Spring Lecture Series Workshop
April 18-20, 2019

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Outline for Today's talk

AR(1) models

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- AR(1) models are of major interest by themselves as simple models for many situations but also as building blocks for more complex time series.

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- For $n = 1$ - each x_t has the same distribution.
- For $n = 2$ - each pair (x_t, x_s) has the same distribution.
- Assume $\mathbb{E}(x_t) = m, \mathbf{V}(x_t) = s$, then for all t, $x_t \sim \mathcal{N}(m, s)$.

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- **This can only happen if $|\phi| < 1$: a characteristic of stationary AR(1) process.**

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- Of course, linear processes can be non-Gaussian as well

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- Using the identity: $p(x_t) = \int p(x_t|x_{t-1})\, p(x_{t-1})\, dx_{t-1}$, we can also prove 'time-reversibility':

$$(x_t \mid x_{t-1}) \sim \mathcal{N}(\phi x_{t-1}, \nu), x_{t-1} \sim \mathcal{N}(0, s) \Rightarrow (x_{t-1} \mid x_t) \sim \mathcal{N}(\phi x_t, \nu)$$

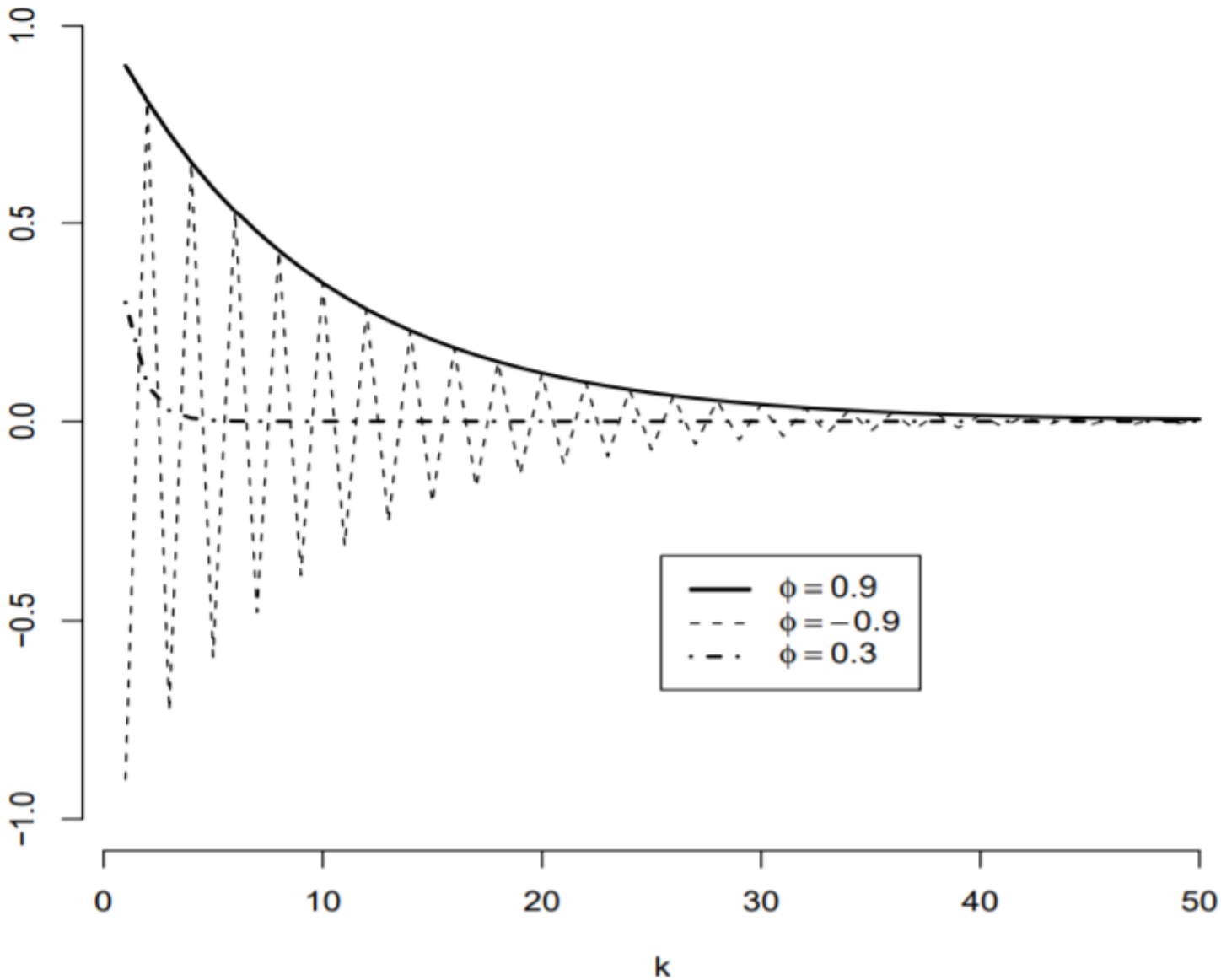

Figure: Auto-correlation functions for AR processes with parameters 0.9, -0.9 and 0.3.

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- Fit the Bayesian model to a real data set. Explore posterior histograms, means, etc.

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- Simplest Hidden Markov Model.
- Stochastic Volatility Model.
- I'll introduce the basic notions and also show the Stan code for fitting a Stochastic Volatility Model.

Simple HMM

- Observed value is now y_t:

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- Signal-to-noise ratio: $s/(s+w)$.

SVM I

- Stochastic Volatility Models (popular tool in quantitative finance).

SVM II

- Take $z_t = \log(y_t^2)/2 = \log(|y_t|)$.

$$z_t = \mu + x_t + \nu_t, \quad \text{where } \nu_t = \log(\kappa_t)/2 \tag{1}$$

$$x_t \sim AR(1 \mid \theta)\ \theta = (\phi, \nu) \tag{2}$$

- Observed z_t = intercept (defining the baseline volatility on the log scale) + a latent AR(1) process x_t (time-correlated changes in volatility)
- like HMM, but the noise is non-Gaussian - log of a χ_1^2.

Synthetic Examples

http://dattahub.github.io/sls-2019/time-series-demo.html

Outline for Today's talk

Part II: Introduction to Modern Bayesian Computing Tools

R-Stan for Time Series Analysis

Bayesian Computation

- For a prior $\pi(\theta)$, and likelihood $L(Y^{(n)} \mid \theta)$, the posterior:

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  2. Use approximating class $q(\boldsymbol{\theta})$, e.g. exponential family, and minimize discrepancy: variational Bayes, Expectation-Propagation etc.

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- Approach 2: MCMC: sequential algorithm to obtain correlated draws from the posterior.

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- How to calculate $\mathbb{E}_{\pi_n}(h(X))$ for arbitrary $h(X)$[1] when π_n is only available up to constants?

[1] measurable

Monte Carlo Integration

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- Goal: Evaluate the expectation (where π is a density):

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- But, **in practice**, independent samples from π might be difficult.
- Or we might not be able to directly sample from π.

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- In the pioneering paper, Metropolis et al. (1953) allowed for serial dependence of the $\mathbf{x}_j^*$ by combining von Neumann's idea of rejection sampling (published in 1951), with concepts from a subject in the theory of stochastic processes called Markov Chains.

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- Thus it is necessary to relax the condition $x^{(t)} \overset{\text{iid}}{\sim} \pi(x)$
- Instead consider samples $x_1^*, x_2^*, \ldots, x_m^*$ that form a time series: a series of draws from $\pi(\mathbf{x})$ in which $\mathbf{x}_j^*$ may depend on $\mathbf{x}_{j'}^*$ for $j' < j$.
- In the pioneering paper, Metropolis et al. (1953) allowed for serial dependence of the $\mathbf{x}_j^*$ by combining von Neumann's idea of rejection sampling (published in 1951), with concepts from a subject in the theory of stochastic processes called Markov Chains.
- Combining Monte Carlo sampling with Markov Chains give rise to the name now used for this technique: **Markov Chain Monte Carlo**.

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- It turns out that (4) still applies when the samples are generated from a **stationary Markov chain** !
- We construct an (ergodic) Markov chain with transition kernel Π chosen to have the same stationary distribution as π. Then, samples from this Markov chain are samples from π if either:
  - We initialize the chain with a draw from π;
  - We run the chain long enough (infinitely long!) so that it converges to π.

  The first is, again, impossible. Let's look more closely at the second.
- What is a Markov Chain?

Markov Chains

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- Intuitively speaking, a Markov chain is a stochastic process unfolding in time in such a way that **the past and future states of the process are independent given the present state**.
- More formally, a stochastic process $\{\theta_t^*, t \in \mathrm{T}\}$, $\mathrm{T} = \{0, 1, \ldots\}$ with state space S is a Markov Chain if, for any set $A \in S$,

$$P(\theta_{t+1}^* \in A \mid \theta_0^*, \theta_1^*, \ldots, \theta_t^*) = P(\theta_{t+1}^* \in A \mid \theta_t^*)$$

Example

- Discrete-time Markov chain with continuous-state space:

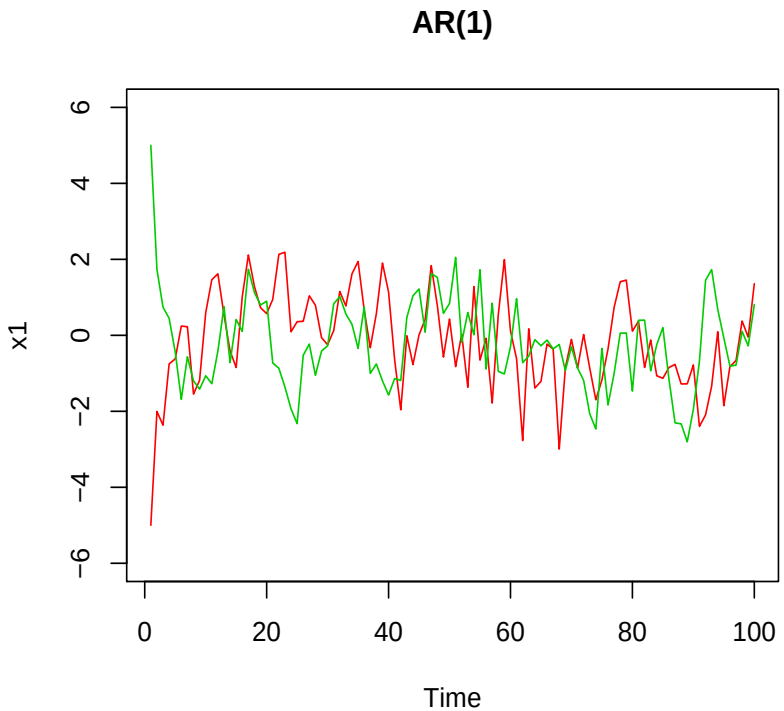
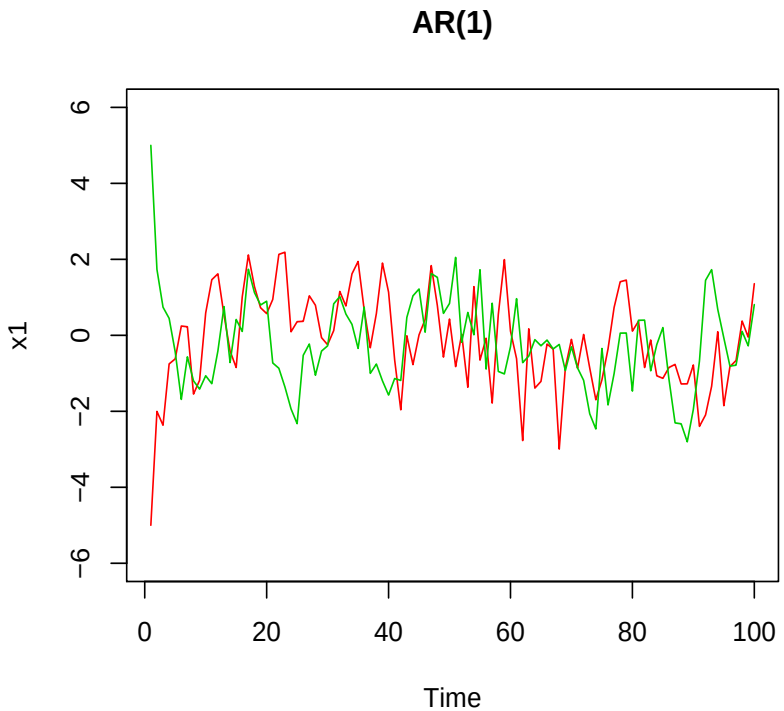


Figure: $\mathrm{AR}(1)$ with different starting points After $5-7$ iterations the chains seemed to have forgotten their starting positions.

Effective Sample Size

Example

- Discrete-time Markov chain with continuous-state space:
- $\theta^*_{t+1} \sim \mathcal{N}(0.5 \times \theta^*_t, 1.0)$ ($\mathrm{AR}(1)$ with lag-1 auto-correlation 0.5)



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- Does this happen for all Markov chain?

Nice Behaviour

Yes, if they satisfy three key properties:

- **Irreducibility**
- **Aperiodicity**
- **Positive Recurrence**

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- **Irreducibility**
- *No matter where it starts, the chain has to reach any other state in a finite number of iterations with positive probability.*
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Yes, if they satisfy three key properties:

- **Irreducibility**
- **Aperiodicity**
- For all states i, the set of all possible **sojourn times**, time to get back to i, can have no divisor bigger than 1. [This is a technical condition, periodic chains are also nice, but aperiodic chains are nicer !]
- **Positive Recurrence**

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Nice Behaviour

Yes, if they satisfy three key properties:

- **Irreducibility**
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- **Positive Recurrence**
- The 'nicest' Markov chains have all three properties.

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Ergodicity

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- Also, if $\sigma_h^2 = \mathbf{V}_\pi[h(\Theta)] < \infty$, Central Limit Theorem holds and convergence occurs geometrically !

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- The Ergodic theorem is silent on these issues: **how long you have to wait for stationarity** and **how long to monitor after that?**
- A third issue is what to use for the initial value θ_0? Intuitively, the closer θ_0 is to the center of your target $p(\theta)$ the less time you have to wait for stationarity.

▸ Effective Sample Size

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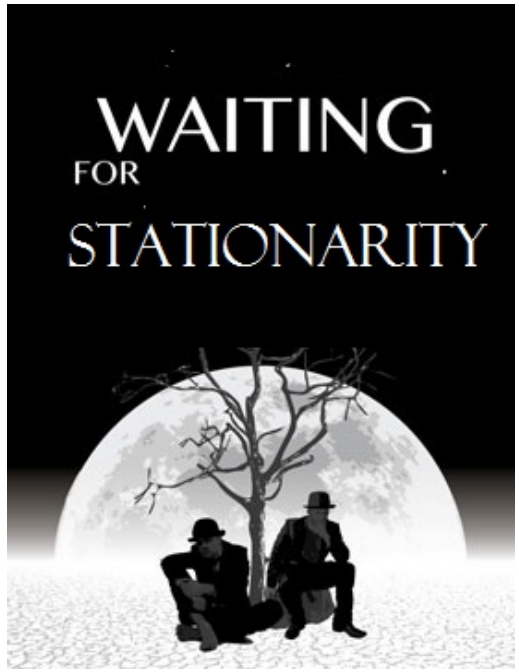

Figure: The Key Question: How long should MCMC run?

Numerical Standard Errors

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- Many references: Geyer (1992), Besag and Green (1993) for ideas.

Charlie Geyer's Advice

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- Second rule of MCMC: variance estimation is not "diagnostic". If chain doesn't converge, then variance estimation is GIGO (garbage in, garbage out).

Multiple Chains: Gelman-Rubin's $\hat{R}$ [2]

- Most approaches for detecting convergence, both formal and informal, rest on the idea of starting multiple Markov chains and observing whether they come together and start to behave similarly (if they do, we can pool the results from each chain).

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- Most approaches for detecting convergence, both formal and informal, rest on the idea of starting multiple Markov chains and observing whether they come together and start to behave similarly (if they do, we can pool the results from each chain).
- It is typically recommended (e.g., Gelman and Rubin, 1992) to use overdispersed initial values, meaning "more variable than the target distribution" i.e., the posterior.

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- The basic idea is to quantify the between-chain and the within-chain variability of a quantity of interest fi the chains have converged, these measures will be similar; otherwise, the between-chain variability will be larger.

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- As $T \to \infty$, $\hat{R} \to 1$; $\hat{R} < 1.05$ is widely accepted as implying convergence for practical purposes.

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Auto-correlation

Auto-correlation: $\rho_l(h) = \text{Correlation}\left(h(\theta^{(t)}), h(\theta^{(t+l)})\right)$.

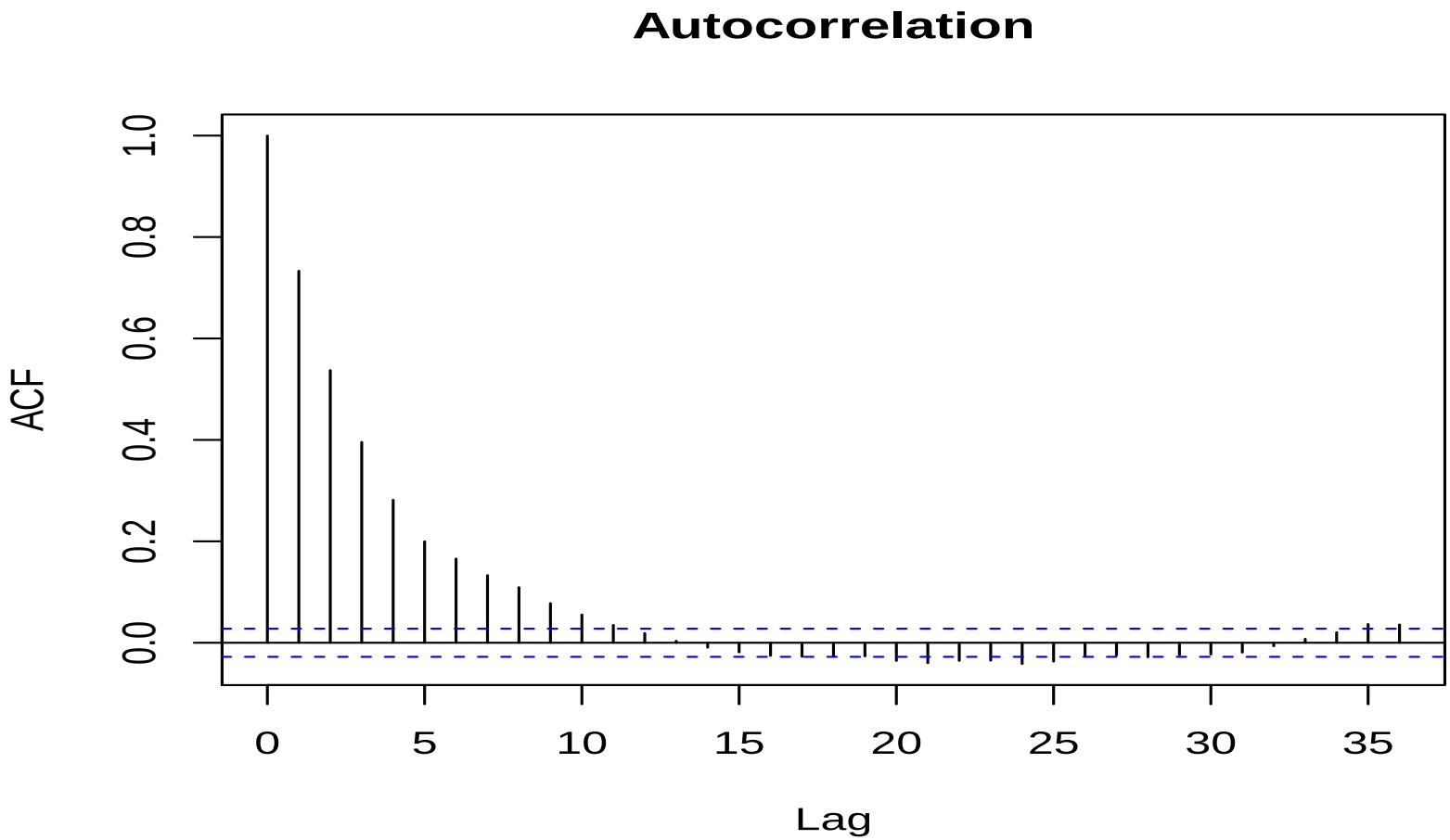

Figure: ACF for the Normal-Cauchy Example.

Numerical Standard Errors: Price of Markov Chain

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- If the nse is finite, then we can make it as small as we like by increasing N. (Long chain!)

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- Ergodic Theorem:

$$\bar{h}_N = \frac{1}{N}\sum_{i=1}^{N} h(\theta^{(t)}) \to \mathbb{E}_\pi(h(\theta)) = \int h(\theta)\pi(\theta)d\theta, \text{ as } N \to \infty$$

- Numerical standard error of $\bar{h}_N$ as $\sqrt{\mathbf{V}_\pi(\bar{h}_N)}$, and for large N:

$$\text{NSE}(\bar{h}_N) \approx \sqrt{\frac{\sigma_h^2}{N}\left\{1 + 2\sum_{l=1}^{N-1}\rho_l(h)\right\}} \doteq \sqrt{\frac{\sigma_h^2}{N_{ess}}}$$

  where $\rho_l(h)$ is the lag-l auto-correlation in $h(\theta^{(t)})$.

- Effective sample size:

$$N_{ess} = \frac{N}{\left\{1 + 2\sum_{l=1}^{N-1}\rho_l(h)\right\}}$$

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- Very roughly: Metropolis-Hastings methods move around the probability space randomly (without knowledge of the underlying geometry) and use a accept-reject step to adjust probabilities accordingly.
- Hamiltonian Monte Carlo gives a particle a random "kick" and samples based on the path of the particle: uses Hamiltonian mechanics to simulate the path of the particle in an energy field induced by the target density π.

Outline for Today's talk

Probabilistic Programming Language

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- Unlike BUGS and JAGS, not restricted to Gibbs sampling or conjugate (exponential family graphical) models.
- Stan is open source
- Built to be fast (about 10 times faster then BUGS according Gelman)
  Named after Stanislaw Ulam, co-inventor of Monte Carlo method.

Stan does ..

- Full Bayesian Inference (via Hamiltonian Monte Carlo)

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- Variational Bayesian Inference (via ADVI)
- Penalized MLE (Bayesian MAP)
- Best thought of as a DSL for specifying a distribution and sampling from it.

Stan: Building Blocks

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- Matrix types: `cov_matrix`, `corr_matrix`, `cholesky_factor_cov`, `cholesky_factor_corr`

Building Blocks

A Stan model is defined by the following five programming blocks:

```
data

transformed data

parameters \\(required)

transformed parameters

model \\(required)

generated quantities
```

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- Additional blocks, e.g., `transformed data`, `generated quantities` are useful for performing additional transformations within Stan.

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- `generated quantities` allows for post-processing the posterior samples.

Outline for Today's talk

First Example

Mean only, Normal 'shocks' $y_t \sim \mathcal{N}(\theta, \sigma^2)$, $t = 1, \ldots, T$.

```
data {
    int<lower=1> T; //error checking for T
    real y[T];
}
parameters {
    real theta;
    real<lower=0> sigma;
}
model {
    y ~ normal(theta, sigma); // vectorized
}
```

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- The R code "fitting-normal-mean-only-stan.R" will read the data-set and fit the Stan model in "normal-mean-only.stan".

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- The stan code "mean-only_normal_predict_new_generated_quantities.stan" is an example of using the "generated quantities" block.

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- Let us look at this `stan` code carefully to understand the basic functional features.

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Now we'll see a couple of advanced models:

- Auto-regressive model (stochastic unit roots)

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- Stochastic volatility models.

Stan: STUR model

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$$y_t = \rho_t y_{t-1} + \varepsilon_t \quad \varepsilon_t \sim \mathcal{N}\left(0, \sigma_1^2\right); \quad |\rho_t| < 1 \Rightarrow \quad \text{stationary}$$

$$\rho_t = e^{\omega_t}$$

$$\omega_t = \varphi_0 + \varphi_1 \omega_{t-1} + \eta_t \quad \eta_t \sim \mathcal{N}\left(0, \sigma_2^2\right)$$

$$= \mu_\omega + \varphi_1 \left(\omega_{t-1} - \mu_\omega\right) + \eta_t \text{ where } \mu_\omega = \frac{\varphi_0}{1 - \varphi_1}$$

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- Stationarity can be re-written in terms of μ_ω, i.e.

$$\begin{aligned} \mu_\omega < 0 \quad &\Leftrightarrow \mathbb{E}\left(\rho_t\right) < 1 \\ \mu_\omega = 0 \quad &\Leftrightarrow \mathbb{E}\left(\rho_t\right) = 1 \\ \mu_\omega > 0 \quad &\Leftrightarrow \mathbb{E}\left(\rho_t\right) > 1 \end{aligned}$$

Stan demo

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  1. "stur.stan".
  2. "stochvol.stan"
- The R code for fitting the two models on the same data-set is "analyse-us-income-stur-stochvol.R"

STUR

```
 data {
    int<lower=1> T;
    vector[T]    y;
  }

parameters {
    real sigma_1;
    real sigma_2;

    vector[T] omega;
    real mu_omega;
    real phi;
}
```

STUR

```
transformed parameters {
    vector[T] rho;
for (t in 1:T) {
    rho[t] = exp(omega[t]);
}
}
```

STUR

```
model {
//  priors
    mu_omega ~ normal(0, 1);
    phi ~ normal(0,1);
    sigma_1 ~ inv_gamma(0.001, 0.001);
    sigma_2 ~ inv_gamma(0.001, 0.001);

// initial conditions as latent data: diffuse prior
    omega[1] ~ normal(0, 1);

for (t in 2:T) {
    y[t] ~ normal(rho[t]*y[t-1], sigma_1);
    omega[t] ~ normal(mu_omega + phi*(omega[t-1]-mu_omega), sigma_2);
}
}
```

Let's look at this code in Stan

Stochastic Volatility

- Stochastic volatility model: latent mean-reverting AR(1) process:

$$r_t \sim \mathcal{N}\left(0, \sigma_t^2\right)$$
$$\sigma_t^2 = \exp\left\{h_t\right\}$$
$$h_{t+1} \sim \mathcal{N}(\mu + \phi(h_t - \mu), \sigma^2)$$

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- Let's see how to code this efficiently in `Stan`.
- This is also available on R package `stochvol`.

Stochastic Volatility

```
data {
  int<lower=0> T;
  vector[T] y;
}
parameters {
  real mu;
  real<lower=-1, upper=1> phi; // Stationary volatility
  real<lower=0> sigma;
  vector[T] h_std;
}
```

Stochastic Volatility

We can generate $\mathcal{N}(0,1)$ and apply location-scale transformations:

$$h_{t+1} \sim \mathcal{N}\left(\mu + \phi\left(h_t - \mu\right), \sigma^2\right).$$

```
transformed parameters {
  vector[T] h;
  h = h_std * sigma;
  h[1] = h[1] / sqrt(1 - phi * phi);
  h = h + mu;
  for(t in 2:T){
    h[t] = h[t] + phi * (h[t-1] - mu);
  }
}
```

Stochastic Volatility

```
model {
  // Priors
  phi ~ uniform(-1, 1);
  sigma ~ cauchy(0, 5);
  mu ~ cauchy(0, 10);
  // Scaled Innovations in h process are IID N(0,1)
  h_std ~ normal(0, 1);
  // Observation likelihood.
  // Note exp(h/2) since Stan uses normal(mean, SD)
  y ~ normal(0, exp(h/2));
}
```

Hands-on activity