# Randomized Algorithms

- Deterministic and (Las Vegas & Monte Carlo) Randomized Algorithms
- Probability Review
- Probabilistic Analysis of deterministic QUICK-SORT Algorithm
- RANDOMIZED-SELECT and RANDOMIZED-QUICK-SORT
- Max-Cut
- Min-Cut
- MAX-3-SAT and Derandomization
- Closest pair
- Hashing, Bloom filters, Streams, Sampling, Reservoir sampling, Sketch

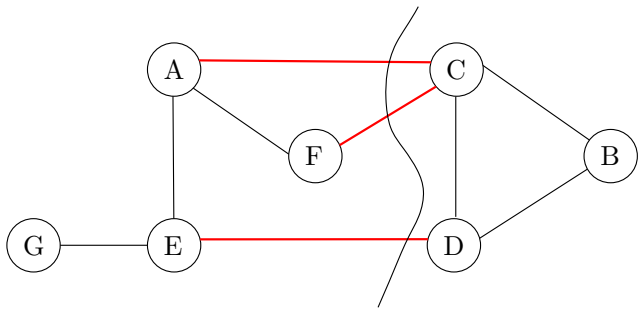
Imdad ullah Khan

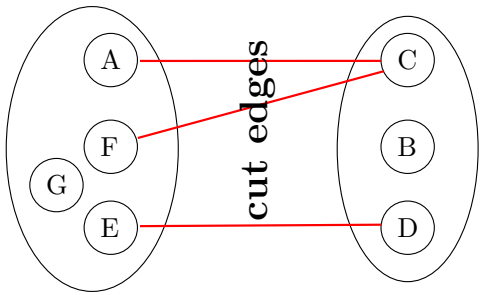
# Cuts in Graphs

- Cuts in graphs are useful structures
- Application in network flows, statistical physics, circuit design, complexity and approximation theory

**A cut in $G$ is a subset $S \subset V$**

- Denoted as $[S, \overline{S}]$
- $S = \emptyset$ and $S = V$ are trivial cuts, we assume that $\emptyset \neq S \neq V$
- A graph on $n$ vertices has $2^n$ cuts
- An edge $(u, v)$ is crossing the cut $[S, \overline{S}]$, if $u \in S$ and $v \in \overline{S}$



# The MIN-CUT($G$) problem

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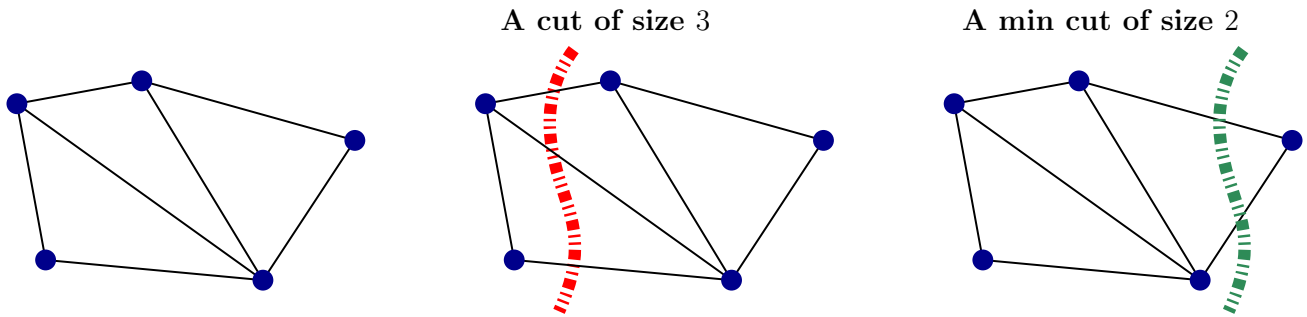
**Size (or cost) of a cut in the number of crossing edges**




- In weighted graph size of cut is the sum of weights of crossing edges

The MIN-CUT($G$) problem: Find a cut in $G$ of minimum size?

# The MIN-CUT($G$) problem

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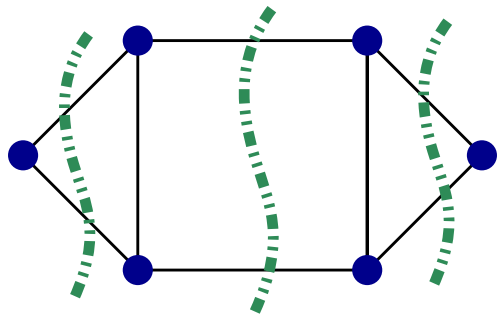


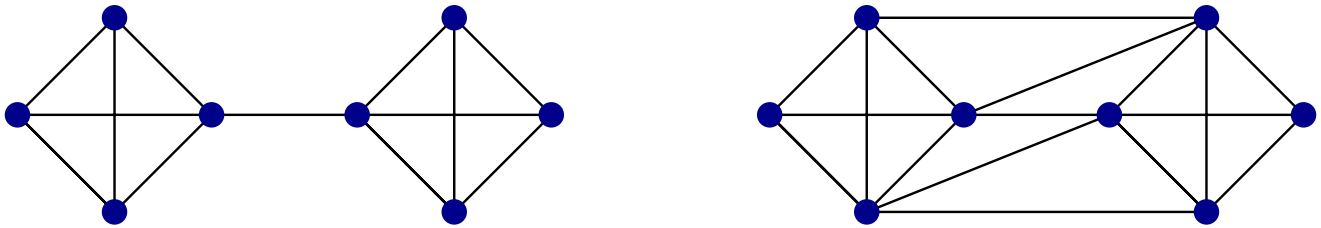
- Min cut does not have to be unique
- size of min-cut is at most the minimum degree of any vertex

# Global Min-Cut

The MIN-CUT($G$) problem: Find a cut in $G$ of minimum size?

Also called Global Min-Cut

Min-cut has applications in network reliability and robustness analysis



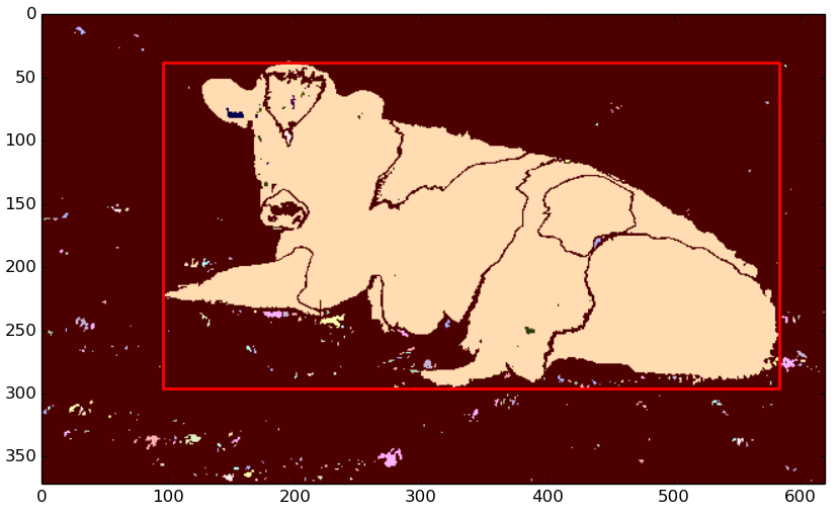
The network on the left is easier to disconnect

Normalized min-cut spectral clustering applied to image segmentation

# Global Min-Cut: Image Segmentation

Separate foreground from background (e.g Aircraft/missile from horizon)


https://stackoverflow.com/



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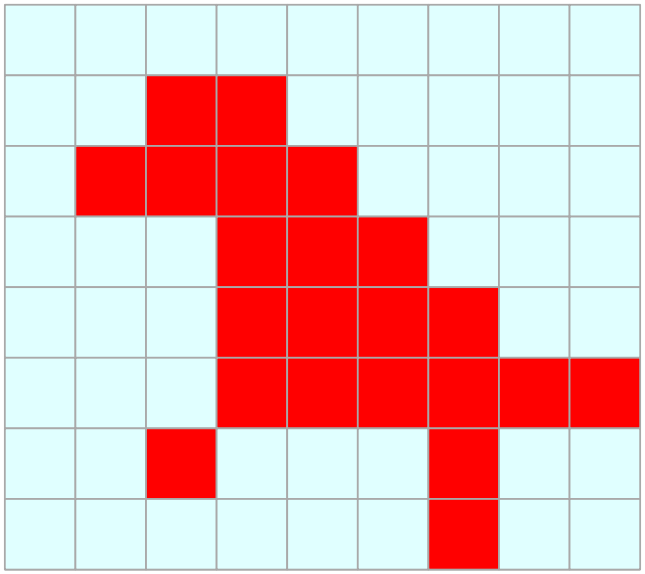

# Global Min-Cut: Image Segmentation

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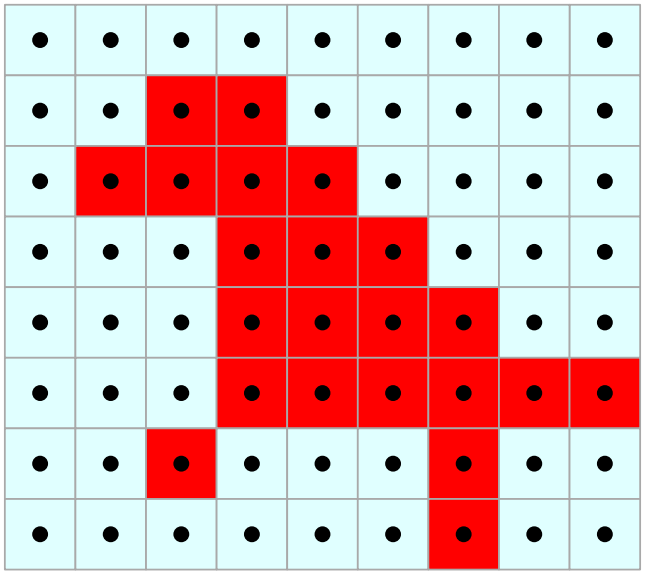
If pixel $(x, y)$ is background/foreground, then so are nearby pixels

Make a graph with nodes for each pixel adjacent to neighboring pixels



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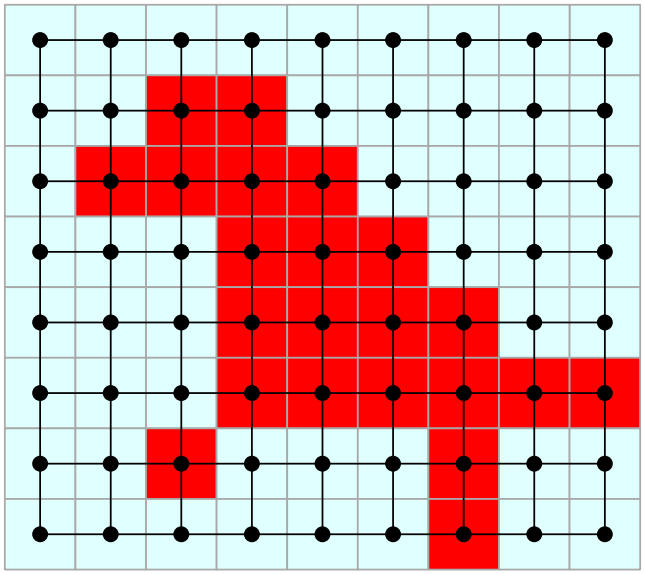

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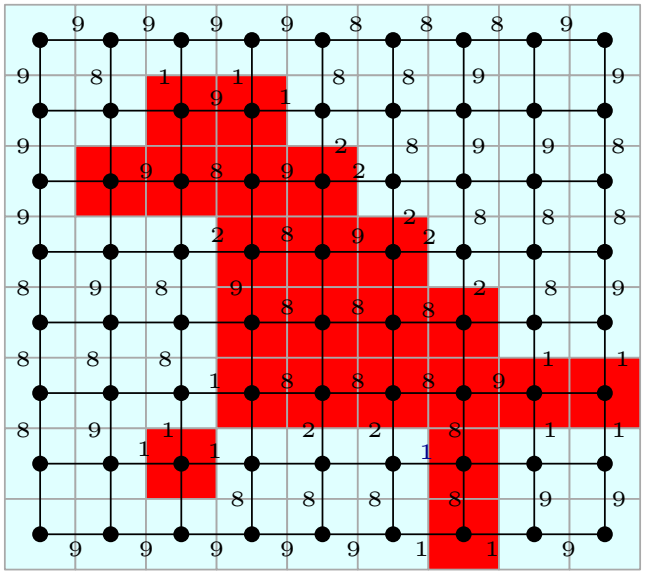
$p_{ij}$ is a "similarity measure" determined by image processing



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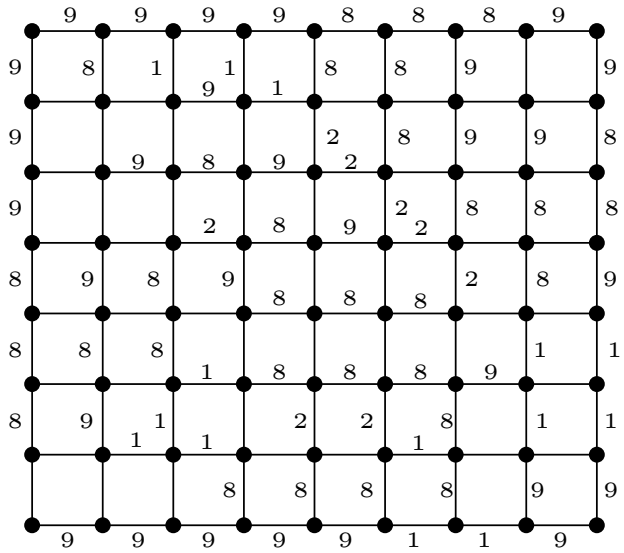

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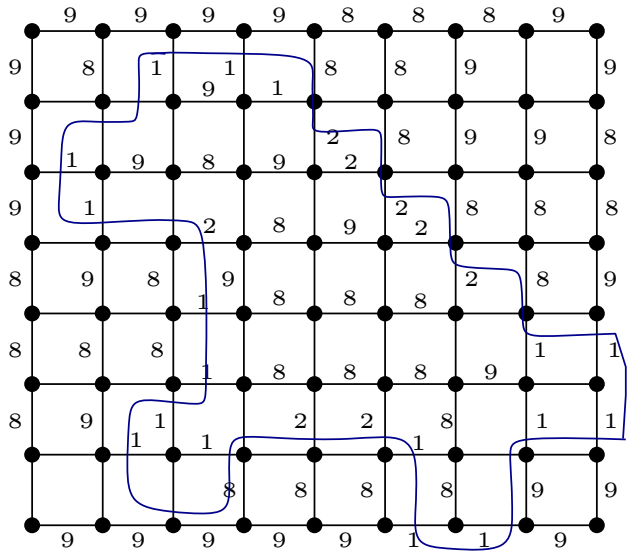


Find a min-cut in this weighted graph

# Global Min-Cut using min $s - t$ cut

**Maximum $s - t$ flow in $G$ is equal to minimum $s - t$ cut**

- Value of the mincut is minimum over all possible $s - t$ cuts in $G$
- **Brute Force Solution:** compute min $s - t$ cut for all pairs of $V$
- $O(n^2)$ calls to min $s - t$ cut (max $s - t$ flow) solver
  - $O(n^2 \cdot m \cdot |f_{max}|)$ ▷ FORD-FULKERSON algorithm
  - $O(n^2 \cdot n \cdot m^2)$ ▷ EDMOND-KARP algorithm
  - $O(n^2 \cdot n^2 \cdot m)$ ▷ DINIC's or push-relabel algorithm
- **Smarter approach:** A fixed node $s$ must appear in one of $S$ or $\overline{S}$. Fix $s$ and find min $s - t$ cut for all $t \in V$
- Only $O(n)$ calls to min $s - t$ cut (max $s - t$ flow) solver

# Algorithms for Min-Cut

Many deterministic algorithms have been proposed

- Stoer-Wagner $O(nm + n^2 \log m)$ time algorithm
- We study a simple randomized algorithm by Karger
- And an elegant extension of it due to Karger and Stein

These algorithms are based on the Edge Contraction Operation

# Types of Graphs: PseudoGraphs and Multigraphs

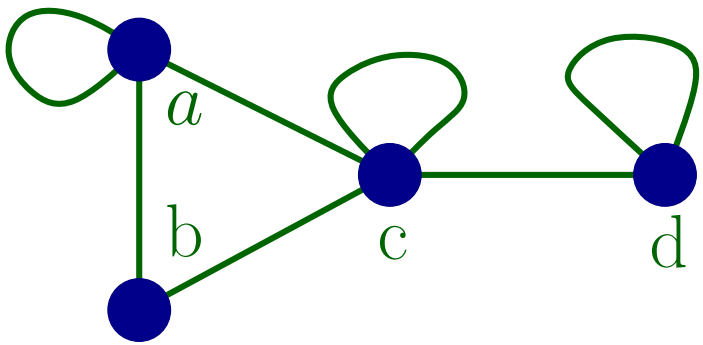
- PseudoGraphs

$G = (V, E)$

$V$ is set of vertices

$E$ is set of edges

(self loops allowed)



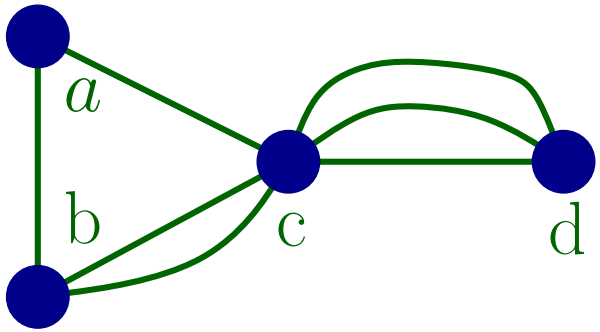

- Multigraphs

$G = (V, E)$

$V$ is set of vertices

$E$ is multi-set of edges

may have self loops too

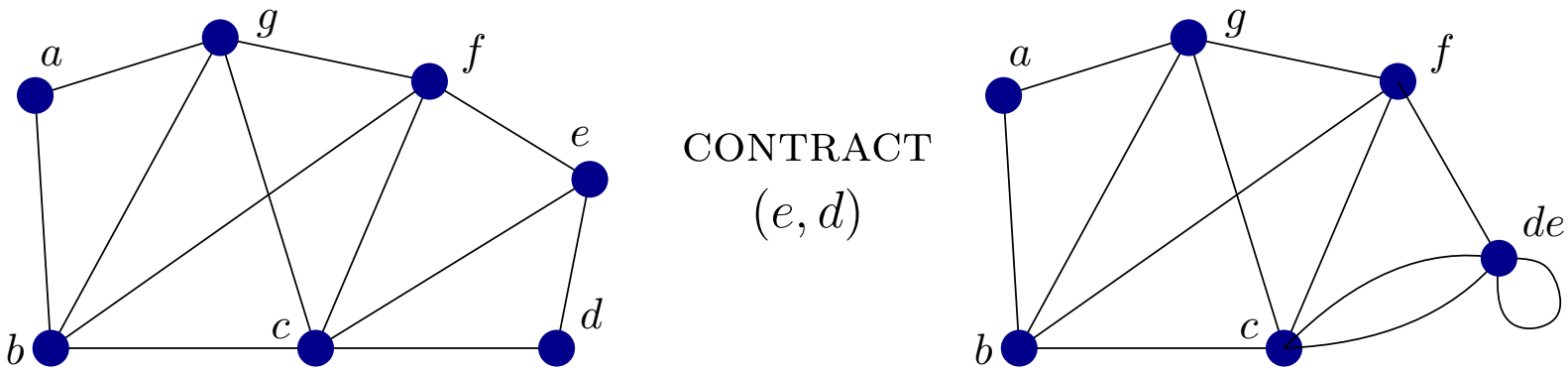


# Edge Contraction

Contraction of an edge $(u, v)$ in $G$ constructs a graph $G \setminus uv$

- $u$ and $v$ become one vertex $uv$
- edge $(u, v)$ becomes a self-loop (we remove it)
- All edges incident on $u$ or $v$ become incident on $uv$

The resulting graph may become a multigraph (we keep all edges)



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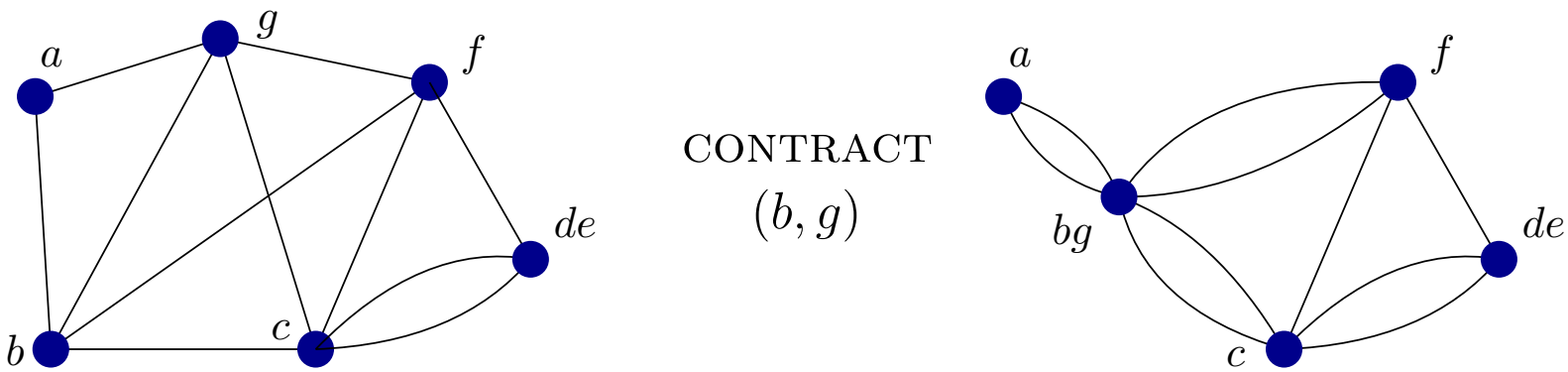

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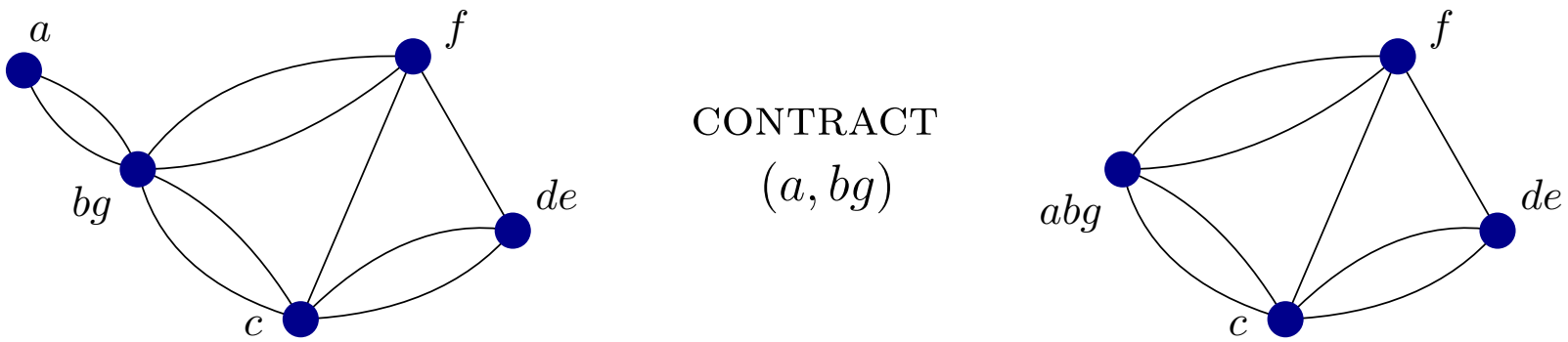

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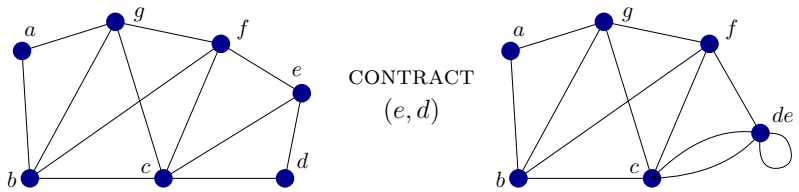
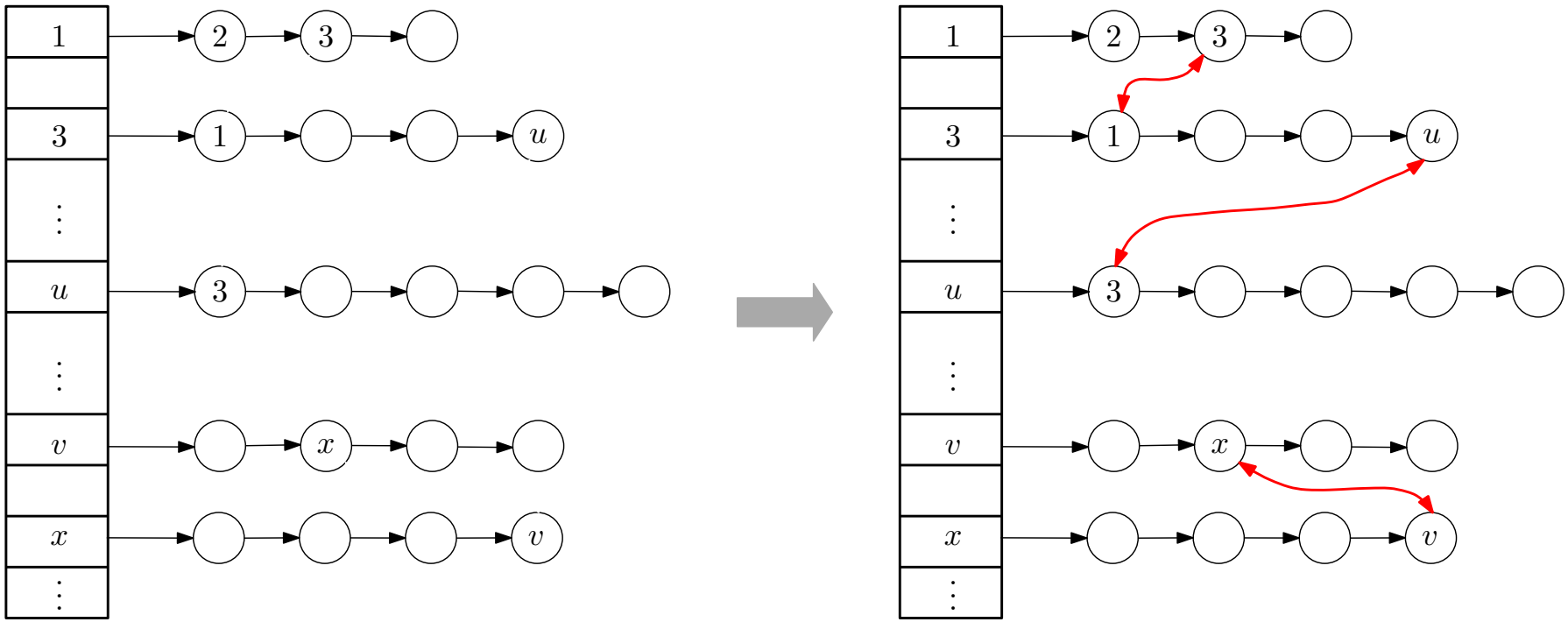



▷ Multigraphs can be saved with multiplicity as edge weight

# Edge Contraction: Runtime

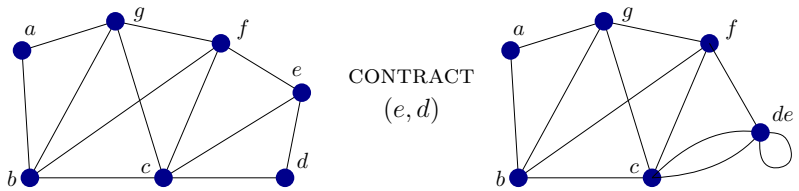
Edge contraction can be performed in $O(n)$ time

- Merge adjacency lists of $u$ and $v$
- Adjacency lists of other vertices can be updated in $O(n)$ time (if we keep corresponding pointers at entries of adjacency lists)



# Edge Contraction

- Contraction of an edge $(u, v)$ in $G$ makes multigraph $G \setminus uv$
- $u$, $v$ merged into $uv$, edges incident on $u$ or $v$ become incident on $uv$




What happens to min cut after contraction?

▷ If the min-cut in $G$ is of size 10, can $G \setminus uv$ have min cut of size 9?

- The min cut in $G \setminus uv$ is at least as large as min cut in $G$
  - Because any cut in $G \setminus uv$ is "actually" a cut in $G$ too
- The converse is not necessarily true


Edge contraction increases min cut if the edge is in all possible min cuts

# Karger's Algorithm

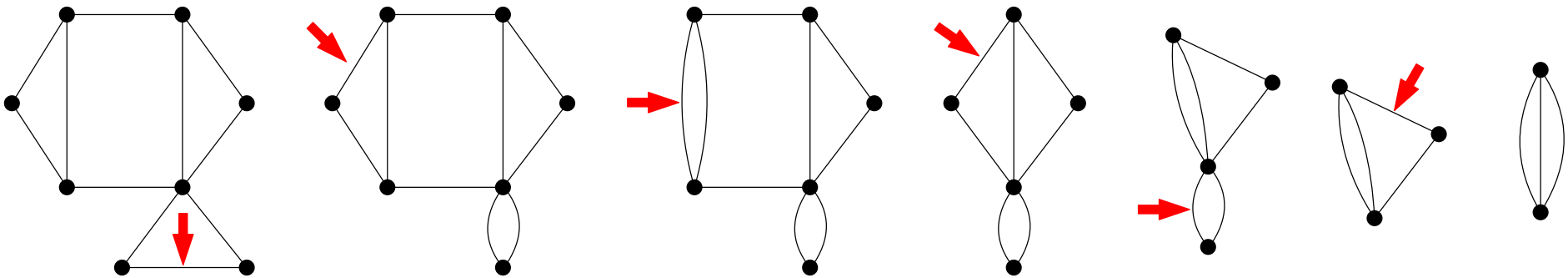
**Algorithm** : Karger's algorithm for mincut $(G)$

**while** there are more than two vertices left in $G$ **do**

Pick a random edge $e = (u, v)$

$G \leftarrow G \setminus uv$

**return** $G$ ▷ the cut induced by the remaining two (super)nodes



A run of Karger algorithm that produces a sub-optimal cut (with 3 edges)

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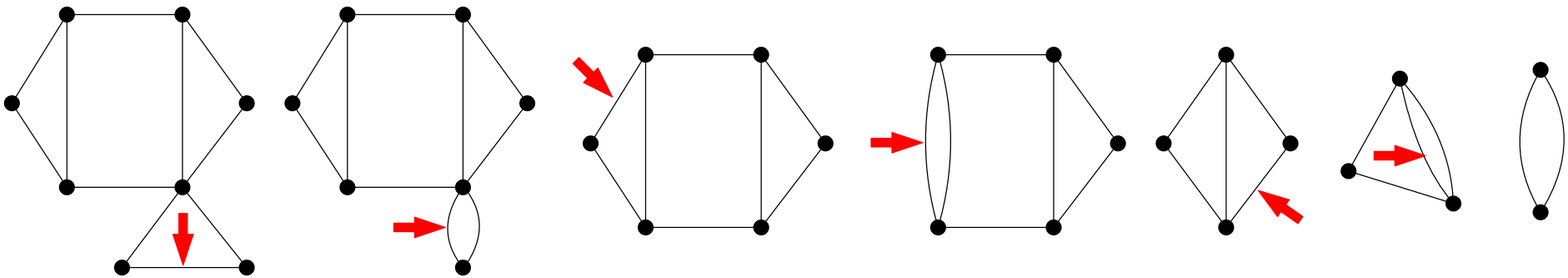

A run of Karger algorithm that produces an optimal cut (with 2 edges)

# Karger's Algorithm: Runtime

**Algorithm** : Karger's algorithm for mincut $(G)$

**while** there are more than two vertices left in $G$ **do**

Pick a random edge $e = (u, v)$

$G \leftarrow G \setminus uv$

**return** $G$ ▷ the cut induced by the remaining two (super)nodes

- With the right data structure a contraction can be done in $O(n)$
- Each contraction reduces the number of vertices by 1
- Number of contraction is $n - 2$
- Total runtime is $O(n^2)$

# Karger's Algorithm: Analysis

The intuition:

- Let $C = [S, \overline{S}]$ be a specific cut
- If during the execution some edge in $C$ is contracted, the algorithm will not output the cut $C$
  - If $(u, v) \in C \leftrightarrow u \in S \wedge v \in \overline{S}$ is contracted, then $u$ and $v$ will belong to the same supernode and $(u, v)$ cannot be a crossing edge
- The algorithm will output $C$ if it never contracts any edge in $C$

Among all cuts, min-cuts have the least probability of having an edge contracted

# Karger's Algorithm: Analysis

Let $G_0 = (V_0, E_0) = G = (V, E)$ ▷ $|V_i| = n_i$, $|E_i| = m_i$

For $0 \leq i \leq n-2$, $G_i = (V_i, E_i)$ : graph after $i$th contraction ▷ $n_i = n - i$

Let $C = [S, \overline{S}]$ be a (specific) min-cut of size $k$

Every vertex has degree $\geq k \implies m_0 \geq {kn_0}/{2}$ ▷ $\because C$ is a min-cut of size $k$

$C$ has survived up to $G_i$, $\implies m_i \geq {kn_i}/{2} = {k(n-1)}/{2}$

$Pr[C \text{ is "killed" in 1st round}] = Pr[\text{an edge in } C \text{ is contracted}] = {k}/{m_0} \leq {2}/{n_0}$

$Pr[C \text{ survives in 1st round}] = Pr[\text{no edge in } C \text{ is contracted}] \geq 1 - {2}/{n_0}$

$Pr[C \text{ survives in } (i+1)\text{th round} \mid C \text{ survived so far}] = 1 - {k}/{m_i} \geq 1 - {2}/{n-i}$

$Pr[C \text{ survives all rounds}] = \prod_{i=0}^{n-3} Pr[C \text{ survives round } i+1 \mid C \text{ survived so far}]$

$Pr[C \text{ survives all rounds}] = Pr[C \text{ is the output}] = \prod_{i=0}^{n-3} \frac{n-i-2}{n-i}$

$Pr[C \text{ is the output}] \geq \frac{n-2}{n} \times \frac{n-3}{n-1} \times \frac{n-4}{n-2} \times \ldots \times \frac{2}{4} \times \frac{1}{3} = \frac{2}{n(n-1)} = {1}/{\binom{n}{2}}$

# Karger's Algorithm: Analysis

Let $G_0 = (V_0, E_0) = G = (V, E)$ ▷ $|V_0| = n$, $|E_0| = m$

Let $C = [S, \overline{S}]$ be a (specific) min-cut of size $k$

$$Pr[C \text{ is the output}] \simeq 1/n^2$$

This probability is very small is it?

- There are $2^m$ cuts, many of them min-cuts, we find one of the min-cuts with probability $1/n^2$
- With repeated trials, we amplify the probability to any desired value

# Karger's Algorithm: Analysis

Let $G_0 = (V_0, E_0) = G = (V, E)$ ▷ $|V_0| = n$, $|E_0| = m$

Let $C = [S, \overline{S}]$ be a (specific) min-cut of size $k$

$$Pr[C \text{ is the output}] \simeq 1/n^2$$

- With repeated trials, we amplify the probability to any desired value

**Algorithm** Good-Min-Cut$(G, M)$

Run Min-Cut$(G)$ $M$ times

Return smallest of these $M$ cuts

**Algorithm** Min-Cut $(G)$

**while** more than two vertices left in $G$ **do**

Pick a random edge $e = (u, v)$

$G \leftarrow G \setminus uv$

**return** $G$

# Karger's Algorithm: Analysis

Let $G_0 = (V_0, E_0) = G = (V, E)$ ▷ $|V_0| = n$, $|E_0| = m$

$C = [S, \overline{S}]$ : a (specific) min-cut of size $k$ ▷ $Pr[C \text{ is the output}] \simeq 1/n^2$

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Run Min-Cut$(G)$ $M$ times

Return smallest of these $M$ cuts

**Algorithm** Min-Cut $(G)$

**while** more than two vertices left in $G$ **do**

Pick a random edge $e = (u, v)$

$G \leftarrow G \setminus uv$

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$Pr[\text{all } M \text{ runs fail to output } C] = \prod_{i=1}^{n} Pr[\text{Run } i \text{ fails}] \leq \left(1 - 1/n^2\right)^M$

$\forall\, x \in \mathbb{R}\ \ (1 + x) < e^x$ ▷ A very useful inequality

$Pr[\text{GOOD-MIN-CUT}(G, M) \text{ fails to output } C] \leq e^{M/n^2}$

$M = cn^2 \log n \implies Pr[\text{GOOD-MIN-CUT}(G, M) \text{ outputs } C] \geq 1 - 1/n^c$

Runtime is $O(n^4 \log n)$

# Karger-Stein Algorithm

**Algorithm** Good-Min-Cut$(G, M)$

Run Min-Cut$(G)$ $M$ times

Return smallest of these $M$ cuts

**Algorithm** : Min-Cut $(G)$

**while** more than two vertices left in $G$ **do**

Pick a random edge $e = (u, v)$

$G \leftarrow G \setminus uv$

**return** $G$

$Pr[C \text{ is "killed" in round } 1] = Pr[\text{an edge in } C \text{ is contracted}] = k/m_0 \leq 2/n$

$Pr[C \text{ is "killed" in round } 2 \mid C \text{ survived round } 1] = k/m_1 \leq 2/n-1$

$Pr[C \text{ is "killed" in rond } (i+1) \mid C \text{ survived so far}] = k/m_i \leq 2/n-i$

$Pr[C \text{ is "killed" in rond } (n-3) \mid C \text{ survived so far}] \leq 2/4$

$Pr[C \text{ is "killed" in rond } (n-2) \mid C \text{ survived so far}] \leq 2/3$

Bound on probability of wrong contraction increases in each round

As $G$ gets smaller, repeat increasingly many times to reduce the error probability

▷ do not waste time repeating the first "few" iterations

# Karger-Stein Algorithm

**Algorithm** Fast-Cut($G$)

**if** $n \leq 6$ **then**
  **return** Min-cut (via brute force)
$t \leftarrow \lceil 1 + n/\sqrt{2} \rceil$
$H_1 \leftarrow \text{CONTRACT}(G, t)$
$H_2 \leftarrow \text{CONTRACT}(G, t)$
$C_1 \leftarrow \text{FAST-CUT}(H_1)$
$C_2 \leftarrow \text{FAST-CUT}(H_2)$
**return** smaller of $C_1$ and $C_2$

**Algorithm** Contract $(G, t)$

**function** CONTRACT$(G, t)$
  **while** more than $t$ vertices left in $G$ **do**
    Pick a random edge $e = (u, v)$
    $G \leftarrow G \setminus uv$
  **return** $G$

- Two independent randomly contracted graphs $H_1$ and $H_2$ from $G$
- When $H_1$ and $H_2$ are small, make 4 random contractions
- and so on
- When graph has less 6 vertices, return min among all $\sim 2^5$ cuts
- Now we cannot chase a fixed minimum cut $C$, as both $X_1$ and $X_2$ could be min cuts (if successful) and we may choose either

# Karger-Stein Algorithm

**Algorithm** Fast-Cut($G$)

**if** $n \leq 6$ **then**
  **return** Min-cut (via brute force)
$t \leftarrow \lceil 1 + n/\sqrt{2} \rceil$
$H_1 \leftarrow \text{CONTRACT}(G, t)$
$H_2 \leftarrow \text{CONTRACT}(G, t)$
$C_1 \leftarrow \text{FAST-CUT}(H_1)$
$C_2 \leftarrow \text{FAST-CUT}(H_2)$
**return** smaller of $C_1$ and $C_2$

**Algorithm** Contract $(G, t)$

**function** CONTRACT$(G, t)$
  **while** more than $t$ vertices left in $G$ **do**
    Pick a random edge $e = (u, v)$
    $G \leftarrow G \setminus uv$
  **return** $G$

Let $T(n)$ be runtime of FAST-CUT$(G)$ with $|V(G)| = n$

$$T(n) = \begin{cases} 2T\left(n/\sqrt{2}\right) + O(n^2) & \text{if } n > 6 \\ O(1) & \text{else} \end{cases}$$

$\mathbf{T(n) = O(n^2 \log n)}$ ▷ master theorem

# Karger-Stein Algorithm: Quality

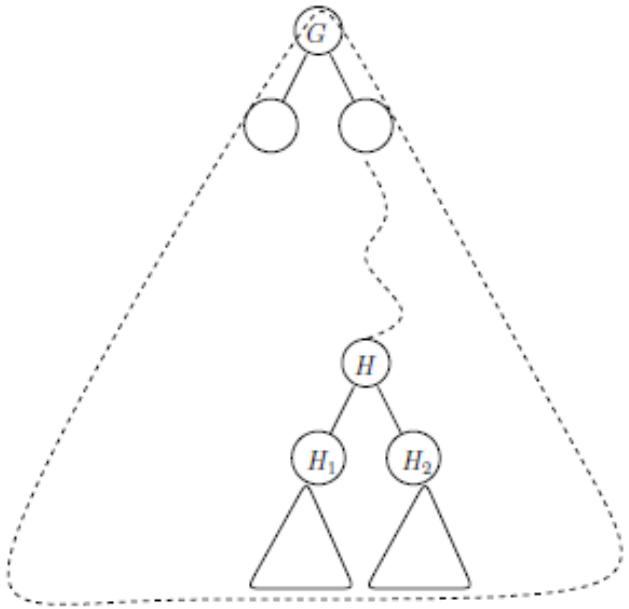
1: **function** FAST-CUT($G$)
2: **if** $n \leq 6$ **then**
3: **return** Min-cut (brute force)
4: $t \leftarrow \lceil 1 + n/\sqrt{2} \rceil$
5: $H_1 \leftarrow$ CONTRACT($G, t$)
6: $H_2 \leftarrow$ CONTRACT($G, t$)
7: $C_1 \leftarrow$ FAST-CUT($H_1$)
8: $C_2 \leftarrow$ FAST-CUT($H_2$)
9: **return** smaller of $C_1$ and $C_2$

**Algorithm** Contract $(G, t)$

**function** CONTRACT($G, t$)
**while** more than $t$ vertices left in $G$ **do**
Pick a random edge $e = (u, v)$
$G \leftarrow G \setminus uv$
**return** $G$

FAST-CUT($G$) succeeds iff

- A min-cut survives the CONTRACT($G, t$) step
- At least one of the FAST-CUT($H_1$) and FAST-CUT($H_2$) finds a min-cut



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1: function FAST-CUT(G)
2:     if n ≤ 6 then
3:         return Min-cut
4:     t ← ⌈1 + n/√2⌉
5:     H1 ← CONTRACT(G, t)
6:     H2 ← CONTRACT(G, t)
7:     C1 ← FAST-CUT(H1)
8:     C2 ← FAST-CUT(H2)
9:     return MIN of C1 and C2
```

Probability a min cut survive CONTRACT($G, t$) step ▷ line 5&6

$$Pr[\text{a cut survives } n-t \text{ contractions}] = \prod_{i=0}^{n-t-1} \frac{n-i-2}{n-i}$$

$$Pr[\text{a cut survives } n-t \text{ contractions}] = \frac{n-2}{n} \times \ldots \times \frac{t}{t+2} \times \frac{t-1}{t+1} = \frac{t(t-1)}{n(n-1)}$$

$$Pr[\text{a cut survives } n-t \text{ contractions}] = t(t-1)/n(n-1) \simeq 1/2 \qquad \triangleright\ t = n/\sqrt{2}$$

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$P(j)$ : prob that FAST-CUT($H$) finds min-cut if $|V(H)| = j$

Probability that FAST-CUT($G$) succeeds: **P(n)**

- A min-cut survives in $H_1$ (line 5) ▷ Prob: $1/2$
- **AND** $C_1$ is a min-cut in $H_1$ (line 7) ▷ Prob: $P(t)$

**OR**

- A min-cut survives in $H_2$ (line 6) ▷ Prob: $1/2$
- **AND** $C_2$ is a min-cut in $H_2$ (line 8) ▷ Prob: $P(t)$

# Karger-Stein Algorithm: Quality

$P(j)$ : prob that FAST-CUT$(H)$ finds min-cut if $|V(H)| = j$

Probability that FAST-CUT$(G)$ succeeds **P(n)**

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**OR**

- A min-cut survives in $H_2$ (line 6) ▷ Prob: $1/2$
- **AND** $C_2$ is a min-cut in $H_2$ (line 8) ▷ Prob: $P(t)$

$$Pr[\text{Branch-i succeeds}] = Pr\left[\begin{array}{l}\text{A min-cut survives in } H_i \text{ (line 5/6)}\\ \textbf{AND } C_i \text{ is min-cut in } H_i \text{ (line 7/8)}\end{array}\right] = \frac{1}{2}\cdot P(t)$$

$$Pr[\text{Branch-i fails}] = 1 - 1/2 P(t) \qquad Pr[\text{Both Branches fail}] = (1 - 1/2 P(t))^2$$

$$Pr[\text{Algo succeeds}] = Pr[\textbf{NOT } \text{Both Branches fail}] \geq 1 - (1 - 1/2 P(t))^2$$

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$$P(n) \geq 1 - (1 - 1/2 P(t))^2 = 1 - (1 - 1/2 P(n/\sqrt{2}))^2 = \Omega(1/\log n)$$

Easily proved via induction

# Karger-Stein Algorithm: Quality

- FAST-CUT($G$) takes $O(n^2 \log n)$ times not much worse than $O(n^2)$ initial version
- Has a success probability $\Omega(1/\log n)$ much better than $\Omega(1/n^2)$ of initial version
- The initial version amplified by $n^2 \log n$ independent trial had runtime $O(n^4 \log n)$ and success probability $\Omega(1 - 1/n^c)$
- FAST-CUT($G$) amplified by $c \log^2 n$ independent trial has runtime $O(n^2 \log^3 n)$ and success probability $\Omega(1 - 1/n^c)$