HW6

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证明. $g(A_1+A_2,B)=\sum\limits_{i,j}\overline{\left(a_{ij}^{(1)}+a_{ij}^{(2)}\right)}b_{ij}=\sum\limits_{i,j}a_{ij}^{(1)}b_{ij}+\sum\limits_{i,j}a_{ij}^{(2)}b_{ij}=$ $g(A_1,B)+g(A_2,B)$

$g(A,B_1+B_2)=\sum\limits_{i,j}a_{ij}\overline{\left(b_{ij}^{(1)}+b_{ij}^{(2)}\right)}=\sum\limits_{i,j}a_{ij}b_{ij}^{(1)}+\sum\limits_{i,j}a_{ij}b_{ij}^{(2)}=$ $g(A,B_1)+g(A,B_2)$

$g(kA,B)=\sum\limits_{i,j}\overline{ka_{ij}}b_{ij}=\overline{k}\sum\limits_{i,j}\overline{a_{ij}}b_{ij}=\overline{k}g(A,B)$

$g(A,kB)=\sum\limits_{i,j}\overline{a_{ij}}kb_{ij}=k\sum\limits_{i,j}\overline{a_{ij}}b_{ij}=kg(A,B)$

故 g 为双线性型 $\forall A\in F, g(A,A)=\sum\limits_{i,j}\overline{a_{ij}}a_{ij}=\sum\limits_{i,j}|a_{ij}|^2\geq 0$ 且，

$g(A,A)=0\iff \forall i,j,a_{i,j}=0\iff A=O$

故 g 为正定 Hermite 型，即 g 为 Hermite 内积

而考虑 $\overline{A^T}B$ 的对角线位置 $c_{ii}=\overline{a_i^T}b_i=\sum\limits_{j}\overline{a_{ji}^T}b_{ji}$

故 $\mathrm{tr}(\overline{A^T}B)=\sum\limits_{i}c_{ii}=\sum\limits_{i,j}\overline{a_{ji}^T}b_{ji}=g(A,B)$

同理 $\mathrm{tr}(B\overline{A^T})=g(A,B)$

即有 $\langle A,B\rangle=$tr$(B\overline{A^T})=$tr$(\overline{A^T}B)$ $\square$

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证明.

$$
\begin{aligned}
\langle f+h, g\rangle &= \int_0^1 \left(\overline{f(t)+h(t)}\right) g(t)\mathrm{d}t \\
&= \int_0^1 \overline{f(t)} g(t)\mathrm{d}t + \int_0^1 \overline{h(t)} g(t)\mathrm{d}t = \langle f, g\rangle + \langle h, g\rangle \\
\langle f, g+h\rangle &= \int_0^1 \overline{f(t)} \left(g(t)+h(t)\right)\mathrm{d}t \\
&= \int_0^1 \overline{f(t)} g(t)\mathrm{d}t + \int_0^1 \overline{f(t)} h(t)\mathrm{d}t = \langle f, g\rangle + \langle f, h\rangle \\
&\langle kf, g\rangle = \int_0^1 \overline{kf(t)} g(t)\mathrm{d}t = \overline{k}\int_0^1 f(t)g(t)\mathrm{d}t = \overline{k}\langle f, g\rangle \\
&\langle f, kg\rangle = \int_0^1 \overline{f(t)} kg(t)\mathrm{d}t = k\int_0^1 f(t)g(t)\mathrm{d}t = k\langle f, g\rangle \\
&\forall f \in C[0,1], \langle f, f\rangle = \int_0^1 f^2(t)\mathrm{d}t \geq 0 \\
&\langle f, f\rangle = 0 \iff \forall t \in [0,1], f(t) \equiv 0
\end{aligned}
$$

故可知 $\langle f, g\rangle$ 为 Hermite 内积 □

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1. $g\circ\varphi(x+z, y) = \langle\varphi(x+z), \varphi y\rangle = \langle\varphi x+\varphi z, \varphi y\rangle =$
   $\langle\varphi x, \varphi y\rangle + \langle\varphi z, \varphi y\rangle = g\circ\varphi(x, y) + g\circ\varphi(z, y)$
   同理，$g\circ\varphi(x, y+z) = g\circ\varphi(x, y) + g\circ\varphi(x, z)$

2. $g\circ\varphi(kx, y) = \langle\varphi(kx), \varphi y\rangle = \langle k\varphi x, \varphi y\rangle = \overline{k}\langle\varphi x, \varphi y\rangle = \overline{k}g\circ\varphi(x, y)$
   同理，$g\circ\varphi(x, ky) = kg\circ\varphi(x, y)$

3. $\forall x \in V_1, g\circ\varphi(x, x) = \langle\varphi x, \varphi x\rangle \geq 0$
   且 $g\circ\varphi(x, x) = 0 \iff \varphi x = 0 \iff x = 0$

故可知 $g\circ\varphi$ 为 Hermite 内积

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证明. 充分性：若 $\alpha=0, \forall\beta\in V, g(\alpha,\beta)=g(-\alpha,\beta)=-g(\alpha,\beta)$，故 $g(\alpha,\beta)=0$
必要性：令 $\beta=\alpha$，则 $g(\alpha,\alpha)=0\Rightarrow\alpha=0$

$\square$

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取标准正交基 $(\varepsilon_{11},\cdots,\varepsilon_{nn})$，其中 ε_{ij} 代表第 i 行第 j 列为 1，其他为 0 的 n^2 维方阵

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(1)

1. $\int_0^1 \overline{1}\cdot 1\mathrm{d}t=1$
   $\int_0^1 \overline{f_n(t)}f_n(t)\mathrm{d}t=2\int_0^1\cos^2(2\pi nt)\mathrm{d}t=\int_0^1(\cos 4\pi nt+1)\mathrm{d}t=1$
   $\int_0^1 \overline{g_m(t)}g_m(t)\mathrm{d}t=2\int_0^1\sin^2(2\pi mt)\mathrm{d}t=\int_0^1(-\cos 4\pi mt+1)\mathrm{d}t=1$

2. $\int_0^1 \overline{1}\cdot f_n(t)\mathrm{d}t=\sqrt{2}\int_0^1\cos 2\pi nt\mathrm{d}t=0$
   $\int_0^1 \overline{1}\cdot g_m(t)\mathrm{d}t=\sqrt{2}\int_0^1\sin 2\pi mt\mathrm{d}t=0$
   $\int_0^1 \overline{f_n(t)}g_m(t)\mathrm{d}t=2\int_0^1\cos(2\pi nt)\sin(2\pi mt)\mathrm{d}t=$
   $\int_0^1\left(\sin(2\pi mt+2\pi nt)+\sin(2\pi mt-2\pi nt)\right)\mathrm{d}t=0$
   $\int_0^1 \overline{f_n(t)}f_m(t)\mathrm{d}t=2\int_0^1\cos(2\pi nt)\cos(2\pi mt)\mathrm{d}t=$
   $\int_0^1\left(\cos(2\pi mt+2\pi nt)+\cos(2\pi mt-2\pi nt)\right)\mathrm{d}t=0$
   $\int_0^1 \overline{g_n(t)}g_m(t)\mathrm{d}t=2\int_0^1\sin(2\pi nt)\sin(2\pi mt)\mathrm{d}t=$
   $\int_0^1\left(\cos(2\pi mt-2\pi nt)-\cos(2\pi mt+2\pi nt)\right)\mathrm{d}t=0$

故综上，$\{1,f_1,f_2,\cdots,g_1,g_2,\cdots\}$ 为 $C[0,1]$ 的标准正交向量组

(2)

$e^{2\pi inx}=\cos 2\pi nx+i\sin 2\pi nx=f_n(x)+ig_n(x)(f_n,g_n$ 同上一问,$f_0=g_0=1)$

由 (1) 可知，当 $m \neq n$ 时，$e^{2\pi inx}$ 和 $e^{2\pi imx}$ 正交
且 $|e^{2\pi inx}| = 1$
故可知 $e^{2\pi inx}$ 为标准正交向量组