# HW8

计 37　张馨元

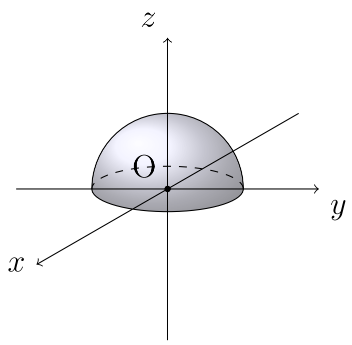
2023010872

## 习题 3.3

### 1

#### (1)

令 $z = \sqrt{R^2 - x^2 - y^2}$ 则可知 $x^2 + y^2 + z^2 = R^2 (z \geq 0)$
曲面为上半球，即：

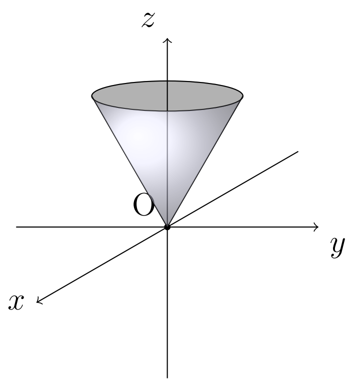



由几何意义知，此二重积分的值为球体积的一半，即：

$$\iint\limits_D \sqrt{R^2 - x^2 - y^2} \mathrm{d}x\mathrm{d}y = \frac{1}{2} \cdot \frac{4\pi R^3}{3} = \frac{2\pi R^3}{3}$$

#### (2)

同样令 $z = \sqrt{x^2 + y^2}$，则有 $z^2 = x^2 + y^2 (z \geq 0)$
此曲面为上半圆锥面，即：

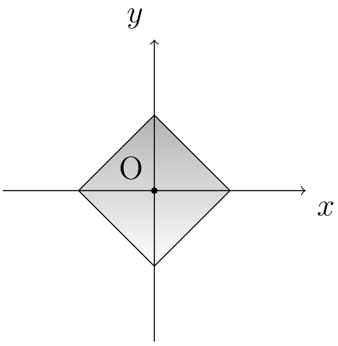



因此，由几何意义知，该积分的值为圆柱体积和圆锥体积之差，即：

$$\iint\limits_D \sqrt{x^2+y^2}\mathrm{d}x\mathrm{d}y = \pi R^2 \cdot R - \frac{1}{3}\pi R^2 \cdot R = \frac{2}{3}\pi R^3$$

**(3)**

区域 $D$ 由四条曲线围成，分别为
$x+y=1, x-y=1, -x+y=1, -x-y=1$:




所求积分为区域 $D$ 的面积，即：

$$\iint\limits_D \mathrm{d}x\mathrm{d}y = (\sqrt{2})^2 = 2$$

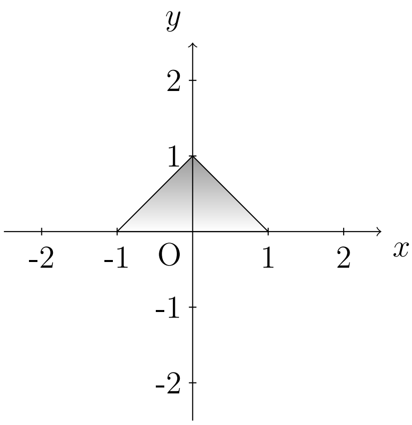
## 3

$$\iint_I \frac{\partial^2 f}{\partial x \partial y}\mathrm{d}x\mathrm{d}y = \int_c^d \mathrm{d}y \int_a^b \frac{\partial^2 f}{\partial x \partial y}\mathrm{d}x = \int_c^d \mathrm{d}y(f_y'(b,y) - f_y'(a,y))$$

$$= f(b,d) - f(b,c) - f(a,d) + f(a,c)$$

## 5

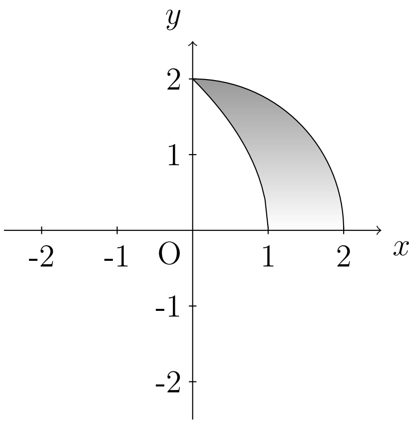
### (1)




则 $D = \{(x,y) \mid 0 \le y \le 1, y-1 \le x \le 1-y\}$
交换积分次序为

$$\int_0^1 \mathrm{d}x \int_{y-1}^{1-y} f(x,y)\mathrm{d}y$$

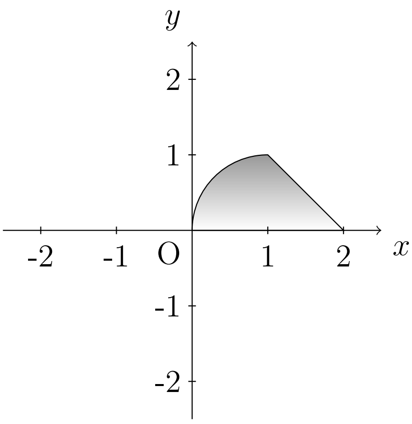
### (2)



则 $D=\left\{(x,y)\mid 0\le y\le 2, 0\le x\le 2\sqrt{1-x}\right\}$
交换积分次序为：

$$\int_0^2 \mathrm{d}y\int_{1-\frac{y^2}{4}}^{\sqrt{4-y^2}} f(x,y)\mathrm{d}x$$

**(3)**




则区域 $D=\left\{(x,y)\mid 1-\sqrt{1-y^2}\le x\le 2-y, 0\le y\le 1\right\}$
交换积分次序为：

$$\int_0^1 \mathrm{d}y\int_{1-\sqrt{1-y^2}}^{2-y} f(x,y)\mathrm{d}x$$

## 6

### (1)

注意到 $xy^2$ 关于 y=0 对称，因此可知：

$$\begin{aligned}\iint\limits_D xy^2\mathrm{d}x\mathrm{d}y &= \int_{-2}^{2}\mathrm{d}y\int_{\frac{y^2}{4}}^{1} xy^2\mathrm{d}x \\ &= 2\int_{0}^{2} y^2\mathrm{d}y\int_{\frac{y^2}{4}}^{1} x\mathrm{d}x \\ &= \int_{0}^{2}(y^2-\frac{y^6}{16})\mathrm{d}y \\ &= (\frac{y^3}{3}-\frac{y^7}{112})\Big|_0^2 \\ &= \frac{32}{21}\end{aligned}$$

### (3)

注意到 $|xy|$ 关于 $x$ 轴和 $y$ 轴均对称，因此可知：

$$\begin{aligned}\iint\limits_D |xy|\,\mathrm{d}x\mathrm{d}y &= 4\int_{0}^{R}\mathrm{d}y\int_{0}^{\sqrt{R^2-y^2}} xy\mathrm{d}x \\ &= 2\int_{0}^{R} y(R^2-y^2)\mathrm{d}y \\ &= \frac{R^4}{2}\end{aligned}$$

### (5)

$$\begin{aligned}\iint\limits_D (x^2+y^2)\mathrm{d}x\mathrm{d}y &= \int_{1}^{4}\mathrm{d}y\int_{y-1}^{y}(x^2+y^2)\mathrm{d}x \\ &= \int_{1}^{4}(\frac{y^3}{3}-\frac{(y-1)^3}{3}+y^2)\mathrm{d}y \\ &= \frac{71}{2}\end{aligned}$$

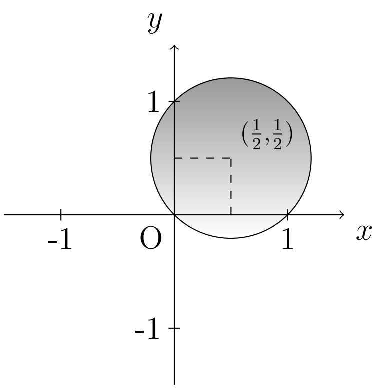
## 12

### (1)

变换到极坐标系下，有 $D'=\left\{(\rho,\varphi)\mid 2\cos^2\varphi\le\rho\le 4\cos^2\varphi,-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\right\}$

$$\begin{aligned}\iint\limits_D(x^2+y^2)\mathrm{d}x\mathrm{d}y&=\iint\limits_{D'}\rho^3\mathrm{d}\rho\mathrm{d}\varphi\\&=2\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_{2\cos\varphi}^{4\cos\varphi}\rho^3\mathrm{d}\rho\\&=120\int_0^{\frac{\pi}{2}}\cos^4\varphi\mathrm{d}\varphi\\&=15\int_0^{\frac{\pi}{2}}\cos4\varphi+60\int_0^{\frac{\pi}{2}}\cos2\varphi\mathrm{d}\varphi+45\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\\&=\frac{45\pi}{2}\end{aligned}$$

### (3)




变换到极坐标系下，有

$D'=\left\{(\rho,\varphi)\mid 0\le\rho\le\sqrt{2}\cos\left(\frac{\pi}{4}-\varphi\right)=\sin\varphi+\cos\varphi,-\frac{\pi}{4}\le\varphi\le\frac{3\pi}{4}\right\}$

故

$$\begin{aligned}\iint\limits_{D}(x+y)\mathrm{d}x\mathrm{d}y &= \iint\limits_{D'}\rho^2(\sin\varphi+\cos\varphi)\mathrm{d}\rho\mathrm{d}\varphi \\ &= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\mathrm{d}\varphi\int_0^{\sin\varphi+\cos\varphi}\rho^2(\sin\varphi+\cos\varphi)\mathrm{d}\rho \\ &= \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}(\sin\varphi+\cos\varphi)^4\mathrm{d}\varphi \\ &= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\mathrm{d}\varphi-\frac{1}{6}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\cos 2\varphi\mathrm{d}\varphi+\frac{2}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\sin 2\varphi\mathrm{d}\varphi \\ &= \frac{\pi}{2}\end{aligned}$$

**(5)**

变换到极坐标下，有 $D'=\left\{(\rho,\varphi)\mid 0\le\rho\le 1,\pi\le\varphi\le\frac{3\pi}{2}\right\}$
故

$$\begin{aligned}\iint\limits_{D}\arctan\frac{y}{x}\mathrm{d}x\mathrm{d}y &= \iint\limits_{D'}\rho\varphi\mathrm{d}\rho\mathrm{d}\varphi \\ &= \int_0^{\frac{\pi}{2}}\varphi\mathrm{d}\varphi\int_0^1\rho\mathrm{d}\rho \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}}\varphi\mathrm{d}\varphi \\ &= \frac{\pi^2}{16}\end{aligned}$$

## 14

**(1)**

令 $\begin{cases} u=xy, u\in[2,4] \\ v=\dfrac{y}{x}, v\in[1,3]\end{cases}$ $\left|\frac{D(x,y)}{D(u,v)}\right|=\left|\frac{D(u,v)}{D(x,y)}\right|^{-1}=\begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x}\end{vmatrix}^{-1}=\frac{x}{2y}=\frac{1}{2v}\neq 0$
故

$$\iint\limits_{D}x^2y^2\mathrm{d}x\mathrm{d}y=\iint\limits_{D'}u^2\left|\frac{D(x,y)}{D(u,v)}\right|\mathrm{d}u\mathrm{d}v=\frac{1}{2}\int_1^3\frac{\mathrm{d}v}{v}\int_2^4u^2\mathrm{d}u=\frac{28}{3}\int_1^3\frac{\mathrm{d}v}{v}=\frac{28\ln 3}{3}$$

## 15

### (1)

令 $\begin{cases} u = a_1x + b_1y + c_1 \\ v = a_2x + b_2y + c_2 \end{cases}$

故 $\left|\frac{D(x,y)}{D(u,v)}\right| = \left|\frac{D(u,v)}{D(x,y)}\right|^{-1} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^{-1} = \frac{1}{a_1b_2-a_2b_1}$

故

$$\begin{aligned} S &= \left|\iint\limits_{D} \mathrm{d}x\mathrm{d}y\right| = \left|\iint\limits_{D'} \left|\frac{D(x,y)}{D(u,v)}\right| \mathrm{d}u\mathrm{d}v\right| = \left|\frac{1}{a_1b_2 - a_2b_1} \iint\limits_{D'} \mathrm{d}u\mathrm{d}v\right| \\ &= \left|\frac{1}{a_1b_2 - a_2b_1}\right| S' = \frac{\pi}{|a_1b_2 - a_2b_1|} \end{aligned}$$