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Global-local mixtures

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SUMMARY

Global-local mixtures are derived from the Cauchy-Schlömilch and Liouville integral transformation identities. We characterize well-known normal-scale mixture distributions including the Laplace or Lasso, logit and quantile as well as new global-local mixtures. We also apply our methodology to convolutions that commonly arise in Bayesian inference. Finally, we conclude with a conjecture concerning bridge and uniform correlation mixtures.

Some key words: Global-local mixture; Scale mixture; Stable laws; Bayes regularization; Lasso; Quantile; Logistic; Cauchy; Convolutions.

1. INTRODUCTION

Many statistical problems involve regularization penalties derived from global-local mixture distributions (Polson et al., 2011; Hans, 2011; Bhadra et al., 2016a). A global-local mixture density, denoted by $p(x_1, \dots, x_p)$, takes the form

$$p(x_1, \dots, x_p) = \int_0^\infty \prod_{i=1}^p p(x_i \mid \tau) p(\tau) d\tau,$$

where $p(x_i \mid \tau) = \int_0^\infty p(x_i \mid \lambda_i, \tau) p(\lambda_i \mid \tau) d\lambda_i$ is a local mixture and $p(x_1, \dots, x_p)$ is a global mixture over $\tau \sim p(\tau)$. There is great interest in analytically calculating $p(x_i \mid \tau)$, and the associated regularization penalty $\phi(x_i, \tau) = -\log p(x_i \mid \tau)$. Convolution mixture of the form $p(x_i \mid \tau) = \int p(x_i - \lambda_i) p(\lambda_i) d\lambda_i$ are also of interest. We show how the Cauchy-Schlömilch and Liouville transformations can be used to derive closed-form global-local mixtures. We start by stating two key integral identities: the Cauchy-Schlömilch transformation

$$\int_0^\infty f\left\{(ax - bx^{-1})^2\right\} dx = \frac{1}{2a} \int_0^\infty f(y^2) dy \ , \qquad (1)$$

and the Liouville transformation

$$\int_0^\infty f\left(ax + \frac{b}{x}\right) x^{-1/2} dx = a^{-1/2} \int_0^\infty f\left(2(ab)^{1/2} + y\right) y^{-1/2} dy, \quad a, b > 0\,. \tag{2}$$

See Boros et al. (2006); Baker (2008); Jones (2014) for further discussion. Identity (1) follows from the simple transformation $t = b/(ax)$:

$$I = \int_0^\infty f\left\{(ax - b/x)^2\right\} dx = \int_0^\infty f\left\{(at - b/t)^2\right\} \frac{b}{at^2} dt.$$

Adding the two terms in the last equality yields $2I = \int_0^\infty f\left\{(at - b/t)^2\right\}\left(1 + \frac{b}{at^2}\right) dt$ and transforming $y = b/t - at$ gives $dy = -a(1 + \frac{b}{at^2})dt$, which yields $I = (2a)^{-1}\int_0^\infty f(y^2)dy$ as required. A useful generalization of the Cauchy-Schlömilch transformation is

$$\int_0^\infty f\left[\{x - s(x)\}^2\right] dx = \int_0^\infty f(y^2) dy \tag{3}$$

where $s(x) = s^{-1}(x)$ is a self-inverse function such as $s(x) = b/x$ or $s(x) = -a^{-1}\log\{1 - \exp(ax)\}$. The Liouville transformation follows in a similar manner. The proof is along the same lines, so, for the sake of brevity, we do not reproduce it here. These identities can be used to construct new global-local mixture distributions. Let $f(x) = 2g\{t(x)\}$ and give $t(x)$ the form $x - s(x)$, where $s : \Re^+ \to \Re^+$ is a self-inverse, onto and monotone decreasing function. Together with the Cauchy-Schlömilch transformation, we have a rather surprising way to represent the resulting $g\{t(x)\}$ as a global-local scale mixture.

Jones (2014) shows that only a few choices of $t(x)$ leads to fully tractable formulae for its inverse $t^{-1} = \Pi$ and the integral $\Pi(y) = \int_{-\infty}^{y} \pi(\omega)d\omega$. Two special choices are the t-distribution with 2 degrees of freedom and the logistic.

$$\begin{aligned}
\Pi_T(y) = (1/2)\{y + (4b + y^2)^{1/2}\}, \Pi_T^{-1}(x) = t_T(x) = x - b/x, \quad b > 0 \\
\Pi_L(y) = a^{-1}\log(1 + e^{ay}), \Pi_L^{-1}(x) = t_L(x) = a^{-1}\log(e^{ax} - 1), \quad a > 0.
\end{aligned}$$

Now, the integral identity in (1) shows that if $f(x)$ with $x \geq 0$ is a density function, so is $g(x) = 2af(|ax - b/x|)$ with $x > 0$. The functions f and g are called mother and daughter density functions, respectively.

Apart from simplifying proofs involving global-local mixtures, the Cauchy-Schlömilch and Liouville transformations can generate new distributions via scale transformations. These transformations can take the form $f(x) = 2g\{t(x)\}$ for certain $f(x)$ under suitable conditions. For example, given a density $f(x)$ we can create a new global-local scale family, $f(ax - b/x)$, by effectively reallocating its probability mass. A particularly useful tool for generating univariate and multivariate random variables is Khintchine's theorem. Khintchine's theorem states that any random variable X with a unimodal, univariate distribution and a mode at zero can be written as a product $X = ZU$, where $U \sim \mathcal{U}(0, 1)$ and Z has the density function $f_Z(z) = -zf_X'(z), z \in \Re$. Bryson & Johnson (1982), and subsequently Jones (2002), discuss how Khintchine's theorem allows one to construct both univariate and multivariate distributions, even with special dependence structure. Jones (2014) develops an extended Khintchine's theorem that further allows one to generate random variables with unimodal densities of the form $2g\{t(x)\}$.

2. Global-local Scale Mixtures

2·1. *Lasso as a normal scale mixture*

The lasso penalty arises as a Laplace global-local mixture (Andrews & Mallows, 1974). A simple transformation proof follows using Cauchy-Schlömilch with $f(x) = e^{-x}$. Starting with the normal integral identity, $\int_0^\infty f(y^2)dy = \int_0^\infty e^{-y^2}dy = \pi^{1/2}/2$, we obtain

$$\int_0^\infty e^{-(ax)^2-(b/x)^2}dx = \int_0^\infty \exp\left\{-ab\left(\frac{a}{b}x^2 + \frac{b}{a}x^{-2}\right)\right\}dx = \frac{\pi^{1/2}}{2a}e^{-2ab} \quad a, b \in \Re.$$

Substituting $t = (a/b)^{1/2}x$ and $c = ab$ yields the Laplace or Lasso penalty:

$$\int_0^\infty e^{-c(t-t^{-1})^2}dt = \frac{1}{2}(\pi/c)^{1/2} \Rightarrow \int_0^\infty e^{-c(t^2+t^{-2})}dt = \frac{1}{2}(\pi/c)^{1/2}e^{-2c}\ .$$

The Laplace density can be viewed as a transformed normal, via $y = t - t^{-1}$.

Remark 1. The usual identity for the lasso can also be obtained from Lévy (1940)

$$\int_0^\infty \frac{a}{(2\pi)^{1/2}t^{3/2}}e^{-\frac{a^2}{2t}}e^{-\lambda t}dt = e^{-a(2\lambda)^{1/2}}\ . \tag{4}$$

For $a = 1$, and $\theta = (2\lambda)^{1/2}$, this can be written as

$$E[\exp\{-\theta^2/(2G)\}] = \exp(-\theta), \quad G \sim \mathcal{G}(1/2, 1/2) \tag{5}$$

Proof. First substitute $t^{-1} = x^2$, which makes the left hand side in (4) equal to

$$\int_0^\infty \frac{a}{(2\pi)^{1/2}t^{3/2}}e^{-\frac{a^2}{2t}}e^{-\lambda t}dt = \left(\frac{2}{\pi}\right)^{1/2} ae^{-a(2\lambda)^{1/2}} \int_0^\infty e^{-(2^{-1/2}ax-\lambda x^{-1})^2}dx = e^{-a(2\lambda)^{1/2}}\ .$$

The last step follows directly from Cauchy-Schlömilch formula. The second relationship (5) follows by fixing $a = 1$, $\theta = (2\lambda)^{1/2}$ and substituting $t = x^{-1}$:

$$\int_0^\infty \frac{a}{(2\pi)^{1/2}t^{3/2}}e^{-\frac{a^2}{2t}}e^{-\lambda t}dt = \frac{1}{(2\pi)^{1/2}}\int_0^\infty e^{-\frac{\theta^2}{2x}}x^{-1/2}e^{-x/2}dx.$$

The left hand side can be identified as $E\left(e^{-\theta^2/(2G)}\right)$ for $G \sim \mathcal{G}(1/2, 1/2)$. □

2·2. *Logit and quantile as global-local mixtures*

Logistic modeling can be viewed within the global-local mixture framework via the Pólya-Gamma distribution (Polson et al., 2013). This leads to efficient Markov chain Monte Carlo algorithms for inference. The two key marginal distributions for the hyperbolic generalized inverse Gaussian (Barndorff-Nielsen et al., 1982) and Pólya-Gamma mixtures are

$$\frac{\alpha^2-\kappa^2}{2\alpha}e^{-\alpha|x-\mu|+\kappa(x-\mu)} = \int_0^\infty \phi(x \mid \mu+\kappa\lambda, \lambda)p_{\text{GIG}}\left(\lambda \mid 1, 0, (\alpha^2-\kappa^2)^{1/2}\right)d\lambda,\ \alpha \geq \kappa \geq 0, \tag{6}$$

$$\frac{1}{B(\alpha,\kappa)}\frac{e^{\alpha(x-\mu)}}{(1+e^{x-\mu})^{\alpha+\kappa}} = \int_0^\infty \phi(x \mid \mu+\kappa\lambda, \lambda)p_{\text{Polya}}(\lambda \mid \alpha, \kappa)d\lambda\ , \tag{7}$$

where $\phi(\mu+\kappa\lambda, \lambda)$ denotes the normal density function with mean $(\mu+\kappa\lambda)$ and variance λ. The functions p_{GIG} and p_{Polya} are the corresponding local mixture densities for the generalized inverse Gaussian and the Pólya-Gamma, respectively. The logit and quantile identities can be derived using Cauchy-Schlömilch identity. Let $f(x) = e^{-x^2/2}$, $a = \alpha$ and $b = |x - \phi|$ in (1)

$$(2/\pi)^{1/2} \int_0^\infty \exp\left\{-\frac{1}{2}\left(\alpha y - \frac{|x-\mu|}{y}\right)^2\right\} dy = \frac{1}{\alpha}(2\pi)^{-1/2}\int_0^\infty e^{-\frac{1}{2}y^2} dy = \frac{1}{\alpha} .$$

Let $\nu = y^2$ and rearrange the constant terms to get the relation

$$\frac{1}{\alpha} e^{-\alpha|x-\mu|} = \frac{1}{(2\pi\nu)^{1/2}} \int_0^\infty \exp\left[-\left\{\frac{(x-\mu)^2}{2\nu} + \frac{\alpha^2}{2}\nu\right\}\right] d\nu .$$

Multiplying by $2^{-1}(\alpha^2 - \kappa^2)e^{\kappa(x-\mu)}$ and completing the square yields

$$\frac{\alpha^2-\kappa^2}{2\alpha}\exp\left(-\alpha|x-\mu| + \kappa(x-\mu)\right) = \int_0^\infty \phi(x \mid \mu+\kappa\nu, \nu)\frac{\alpha^2-\kappa^2}{2}\exp\left(-\frac{\alpha^2-\kappa^2}{2}\nu\right) d\nu.$$

The mixing distribution is exponential with rate parameter $(\alpha^2 - \kappa^2)/2$, a special case of the generalized inverse Gaussian distribution introduced by Etienne Halphen circa 1941 (Seshadri, 2004). The density with parameters $(\lambda, \delta, \gamma)$ has the form

$$p_{\mathrm{GIG}}(x \mid \lambda, \delta, \gamma) = \frac{(\gamma/\delta)^\lambda}{2K_\lambda(\delta\gamma)} x^{\lambda-1}\exp\left\{-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)\right\}, \quad x, \lambda, \delta > 0, p \in \Re ,$$

where K_λ is the modified Bessel function of the second kind. The Liouville formula can be used to show that the above is a valid probability density function. When δ or γ is zero, the normalizing constant takes the limit values given by $K_\lambda(u) \asymp \Gamma(|\lambda|)2^{|\lambda|-1}u^{|\lambda|}$ for $\lambda > 0$. If $\delta = 0$, the generalized inverse Gaussian is identical to a gamma distribution:

$$p_{\mathrm{GIG}}(x \mid \lambda, \delta=0, \gamma) = \frac{\alpha^\lambda}{\Gamma(\lambda)} x^{\lambda-1}\exp(-\alpha x) , \quad \text{for} \quad x > 0, \alpha = \gamma^2/2.$$

We now present a simple proof for the Pólya-Gamma mixture in (7). First, write κ for $a - b/2$:

$$\frac{(e^\psi)^a}{(1+e^\psi)^b} = 2^{-b}e^{\kappa\omega}\int_0^\infty e^{-\omega\psi^2/2}p(\omega)d\omega , \tag{8}$$

where $\omega \sim \mathrm{PG}(b, 0)$, the Pólya-Gamma distribution with density is

$$p(\omega \mid b, 0) = \frac{2^{b-1}}{\Gamma(b)}\sum_{n=0}^\infty (-1)^n \frac{\Gamma(n+b)}{\Gamma(n+1)}\frac{2n+b}{(2\pi)^{1/2}\omega^{3/2}}\exp\left(-\frac{(2n+b)^2}{8\omega}\right).$$

The logit function corresponds to $a = 0, b = 1$ in (8). Cauchy-Schlömilch yields

$$\frac{1}{1+e^\psi} = \frac{1}{2}e^{-\frac{1}{2}\psi}\int_0^\infty e^{-(\psi^2\omega)/2}p(\omega)d\omega, \quad p(\omega) = \sum_{n=0}^\infty (-1)^n \frac{2n+1}{(2\pi\omega^3)^{1/2}} e^{-(2n+1)^2/(8\omega)} . \tag{9}$$

To show (9), write the left-hand side interchanging integral and summation:

$$I = \frac{1}{2}e^{-\psi/2}\sum_{n=0}^\infty (-1)^n \frac{2n+1}{(2\pi)^{1/2}}\int_0^\infty \exp\left[-\left\{\frac{\psi^2}{2}\omega + \frac{(2n+1)^2}{8\omega}\right\}\right]\frac{1}{\omega^{3/2}}d\omega .$$

Using the change of variable $\omega = t^{-2}$ gives

$$I = \sum_{n=0}^{\infty}(-1)^n e^{-(n+1)\psi} \frac{2n+1}{(2\pi)^{1/2}} \left\{ \int_0^\infty \exp\left\{ -\frac{1}{2}\left(\frac{(2n+1)t}{2} - \frac{\psi}{t} \right)^2 \right\} dt \right\} .$$

Applying Cauchy-Schlömilch to the inner integral yields

$$\int_0^\infty \exp\left[-\frac{1}{2}\left\{ \frac{(2n+1)t}{2} - \frac{\psi}{t} \right\}^2 \right] dt = \int_0^\infty e^{-y^2/2} dy \frac{1}{2n+1} = \frac{(2\pi)^{1/2}}{2n+1} ,$$

which implies $I = \sum_{n=0}^{\infty} (-1)^n \exp\{-(n+1)\psi\} = \{1 + \exp(\psi)\}^{-1}$.

Remark 2. When $\alpha = \kappa$, we have the limiting result $(\alpha^2 - \kappa^2)^{-1} p_{\text{GIG}}\{1, 0, (\alpha^2 - \kappa^2)^{1/2}\} \equiv 1$, or equivalently in terms of densities, with a marginal improper uniform prior, $p(\lambda) = 1$,

$$\int_0^\infty \phi(b \mid -a\lambda, c\lambda) d\lambda = a^{-1} \exp\left\{ -2\max(ab/c, 0) \right\} . \tag{10}$$

This pseudo-likelihood represents support vector machines as a global-local mixture. The identity for quantile regression, which is a limiting case of the above identities by applying Fatou-Lebesgue theorem, is the following:

$$c^{-1} \exp\{2c^{-1}\rho_q(b)\} = \int_0^\infty \phi(b \mid \lambda - 2\tau\lambda, c\lambda) \mathrm{e}^{-2\tau(1-\tau)\lambda} d\lambda, c, \tau > 0,$$

where $\rho_q(b) = |b|/2 + (q - 1/2)b$ is the check-loss function (Polson & Scott, 2013).

Polson et al. (2011) derive this as a direct consequence of the Lasso identity $\int_0^\infty p/(2\pi\lambda)^{1/2} \exp\left\{ -1/2\left(p^2\lambda + q^2\lambda^{-1} \right) \right\} d\lambda = e^{-|pq|}$. Applying Liouville formula yields

$$\int_0^\infty f\left(ax + \frac{b}{x} \right) x^{-1/2} dx = a^{-1/2} \int_0^\infty f\left\{ 2(ab)^{1/2} + y \right\} y^{-1/2} dy, \quad a, b > 0.$$

Setting $f(x) = e^{-x}$, $a = p^2/2$, and $b = q^2/2$ we get

$$\int_0^\infty \frac{e^{-1/2(p^2\lambda + q^2\lambda^{-1})}}{\lambda^{1/2}} d\lambda = \frac{2^{\frac{1}{2}}}{p} \int_0^\infty e^{-|pq| + y} y^{-\frac{1}{2}} dy = \frac{2^{\frac{1}{2}} e^{-|pq|}}{p} \int_0^\infty e^{-y} y^{-\frac{1}{2}} dy = \frac{(2\pi)^{\frac{1}{2}} e^{-|pq|}}{p} .$$

Hans (2011) shows that an elastic-net regression can be recast as a global-local mixture with a mixing density belonging to the orthant-normal family of distributions. The orthant-normal prior on a single regression coefficient, β, given hyper-parameters λ_1 and λ_2, has a density function with the following form:

$$p(\beta \mid \lambda_1, \lambda_2) = \begin{cases} \phi(\beta \mid \frac{\lambda_1}{2\lambda_2}, \frac{\sigma^2}{\lambda_2}) / 2\Phi(\frac{-\lambda_1}{2\sigma\lambda_2^{1/2}}), & \beta < 0 \\ \phi(\beta \mid \frac{-\lambda_1}{2\lambda_2}, \frac{\sigma^2}{\lambda_2}) / 2\Phi(\frac{-\lambda_1}{2\sigma\lambda_2^{1/2}}) & \beta \geq 0 \end{cases} . \tag{11}$$

3. CONVOLUTION MIXTURES

Another interesting area of application is convolution mixtures and marginal densities for location-scale mixture problems. We show that the Cauchy convolution (Pillai & Meng, 2016) and Inverse-Gamma convolution can be derived similarly (Polson & Scott, 2012). Bhadra et al. (2016b) shows that the regularly varying tails of half-Cauchy priors work well for low-dimensional functions of normal vector mean, where flat priors give poorly calibrated inference.

LEMMA 1. *Let* $X_i \sim \mathcal{C}(0,1)$ $(i=1,2)$ *be Cauchy distributed random variates, then* $Z = w_1X_1 + w_2X_2 \sim \mathcal{C}(0, w_1+w_2)$. *where* $w_1, w_2 > 0$.

LEMMA 2. *Let* $X_i \sim \mathcal{IG}(\alpha t_i, \alpha t_i^2)$ $(i=1,2)$, *then* $Z = X_1 + X_2 \sim \mathcal{IG}(\alpha(t_1+t_2), \alpha(t_1^2 + t_2^2)$, *where* $\alpha, t_1, t_2 \geq 0$, *and* $\mathcal{IG}(\alpha t, \alpha t^2)$ *denotes the inverse-Gaussian density given by*

$$f(t) = \frac{t\alpha^{1/2}e^t}{(2\pi)^{1/2}x^{3/2}} \exp\left(-\frac{\alpha t^2}{2x} - \frac{x}{2\alpha}\right); \quad x \geq 0$$

Both of these results follow from a straight-forward application of the Cauchy-Schlömilch transformation. We give the proof for the Cauchy convolution identity below.

Proof. Exploiting symmetry and the Lagrange identity $(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2$, leads to the convolution density

$$\begin{aligned} f_Z(z) &= 2\int_0^\infty \frac{1}{\pi w_1(1+x^2/w_1^2)} \frac{1}{\pi w_2\{1+(z-x)^2/w_2^2\}} dx, \\ &= \frac{2}{\pi^2 w_1 w_2} \int_0^\infty \frac{1}{\{1+w_1^{-1}w_2^{-1}x(z-x)\}^2 + \{w_2^{-1}z - (w_1^{-1}+w_2^{-1})x\}^2} dx. \end{aligned}$$

Transforming x to $x + w_2^{-1}z(w_1^{-1}+w_2^{-1})^{-1}$ and letting $a = 1 + z^2(w_1+w_2)^{-2}$, $b = (w_1w_2)^{-1}$, $c = z(w_2-w_1)\{(w_1+w_2)w_1w_2\}^{-1}$, $d = z(w_2-w_1)\{(w_1+w_2)w_1w_2\}^{-1}$ gives

$$\begin{aligned} f_Z(z) &= \frac{2}{\pi^2 w_1 w_2} \int_0^\infty \left[\left\{1 + \frac{z^2}{(w_1+w_2)^2} - \frac{x^2}{w_1w_2} + xz\frac{w_2-w_1}{(w_1+w_2)w_1w_2}\right\}^2 + x^2\left(\frac{w_1+w_2}{w_1w_2}\right)^2\right]^{-1} dx \\ &= \frac{2}{\pi^2 w_1 w_2} \int_0^\infty \frac{dx}{(a - bx^2 + cx)^2 + x^2d^2} = \frac{2}{\pi^2 w_1 w_2} \int_0^\infty \frac{dx/x^2}{(a/x - bx + c)^2 + d^2}. \end{aligned}$$

If we let $y = x^{-1}$ and apply the Cauchy-Schlömilch transformation, we arrive at

$$f_Z(z) = \frac{2}{\pi w_1 w_2} \int_0^\infty \frac{dy}{2a(y^2+d^2)} = \frac{1}{\pi w_1 w_2} \frac{1}{ad} = \frac{1}{\pi(w_1+w_2)} \frac{1}{1+z^2/(w_1+w_2)^2}.$$

A simple induction argument proves that the sum of any number of independent Cauchy random variates is also another Cauchy. □

One can also use the characteristic function of $X \sim \mathcal{C}(\mu, \sigma)$, $\psi_X(t) = \exp(it\mu - |t|\sigma^2)$, and the relation $\psi_{X+Y}(t) = \psi_X(t)\psi_Y(t)$ to derive the result in just one step. For $X = \sum_{i=1}^p \omega_i C_i$ and $C_i \sim \mathcal{C}(0,1)$, when $\sum_{i=1}^p \omega_i = 1$ we have $\phi_X(t) = \exp\left(-\sum_{i=1}^p \omega_i |t|\right) = \exp(-|t|) = \phi_C(t)$, where $C \sim \mathcal{C}(0,1)$.

The most general result in this category is due to Pillai & Meng (2016), who they showed the following: Let $(X_1, \cdots, X_m)$ and $(Y_1, \cdots, Y_m)$ be independent and identically distributed $\mathcal{N}(0, \Sigma)$ for an arbitrary positive definite matrix Σ, then $Z = \sum_{j=1}^m w_j X_j / Y_j \sim \mathcal{C}(0,1)$, as long as $(w_1, \cdots, w_m)$ is independent of (X, Y), $w_j \geq 0$ $(j = 1, \cdots, m)$ and $\sum_{j=1}^m w_j = 1$.

4. DISCUSSION

The Cauchy-Schlömilch and Liouville transformations not only guarantee an simple normalizing constant for $f(\cdot)$, it also establishes the wide class of unimodal densities as global-local scale mixtures. Global-local scale mixtures that are conditionally Gaussian hold a special place in statistical modeling and can be rapidly fit using an expectation-maximization algorithm as pointed out by Polson & Scott (2013). Palmer et al. (2011) provides a similar tool for modeling multivariate dependence by writing general non-Gaussian multivariate densities as multivariate Gaussian scale mixtures.

We end our paper with conjectures that two other remarkable identities arise as corollaries of such transformation identities. The first one is a recent result by Zhang et al. (2014) that proves a uniform correlation mixture of a bivariate Gaussian density with unit variance is a function of the maximum norm:

$$\int_{-1}^{1} \frac{1}{4\pi(1-\rho^2)^{1/2}} \exp\left\{-\frac{x_1^2+x_2^2-2\rho x_1 x_2}{2(1-\rho^2)}\right\} d\rho = \frac{1}{2}\left\{1-\Phi(||x||_\infty)\right\} \ , \tag{12}$$

where $\Phi(\cdot)$ is the standard normal distribution function and $||x||_\infty = \max\{x_1, x_2\}$. The bivariate density on the right side of (12) was introduced by Bryson & Johnson (1982) as uniform mixtures of a chi random variate with 3 degrees of freedom, but the representation as a uniform correlation mixture is a new find. We make a couple of remarks connected to the Erdelyi's integral identity, which is key to the proof of the uniform correlation mixture (12).

THEOREM 1. *Erdelyi's identity, defined by*

$$\int_{1/2}^{\infty} \frac{e^{-x^2 z}}{4\pi z(2z-1)^{1/2}} dz = \frac{1}{2}\left\{1-\Phi(x)\right\}, \quad x \geq 0 \tag{13}$$

follows from the Laplace transformation $(1+u)^{-1} = \int_0^\infty \exp\{-v(1+u)\}dv$.

Proof. Apply the transform $u = 2z - 1$ to the left hand side of (13), denoted by I, to obtain

$$I = \int_0^\infty \frac{e^{-x^2/(2(1+u))}}{4\pi u^{1/2}(1+u)} du \ .$$

Using the Laplace transformation $(1+u)^{-1} = \int_0^\infty e^{-v(1+u)}dv$,

$$\begin{aligned} I &= \int_0^\infty \frac{e^{-x^2/(2(1+u))}}{4\pi u^{1/2}} \int_0^\infty e^{-v(1+u)} dv du = \int_{v=0}^\infty \int_{u=0}^\infty \frac{e^{-(\frac{x^2}{2}+v)(1+u)}}{4\pi u^{1/2}} dv du \\ &= \int_{v=0}^\infty \frac{1}{4\pi} e^{-(\frac{x^2}{2}+v)} \int_{u=0}^\infty u^{-1/2} e^{-(\frac{x^2}{2}+v)u} du dv = \int_{v=0}^\infty \frac{e^{-\frac{1}{2}(x^2+2v)}}{2(2\pi)^{1/2}} \frac{1}{(x^2+2v)^{1/2}} dv \ , \end{aligned}$$

and letting $z^2 = x^2 + 2v$ we get

$$I = \frac{1}{2}\int_{z=|x|}^\infty \frac{1}{(2\pi)^{1/2}} e^{-\frac{1}{2}z^2} dz = \frac{1}{2}\left\{1-\Phi(|x|)\right\} \ .$$

□

The second candidate is the symmetric stable distribution, defined by its characteristic function $\phi(t) = \exp(-|t|^\alpha)$ for $0 < \alpha \leq 2$. It admits a normal scale mixture representation with mixing density

$$f(v) = \frac{1}{2} s_{\alpha/2} \frac{v}{2}, \quad v > 0$$

$$e^{-|x|^\alpha} = \int_0^\infty e^{-xs} g(s) ds, \quad g(s) = \sum_{j=1}^\infty (-1)^j \frac{s^{-j\alpha-1}}{j!\Gamma(-\alpha j)} \ ,$$

when $s_{\alpha/2}$ is the density of the positive stable distribution with index $\alpha/2$. An important application of this representation is found in Bayesian bridge regression (Polson et al., 2014). Regularization, in this case, is an outcome of a normal scale mixture with respect to an α-stable random variable. We conjecture that these two results follow from the Cauchy-Schlömilch formula (1). Other potential applications include using Liouville formula to recognize and generate global-local mixtures, and to calculate higher-order closed-form moments $E(X^n)$ for random variables X that admit a global-local representation.

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