# HW11

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## 习题 4.5

### 1

#### (2)

将此积分拆为上下两个半球的积分:$\oiint\limits_{S^+} z\mathrm{d}x \wedge \mathrm{d}y = \iint\limits_{S_1} z\mathrm{d}x\mathrm{d}y - \iint\limits_{S_2} z\mathrm{d}x\mathrm{d}y$

令

$$\begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi, 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi \\ z = R\cos\theta + R \end{cases}$$

则 $\left|\dfrac{D(x,y)}{D(u,v)}\right| = R^2\sin\theta\,|\cos\theta|$

$$\begin{aligned}
\iint\limits_{S_1} z\mathrm{d}x\mathrm{d}y &= \iint\limits_{S_1'} (R\cos\theta+R)R^2\sin\theta\cos\theta\mathrm{d}\theta\mathrm{d}\varphi \\
&= \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta R^3\sin\theta\cos\theta(\cos\theta+1) \\
&= \frac{5}{3}\pi R^3 \\
\iint\limits_{S_2} z\mathrm{d}x\mathrm{d}y &= \iint\limits_{S_2'} (R\cos\theta+R)R^2\sin\theta(-\cos\theta)\mathrm{d}\theta\mathrm{d}\varphi \\
&= -\int_0^{2\pi}\mathrm{d}\varphi\int_{\frac{\pi}{2}}^{\pi}\mathrm{d}\theta R^3\sin\theta\cos\theta(\cos\theta+1) \\
&= \frac{1}{3}\pi R^3
\end{aligned}$$

故

$$\oiint\limits_{S^+} z\mathrm{d}x\wedge\mathrm{d}y = \frac{4}{3}\pi R^3$$

# 3

## (1)

由对称性可知，只需考虑正方体 $x=0$ 和 $x=1$ 的两个面
对这两个面而言，$\cos\beta=\cos\gamma\equiv\dfrac{\pi}{2}$，因此只需考虑 $x\mathrm{d}y\wedge\mathrm{d}z$ 项

$$\begin{aligned}
\iint\limits_{S^+_{x=0}} x\mathrm{d}y\wedge\mathrm{d}z &= 0 \\
\iint\limits_{S^+_{x=1}} x\mathrm{d}y\wedge\mathrm{d}z &= \iint\limits_{S^+_{x=1}} x\mathrm{d}y\mathrm{d}z \\
&= \iint\limits_{S^+_{x=1}} \mathrm{d}y\mathrm{d}z \\
&= 1
\end{aligned}$$

故 $I = 3\times(0+1) = 3$

**(2)**

由于 $\cos\gamma \equiv \dfrac{\pi}{2}$，故 $\iint\limits_{S^+} z\mathrm{d}x\wedge\mathrm{d}y = 0$

由对称性可知

$$\begin{aligned}
I &= 2\iint\limits_{S^+} x^2\mathrm{d}y\wedge\mathrm{d}z \\
&= 2\iint\limits_{S^+_{x\geq 0}} x^2\mathrm{d}y\mathrm{d}z - 2\iint\limits_{S^+_{x\leq 0}} x^2\mathrm{d}y\mathrm{d}z
\end{aligned}$$

注意到 $x \geq 0$ 和 $x \leq 0$ 两片区域完全对称，故可知 $I = 0$

**(3)**

对于上表面 $\cos\alpha = \cos\beta \equiv \dfrac{\pi}{2}$

$$\begin{aligned}
\iint\limits_{S^+_{上}} (x-y)\mathrm{d}x\wedge\mathrm{d}y &= \int_{-h}^{h}\mathrm{d}y\int_{-\sqrt{h^2-y^2}}^{\sqrt{h^2-y^2}}(x-y)\mathrm{d}x \\
&= -2\int_{-h}^{h} y\sqrt{h^2-y^2}\mathrm{d}y \\
&= 0
\end{aligned}$$

故只需考虑侧面，则

$$\begin{aligned}
\iint\limits_{S^+} (x-y)\mathrm{d}x\wedge\mathrm{d}y &= \iint\limits_{S^+} (y-x)\mathrm{d}x\mathrm{d}y \\
&= \int_{-h}^{h}\mathrm{d}x\int_{-\sqrt{h^2-x^2}}^{\sqrt{h^2-x^2}}\mathrm{d}y\,(y-x) \\
&= -2\int_{-h}^{h} x\sqrt{h^2-x^2}\mathrm{d}x \\
&= 0
\end{aligned}$$

由对称性可知 $\iint\limits_{S^+} (y-z)\,\mathrm{d}y\wedge\mathrm{d}z = \iint\limits_{S^+} (z-x)\mathrm{d}z\wedge\mathrm{d}x$

$\iint\limits_{S^+} (y-z)\mathrm{d}y\wedge\mathrm{d}z$ 可以拆分为 $x \geq 0$ 和 $x \leq 0$ 两部分：

$$\iint\limits_{S^+} (y-z)\mathrm{d}y\wedge\mathrm{d}z = \iint\limits_{S^+_1} (y-z)\mathrm{d}y\mathrm{d}z - \iint\limits_{S^+_2} (y-z)\mathrm{d}y\mathrm{d}z$$

这两部分完全对称，故 $\iint\limits_{S^+}(y-z)\,\mathrm{d}y\wedge\mathrm{d}z=\iint\limits_{S^+}(z-x)\mathrm{d}z\wedge\mathrm{d}x=0$

故 $I=0$

**(4)**

一共五个面

考虑 $z=0$：$\cos\alpha=\cos\beta\equiv\dfrac{\pi}{2},\iint\limits_{S_1^+}y^2z\mathrm{d}x\wedge\mathrm{d}y=0$，故 $I_1=0$

考虑 $x=0$：$\cos\alpha=\cos\gamma\equiv\dfrac{\pi}{2},\iint\limits_{S_2^+}x^2y\mathrm{d}z\wedge\mathrm{d}x=0$，故 $I_2=0$

考虑 $y=0$：$\cos\beta=\cos\gamma\equiv\dfrac{\pi}{2},\iint\limits_{S_3^+}z^2x\mathrm{d}z\wedge\mathrm{d}x=0$，故 $I_3=0$

考虑抛物面，向上为正方向，故

$\mathrm{d}\vec{S}=\left(-z_x',-z_y',1\right)\mathrm{d}x\mathrm{d}y=(-2x,-2y,1)\,\mathrm{d}x\mathrm{d}y$

故

$$\begin{aligned}
I_4&=\iiint\limits_{S_2^+}\left(z^2x,x^2y,y^2z\right)\cdot(-2x,-2y,1)\,\mathrm{d}x\mathrm{d}y\\
&=\iiint\limits_{S_2^+}\left(-2x^2z^2-2x^2y^2+y^2z\right)\mathrm{d}x\mathrm{d}y\\
&=\int_0^1\mathrm{d}x\int_0^{\sqrt{1-x^2}}\left(-2x^6-2x^2y^4-4x^4y^2-x^2y^2+y^4\right)\mathrm{d}y\\
&=-\frac{1}{15}\int_0^1\left(16x^6+17x^2-3\right)\sqrt{1-x^2}\mathrm{d}x\\
&=-\frac{\pi}{24}
\end{aligned}$$

考虑柱面，$\cos\gamma=\dfrac{\pi}{2}$

$$\iiint\limits_{S_3^+}z^2x\mathrm{d}y\wedge\mathrm{d}z=\int_0^1z^2\mathrm{d}z\int_0^1\sqrt{1-y^2}\mathrm{d}y=\frac{\pi}{12}$$

$$\iiint\limits_{S_3^+}x^2y\mathrm{d}z\wedge\mathrm{d}x=\int_0^1\mathrm{d}z\int_0^1x^2\sqrt{1-x^2}\mathrm{d}x=\frac{\pi}{16}$$

故 $I_5=\dfrac{7\pi}{48}$

故 $I=I_1+I_2+I_3+I_4+I_5=\dfrac{5\pi}{48}$

**(5)**

$$
\begin{aligned}
\iint\limits_{S^+} z^2\mathrm{d}x\wedge\mathrm{d}y &= \iint\limits_{x^2+y^2\leq Rx}\left(R^2-x^2-y^2\right)\mathrm{d}x\mathrm{d}y\\
&=\iint\limits_{D_{r\theta}}\left(R^2-r^2\right)r\mathrm{d}r\mathrm{d}\theta\\
&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{R\cos\theta}\left(R^2-r^2\right)r\mathrm{d}r\\
&=\frac{5R^4}{32}\pi
\end{aligned}
$$

## 4

令

$$
\begin{cases}
x=R\sin\theta\cos\varphi\\
y=R\sin\theta\sin\varphi\\
z=R\cos\theta,0\leq\theta\leq\dfrac{\pi}{2},0\leq\varphi\leq 2\pi
\end{cases}
$$

故 $\mathrm{d}\vec{\boldsymbol{S}}=-\left(A,B,C\right)\mathrm{d}\theta\mathrm{d}\varphi=$
$-\left(R^2\sin^2\theta\cos\varphi,R^2\sin^2\theta\sin\varphi,R^2\sin\theta\cos\varphi\right)\mathrm{d}\theta\mathrm{d}\varphi$

$$
\begin{aligned}
\iint\limits_{S^+}\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{S}} = -\iint\limits_{S^+}&\left(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta\right)\cdot\\
&\left(R^2\sin^2\theta\cos\varphi,R^2\sin^2\theta\sin\varphi,R^2\sin\theta\cos\varphi\right)\mathrm{d}\theta\mathrm{d}\varphi\\
&=-R^2\iint\limits_{S^+}\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\\
&=-R^2\int_0^{\frac{\pi}{2}}\sin\theta\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\\
&=-2\pi R^2
\end{aligned}
$$

# 习题 4.6

## 1

### (2)

$$
\begin{aligned}
&\oint\limits_{L^+} (x+y)\,\mathrm{d}x + (x-y)\,\mathrm{d}y \\
&= \iint\limits_{D} \left( \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) \mathrm{d}x\mathrm{d}y \\
&= \iint\limits_{D} (1-1)\mathrm{d}x\mathrm{d}y \\
&= 0
\end{aligned}
$$

### (3)

$$
\begin{aligned}
&\oint\limits_{L^+} \left(x^2+y\right)\mathrm{d}x - \left(x-y^2\right)\mathrm{d}y \\
&= \iint\limits_{D} \left( \frac{\partial Y}{\partial x} + \frac{\partial X}{\partial y} \right) \mathrm{d}x\mathrm{d}y \\
&= -2\iint\limits_{D} \mathrm{d}x\mathrm{d}y \\
&= -2\pi ab
\end{aligned}
$$

## 2

### (1)

$$\begin{aligned}
&\oint_{L^+} \frac{x+y}{x^2+y^2}\mathrm{d}x - \frac{x-y}{x^2+y^2}\mathrm{d}y \\
&= \iint_D \left(\frac{\partial Y}{\partial x} + \frac{\partial X}{\partial y}\right)\mathrm{d}x\mathrm{d}y \\
&= 0
\end{aligned}$$

### (2)

令

$$\begin{cases} x = a\cos\theta \\ y = a\sin\theta, \theta \in [0, 2\pi] \end{cases}$$

故

$$\int_{L^+} \frac{x+y}{x^2+y^2}\mathrm{d}x - \frac{x-y}{x^2+y^2}\mathrm{d}y = -\int_0^{2\pi} \mathrm{d}\theta = -2\pi$$

### (3)

考虑 $D^* = \{(x,y)|x^2+y^2 \le \epsilon^2\} \subset D$，正方向为顺时针
由 Green 公式可知

$$\int_{\partial(D-D^*)} \frac{(x+y)\mathrm{d}x + (y-x)\mathrm{d}y}{x^2+y^2} = 0$$

同上一问可知

$$\int_{\partial D^*} \frac{(x+y)\mathrm{d}x + (y-x)\mathrm{d}y}{x^2+y^2} = 2\pi$$

故

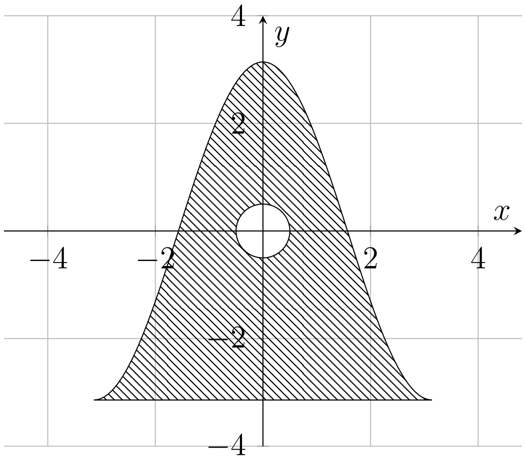
$$\int_{\partial D} \frac{(x+y)\mathrm{d}x + (y-x)\mathrm{d}y}{x^2+y^2} = -2\pi$$

**(4)**

也考虑 $D^* = \{(x,y)|x^2+y^2 \le \epsilon^2\} \subset D$，正方向为顺时针，同上一问可求得

$$\int\limits_{\partial D} \frac{(x+y)\mathrm{d}x + (y-x)\mathrm{d}y}{x^2+y^2} = -2\pi$$

**(5)**




考虑如图所示的区域，外侧方向为顺时针，内侧方向为逆时针

由 Green 公式可知，$I_{\partial(D-D^*)} = 0$

同第 (2) 可知，$I_{\partial D^*} = -2\pi$

而对于下面这条直线，有

$$\int\limits_{L'} = \int_{\pi}^{-\pi} \frac{(x-\pi)\,\mathrm{d}x}{x^2+\pi^2} = \frac{\pi}{2}$$

故 $I = \dfrac{3\pi}{2}$

## 4

### (1)

$$
\begin{aligned}
S &= -\oint\limits_{L^+} y\mathrm{d}x = 3a^2\int_0^{2\pi}\sin^4 t\cos^2 t\mathrm{d}t \\
&= -12a^2\int_0^{\frac{\pi}{2}}\left(\sin^6 t-\sin^4 t\right)\mathrm{d}t \\
&= \frac{3\pi a^2}{8}
\end{aligned}
$$

## 10

### (1)

$\dfrac{\partial u}{\partial x} = x^2 - y$

故 $u = \int\left(x^2-y\right)\mathrm{d}x = \dfrac{1}{3}x^3 - yx + C(y)$

故 $\dfrac{\partial u}{\partial y} = -x + C'(y) = -x - \sin^2 y$

故 $C(y) = \int\sin^2 y\mathrm{d}y = \dfrac{\sin 2y}{4} - \dfrac{y}{2} + C'$

故 $u = \dfrac{x^3}{3} - yx + \dfrac{\sin 2y}{4} - \dfrac{y}{2} + C'$

故方程的解为 $\dfrac{x^3}{3} - yx + \dfrac{\sin 2y}{4} - \dfrac{y}{2} = C$

### (2)

$\dfrac{\partial u}{\partial x} = e^y$

故 $u = xe^y + C(y)$

故 $\dfrac{\partial u}{\partial y} = xe^y + C'(y) = xe^y - 2y$

故 $C(y) = \int -2y\mathrm{d}y = -y^2 + C'$

故 $u = xe^y - y^2 + C'$

故方程的解为 $xe^y - y^2$

## 11

### (4)

$$\mathrm{d}\left(x-y\right)=\frac{\mathrm{d}\left(x+y\right)}{x+y}$$

故解为 $x-y=\ln\left(x+y\right)+C$