Approximation Algorithms

- Approximation Algorithms for Optimization Problems: Types
- Absolute Approximation Algorithms
- Inapproximability by Absolute Approximate Algorithms
- Relative Approximation Algorithm
- InApproximability by Relative Approximate Algorithms
- Polynomial Time Approximation Schemes
- Fully Polynomial Time Approximation Schemes

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Inapproximability

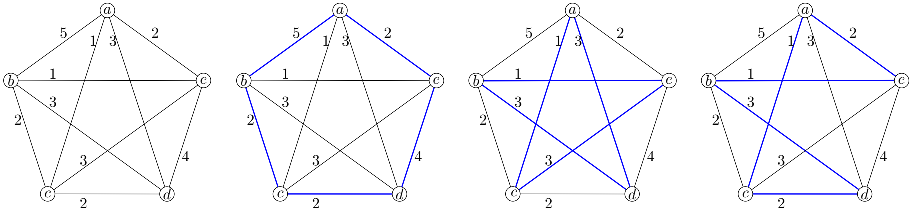
All NP-COMPLETE problems are somewhat equivalent with respect to polynomial time reduction

But in terms of approximability, they could substantially differ

- The PLANAR k-COLOR has a 1-absolute approximation algorithm
- INDEPENDENT-SET has no k-absolute approximation algorithm
- VERTEX-COVER has an 2-approximate algorithm
- VERTEX-COVER has no $4/3$-approximation unless $\mathrm{P} = \mathrm{NP}$
- TSP has no constant factor approximation unless $\mathrm{P} = \mathrm{NP}$

Traveling Salesman Problem

Given a complete graph G on n vertices with edge weights $w : E \mapsto \mathbb{R}$, a TSP tour is a Hamiltonian cycle in G



K_5 with edge weights

A TSP tour of length 15

A TSP tour of length 11

A TSP tour of length 9

Traveling Salesman Problem TSP(G): Find TSP tour of minimum length?

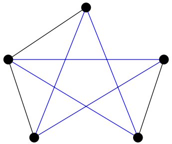
TSP: Impossibility of Relative Approximation

If $\mathrm{P} \neq \mathrm{NP}$, then for any $\alpha > 1$, there is no α-approximation for TSP

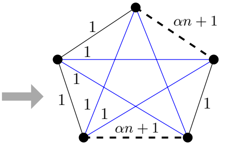
Reduce HAM-CYCLE(G) problem to α-APPROXIMATE-TSP(G) problem

- Given an instance $G = (V, E)$ of HAM-CYCLE(G), $|V| = n$
- Make a complete graph G' on n vertices with weights

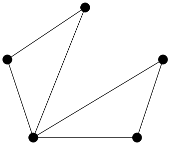
$$w(v_i, v_j) = \begin{cases} 1 & \text{if } (v_i, v_j) \in E(G) \\ \alpha n + 1 & \text{else} \end{cases}$$



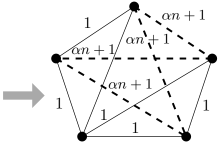
Hamiltonian cycle in G shown in blue



TSP tour in G' of length $5 = n$ shown in blue



No Hamiltonian cycle in G



Any TSP tour must use an edge of weight $\alpha n + 1$

If $\mathrm{P} \neq \mathrm{NP}$, then for any $\alpha > 1$, there is no α-approximation for TSP

Reduce HAM-CYCLE(G) problem to α-APPROXIMATE-TSP(G) problem

- Suppose there is an algorithm $\mathcal{A}$ for the α-APPROXIMATE-TSP(G) problem
- For instance $G = (V, E)$ of HAM-CYCLE(G), $|V| = n$, make $K_n = G'$ with

$$w(v_i, v_j) = \begin{cases} 1 & \text{if } (v_i, v_j) \in E(G) \\ \alpha n + 1 & \text{else} \end{cases}$$

1. If G has a HAM-CYCLE, then OPT-TSP$(G') = n$
2. If G has no HAM-CYCLE, then OPT-TSP$(G') \geq \alpha n + 1$

- Run $\mathcal{A}$ on G'
- If $\mathcal{A}$ returns a TSP tour of length $\leq \alpha n$, output **Yes** for HAM-CYCLE(G)

We solved HAM-CYCLE(G) in poly-time and proved $\mathrm{P} = \mathrm{NP}$

Distance Metric and Metric Space

A distance function takes two objects and returns a real number

A distance function, $d(u, v)$ is a **distance metric** if it has 4 properties

1. $d(u, v) \geq 0$ $\triangleright$ Non-negativity
   - it doesn't make sense to have distance of -3
2. $d(u, v) = 0 \iff u = v$ $\triangleright$ Indiscernibility
3. $d(u, v) = d(v, u)$ $\triangleright$ Symmetry
4. $d(u, w) \leq d(u, v) + d(v, w)$ $\triangleright$ Triangle Inequality
   - direct distance is shorter than the distance via an intermediate point

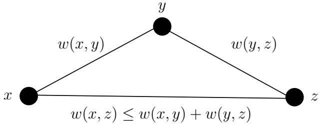
Euclidean or metric-TSP

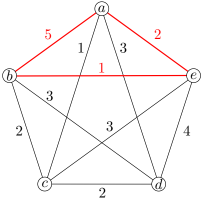
Given a complete graph G on n vertices with metric edge weights $w : E \mapsto \mathbb{R}^+$, a TSP tour is a Hamiltonian cycle in G

$\triangleright$ For all vertices x, y, z $\quad w(x,z) \leq w(x,y) + w(y,z)$

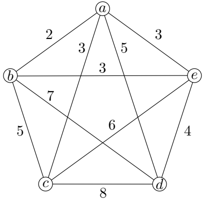
Euclidean-TSP: When

- vertices are points in a plane
- distance is Euclidean distance





Not a metric-TSP instance



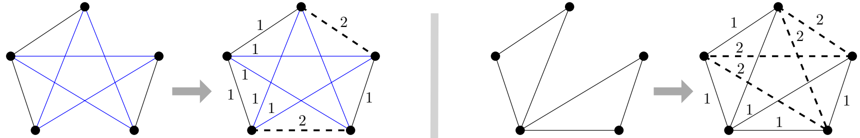
A metric-TSP instance

METRIC-TSP is NP-HARD   HAM-CYCLE(G) $\leq_p$ METRIC-TSP(G', k)

- METRIC-TSP(G', k) requires weighted graph and a number k
- Given an instance $G = (V, E)$ of HAM-CYCLE(G), $|V| = n$
- Make a complete graph on n vertices G' with weights as follows

$$w(v_i, v_j) = \begin{cases} 1 & \text{if } (v_i, v_j) \in E(G) \\ 2 & \text{else} \end{cases}$$

▷ w induces a distance metric



Hamiltonian cycle in G shown in blue

TSP tour in G' of length 5 shown in blue

No Hamiltonian cycle in G

No TSP tour of length 5 in G'

G has a Hamiltonian cycle iff G' has a TSP tour of length $k = n$

Lower Bound for METRIC-TSP

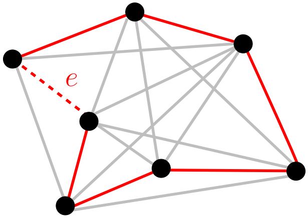
If C is HAM-CYCLE and T^* is a MST in G, then

$$w(T^*) \leq w(C)$$

Proof: Let C be a HAM-CYCLE and $e \in C$ be any edge

$T = C \setminus e$ is a spanning tree in G ▷ C is a HAM-CYCLE

$w(T^*) \leq w(T) \leq w(C \setminus e) \leq w(C)$ ▷ T^* is a MST

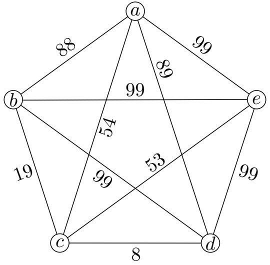


2-approximation for METRIC-TSP

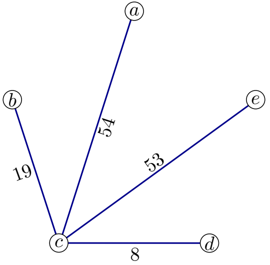
We use a spanning tree T to find a TSP tour C

Suppose each edge in T is duplicated, i.e. can be used twice

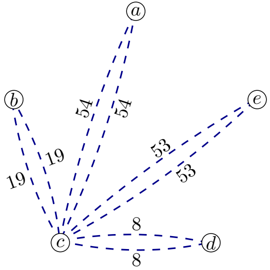
Consider some vertex s to be the root of the T



A metric-TSP instance



An MST T



T rooted at c and edges duplicated

Eulerian Graphs

Euler Circuit

A closed walk in G containing every edge of G

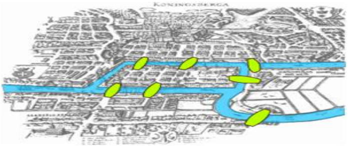
Euler Path

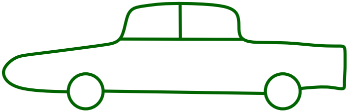
A walk in G containing every edge of G

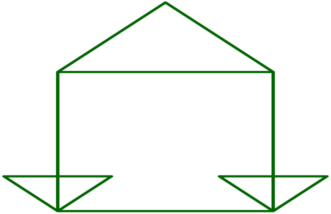
Eulerian Graphs

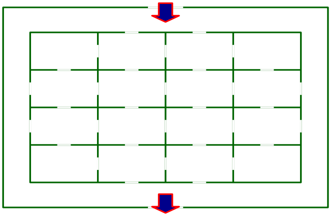
Tour this city/building passing each bridge/door exactly once

Draw the picture without lifting pencil or retracing









Which graphs has Euler Path/Circuit?

Eulerian Graphs

Euler Circuit

A closed walk in G containing every edge of G

Euler Path

A walk in G containing every edge of G

Theorem

G has an Euler circuit if and only if every vertex has even degree

Theorem

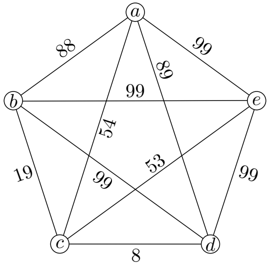
G has an Euler path if and only if it has exactly two vertices of odd degree

2-approximation for METRIC-TSP

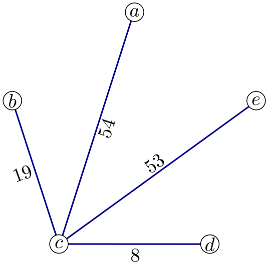
We use a spanning tree T to find a TSP tour C

Suppose each edge in T is duplicated, i.e. can be used twice

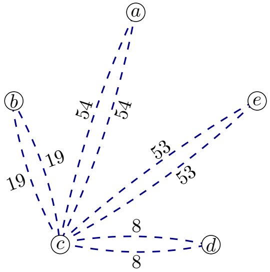
Consider some vertex s to be the root of the T



A metric-TSP instance



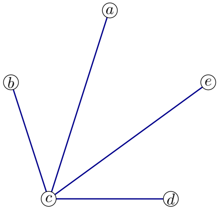
An MST T



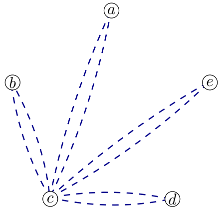
T rooted at c and edges duplicated

2-approximation for METRIC-TSP

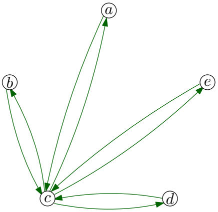
- Let L be an Euler tour on $T^{(*)}$ starting from s (the root)
- List vertices in order of L ▷ with repetitions



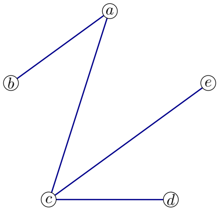
An MST T



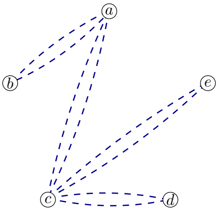
T rooted at c and edges duplicated



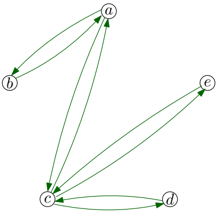
L $c, a, c, b, c, d, c, e, c$



An MST T



T rooted at d and edges duplicated

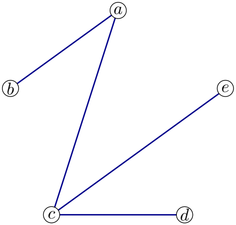


L $d, c, e, c, a, b, a, c, d$

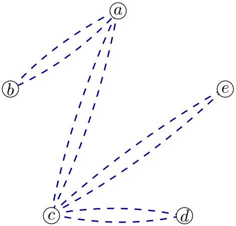
2-approximation for METRIC-TSP

- Let L be an Euler tour on $T^{(*)}$ starting from s (the root)
- List vertices in order of L ▷ with repetitions
- $w(L) = \sum_{e \in L} w(e)$
- Let C^* be an optimal TSP tour in G

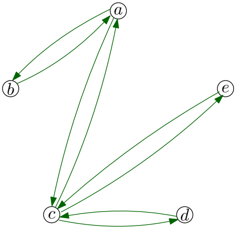
$$w(L) = 2w(T) \leq 2w(C^*) \qquad \triangleright \because w(T) \leq w(C^*)$$



An MST T



T rooted at d and edges duplicated



L $d, c, e, c, a, b, a, c, d$

2-approximation for METRIC-TSP

C^* : an optimal TSP tour $\quad T$: an MST $\quad L$: EULER tour on $T^{(*)}$

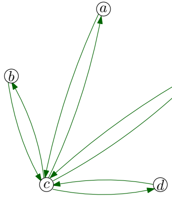
$$w(L) = 2w(T) \leq 2w(C^*)$$

Is $L = C^*$? $\quad \triangleright$ C^* visits each vertex once (except first)

Short-circuit

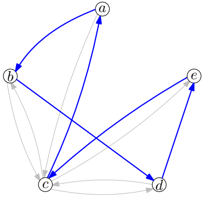
L to get a TSP tour C

- Keep only first occurrence of a vertex
- instead of revisiting, visit the *next* vertex
- keep the last vertex, the root repeated



L_1 $c, a, c, b, c, d, c, e, c$

$c \quad a \quad c \quad b \quad c \quad d \quad c \quad e \quad c$



C_1 = SHORT-CIRCUITED L_1

$c \quad a \quad c \quad b \quad c \quad d \quad c \quad e \quad c$

2-approximation for METRIC-TSP

C^* : an optimal TSP tour T : an MST L : EULER tour on $T^{(*)}$

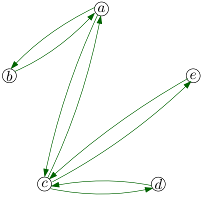
$$w(L) \;=\; 2w(T) \;\leq\; 2w(C^*)$$

Is $L \;=\; C^*$? $\triangleright$ C^* visits each vertex once (except first)

Short-circuit

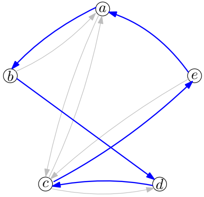
L to get a TSP tour C

- Keep only first occurrence of a vertex
- instead of revisiting, visit the *next* vertex
- keep the last vertex, the root repeated



L_2 $d, c, e, c, a, b, a, c, d$

$d \quad c \quad e \quad c \quad a \quad b \quad a \quad c \quad d$



$C_2 =$ SHORT-CIRCUITED L_2

$d \quad c \quad e \quad c \quad a \quad b \quad a \quad c \quad d$

2-approximation for METRIC-TSP

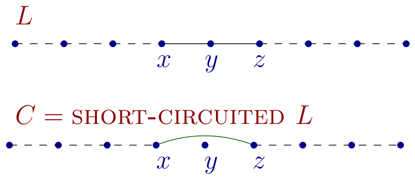
C^* : an optimal TSP tour T : an MST L : EULER tour on $T^{(*)}$

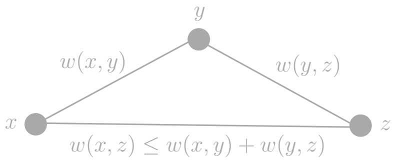
$$w(L) = 2w(T) \leq 2w(C^*)$$

Short-circuit

L to get a TSP tour C

- Keep only first occurrence of a vertex
- instead of revisiting, visit the *next* vertex
- keep the last vertex, the root repeated





$$w(C) \leq w(L) = 2w(T) \leq 2w(C^*)$$

2-approximation for METRIC-TSP

Algorithm DOUBLE-TREE-TSP(G)

$T \leftarrow$ MST(G) ▷ e.g. KRUSKAL algorithm

$T^{(*)} \leftarrow$ DUPLICATE edges of T ▷ every vertex has even degree

$L \leftarrow$ EULER tour of $T^{(*)}$ ▷ Fleury or Hierholzer algorithm

$C \leftarrow$ SHORT-CIRCUIT(L)

return C

Runtime is clearly polynomial

$$w(C) \leq w(L) = 2w(T) \leq 2w(C^*)$$

DOUBLE-TREE-TSP(G) is a 2-approximation for METRIC-TSP

Can we do better?

1.5-approximation for METRIC-TSP

Christofides Algorithm is 1.5−approximation for METRIC-TSP

- The factor 2 appeared because of duplicating all edges of T
- We needed all vertices to have even degree for an Euler tour
- Why alter vertices with already even degree?
- Is there a less costly way to make degrees of select vertices even?

Hand-shaking lemma

The number of odd-degree vertices in T is even

1.5-approximation for METRIC-TSP

C^* : an optimal TSP tour, T : an MST

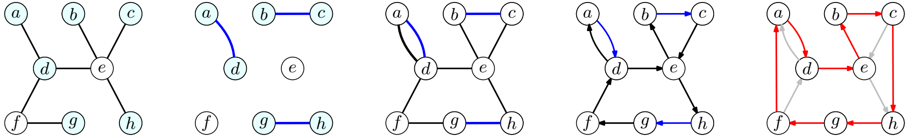
- Let O be the set of odd degree vertices in T ▷ $|O|$ is even
- M : min-cost perfect matching in subgraph induced by vertices in O
  - e.g. Micali and Vazirani algorithm ▷ $O(n^{2.5})$

1.5-approximation for METRIC-TSP

C^* : an optimal TSP tour, T : an MST

- Let O be the set of odd degree vertices in T
- M : min-cost perfect matching in subgraph induced by vertices in O

1. Combine edges of M and T to get a multigraph H
2. Find an Euler tour L in H ▷ Each vertex in H has even degree
3. Obtain C by short-circuiting L

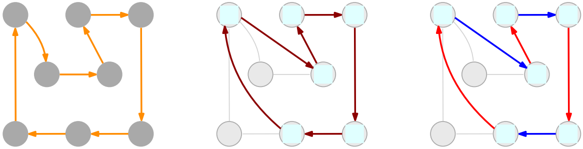


Spanning tree T | Matching M among odd degree vertices | $H = M \cup T$ | EULER tour L | C : SHORT-CUT L

$$w(C) \leq w(L) = w(T) + w(M) \leq w(C^*) + w(M)$$

1.5-approximation for METRIC-TSP

$$w(C) \leq w(L) = w(T) + w(M) \leq w(C^*) + w(M)$$



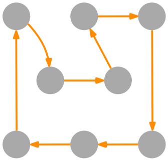
A HAM-CYCLE C | C SHORT-CUT to even number of vertices, C' | C' decomposed into 2 perfect matchings

Let C be a HAM-CYCLE and $U \subseteq V$ with $|U|$ is even. For a min-cost perfect matching M on U, $w(M) \leq 1/2 w(C)$

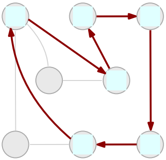
- SHORT-CIRCUIT C to C' on vertices in U
- $w(C') \leq w(C)$ ▷ triangle inequality
- C' can be decomposed into two perfect matchings on U
- M is a min-cost perfect matching, we get $w(M) \leq 1/2 w(C')$

1.5-approximation for METRIC-TSP

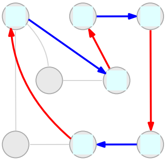
$$w(C) \leq w(L) = w(T) + w(M) \leq w(C^*) + w(M)$$



A HAM-CYCLE C



C SHORT-CUT to even number of vertices, C'



C' decomposed into 2 perfect matchings

Let C be a HAM-CYCLE and $U \subseteq V$ with $|U|$ is even. For a min-cost perfect matching M on U, $w(M) \leq {}^1\!/_2 w(C)$

$$w(C) \leq w(L) = w(T) + w(M) \leq w(C^*) + w(M) \leq (1 + {}^1\!/_2) w(C^*)$$

The Christofides algorithm is a 1.5-approximation for METRIC-TSP