Global-local mixtures

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SUMMARY

Global-local mixtures are derived from the Cauchy-Schlömilch and Liouville integral transformation identities. We characterize well-known normal-scale mixture distributions including the Laplace or Lasso, logit and quantile as well as new global-local mixtures. We also apply our methodology to convolutions that commonly arise in Bayesian inference. Finally, we conclude with a conjecture concerning bridge and uniform correlation mixtures.

Some key words: Global-local mixture, Scale mixture, Stable laws, Bayes regularization, Lasso, Quantile, Logistic, Cauchy, Convolutions.

1. INTRODUCTION

Many statistical problems involve regularization penalties derived from global-local mixture distributions (Polson & Scott, 2011; Hans, 2011; Bhadra et al., 2015a). A global-local mixture density, denoted by $p(x_1, \ldots, x_p)$, takes the form

$$p(x_1, \ldots, x_p) = \int_0^\infty \prod_{i=1}^p p(x_i \mid \tau) p(\tau) d\tau \quad \text{where} \quad p(x_i \mid \tau) = \int_0^\infty p(x_i \mid \lambda_i, \tau) p(\lambda_i \mid \tau) d\lambda_i,$$

where $p(x_i \mid \tau)$ is a local mixture and $p(x_1, \ldots, x_p)$ is a global mixture over $\tau \sim p(\tau)$. There is great interest in analytically calculating $p(x_i \mid \tau)$, and the associated regularization penalty $\phi(x_i, \tau) = -\log p(x_i \mid \tau)$. Convolution mixture of the form $p(x_i \mid \tau) = \int p(x_i - \lambda_i) p(\lambda_i) d\lambda_i$ are also of interest. We show how Cauchy-Schlömilch and Liouville transformations derive closed-form global-local mixtures. We start by stating the two key integral identities, the Cauchy-Schlömilch transformation:

$$\int_0^\infty f\left\{(ax - bx^{-1})^2\right\} dx = \frac{1}{2a} \int_0^\infty f(y^2) dy, \tag{1}$$

and the Liouville transformation:

$$\int_0^\infty f\left(ax + \frac{b}{x}\right)\frac{dx}{\sqrt{x}} = \frac{1}{\sqrt{a}}\int_0^\infty f\left(2\sqrt{ab} + y\right)\frac{dy}{\sqrt{y}}, a, b > 0. \qquad (2)$$

See Boros et al. (2006); Baker (2008); Amdeberhan et al. (2010); Jones (2014) for further discussion. Equation (1) follows from the simple transformation $t = b/ax$:

$$I = \int_0^\infty f\left\{(ax - b/x)^2\right\} dx = \int_0^\infty f\left\{(at - b/t)^2\right\} \frac{b}{at^2} dt.$$

Adding yields $2I = \int_0^\infty f\left\{(at - b/t)^2\right\}\left\{1 + b(at)^{-2}\right\} dt$ and transforming $y = b/t - at$ gives $dy = -a(1 + b/at^2)dt$, which yields $I = (2a)^{-1}\int_0^\infty f(y^2)dy$ as required. A useful generalization of the Cauchy-Schlömilch transformation is given by:

$$\int_0^\infty f\left[\{x - s(x)\}^2\right] dx = \int_0^\infty f(y^2)dy \qquad (3)$$

where $s(x) = s^{-1}(x)$ is a self-inverse function such as $s(x) = b/x$ or $s(x) = -a^{-1}\log\{1 - \exp(ax)\}$. The Liouville transformation follows in a similar manner. These identities can be used to construct new global-local mixture distributions. Let $f(x) = 2g\{t(x)\}$ where $t(x)$ has form $x - s(x)$ where $s : \Re^+ \to \Re^+$ is a self-inverse, onto, monotone decreasing function, with $s\{s(x)\} = x$. Together with the Cauchy-Schlömilch transformation, we have a rather surprising way to represent the resulting $g\{t(x)\}$ as a global-local scale mixture.

Jones (2014) shows that only a few choices of $t(x)$ leads to fully tractable formulae for its inverse $t^{-1} = \Pi$ and the integral $\Pi(y) = \int_{-\infty}^y \pi(\omega)d\omega$. Two special choices are the t distribution with 2 degrees of freedom and the logistic.

$$t(2) : \Pi_T(y) = (1/2)(y + \sqrt{4b + y^2}) \Rightarrow \Pi_T^{-1}(x) = t_T(x) = x - (b/x)$$
$$\text{Logistic} : \Pi_L(y) = a^{-1}\log(1 + e^{ay}) \Rightarrow \Pi_L^{-1}(x) = t_L(x) = a^{-1}\log(e^{ax} - 1)$$

Now, the integral identity (1) shows that if $f(x), x \ge 0$ is a density function, so is $g(x) = 2af(|ax - b/x|), x > 0$, and $f(\cdot)$ and $g(\cdot)$ are called the mother and the daughter pdf. Chaubey et al. (2010) provide a one-to-one correspondence between f and g.

Apart from simplifying proofs for global-local mixtures, Cauchy-Schlömilch and Liouville transformations can generate new distributions by a suitable scale transformation $f(x) = 2g\{t(x)\}$ of a simple baseline function f under some conditions, and given a density $f(x)$ we can create a new global-local scale family $f(ax - bx^{-1})$, by reallocating its probability mass. A particularly useful tool for generating univariate and multivariate random variables is Khintchine's theorem for unimodality of univariate distributions (Bryson & Johnson, 1982). Khintchine's theorem states that any random variable X with a mode at zero can be written as a product $X = ZU$, where $U \sim U(0, 1)$ and Z has the density function $f_Z(z) = -zf_X'(z)$. Bryson & Johnson (1982) and successively Jones (2012) discuss how Khintchine's theorem allows us to construct both univariate and multivariate distributions, even with special dependence structure.Jones (2014) develops an extended Khintchine's theorem that further lets one generate random variable with unimodal density of the form $2g(t(x))$.

The rest of the paper is organized as follows: §2 derives scale mixture results for the Lasso, quantile and logistic regression, §3 for convolutions of densities via mixtures and finally §4 concludes with two open problems.

2. GLOBAL-LOCAL SCALE MIXTURES

2·1. *Lasso as a Normal Scale Mixture*

The lasso penalty arises as a Laplace global-local mixture (Andrews & Mallows, 1974). A simple transformation proof follows using Cauchy-Schlömilch with $f(x) = \exp(-x)$. Starting with the normal integral identity, $\int_0^\infty f(y^2)dy = \int_0^\infty e^{-y^2}dy = \pi^{1/2}/2$, we obtain:

$$\int_0^\infty e^{-a^2x^2 - b^2/x^2}dx = \int_0^\infty e^{-ab\left(\frac{a}{b}x^2 + \frac{b}{a}x^{-2}\right)}dx = \frac{\pi^{1/2}}{2a}e^{-2ab}.$$

Substituting $t = (a/b)^{1/2}x$ and $c = ab$ yields the Laplace or Lasso penalty.

$$\int_0^\infty e^{-c(t-t^{-1})^2}dt = \frac{1}{2}(\pi/c)^{1/2} \Rightarrow \int_0^\infty e^{-c(t^2+t^{-2})}dt = \frac{1}{2}(\pi/c)^{1/2}e^{-2c}. \qquad 70$$

The Laplace density can be viewed as a transformed normal, via $y = t - t^{-1}$.

Remark 1. The usual identity for lasso can also be obtained from Lévy (1940):

$$\int_0^\infty \frac{a}{\sqrt{(2\pi)}t^{3/2}}e^{-a^2/2t}e^{-\lambda t}dt = e^{-a\sqrt{(2\lambda)}} \qquad (4)$$

For $a = 1$, and $\theta = (2\lambda)^{1/2}$, this can be written as:

$$E\left[\exp\{-\theta^2/2G\}\right] = \exp(-\theta), \quad \text{where } G \sim \text{Ga}(1/2, 1/2) \qquad (5)$$

Proof. First substitute $t^{-1} = x^2$, which makes the left hand side in (4) equal to:

$$\int_0^\infty \frac{a}{\sqrt{(2\pi)}t^{3/2}}e^{-a^2/2t}e^{-\lambda t}dt = \sqrt{\frac{2}{\pi}}ae^{-a\sqrt{(2\lambda)}}\int_0^\infty e^{-\left(\frac{a}{\sqrt{2}}x - \lambda x^{-1}\right)^2}dx = \exp(-a\sqrt{(2\lambda)}). \qquad 75$$

The last step follows directly from Cauchy-Schlömilch formula. The second relationship (5) follows by fixing $a = 1$, $\theta = (2\lambda)^{1/2}$ and substituting $t = x^{-1}$ to give:

$$\int_0^\infty \frac{a}{\sqrt{(2\pi)}t^{3/2}}e^{-a^2/2t}e^{-\lambda t}dt = \int_0^\infty e^{-\frac{\theta^2}{2x}}\sqrt{\left(\frac{1}{2\pi}\right)}x^{-\frac{1}{2}}e^{-\frac{1}{2}x}dx.$$

The left hand side can be identified as $\mathbb{E}\left[\exp\{-\theta^2/2G\}\right]$ for $G \sim \text{Ga}(1/2, 1/2)$. □

2·2. *Logit and Quantile as Global-local Mixtures*

Logistic modeling can be viewed within the global-local mixture framework via the Pólya–Gamma distribution (Polson et al., 2013). This leads to efficient MCMC algorithms for inference. The two key marginal distributions for the hyperbolic-GIG and Z-Pólya mixtures (Polson et al., 2013; Barndorff-Nielsen et al., 1982), with $\phi(\mu + \kappa\lambda, \lambda)$ denoting a normal density, are:

$$\frac{\alpha^2 - \kappa^2}{2\alpha}e^{-\alpha|x-\mu|+\kappa(x-\mu)} = \int_0^\infty \phi(x \mid \mu + \kappa\lambda, \lambda)p_{\text{GIG}}(\lambda \mid 1, 0, \sqrt{(\alpha^2 - \kappa^2)})d\lambda, \qquad (6)$$

$$\frac{1}{B(\alpha, \kappa)}\frac{e^{\alpha(x-\mu)}}{(1 + e^{x-\mu})^{\alpha+\kappa}} = \int_0^\infty \phi(x \mid \mu + \kappa\lambda, \lambda)p_{\text{Polya}}(\lambda \mid \alpha, \kappa)d\lambda. \qquad (7)$$

The corresponding local mixture densities are Generalized Inverse-Gaussian and the Pólya -Z, denoted by p_{GIG} and p_{Polya} respectively. Rather surprisingly, the logit and quantile identities can be derived by Cauchy-Schlömilch transformation.

Let $f(x) = e^{-x^2/2}$, $a = \alpha$, $b = |x - \phi|$ in (1), then we have:

$$\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{1}{2}\left(\alpha y - \frac{|x-\mu|}{y}\right)^2} dy = \frac{1}{\alpha} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}y^2} dy = \frac{1}{\alpha}$$

Letting $\nu = y^2$, re-arrange the constant terms to get the relation:

$$\frac{1}{\alpha} e^{-\alpha|x-\mu|} = \int_0^\infty e^{-\left(\frac{(x-\mu)^2}{2\nu} + \frac{\alpha^2}{2}\nu\right)} \frac{1}{\sqrt{(2\pi\nu)}} d\nu.$$

Multiplying by $2^{-1}(\alpha^2 - \kappa^2) \exp\{\kappa(x - \mu)\}$, and completing the square yields

$$\frac{\alpha^2 - \kappa^2}{2\alpha} e^{-\alpha|x-\mu|+\kappa(x-\mu)} = \int_0^\infty \phi(x \mid \mu + \kappa\nu, \nu) \frac{\alpha^2 - \kappa^2}{2} e^{-\frac{\alpha^2 - \kappa^2}{2}\nu} d\nu.$$

The mixing distribution is exponential with rate parameter $(\alpha^2 - \kappa^2)/2$, a special case of the Generalized inverse Gaussian distribution introduced by Etienne Halphen circa 1941 (Seshadri, 1997). The density with parameters $(\lambda, \delta, \gamma)$ has the form:

$$p_{\mathrm{GIG}}(x \mid \lambda, \delta, \gamma) = \frac{(\gamma/\delta)^\lambda}{2K_\lambda(\delta\gamma)} x^{\lambda-1} \exp\left\{-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)\right\}; \ x > 0, \lambda > 0, \delta > 0, p \in \Re,$$

where K_λ is the modified Bessel function of the second kind. The Liouville formula directly applies here to show its a valid probability density function. When δ or γ is zero, the normalizing constant is to be replaced by the corresponding limits as $K_\lambda(u) \sim \Gamma(|\lambda|)2^{|\lambda|-1}u^{|\lambda|}$ when $\lambda \neq 0$. It turns out that if $\delta = 0$, GIG is identical to a Gamma distribution:

$$p_{\mathrm{GIG}}(x \mid \lambda, \delta = 0, \gamma) = \frac{(\alpha)^\lambda}{\Gamma(\lambda)} x^{\lambda-1} \exp\{-\alpha x\}; \ x > 0, \alpha = \gamma^2/2.$$

We now present a simple proof for the Pólya–Gamma mixture in (7). First, write κ for $a - b/2$:

$$\frac{(e^\psi)^a}{(1 + e^\psi)^b} = 2^{-b} e^{\kappa\omega} \int_0^\infty e^{-\omega\psi^2/2} p(\omega) d\omega, \tag{8}$$

where $\omega \sim PG(b, 0)$, the Pólya–Gamma distribution with density given by:

$$p(\omega \mid b, 0) = \frac{2^{(b-1)}}{\Gamma(b)} \sum_{n=0}^\infty (-1)^n \frac{\Gamma(n + b)}{\Gamma(n + 1)} \frac{2n + b}{\sqrt{(2\pi)\omega^{3/2}}} \exp\left\{-\frac{(2n + b)^2}{8\omega}\right\}.$$

The logit function corresponds to $a = 0, b = 1$ in (8). Cauchy-Schlömilch yields:

$$\frac{1}{1 + e^\psi} = \frac{1}{2} e^{-\frac{1}{2}\psi} \int_0^\infty e^{-\frac{\psi^2}{2}\omega} p(\omega) d\omega, \ \text{where } p(\omega) = \sum_{n=0}^\infty (-1)^n \frac{2n + 1}{\sqrt{(2\pi)\omega^{3/2}}} e^{-\frac{(2n+1)^2}{8\omega}}. \tag{9}$$

To show (9), write the left-hand side interchanging integral and summation:

$$I = \frac{1}{2} e^{-\psi/2} \sum_{n=0}^\infty (-1)^n \frac{2n + 1}{\sqrt{(2\pi)}} \int_0^\infty e^{-\left(\frac{\psi^2}{2}\omega + \frac{(2n+1)^2}{8\omega}\right)} \frac{1}{\omega^{3/2}} d\omega.$$

Using the change of variable $\omega = t^{-2}$ gives

$$I = \sum_{n=0}^\infty (-1)^n \left[e^{-(n+1)\psi} \frac{(2n + 1)}{\sqrt{(2\pi)}} \left\{\int_0^\infty e^{-\frac{1}{2}\left(\frac{(2n+1)t}{2} - \frac{\psi}{t}\right)^2} dt\right\}\right].$$

Applying Cauchy-Schlömilch to the inner integral yields:

$$\int_0^\infty e^{-\frac{1}{2}\left(\frac{(2n+1)t}{2}-\frac{\psi}{t}\right)^2}dt = \int_0^\infty e^{-\frac{1}{2}y^2}dy\frac{1}{2n+1} = \frac{\sqrt{(2\pi)}}{2n+1},$$

which implies $I = \sum_{n=0}^\infty (-1)^n \exp\{-(n+1)\psi\} = \{1+\exp(\psi)\}^{-1}$.

Remark 2. When $\alpha = \kappa$, we have the limiting result $(\alpha^2 - \kappa^2)^{-1}p_{\mathrm{GIG}}\{1, 0, \sqrt{(\alpha^2-\kappa^2)}\} \equiv 1$, or equivalently in terms of densities, with a marginal improper uniform prior $p(\lambda) = 1$,

$$\int_0^\infty \phi(b \mid -a\lambda, c\lambda)d\lambda = a^{-1}\exp(-2\max(ab/c, 0)). \tag{10}$$

This pseudo-likelihood represents support vector machines as a global-local mixture.

Polson & Scott (2011) derive this as a direct consequence of the Lasso identity $\int_0^\infty p/(2\pi\lambda)^{\frac{1}{2}} \exp\{-\frac{1}{2}(p^2\lambda + q^2\lambda^{-1})\}d\lambda = \exp(-|pq|)$. We apply Liouville formula:

$$\int_0^\infty f\left(ax + \frac{b}{x}\right)\frac{dx}{\sqrt{x}} = \frac{1}{\sqrt{a}}\int_0^\infty f\left(2\sqrt{ab}+y\right)\frac{dy}{\sqrt{y}}, a, b > 0.$$

Setting $f(x) = e^{-x}, a = p^2/2, b = q^2/2$, we get :

$$\int_0^\infty \lambda^{-\frac{1}{2}}e^{-\frac{1}{2}(p^2\lambda+q^2\lambda^{-1})}d\lambda = \frac{2^{\frac{1}{2}}}{p}\int_0^\infty e^{-|pq|+y}y^{-\frac{1}{2}}dy = \frac{2^{\frac{1}{2}}e^{-|pq|}}{p}\int_0^\infty e^{-y}y^{-\frac{1}{2}}dy = \frac{(2\pi)^{\frac{1}{2}}e^{-|pq|}}{p}.$$

Hans (2011) shows elastic-net regression can be recast as aglobal-local mixture where the mixing density belongs to a 'orthant-normal' family of distributions. The orthant-normal prior on a single regression coefficient β given hyper-parameters λ_1 and λ_2 is given by:

$$p(\beta \mid \lambda_1, \lambda_2) = \begin{cases} \phi(\beta \mid \frac{\lambda_1}{2\lambda_2}, \frac{\sigma^2}{\lambda_2})/2\Phi(\frac{-\lambda_1}{2\sigma\sqrt{\lambda_2}}) & \text{if } \beta < 0 \\ \phi(\beta \mid \frac{-\lambda_1}{2\lambda_2}, \frac{\sigma^2}{\lambda_2})/2\Phi(\frac{-\lambda_1}{2\sigma\sqrt{\lambda_2}}) & \text{if } \beta \geq 0 \end{cases}$$

3. Convolution mixtures

Another interesting area of application is convolution mixtures and marginal densities for location-scale mixture problems. We show that the Cauchy convolution (Pillai & Meng, 2015) and Inverse-Gamma convolution can be derived similarly (Gelman et al., 2006; Polson & Scott, 2012; Bhadra et al., 2015a). In a recent article, Bhadra et al. (2015b) shows that the regularly varying tails of half-Cauchy priors are also key to learning many-to-one functions of normal vector mean, where the flat prior gives poorly calibrated inference.

LEMMA 1. *Let $X_i \sim \mathrm{Cauchy}(0, 1)$ for $i = 1, 2$, $Z \doteq w_1 X_1 + w_2 X_2 \sim C(0, w_1 + w_2)$.*

LEMMA 2. *Let $X_i \sim \mathrm{IG}(\alpha t_i, \alpha t_i^2)$ for $i = 1, 2$, $Z \doteq X_1 + X_2 \sim \mathrm{IG}(\alpha(t_1 + t_2), \alpha(t_1^2 + t_2^2)$, where $\mathrm{IG}(\alpha t, \alpha t^2)$ denotes the inverse-Gaussian density given by:*

$$f(t) = \frac{t\sqrt{\alpha}e^t}{\sqrt{(2\pi)}x^{3/2}}\exp\left(-\frac{\alpha t^2}{2x} - \frac{x}{2\alpha}\right)1_{(0,\infty)}(x)$$

Both of these results will follow from a straight-forward application of the Cauchy-Schlömilch transformation. We show the proof for the Cauchy convolution identity below:

Proof. Exploiting the symmetry and the Lagrange identity $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$, leads us to the the convolution density as:

$$f_Z(z) = 2 \int_0^\infty \frac{1}{\pi w_1 \{(1 + x^2/w_1^2)\}} \frac{1}{\pi w_2 \{1 + (z-x)^2/w_2^2\}} dx,$$

$$= \frac{2}{\pi^2 w_1 w_2} \int_0^\infty \frac{1}{\{1 + w_1^{-1} w_2^{-1} x(z-x)\}^2 + \{w_2^{-1} z - (w_1^{-1} + w_2^{-1})x\}^2} dx.$$

Now transform $x \mapsto x + w_2^{-1} z (w_1^{-1} + w_2^{-1})^{-1}$, with $a = 1 + z^2 (w_1 + w_2)^{-2}$, $b = (w_1 w_2)^{-1}$, $c = z(w_2 - w_1)\{(w_1 + w_2)w_1 w_2\}^{-1}$, $d = z(w_2 - w_1)\{(w_1 + w_2)w_1 w_2\}^{-1}$:

$$f_Z(z) = \frac{2}{\pi^2 w_1 w_2} \int_0^\infty \left[\left\{ 1 + \frac{z^2}{(w_1 + w_2)^2} - \frac{x^2}{w_1 w_2} + xz \frac{w_2 - w_1}{(w_1 + w_2)w_1 w_2} \right\}^2 + x^2 \left(\frac{w_1 + w_2}{w_1 w_2} \right)^2 \right]^{-1} dx$$

$$= \frac{2}{\pi^2 w_1 w_2} \int_0^\infty \frac{dx}{(a - bx^2 + cx)^2 + x^2 d^2} = \frac{2}{\pi^2 w_1 w_2} \int_0^\infty \frac{dx/x^2}{(a/x - bx + c)^2 + d^2}.$$

Transforming $y = x^{-1}$, and applying Cauchy-Schlömilch transformation we arrive at:

$$f_Z(z) = \frac{2}{\pi w_1 w_2} \int_0^\infty \frac{dy}{2a(y^2 + d^2)} = \frac{1}{\pi w_1 w_2} \frac{1}{ad} = \frac{1}{\pi (w_1 + w_2)} \frac{1}{1 + z^2/(w_1 + w_2)^2}.$$

A simple induction argument proves that the sum of any number of independent Cauchy random variates is also another Cauchy.   □

One can also use the characteristic function for $X \sim C(\mu, \sigma)$, $\psi_X(t) = \exp(it\mu - |t|\sigma^2)$ and the relation $\psi_{X+Y}(t) = \psi_X(t)\psi_Y(t)$ to derive the result in just one step. For $X = \sum_{i=1}^p \omega_i C_i$ where $C_i \sim C(0, 1)$ and $\sum_{i=1}^p \omega_i = 1$, we have $\phi_X(t) = \exp(-\sum_{i=1}^p \omega_i |t|) = \exp(-|t|) = \phi_C(t)$, where $C \sim C(0, 1)$. The most general result in this category is due to Pillai & Meng (2015), where they showed the following: Let $(X_1, \cdots, X_m)$ and $(Y_1, \cdots, Y_m)$ be i.i.d. $\mathcal{N}(0, \Sigma)$ for an arbitray positive definite matrix Σ, then $Z = \sum_{j=1}^m w_j X_j (Y_j)^{-1} \sim \text{Cauchy}(0, 1)$, as long as $(w_1, \cdots, w_m)$ is independent of (X, Y), $w_j \geq 0$, $j = 1, \cdots, m$ and $\sum_{j=1}^m w_j = 1$.

4. DISCUSSION

The Cauchy-Schlömilch and Liouville transformations not only guarantee an 'astonishingly simple' normalizing constant for $f(\cdot)$, it also establishes the wide class of unimodal densities as global-local scale mixtures. Global-local scale mixtures with conditional Gaussianity hold a special place in statstical modeling and can be rapidly fit using an expectation-maximization algorithm as pointed out by Polson & Scott (2013). Palmer, Kreutz-Delgado and Makeig (2011) provides a similar tool for modeling multivariate dependence by writing general non-Gaussian multivariate densities as multivariate Gaussian scale mixtures. Our future goal is to extend the Cauchy-Schlömilch transformation to express the wide multivariate Gaussian scale mixture models as global-local mixtures that also facilitate easy computation.

We end our paper with conjectures that two other remarkable identities arise as corollaries of such transformation identities. The first one is a recent result by Zhang et al. (2014) that proves the uniform correlation mixture of the bivariate Gaussian density with unit variance is a function of the maximum norm:

$$\int_{-1}^1 \frac{1}{2} \frac{1}{2\pi\sqrt{1 - \rho^2}} \exp\left\{ -\frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{2(1 - \rho^2)} \right\} d\rho = \frac{1}{2}(1 - \Phi(\|x\|_\infty)), \qquad (11)$$

where $\Phi(\cdot)$ is the standard normal distribution function and $||x||_\infty = \max\{x_1, x_2\}$. The bivariate density on the right side of (11) was introduced before in Bryson & Johnson (1982) as uniform mixtures of a χ_3 random variate, but the representation as a uniform correlation mixture is a surprising new find. We make a couple of remarks connected to the Erdelyi's integral identity, which is key to the proof of the uniform correlation mixture (11):

$$\int_{\frac{1}{2}}^\infty \frac{e^{-x^2 z}}{4\pi z \sqrt{2z-1}} dz = \frac{1}{2}(1 - \Phi(x)); x \geq 0 \qquad (12)$$

Erdelyi's identity (12) follows from the Laplace transformation $(1+u)^{-1} = \int_0^\infty e^{-v(1+u)} dv$.

Proof. Let $u = 2z - 1$ on the left hand side of (12), denoted by I, to obtain:

$$I = \int_0^\infty \frac{e^{-x^2/2(1+u)}}{4\pi\sqrt{u}(1+u)} du.$$

Using the Laplace transformation $(1+u)^{-1} = \int_0^\infty e^{-v(1+u)} dv$:

$$I = \int_0^\infty \frac{e^{-x^2/2(1+u)}}{4\pi\sqrt{u}} \int_0^\infty e^{-v(1+u)} dv\,du = \int_{v=0}^\infty \int_{u=0}^\infty \frac{e^{-(\frac{x^2}{2}+v)(1+u)}}{4\pi\sqrt{u}} dv\,du$$

$$= \int_{v=0}^\infty \frac{1}{4\pi} e^{-(\frac{x^2}{2}+v)} \int_{u=0}^\infty u^{-1/2} e^{-(\frac{x^2}{2}+v)u} du\,dv = \int_{v=0}^\infty \frac{e^{-\frac{1}{2}(x^2+2v)}}{2(2\pi)^{1/2}} \frac{1}{(x^2+2v)^{1/2}} dv$$

Transforming $z^2 = x^2 + 2v$, we get:

$$I = \frac{1}{2} \int_{z=|x|}^\infty \frac{1}{(2\pi)^{1/2}} e^{-\frac{1}{2}z^2} dz = \frac{1}{2}(1 - \Phi(|x|)).$$

Secondly, we note that the Erdelyi's identity (1) is same as the identity (9.2) in (Amdeberhan et al., 2010) given below:

$$\int_0^\infty \frac{e^{-\mu^2(x^2+\beta^2)}}{x^2+\beta^2} dx = \frac{\pi}{2\beta}\{1 - \text{erf}(\mu\beta)\}.$$

Using $\beta = 2^{-1/2}$, $x^2 + 1/2 = z$, the above identity reduces to (12) in terms of μ.

The second candidate is the symmetric stable distribution, defined by its characteristic function $\phi(t) = \exp\{-|t|^\alpha\}; \alpha \in (0, 2]$, also admits a normal scale mixture representation as where the mixing density is:

$$f(v) = \frac{1}{2} s_{\alpha/2}\left(\frac{v}{2}\right), v > 0$$

$$\exp(-|x|^\alpha) = \int_0^\infty \exp(-xs)g(s)ds, \ g(s) = \sum_{j=1}^\infty (-1)^j \frac{s^{-j\alpha-1}}{j!\Gamma(-\alpha j)}.$$

where $s_{\alpha/2}$ is the density of the positive stable distribution with index $\alpha/2$. An important application of this is the Bayesian bridge regression by Polson et al. (2014), where the regularization is an outcome of a scale mixture of normal distributions with respect to an α-stable random variable. We conjecture that these two results follow as upshots of the Cauchy-Schlömilch formula (1). Other fruitful areas could be unearthed by applications of the Liouville formula to recog-

nize and generate global-local mixtures, and other applications such as calculating higher-order 'closed-form' moments $E(X^n)$ for random variables X that admit a global-local representation.

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