# HW4

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## 习题 1.5

### 3

#### (1)

$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \frac{2vx-uy}{u^2+v^2} = \frac{x^2y-y^3}{x^4+y^4+3x^2y^2}$
$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = \frac{2vy-ux}{u^2+v^2} = \frac{xy^2-x^3}{x^4+y^4+3x^2y^2}$

#### (3)

令 $u = x^2 - y^2,\ v = e^{xy}$
$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 2x\frac{\partial f}{\partial u} + ye^{xy}\frac{\partial f}{\partial v}$

#### (5)

$\frac{\partial z}{\partial x} = y - \frac{y}{x^2}f(xy) + \frac{y^2}{x}f'(xy)$
$\frac{\partial z}{\partial y} = x + \frac{1}{x}f(xy) + yf'(xy)$

### 9

$J_{f\circ g} = \begin{bmatrix} \frac{y^2-x^2-2xy}{(x^2+y^2)^2} & \frac{x^2-y^2-2xy}{(x^2+y^2)^2} \\ \frac{y^3-3x^2y}{(x^2+y^2)^3} & \frac{x^3-3xy^2}{(x^2+y^2)^3} \\ \frac{-x^2y-y^3}{x^2(x^2+y^2)} & \frac{x^3+xy^2}{x^2(x^2+y^2)} \end{bmatrix}$

$$\mathrm{d}\boldsymbol{Y}=J_{f\circ g}\mathrm{d}\boldsymbol{X}=\begin{bmatrix}\frac{y^2-x^2-2xy}{(x^2+y^2)^2}\mathrm{d}x+\frac{x^2-y^2-2xy}{(x^2+y^2)^2}\mathrm{d}y\\ \frac{y^3-3x^2y}{(x^2+y^2)^3}\mathrm{d}x+\frac{x^3-3xy^2}{(x^2+y^2)^3}\mathrm{d}y\\ \frac{-x^2y-y^3}{x^2(x^2+y^2)}\mathrm{d}x+\frac{x^3+xy^2}{x^2(x^2+y^2)}\mathrm{d}y\end{bmatrix}$$

**(2)**

$$J_{f\circ g}=J_fJ_g=2\begin{bmatrix}u_1 & u_2\\ u_1 & -u_2\end{bmatrix}\begin{bmatrix}\frac{x}{x^2+y^2} & \frac{y}{x^2+y^2}\\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2}\end{bmatrix}=$$
$$2\begin{bmatrix}\frac{x\ln\sqrt{x^2+y^2}-y\arctan\frac{y}{x}}{x^2+y^2} & \frac{y\ln\sqrt{x^2+y^2}+x\arctan\frac{y}{x}}{x^2+y^2}\\ \frac{x\ln\sqrt{x^2+y^2}+y\arctan\frac{y}{x}}{x^2+y^2} & \frac{y\ln\sqrt{x^2+y^2}-x\arctan\frac{y}{x}}{x^2+y^2}\end{bmatrix}$$
$$\mathrm{d}\boldsymbol{Y}=J_{f\circ g}\mathrm{d}\boldsymbol{X}=2\begin{bmatrix}\frac{x\ln\sqrt{x^2+y^2}-y\arctan\frac{y}{x}}{x^2+y^2}\mathrm{d}x+\frac{y\ln\sqrt{x^2+y^2}+x\arctan\frac{y}{x}}{x^2+y^2}\mathrm{d}y\\ \frac{x\ln\sqrt{x^2+y^2}+y\arctan\frac{y}{x}}{x^2+y^2}\mathrm{d}x+\frac{y\ln\sqrt{x^2+y^2}-x\arctan\frac{y}{x}}{x^2+y^2}\mathrm{d}y\end{bmatrix}$$

# 习题 1.6

## 3

**(1)**

$\frac{\partial f}{\partial x}=f_1'(a-c\frac{\partial z}{\partial x})-bf_2'\frac{\partial z}{\partial x}=0$
$\frac{\partial f}{\partial y}=-cf_1'\frac{\partial z}{\partial y}+f_2'(a-b\frac{\partial z}{\partial y})=0$
故 $\frac{\partial z}{\partial x}=\frac{af_1'}{cf_1'+bf_2'}$，$\frac{\partial z}{\partial y}=\frac{af_2'}{cf_1'+bf_2'}$
故 $c\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=\frac{acf_1'+abf_2'}{cf_1'+bf_2'}=a$

**(2)**

左右同时对 $x$ 求偏导：
$\frac{\partial z}{\partial x}=f(x^2-z^2)+2x(x-z\frac{\partial z}{\partial x})f'(x^2-z^2)$
故 $\frac{\partial z}{\partial x}=\frac{f+2x^2f'}{1+2xzf'}$
左右同时对 $y$ 求偏导：
$1+\frac{\partial z}{\partial y}=-2xzf'\frac{\partial z}{\partial y}$
故 $\frac{\partial z}{\partial y}=-\frac{1}{1+2xz(x^2-z^2)f'(x^2-y^2)}$
故 $x\frac{\partial z}{\partial x}+z\frac{\partial z}{\partial y}=\frac{xf+2x^3f'-z}{1+2xzf'}=\frac{y+2x^3f'}{1+2xzf'}$

## 6

$\frac{\partial(F_1,F_2)}{\partial(y,z)} = \begin{bmatrix} 1 & 1+2z \\ 2y & 1+3z^2 \end{bmatrix}$, $\frac{D(F_1,F_2)}{D(y,z)}\bigg|_{(-1,1,0)} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \neq 0$

故 $\begin{pmatrix} y \\ z \end{pmatrix} = f(x)$ 存在

$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$

故 $\frac{\partial(y,z)}{\partial x}\bigg|_{(-1,1,0)} = -\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

故 $y'(-1) = 0,\ z'(-1) = -1$

## 9

### (1)

$J_f = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$

若 $x \neq y$, 则 $J_{f^{-1}} = J_f^{-1} = \frac{1}{2(x^2+y^2)}\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$

$|J_{f^{-1}}| = \frac{1}{4(x^2+y^2)}$

### (3)

$J_f = \begin{bmatrix} 3x^2 & -3y^2 \\ y^2 & 2xy \end{bmatrix}$

若 $y \neq 0 \wedge 2x^3 + y^2 \neq 0$, 则 $J_{f^{-1}} = J_f^{-1} = \frac{1}{3y(2x^3+y^3)}\begin{bmatrix} 2xy & 3y^2 \\ -y^2 & 3x^2 \end{bmatrix}$

$|J_{f^{-1}}| = \frac{1}{3y(2x^3+y^3)}$

### (5)

$J_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

若 $ad \neq bc$，则 $J_{f^{-1}} = J_f^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$|J_{f_{-1}}| = \frac{1}{ad-bc}$

### 10

#### (1)

$J_{g\circ f} = \frac{\partial(u,v)}{\partial(\xi,\eta)}\frac{\partial(\xi,\eta)}{\partial(x,y)} = 2\begin{bmatrix} \xi & -\eta \\ \eta & \xi \end{bmatrix}\begin{bmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{bmatrix} = \begin{bmatrix} u & -v \\ v & u \end{bmatrix}$

故当 $(x,y) = (1,0)$ 时，$|J_{g\circ f}| = \begin{vmatrix} e^2 & 0 \\ 0 & e^2 \end{vmatrix} = e^4 \neq 0$

故 $g\circ f$ 在 $(1,0)$ 处可逆

## 习题 1.7

### 1

### (1)

切平面的法向量为 $(2x,2y,-1)\Big|_{(1,2,5)} = (2,4,-1)$

故切平面方程为 $2(x-1)+4(y-2)-(z-5)=0$

法线方程为 $\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-5}{-1}$

#### (4)

切平面的法向量为 $(\frac{2x}{a^2},\frac{2y}{b^2},\frac{2x}{c^2})\Big|_{(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}})} = (\frac{2}{\sqrt{3}a},\frac{2}{\sqrt{3}b},\frac{2}{\sqrt{3}c})$

故切平面方程为 $\frac{2}{\sqrt{3}a}(x-\frac{a}{\sqrt{3}})+\frac{2}{\sqrt{3}b}(y-\frac{b}{\sqrt{3}})+\frac{2}{\sqrt{3}c}(c-\frac{c}{\sqrt{3}})=0$

法线方程为 $\sqrt{3}ax-a^2 = \sqrt{3}bx-b^2 = \sqrt{3}cx-c^2$

## 3

$x^2+2y^2+3z^2=21$ 和 $x+4y+6z=0$ 在任一点的法向量分别为 $(x,2y,3z)$ 和 $(1,4,6)$

前者的切平面平行于后者，意味着两个法向量平行，即 $\frac{x}{1}=\frac{2y}{4}=\frac{3z}{6}$

同时有 $x^2+2y^2+3z^2=21$

可解得 $(x,y,z)=(1,2,2)\vee(x,y,z)=(-1,-2,-2)$

故切平面可以表示为 $x+4y+6z=\pm21$

## 5

可求得两个曲面的法向量分别为 $(2x,2y,2z)\Big|_{(1,-2,1)}=(2,-4,2)\parallel(1,-2,1)$ 和 $(1,1,1)$

则切向量为 $(1,-2,1)\times(1,1,1)=(-3,0,3)\parallel(-1,0,1)$

故切线方程为

$$\begin{cases}x+z=2\\ y=-2\end{cases}$$

法平面为 $x=z$

## 6

证明. $\forall t\in\mathbb{R}$, 螺旋线的切向量为 $(-a\sin t,a\cos t,b)$

其与 $z$ 轴的夹角为 $\arccos\frac{(-a\sin t,a\cos t,b)\cdot(0,0,1)}{||(-a\sin t,a\cos t,b)||}=\frac{b}{\sqrt{a^2+b^2}}$，为定值

□