HW9

计 37　张馨元

2023010872

习题 3.4

3

(1)

此积分等价于底面半径和高均为 H 的圆锥的体积，即:

$$\iiint\limits_{\Omega} 1\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{\pi H^3}{3}$$

(2)

此积分相当于整一个三条边两两垂直的四面体的体积，故

$$\iiint\limits_{\Omega} 1\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{6}$$

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(1)

$$
\begin{aligned}
&\Omega = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le xy\} \\
&\iiint\limits_{\Omega} xy^2z^3 \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \int_0^{xy} xy^2z^3 \mathrm{d}z \\
&= \frac{1}{4}\int_0^1 x^5 \mathrm{d}x \int_0^x y^6 \mathrm{d}y \\
&= \frac{1}{28}\int_0^1 x^{12} \mathrm{d}x \\
&= \frac{1}{364}
\end{aligned}
$$

(3)

$$
\begin{aligned}
&\Omega = \left\{(x,y,z) \mid x \le 0 \le \sqrt{y}, 0 \le y \le \frac{\pi}{2}, 0 \le z \le \frac{\pi}{2} - y\right\} \\
&\iiint\limits_{\Omega} x\cos(y+z) \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^{\frac{\pi}{2}} \mathrm{d}y \int_0^{\sqrt{y}} \mathrm{d}x \int_0^{\frac{\pi}{2}-y} x\cos(y+z) \mathrm{d}z \\
&= \int_0^{\frac{\pi}{2}} (1-\sin y) \mathrm{d}y \int_0^{\sqrt{y}} x \mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} (1-\sin y)\frac{y}{2} \mathrm{d}y \\
&= \frac{\pi^2 - 8}{16}
\end{aligned}
$$

(5)

变换到柱坐标系下，有：

$$
\begin{aligned}
&\Omega' = \{(r,\theta,z) \mid 0 \le r \le z, 0 \le \theta \le 2\pi, 0 \le z \le 4\} \\
&\iiint\limits_{\Omega} \frac{\sin z}{z} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_{\Omega'} \frac{\sin z}{z} r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z \\
&= \int_0^4 \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_0^z \frac{\sin z}{z} r\mathrm{d}r \\
&= \pi \int_0^4 z\sin z\mathrm{d}z \\
&= \pi(\sin 4 - 4\cos 4)
\end{aligned}
$$

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(3)

$$
\begin{aligned}
&\Omega = \left\{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 1-x, 0 \le z \le x^2+y^2\right\} \\
&\iiint\limits_{\Omega} \frac{z}{x^2+y^2} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^{x^2+y^2} \frac{z}{x^2+y^2}\mathrm{d}z \\
&= \frac{1}{2}\int_0^1 \mathrm{d}x \int_0^{1-x} (x^2+y^2)\mathrm{d}y \\
&= \frac{1}{2}\int_0^1 (x^2(1-x) + \frac{1}{3}(1-x)^3)\mathrm{d}x \\
&= \frac{1}{12}
\end{aligned}
$$

(4)

变换到球坐标系下，有：

$$
\begin{aligned}
&\Omega' = \left\{(r,\theta,\varphi) \mid 0 \le r \le a, 0 \le \theta \le \frac{\pi}{2}, 0 \le \varphi \le \frac{\pi}{2}\right\} \\
&\iiint\limits_{\Omega} x e^{\frac{x^2+y^2+z^2}{a^2}} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_{\Omega'} r^3 \sin^2\theta \cos\varphi e^{\frac{r^2}{a^2}} \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \\
&= \int_0^{\frac{\pi}{2}} \cos\varphi \mathrm{d}\varphi \int_0^{\frac{\pi}{2}} \sin^2\theta \mathrm{d}\theta \int_0^a r^3 e^{\frac{r^2}{a^2}} \mathrm{d}r \\
&= \sin\varphi \Big|_0^{\frac{\pi}{2}} \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} (1-\cos 2\theta) \mathrm{d}\theta \int_0^{a^2} t^{\frac{3}{2}} e^{\frac{t}{a^2}} \mathrm{d}\sqrt{t} \\
&= \frac{\pi}{8} \int_0^{a^2} t e^{\frac{t}{a^2}} \mathrm{d}t \\
&= \frac{a^2\pi}{8} \left(t e^{\frac{t}{a^2}} \Big|_0^{a^2} - \int_0^{a^2} e^{\frac{t}{a^2}} \mathrm{d}t \right) \\
&= \frac{a^2\pi}{8} \left[a^2 e - a^2(e-1)\right] \\
&= \frac{a^4\pi}{8}
\end{aligned}
$$

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(1)

更换到椭球坐标系下，有：

$$\Omega' = \{(r,\theta,\varphi) \mid 0 \le r \le 1, 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi\}$$

$$\iiint\limits_{\Omega} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{a^2}-\frac{z^2}{c^2}}\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_{\Omega'} \sqrt{1-r^2}abcr^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$$

$$= abc\int_0^1 r^2\sqrt{1-r^2}\mathrm{d}r\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\sin\theta\mathrm{d}\theta$$

$$= 4\pi abc\int_0^1 r^2\sqrt{1-r^2}\mathrm{d}r$$

$$= 4\pi abc\int_0^{\frac{\pi}{2}}\sin^2\theta\sqrt{1-\sin^2\theta}\cos\theta\mathrm{d}\theta$$

$$= 4\pi abc\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\mathrm{d}\theta$$

$$= \pi abc\int_0^{\frac{\pi}{2}}\sin^2 2\theta\mathrm{d}\theta$$

$$= \frac{\pi abc}{2}\int_0^{\pi}\sin^2\varphi\mathrm{d}\varphi$$

$$= \frac{\pi abc}{4}\int_0^{\pi}(1-\cos 2\varphi)\mathrm{d}\varphi$$

$$= \frac{\pi^2 abc}{4}$$

习题 3.5

1

考虑 $z \ge 0$ 的部分，即 $z = \sqrt{a^2-x^2}$
则 $\sqrt{1+(z'_x)^2+(z'_y)^2} = \frac{a}{\sqrt{a^2-x^2}}$

故

$$
\begin{aligned}
S' &= \iint\limits_{x^2+y^2\le a^2} \frac{a}{\sqrt{a^2-x^2}}\mathrm{d}x\mathrm{d}y \\
&= \int_{-a}^{a}\mathrm{d}x\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\frac{a}{\sqrt{a^2-x^2}}\mathrm{d}y \\
&= 4a^2
\end{aligned}
$$

故 $S=2S'=8a^2$

(2)

由于锥面 $z=\sqrt{x^2+y^2}$ 在 $z^2=2x$ 内，可知 $\sqrt{x^2+y^2}\le\sqrt{2x}$
故锥面向 Oxy 平面的投影为 $(x-1)^2+y^2\le 1$，是半径为 1 的圆
又有 $\sqrt{1+(z_x')^2+(z_y')^2}=\sqrt{2}$
故

$$
\begin{aligned}
S &= \iint\limits_{(x-1)^2+y^2\le 1}\sqrt{2}\mathrm{d}x\mathrm{d}y \\
&= \sqrt{2}(\pi\cdot 1^2) \\
&= \sqrt{2}\pi
\end{aligned}
$$

2

(1)

$\rho(x,y,z)=1$
易知 $M=\frac{1}{8}\frac{4\pi abc}{3}=\frac{\pi abc}{6}$
$M_{xy}=\iiint\limits_{\Omega} z\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint\limits_{\Omega'} cr\cos\theta abcr^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi=$
$abc^2\int_0^1 r^3\mathrm{d}r\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi=\frac{abc^2\pi}{16}$
故 $z_c=\frac{M_{xy}}{M}=\frac{3c}{8}$
x,y 坐标同理
故质心坐标为 $\left(\frac{3a}{8},\frac{3b}{8},\frac{3c}{8}\right)$

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(2)

联立 $x^2+y^2+z^2=2, x^2+y^2=z^2, z\geq 0$，可得 $z=1$

故平面 $z=1$ 将物体分为两部分，下方为圆锥，上方为球面的一部分，

$J=J_1+J_2$

$$
\begin{aligned}
J_1 &= \iiint\limits_{x^2+y^2\leq z^2, 0\leq z\leq 1} (x^2+y^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \iiint\limits_{\Omega'} r^2\cdot r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z \\
&= \int_0^1 \mathrm{d}z \int_0^z r^3\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \\
&= \frac{\pi}{10} \\
J_2 &= \iiint\limits_{x^2+y^2+z^2\leq 2, z\geq 1} (x^2+y^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \iiint\limits_{\Omega''} r^2\cdot r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z \\
&= \int_1^{\sqrt{2}} \mathrm{d}z \int_0^{\sqrt{2-z^2}} r^3\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \\
&= \frac{16\sqrt{2}\pi}{15} \\
J = J_1+J_2 &= \frac{16\sqrt{2}\pi}{15}+\frac{\pi}{10}
\end{aligned}
$$