HW11

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习题 4.7

3

(2)

$$
\begin{aligned}
I &= \iiint\limits_{x^2+y^2\leq 1, 0\leq z\leq 1} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \iiint\limits_{x^2+y^2\leq 1, 0\leq z\leq 1} (y - z + 0 + 0)\, \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_0^1 \mathrm{d}x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \mathrm{d}y \int_0^1 (y - z)\mathrm{d}z \\
&= -\frac{\pi}{4}
\end{aligned}
$$

(3)

$$
\begin{aligned}
I &= \iiint\limits_{x\geq 0, y\geq 0, z\geq 0, x+y+z\leq 1} (0 + 0 + 3) \\
&= 3V_{tetrahedron} \\
&= \frac{1}{2}
\end{aligned}
$$

4

(2)

$$\Phi = \iiint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1} (1+1+1)\mathrm{d}x\mathrm{d}y\mathrm{d}z = 4\pi abc$$

5

(1)

$$I = \iint\limits_{D} (-1,-1,-1)\cdot\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)\mathrm{d}S = -\sqrt{3}\pi R^2$$

6

(2)

令 $v = x+z$，则原微分等价于
$\dfrac{y\mathrm{d}v - v\mathrm{d}y}{v^2+y^2} = \mathrm{d}\left(\arctan\dfrac{v}{y}\right) = \mathrm{d}\left(\arctan\dfrac{x+z}{y}\right) = \mathrm{d}u$
即 $u = \arctan\dfrac{x+z}{y} + C$

7

(1)

易知 $(y+z)\,\mathrm{d}x + (z+x)\mathrm{d}y + (x+y)\mathrm{d}z = \mathrm{d}(xy+xz+yz)$
故此积分曲线与路径无关
故 $I = (xy+xz+yz)\Big|_{(0,0,0)}^{(1,2,1)} = 5$

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(4)

证明. $\mathrm{grad}(u) = \left(\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial u}{\partial z}\right)$

由于 u 是 $\mathbb{R}^3$ 中的光滑函数
故 $\operatorname{rot}(\operatorname{grad}(u))=\left(\frac{\partial^2 u}{\partial y\partial z}-\frac{\partial^2 u}{\partial z\partial y},\frac{\partial^2 u}{\partial z\partial x}-\frac{\partial^2 u}{\partial x\partial z},\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y\partial x}\right)=\mathbf{0}$ □

(5)

证明. 设 $A=(X,Y,Z)$
$\operatorname{rot}(A)=\left(Z'_y-Y'_z,X'_z-Z'_x,Y'_x-X'_y\right)$
$\operatorname{div}(\operatorname{rot}(A))=Z''_{yx}-Y''_{zx}+X''_{zy}-Z''_{xy}+Y''_{xz}-X''_{yz}=0$ □

习题 5.1

2

证明. $S_{2n}=S_{2n+1}-u_{2n+1}$
故 $\lim\limits_{n\to\infty}S_{2n}=\lim\limits_{n\to\infty}S_{2n+1}-\lim\limits_{n\to\infty}u_{2n+1}=\lim\limits_{n\to\infty}S_{2n+1}$
而 $\{S_n\}=\{S_{2n}\}\cup\{S_{2n+1}\}$
故 $\lim\limits_{n\to\infty}S_n$ 存在，即 $\sum\limits_{n=1}^{\infty}u_n$ 收敛 □

6

(1)

$S=400\sum\limits_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$，例题已证，此级数收敛，且

$$S=400\cdot\left(\frac{1}{1-\frac{1}{4}}-1\right)=\frac{400}{3}$$

(2)

$\lim\limits_{n\to\infty}u=\lim\limits_{n\to\infty}\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}}=1$，故此级数不收敛

(3)

$$u_n=\frac{1}{4}\left(\frac{1}{2n-1}-\frac{1}{2n+3}\right),S_n=\sum_{i=1}^{n}u_n=\frac{1}{3}-\frac{1}{4}\left(\frac{1}{2n+1}+\frac{1}{2n+3}\right)$$

故 $\lim\limits_{n\to\infty} S_n = \dfrac{1}{3}$
故此级数收敛，其值为 $\dfrac{1}{3}$

(4)

$u_n = \dfrac{1}{2}\left(\dfrac{1}{n} - \dfrac{1}{n+1}\right) - \dfrac{1}{2}\left(\dfrac{1}{n+1} - \dfrac{1}{n+2}\right)$
故 $S_n = \dfrac{1}{4} - \dfrac{1}{2(n+1)(n+2)}$
故 $\lim\limits_{n\to\infty} S_n = \dfrac{1}{4}$，故此级数收敛，值为 $\dfrac{1}{4}$

(5)

$\lim\limits_{n\to\infty} \dfrac{n^3}{n^3+n} = \dfrac{1}{2}$，但 $\lim\limits_{n\to\infty}(-1)^n$ 不存在，故 $\lim\limits_{n\to\infty} \dfrac{(-1)^n n^3}{n^3+n}$，不存在
故此级数发散

(6)

$$
\begin{aligned}
u_n &= \frac{1}{\sqrt{n+2}+\sqrt{n+1}} - \frac{1}{\sqrt{n+1}+\sqrt{n}} \\
S_n &= \frac{1}{\sqrt{n+2}+\sqrt{n+1}} - \frac{1}{\sqrt{2}+1} \\
\lim_{n\to\infty} S_n &= 1-\sqrt{2}
\end{aligned}
$$

故此级数收敛，值为 $1-\sqrt{2}$

习题 5.2

(1)

$\lim\limits_{x\to+\infty} \dfrac{x^2}{(2x-1)2^{x-1}} = \lim\limits_{x\to+\infty} \dfrac{x}{2^x} = 0$，故 $\lim\limits_{n\to\infty} \dfrac{n^2}{(2n-1)2^{n-1}} = 0$
由比阶判别法知此级数收敛

(3)

$$\lim_{n\to+\infty}\frac{n}{\ln n}=+\infty$$，由比阶判别法知此级数发散

(5)

$$\lim_{n\to\infty}n^2\left(\frac{1+n^2}{1+n^3}\right)^2=1$$，由比阶判别法知此级数收敛

(7)

$$\lim_{n\to\infty}n^{\frac{3}{2}}\cdot\frac{1}{\sqrt{n}}\ln\left(\frac{n+1}{n-1}\right)=\lim_{n\to\infty}n\ln\left(1+\frac{2}{n-1}\right)=\lim_{n\to\infty}\frac{2n}{n-1}=2$$

故由比阶判别法知此级数收敛

2

(1)

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{1}{n(2n+1)}=0$$

由比率判别法知此级数收敛

(3)

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{3}{n(1+\frac{1}{n})^n}=0$$

由比率判别法知此级数收敛

(5)

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)^3}{3n^3}=\frac{1}{3}$$

由比率判别法知此级数收敛