \phi\sin(\omega t)$ was modeled with an amplitude $\phi = 0.1$ and frequency $\omega = 5$. The resulting behavior, as illustrated in Figure 5,6 confirmed that the predefined sliding surfaces $s_e = 0$ and $s_x = 0$ were indeed attained within a finite timeframe, consistent with the anticipations of Theorem 5. Figure 5 visualizes the attenuation of perturbations, highlighting that the error terms $e_{1,k}$ scale with h^2 and $e_{2,k}$ with h. Notably, the origin of the system's state space is reached after an infinite sequence of control signal switchings, which is characteristic of continuous-time twisting controllers, observable in Figure 5. It is worth mentioning that, although no formal proof of convergence is provided for the discrete-time variants of these twisting algorithms, the adopted backward Euler method is adept at managing the accumulation of events, also known as Zeno behavior, which is why such approaches are sometimes termed event-capturing methods.

IV. Conclusion

This study introduces a discrete-time implementation of sliding-mode control systems, employing an implicit Euler method and encompassing zero-order-hold discretization. These controllers, characterized by their simplicity, project onto the $[-1, 1]$ interval and are derivable from basic quadratic programs. Significantly, they encapsulate the intrinsic multivalued aspect of their analogues in continuous-time, thus precluding abrupt switching and excessive gain dynamics. Analysis confirms that stabilization on sliding manifolds is smooth under disturbance-free conditions, attributable to chatter-free controllers. Conversely, when disturbances are present, their influence is diminished by factors of h or h^2. A salient feature of these controllers is their independence from the sampling period's magnitude, which may be substantial. Additionally, the controllers maintain the desirable trait that their sliding surfaces are analogous in both continuous and discrete-time settings. Empirical validation is provided through extensive simulation results. Prospective research will address the proof of origin convergence within a bounded number of steps for discrete-time twisting and super-twisting algorithms, augmenting the numerical simulations documented in this paper. Future extensions to networked control systems are also considered.

Research in sliding-mode control, particularly for systems grappling with mismatched uncertainties, has advanced. Investigations have also been extended to the numerical analysis of certain optimal control challenges, which are typified by nonlinear variable-structure systems [9]. Additionally, experimental studies have been conducted to compare current methodologies aimed at reducing chattering [7]

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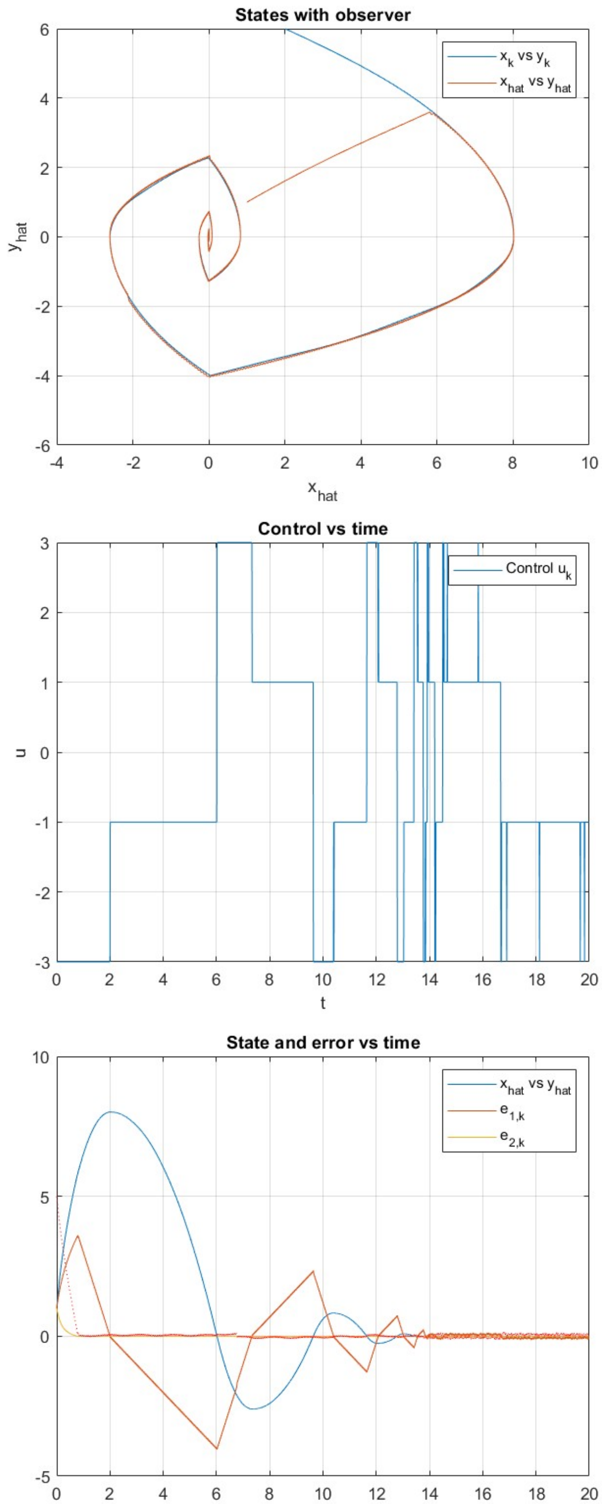

Fig. 5. For System(78)

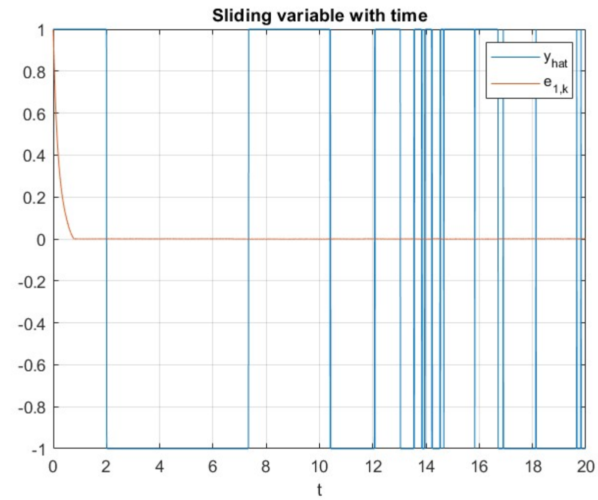

Fig. 6. For System(78),Sliding variable vs time

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