HW6

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习题 2.1

4

(1)

一致收敛

证明. $\exists t > 1$,s.t.$\forall x > t, x^s \le x^b \le e^{\frac{1}{2}x}$

则 $\int_1^{+\infty} x^s e^{-x}\mathrm{d}x = \int_1^t x^s e^{-x}\mathrm{d}x + \int_t^{+\infty} x^s e^{-x}\mathrm{d}x \le$

$\int_1^t x^b e^{-x}\mathrm{d}x + \int_t^{+\infty} e^{-\frac{1}{2}x}\mathrm{d}x = c + \int_t^{+\infty} e^{-\frac{1}{2}x}\mathrm{d}x$($c$ 为常数)

易知 $\int_t^{+\infty} e^{-\frac{1}{2}x}\mathrm{d}x$ 收敛，故有 $\int_1^{+\infty} x^s e^{-x}\mathrm{d}x(a \le y \le b)$ 一致收敛 □

(2)

一致收敛

证明. 令 $f(x,y) = \cos(yx), g(x,y) = \frac{1}{1+x^2}$

显然有 $\lim\limits_{x\to\infty} g(x,y) = 0, \forall y \in \mathbb{R}$

而又有 $|f(x,y)| \le 2, \forall x \in \mathbb{R}, \forall y \in \mathbb{R}$，即 $f(x,y)$ 关于 $y \in \mathbb{R}$ 一致有界

由 Dirichlet 判别法知 $\int_{-\infty}^{+\infty} \frac{\cos(yx)}{1+x^2}(-\infty \le y \le +\infty)$ 一致收敛 □

(4)

一致收敛

证明. 令 $f(x,t)=e^{-tx}, g(x,t)=\sin x$

$|e^{-tx}|\le e^{-t_0x}$ 且 $\int_0^{+\infty} e^{-t_0x}\mathrm{d}x$ 收敛

故知 $\int_0^{+\infty} e^{-tx}$ 一致收敛

又有 $|g(x,t)|\le 2$，一致有界

由 Abel 判别法知 $\int_0^{+\infty} e^{-tx}\sin x\mathrm{d}x(0<t_0\le t<+\infty)$ 一致收敛 □

习题 2.2

1

(1)

$$\begin{aligned}\lim_{a\to 0}\int_{-1}^{1}\sqrt{x^2+a^2}\mathrm{d}x &= \int_{-1}^{1}\lim_{a\to 0}(\sqrt{x^2+a^2})\mathrm{d}x\\ &= \int_{-1}^{1}|x|\mathrm{d}x\\ &= 2\int_{0}^{1}x\mathrm{d}x\\ &= x^2\Big|_0^1\\ &= 1\end{aligned}$$

(2)

$$\begin{aligned}\lim_{a\to 0}\int_{0}^{3}x^2\cos ax\mathrm{d}x &= \int_{0}^{3}\lim_{a\to 0}(x^2\cos ax)\mathrm{d}x\\ &= \int_{0}^{3}x^2\mathrm{d}x\\ &= \frac{x^3}{3}\Big|_0^3\\ &= 9\end{aligned}$$

2

(1)

$$
\begin{aligned}
F'(x) &= \int_x^{x^2} \frac{\mathrm{d}(e^{-xy^2})}{\mathrm{d}x}\mathrm{d}y + e^{-x(x^2)^2}\cdot 2x - e^{-x(x^2)} \\
&= -\int_x^{x^2} y^2 e^{-xy^2}\mathrm{d}y + 2xe^{-x^5} - e^{-x^3}
\end{aligned}
$$

(2)

$\frac{\mathrm{d}}{\mathrm{d}y}\frac{\sin(yx)}{x} = \cos(yx)$
故 $F'(y) = \int_{a+y}^{b+y}\cos(yx)\mathrm{d}x + \frac{\sin y(b+y)}{b+y} - \frac{\sin y(a+y)}{a+y} =$
$\frac{1}{y}[\sin y(a+y) - \sin y(b+y)] + \frac{\sin y(b+y)}{b+y} - \frac{\sin y(a+y)}{a+y} = \frac{a\sin y(a+y)}{y(a+y)} - \frac{b\sin y(b+y)}{y(b+y)}$

(3)

$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\ln(1+tx)}{x} = \frac{1}{1+tx}$
故 $F'(t) = (t+\frac{1}{t})\ln(1+t^2)$

(4)

$F'(t) = \int_0^t (f_1' - f_2')\mathrm{d}x + f(2t, 0)$

5

(1)

令 $F(y) = \int_0^1 \frac{\arctan yx}{x}\frac{1}{\sqrt{1-x^2}}\mathrm{d}x$
则 $F'(y) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}y}(\frac{\arctan yx}{x})\frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \int_0^1 \frac{1}{1+x^2y^2}\frac{1}{\sqrt{1-x^2}}\mathrm{d}x =$
$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin^2\theta y^2}\frac{1}{\sqrt{1-\sin^2\theta}}\mathrm{d}\sin\theta = \int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{1+\sin^2\theta^2 y^2} = \int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sin^2\theta+\cos^2\theta+\sin^2\theta y^2} =$
$\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\tan\theta}{1+\tan^2\theta(y^2+1)} = \int_0^{+\infty}\frac{\mathrm{d}t}{1+t^2(y^2+1)} = \frac{\arctan[t^2(y^2+1)]}{\sqrt{y^2+1}}\Bigg|_0^{+\infty} = \frac{\pi}{2\sqrt{y^2+1}}$
故所求为
$F(1) = \int_0^1 F'(y)\mathrm{d}y = \frac{\pi}{2}\int_0^1\frac{1}{\sqrt{y^2+1}}\mathrm{d}y = \frac{\pi}{2}\ln(\sqrt{y^2+1}+y)\Bigg|_0^1 = \frac{\pi}{2}(\ln\sqrt{2}+1)$