Elements of Probability

Sample Space and Events

An experiment is any process that produces an observation or **outcome**.

The **set of all possible outcomes** of an experiment is called the **sample space** and is denoted by S.

Examples: (a) If the experiment consists of flipping two coins and noting whether they land heads or tails, then

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

The outcome is (H, H) if both coins land heads, (H, T) if the first coin lands heads and the second tails, (T, H) if the first is tails and the second is heads, and (T, T) if both coins land tails.

(b) If the outcome of the experiment is the order of finish in a race among 7 horses having positions 1, 2, 3, 4, 5, 6, 7, then

$$S = \{\text{all orderings of } 1, 2, 3, 4, 5, 6, 7\}$$

The outcome (4, 1, 6, 7, 5, 3, 2) means, for instance, that the number 4 horse comes in first, the number 1 horse comes in second, and so on.

(c) Consider an experiment that consists of rolling two six-sided dice and noting the sides facing up. Calling one of the dice die 1 and the other die 2, we can represent the outcome of this experiment by the pair of upturned values on these dice. If we let (i, j) denote the outcome in which die 1 has value i and die 2 has value j, then the sample space of this experiment is

$$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$$

Any set of outcomes of an experiment is called an **event**. That is, an event is a subset of the sample space. Events will be denoted by the capital letters A, B, C, and so on.

In example (b), if

$$A = \{(H, H), (H, T)\},$$

then A is the event that the first coin lands on heads.

In example (c) if

$$A = \{\text{all outcomes in } S \text{ starting with } 2\}$$

then A is the event that horse number 2 wins the race.

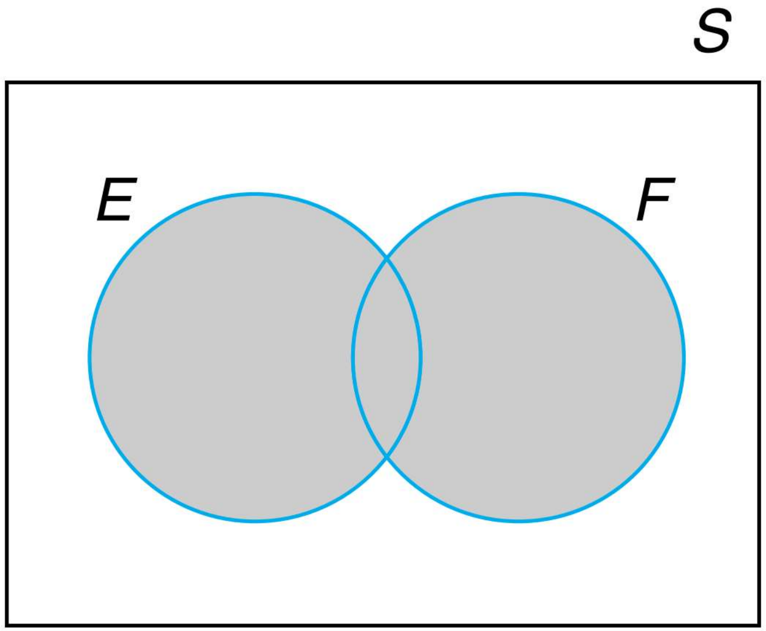
Simple & Compound Events

An event is simple if it consists of exactly one outcome and compound if it consists of more than one outcome.

Occurrence of an Event

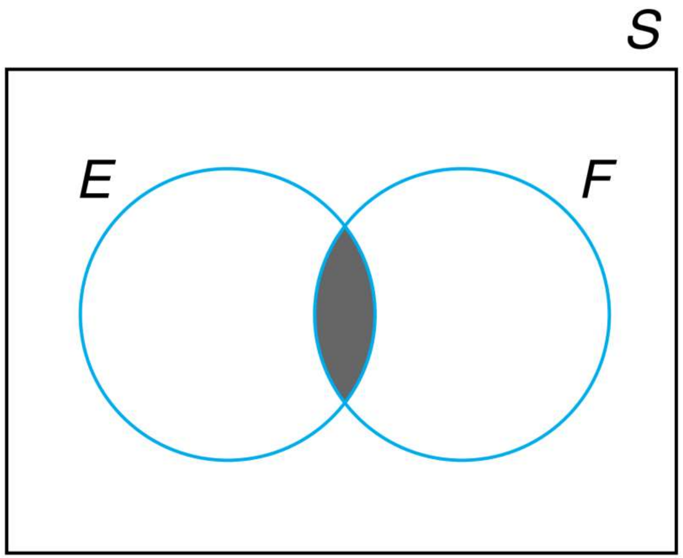
When an experiment is performed, a particular event A is said to occurred if the resulting experimental outcome is contained in A. In general, exactly one simple event will occur, but many compound events can occur simultaneously.

For any two events E and F of a sample space S, we define the new event $E \cup F$, called the union of the events E and F, to consist of all outcomes that are either in E or in F or in both E and F. That is, the event $E \cup F$ will occur if either E or F occurs.



Shaded region: $E \cup F$

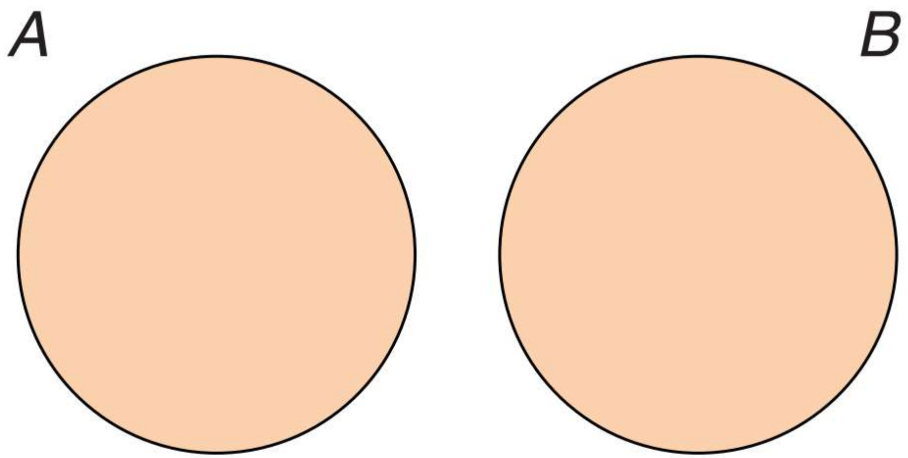
Similarly, for any two events E and F, we may also define the new event EF, sometimes written as E ∩ F, called the intersection of E and F, to consist of all outcomes that are in both E and F. That is, the event EF will occur only if both E and F occur.



shaded region: *EF*

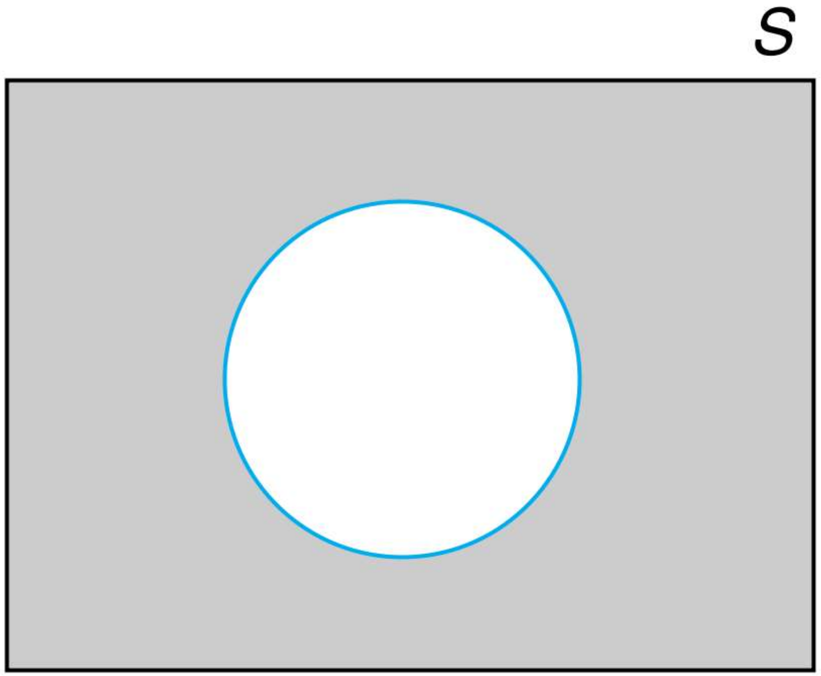
An event consisting of no outcome is called null event, and we denote it by $\emptyset$.

If the intersection of A and B is the null event, then since A and B cannot simultaneously occur, we say that A and B are disjoint, or mutually exclusive.



A and B are disjoint events.

For any event E we define the event E^C, called the complement of E, to consist of all outcomes in the sample space that are not in E. That is, E^C will occur if and only if E does not occur.



Shaded region: E^c

We say that events A, B, and C are disjoint if no two of them can simultaneously occur.

Commutative law	$E \cup F = F \cup E$	$EF = FE$
Associative law	$(E \cup F) \cup G = E \cup (F \cup G)$	$(EF)G = E(FG)$
Distributive law	$(E \cup F)G = EG \cup FG$	$EF \cup G = (E \cup G)(F \cup G)$

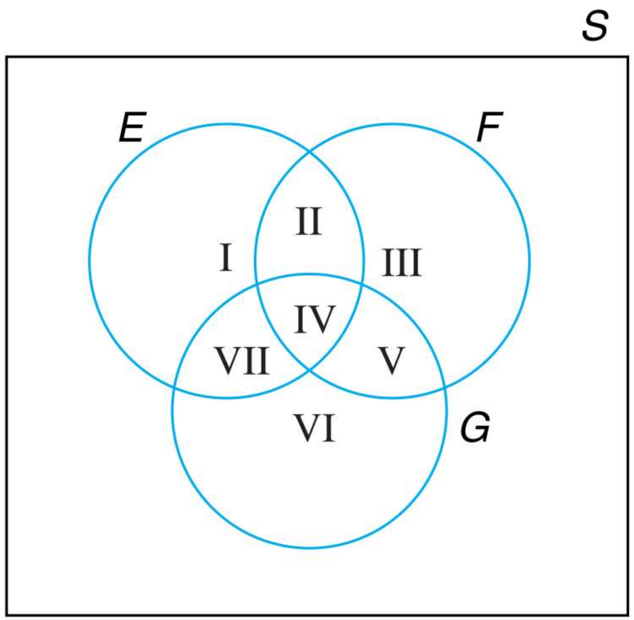
DeMorgan's laws

$$(E \cup F)^c = E^c F^c$$

$$(EF)^c = E^c \cup F^c$$

Practice Problem

For the following Venn diagram, describe in terms of E, F, and G the events denoted in the diagram by the Roman numerals I through VII.



The word probability is a commonly used term that relates to the chance that a particular event will occur when some experiment is performed.

Axioms of Probability

Suppose that for each event E of an experiment having a sample space S there is a number, denoted by P (E), that is in accord with the following three axioms:

Axiom 1: $0 \leq P(E) \leq 1$

Axiom 2: $P(S) = 1$

Axiom 3: For any sequence of mutually exclusive events $E_1, E_2, \ldots$(that is, events for which $E_i E_j = \emptyset$ when $i \neq j$),

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i), \qquad n = 1, 2, \ldots, \infty$$

We call $P(E)$ the probability of the event E.

These axioms will now be used to prove two simple propositions concerning probabilities. We first note that E and E^c are always mutually exclusive, and since $E \cup E^c = S$, we have by Axioms 2 and 3 that

$$1 = P(S) = P(E \cup E^c) = P(E) + P(E^c)$$

Or equivalently, we have the following:

PROPOSITION 3.4.1

$$P(E^c) = 1 - P(E)$$

PROPOSITION 3.4.2

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

Proof

This proposition is most easily proven by the use of a Venn diagram as shown in Figure 3.4.

We can write $E \cup F = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$

$$\Rightarrow P(E \cup F) = P(\mathrm{I} \cup \mathrm{II} \cup \mathrm{III})$$

As the regions I, II, and III are mutually exclusive, so by third axiom

$$P(E \cup F) = P(\mathrm{I}) + P(\mathrm{II}) + P(\mathrm{III}) \text{ — (i)}$$

From Venn diagram and using axiom 3 we have

$P(E) = P(\mathrm{I}) + P(\mathrm{II})$ or $P(\mathrm{I}) + P(\mathrm{II}) = P(E)$ —— (ii)

$P(F) = P(\mathrm{II}) + P(\mathrm{III})$ $\Rightarrow$ $P(\mathrm{III}) = P(F) - P(\mathrm{II})$ —— (iii)

Substituting values from eq (ii) and (iii) in eq (i) we get

$P(E \cup F) = P(E) + P(F) - P(\mathrm{II})$

$\Rightarrow \boxed{P(E \cup F) = P(E) + P(F) - P(EF)}$

proved

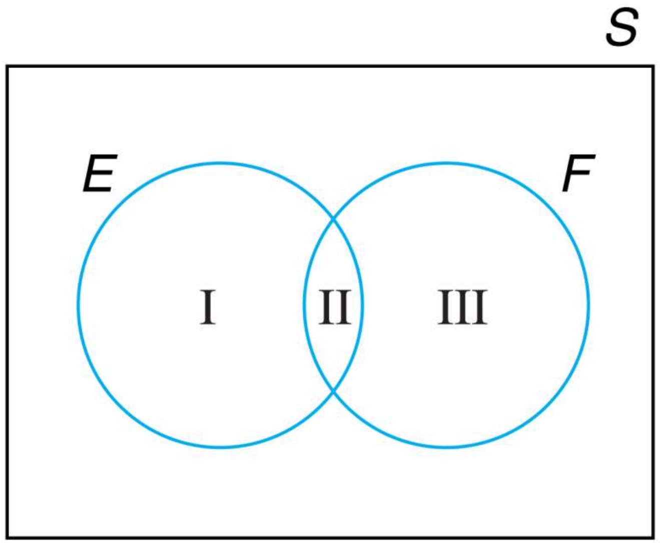


FIGURE 3.4 *Venn diagram.*

Odds of an Event

The *odds* of an event A is defined by

$$\frac{P(A)}{P(A^c)} = \frac{P(A)}{1 - P(A)}$$

Thus the odds of an event A tells how much more likely it is that A occurs than that it does not occur. For instance, if $P(A) = 3/4$, then $P(A)/(1 - P(A)) = 3$, so the odds are 3. Consequently, it is 3 times as likely that A occurs as it is that it does not.

Sample spaces having equally likely outcomes

For a large number of experiments, it is natural to assume that each point in the sample space is equally likely to occur. That is, for many experiments whose sample space S is a finite set, say $S = \{1, 2, \ldots, N\}$, it is often natural to assume that

$$P(\{1\}) = P(\{2\}) = \cdots = P(\{N\}) = p \quad \text{(say)}$$

As we can write $S = \{1\} \cup \{2\} \ldots \cup \{N\}$

$$\Rightarrow P(S) = P(\{1\} \cup \{2\} \ldots \cup \{N\}) \text{ —— (i)}$$

Since $\{1\}, \{2\}, \ldots, \{N\}$ are mutually exclusive. This implies

$$P(\{1\} \cup \{2\} \ldots \cup \{N\}) = P(\{1\}) + P(\{2\}) + \cdots + P(\{N\})$$

$$= P + P + \ldots + P$$

$$\Rightarrow \boxed{P(\{1\} \cup \{2\} \ldots \cup \{N\}) = NP} \quad \text{also} \quad P(S) = 1$$

Substituting values in (i) we get

$$1 = NP \Rightarrow P = \frac{1}{N}$$

$$\boxed{P(\{1\}) = P(\{2\}) = \dots = P(\{N\}) = \frac{1}{N}}$$

which shows that

$$\boxed{P(\{i\}) = p = 1/N}$$

Consider an event having n number of points denoted by 1, 2, ..., n. We can write $P(E) = P(\{1\} \cup P\{2\} \dots \cup \{n\})$

$$\Rightarrow P(E) = P(\{1\}) + P(\{2\}) + \dots + P\{n\})$$

$$= \frac{1}{N} + \frac{1}{N} + \dots + \frac{1}{N} = \frac{n}{N} = \frac{\text{Number of Points in } E}{\text{Number of Point in } S}$$

Thus we have

$$\boxed{P(E) = \frac{\text{Number of points in } E}{N}}$$

In words, if we assume that each outcome of an experiment is equally likely to occur, then the probability of any event E equals the proportion of points in the sample space that are contained in E.

Thus, to compute probabilities it is often necessary to be able to effectively count the number of different ways that a given event can occur.

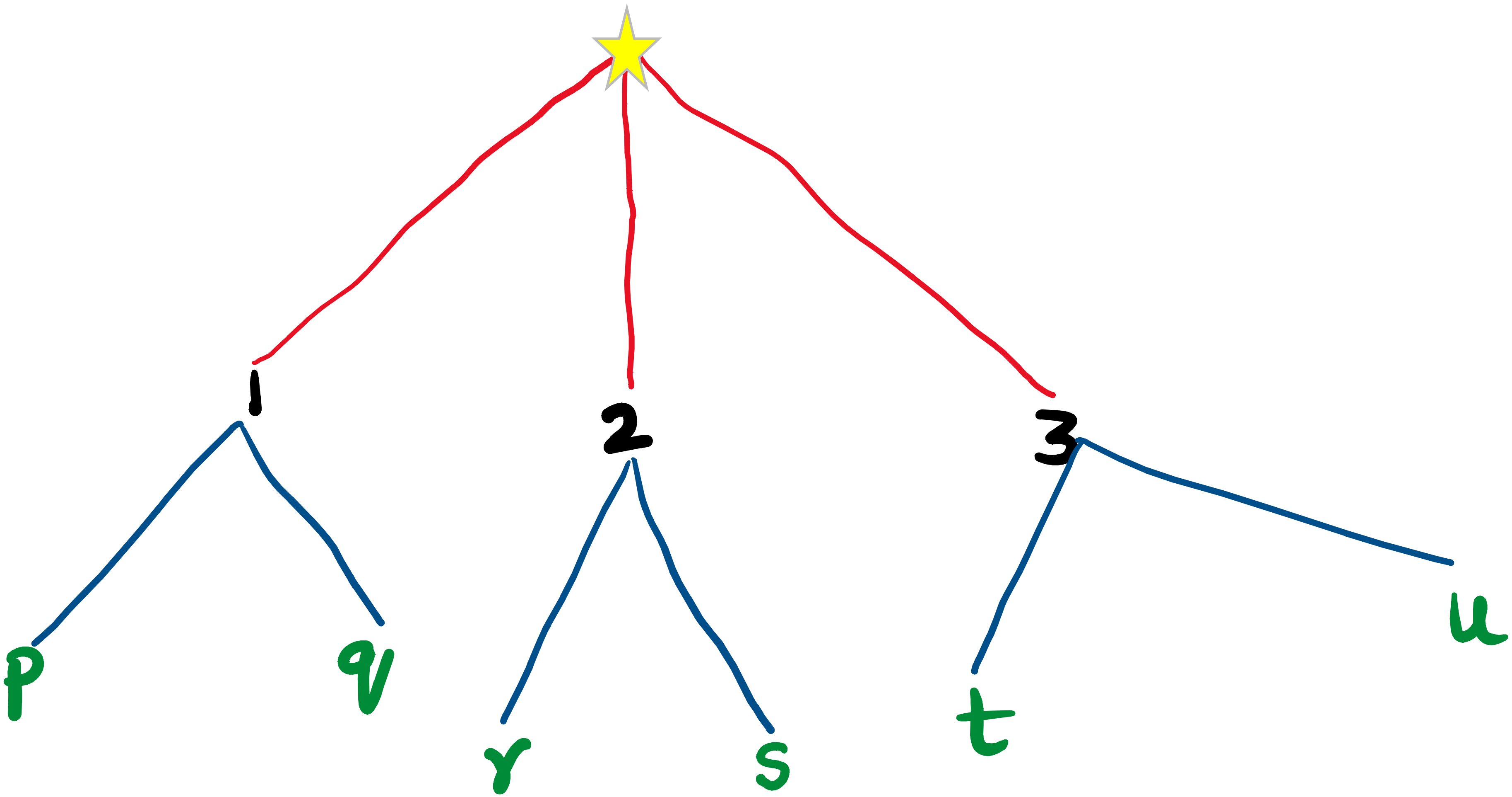
How to count without counting?

The mathematical theory of counting is formally known as *combinatorial analysis*.

The basic principle of counting (Product Rule of Counting)

If an experiment has two parts, where the first part can result in one of m outcomes and the second part can result in one of n outcomes regardless of the outcome of the first part, then the total number of outcomes for the experiment is $m \times n$.

The above procedure is *counting with steps*.



$3 \times 2 = 6$ possible choices. These are 1p, 1q, 2r, 2s, 3t & 3u.

Example

Two 6-sided dice, with faces numbered 1 through 6, are rolled. How many possible outcomes of the roll are there?

Solution:

Since the first die can come up with 6 possible values and the second die similarly can have 6 possible values (regardless of what appeared on the first die), the total number of potential outcomes is = $6 \times 6 = 36$.

These possible outcomes are explicitly listed in the next slide as a series of pairs, denoting the values rolled on the pair of dice:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Problem

A football tournament consists of 14 teams, each of which has 11 players. If one team and one of its players are to be selected as team and player of the year, how many different choices are possible?

Solution

Selecting the team can be regarded as the outcome of the first experiment and the subsequent choice of one of its player as the outcome of the second experiment, so there are $14 * 11 = 154$ possibilities.

Problem

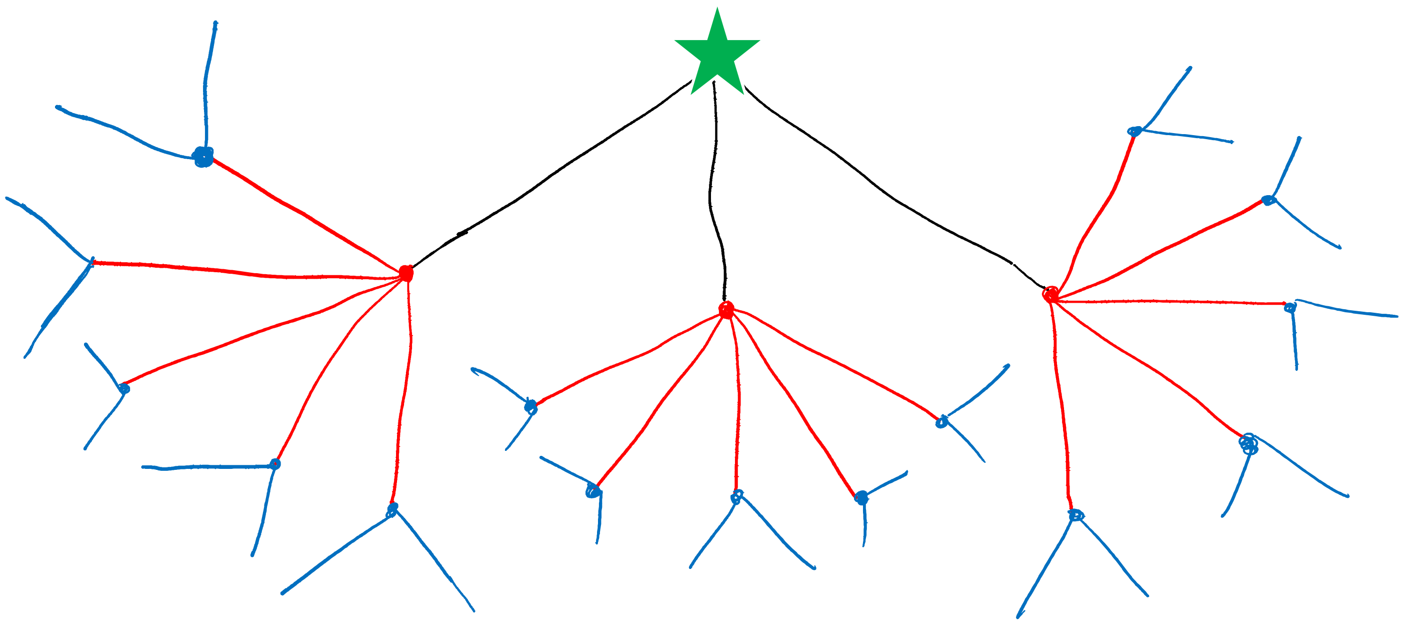
A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

Solution

By regarding the choice of the woman as the outcome of the first experiment and the subsequent choice of one of her children as the outcome of the second experiment, we see from the basic principle that there are $10 * 3 = 30$ possible choices.

The generalized basic principle of counting

If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 \times n_2 \times \cdots \times n_k$ ways.



According to generalized basic rule of Counting we have

$3 \times 5 \times 2 = 30$ possible outcomes.

Problem

A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible?

Solution

We may regard the choice of a subcommittee as the combined outcome of the four separate experiments of choosing a single representative from each of the classes. It then follows from the generalized version of the basic principle that there are $3 * 4 * 5 * 2 = 120$ possible subcommittees.

Practice Problems

(a) How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

(b) How many license plates would be possible if repetition among letters or numbers were prohibited?

How many functions defined on n points are possible if each functional value is either 0 or 1?

Suppose you have 9 long sleeves shirts and 7 short sleeves shirts

How many choices do you have for a shirt?

$$9 + 7 = 16$$

Sum Rule

If a task can be done either in one of n_1 ways or in one of n_2 different ways, then there are $n_1 + n_2$ ways to do the task

Sum Rule

If A and B are disjoint sets, then

$$|A \cup B| = |A| + |B|$$

Suppose you have to choose a project from 4 software development projects or from 5 research projects.

How many choices do you have?

Permutations

How many different ordered arrangements of the letters a, b, and c are possible? By direct enumeration we see that there are 6, namely, *abc, acb, bac, bca, cab*, and *cba*. Each arrangement is known as a *permutation*. Thus, there are 6 possible permutations of a set of 3 objects. This result could also have been obtained from the basic principle, since the first object in the permutation can be any of the 3, the second object in the permutation can then be chosen from any of the remaining 2, and the third object in the permutation is then the remaining 1. Thus, there are $3 \cdot 2 \cdot 1 = 6$ possible permutations.

General Result. Suppose you have n objects. The number of permutations of these n objects is given by

$$n\,(n-1)\,(n-2)\cdots 3\cdot 2\cdot 1 = n!$$

Problem

How many different batting orders are possible for a baseball team consisting of 9 players?

Solution

There are $9! = 362{,}880$ possible batting orders.

Problem

Ms. Jones has 10 books that she is going to put on her bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Ms. Jones wants to arrange her books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

Solution

There are 4! 3! 2! 1! arrangements such that the mathematics books are first in line, then the chemistry books, then the history books, and then the language book. Furthermore, there are 4! possible orderings of the subjects, the desired answer is 4! 4! 3! 2! 1! = 6912.

General Result. Suppose you have n objects. The number of different permutations of these n objects of which n_1 are alike, n_2 are alike,..., n_r are alike is given by

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

Practice Problem How many different letter arrangements can be formed from the letters EEPPPR?

Problem

A chess tournament has 10 competitors, of which 4 are Russian, 3 are from the United States, 2 are from Great Britain, and 1 is from Brazil. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?

Solution

There are $\dfrac{10!}{4!\,3!\,2!\,1!} = 12{,}600$ possible outcomes.

Problem

How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags if all flags of the same color are identical?

Solution

There are $\dfrac{9!}{4!\,3!\,2!} = 1260$ different signals.

Combinations

We are often interested in determining the number of different groups of r objects that could be formed from a total of n objects. For instance, how many different groups of **3** could be selected from the **5** items A, B, C, D, and E?

To answer this question, reason as follows: Since there are 5 ways to select the initial item, 4 ways to then select the next item, and 3 ways to select the final item, there are thus $5 \cdot 4 \cdot 3$ ways of selecting the group of 3 when the order in which the items are selected is relevant. However, since every group of 3—say, the group consisting of items A, B, and C—will be counted 6 times (that is, all of the permutations ABC, ACB, BAC, BCA, CAB, and CBA will be counted when the order of selection is relevant), it follows that the total number of groups that can be formed is

$$\frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$$

In general, as $n(n-1) \cdots (n-r+1)$ represents the number of different ways that a group of r items could be selected from n items when the order of selection is relevant, and as each group of r items will be counted $r!$ times in this count, it follows that the number of different groups of r items that could be formed from a set of n items is

$$\frac{n(n-1) \cdots (n-r+1)}{r!} = \frac{n!}{(n-r)!\, r!}$$

Summary

Notation and terminology

We define $\binom{n}{r}$, for $r \leq n$, by

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

and say that $\binom{n}{r}$ represents the number of possible combinations of n objects taken r at a time.†

†By convention, $0!$ is defined to be 1. Thus, $\binom{n}{0} = \binom{n}{n} = 1$. We also take $\binom{n}{i}$ to be equal to 0 when either $i < 0$ or $i > n$.

Thus, $\binom{n}{r}$ represents the number of different groups of size r that could be selected from a set of n objects when the order of selection is not considered relevant.

Equivalently, $\binom{n}{r}$ is the number of subsets of size r that can be chosen from a set of size n.

Problem

A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

Solution

There are $\binom{20}{3} = \dfrac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140$ possible committees.

The binomial theorem

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Problem Expand $(x + y)^3$.

Solution

$$(x + y)^3 = \binom{3}{0} x^0 y^3 + \binom{3}{1} x^1 y^2 + \binom{3}{2} x^2 y^1 + \binom{3}{3} x^3 y^0$$
$$= y^3 + 3xy^2 + 3x^2 y + x^3$$

Problem Expand $(x+y)^5$.

Solution $$(x+y) = \sum_{k=0}^{5}\binom{5}{k}x^k y^{5-k}$$

$$\Rightarrow (x+y)^5 = \binom{5}{0}x^0y^5 + \binom{5}{1}x^1y^4 + \binom{5}{2}x^2y^3 + \binom{5}{3}x^3y^2 + \binom{5}{4}x^4y^1 + \binom{5}{5}x^5y^0$$

$$\Rightarrow \boxed{(x+y)^5 = y^5 + 5xy^4 + 10x^2y^3 + 10x^3y^2 + 5x^4y^1 + x^5}$$ Ans

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Problem

Two balls are “randomly drawn” from a bowl containing 6 white and 5 black balls. What is the probability that one of the drawn balls is white and the other black?

Solution

If we regard the order in which the balls are selected as being significant, then as the first drawn ball may be any of the 11 and the second any of the remaining 10, it follows that the sample space consists of $11 \cdot 10 = 110$ points. Furthermore, there are $6 \cdot 5 = 30$ ways in which the first ball selected is white and the second black, and similarly there are $5 \cdot 6 = 30$ ways in which the first ball is black and the second white.

Thus the probability that one of the drawn balls is white and the other black is

$$\frac{30 + 30}{110} = \frac{6}{11}$$

Problem

A class in probability theory consists of 6 boys and 4 girls. An exam is given and the students are ranked according to their performance. If no two students obtain the same score then (a) how many different rankings are possible? (b) If all rankings are considered equally likely, what is the probability that boys receive the top 4 scores (c) what is the probability that girls receive the top 4 scores?

Solution

(a) Because each ranking corresponds to a particular ordered arrangement of the 10 people, we see the answer to this part is $10! = 3{,}628{,}800$.

(b) In the top 4 positions 4 boys can be place in 6.5.4.3 ways. In the remaining 6 positions 2 boys and 4 girls can be place in 6! ways, it follows from basic principle there are 6.5.4.3*6!= possible rankings in which the boys receive top 4 scores. Hence the desired probability is

$$\frac{6.5.4.3*6!}{10!} = 0.071$$

(c) Because there are 4! possible rankings of the girls among themselves and 6! possible rankings of the boys among themselves, it follows from the basic principle that there are (6!)(4!) = (720)(24) = 17, 280 possible rankings in which the girls receive the top 4 scores. Hence, the desired probability is

$$\frac{6!4!}{10!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{10 \cdot 9 \cdot 8 \cdot 7} = \frac{1}{210} = \mathbf{0.005}$$

Problem

A committee of size 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

SOLUTION Let us assume that "randomly selected" means that each of the $\binom{15}{5}$ possible combinations is equally likely to be selected. Hence, since there are $\binom{6}{3}$ possible choices of 3 men and $\binom{9}{2}$ possible choices of 2 women, it follows that the desired probability is given by

$$\frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}} = \frac{240}{1001} \quad \blacksquare$$

Problem

If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year?

Solution

Because each person can celebrate his or her birthday on any one of 365 days, there are a total of 365^n possible outcomes. Furthermore, there are $(365)(364)(363)\cdot(365-n+1)$ possible outcomes that result in no two of the people having the same birthday. This is so because the first person could have any one of 365 birthdays, the next person any of the remaining 364 days, the next any of the remaining 363, and so on. Hence, assuming that each outcome is equally likely, we see that the desired probability is

$$\frac{(365)(364)(363)\cdots(365-n+1)}{(365)^n}$$

Problem

The English alphabet has 5 vowels and 21 consonants. How many words with two different vowels and 2 different consonants can be formed from the alphabet?

Solution

Given : 5 vowels and 21 consonants.

select : 2 vowels and 2 consonants.

We can choose 2 vowels from 5 in $\binom{5}{2}$ways = 10 ways.

We can choose 2 consonants from 21 in $\binom{21}{2}$ ways = 210 ways.

Now these selected 4 letters can be arranged in 4! ways = 24 ways.

So by basic principle of counting,

Total number of ways(words) = 10.210.24 = 50400.

Conditional Probabilities

Suppose that one rolls a pair of dice. The sample space S of this experiment can be taken to be the following set of 36 outcomes.

$$S = \{(i, j), i = 1, 2, 3, 4, 5, 6, j = 1, 2, 3, 4, 5, 6\}$$

Suppose further that we observe that the first die lands on side 3. Then, given this information, what is the probability that the sum of the two dice equals 8?

So let E is the event that the sum of the dice is 8 and F is the event that the first die is a 3, then the probability of E given that F has occurred is called the conditional probability of E given that F has occurred, and is denoted by **P(E | F).**

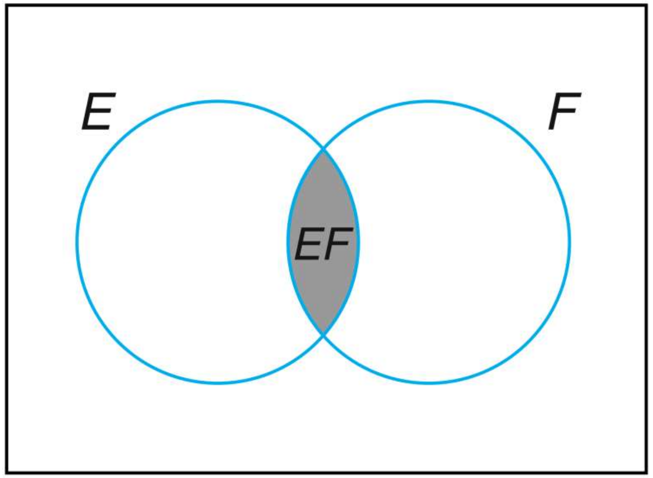
$F = \{$ first die is a 3 $\}$

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$E = \{$ sum of the dice equals 8 $\}$

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Note that above Equation is well defined only when $P(F) > 0$ and hence $P(E|F)$ is defined only when $P(F) > 0$.



Problem

A bin contains 5 defective (that immediately fail when put in use), 10 partially defective (that fail after a couple of hours of use), and 25 acceptable transistors. A transistor is chosen at random from the bin and put into use. If it does not immediately fail, what is the probability it is acceptable?

Solution

Since the transistor did not immediately fail, we know that it is not one of the 5 defectives and so the desired probability is:

$$P\{\text{acceptable} \mid \text{not defective}\}$$

$$= \frac{P\{\text{acceptable, not defective}\}}{P\{\text{not defective}\}}$$

$$= \frac{P\{\text{acceptable}\}}{P\{\text{not defective}\}}$$

where the last equality follows since the transistor will be both acceptable and not defective if it is acceptable.

$$P\{\text{acceptable}|\text{not defective}\} = \frac{25/40}{35/40} = 5/7$$

Problem

The organization that Jones works for is running a father–son dinner for those employees having at least one son. Each of these employees is invited to attend along with his youngest son. If Jones is known to have two children, what is the conditional probability that they are both boys given that he is invited to the dinner? Assume that the sample space S is given by $S = \{(b, b), (b, g), (g, b), (g, g)\}$ and all outcomes are equally likely [(b, g) means, for instance, that the younger child is a boy and the older child is a girl].

SOLUTION The knowledge that Jones has been invited to the dinner is equivalent to knowing that he has at least one son. Hence, letting B denote the event that both children are boys, and A the event that at least one of them is a boy, we have that the desired probability $P(B|A)$ is given by

$$\begin{aligned} P(B|A) &= \frac{P(BA)}{P(A)} \\ &= \frac{P(\{(b,b)\})}{P(\{(b,b),(b,g),(g,b)\})} \\ &= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} \end{aligned}$$

Problem

Ninety percent of flights depart on time. Eighty percent of flights arrive on time. Seventy-five percent of flights depart on time and arrive on time.

(a) Jhon is meeting Ana's flight, which departed on time. What is the probability that Ana will arrive on time?

(b) Jhon has met Ana, and she arrived on time. What is the probability that her flight departed on time?

Solution. Denote the events,

$$A = \{\text{arriving on time}\}, \quad D = \{\text{departing on time}\}.$$

We have:

$$\boldsymbol{P}\{A\} = 0.8, \quad \boldsymbol{P}\{D\} = 0.9, \quad \boldsymbol{P}\{A \cap D\} = 0.75.$$

(a) $\boldsymbol{P}\{A \mid D\} = \dfrac{\boldsymbol{P}\{A \cap D\}}{\boldsymbol{P}\{D\}} = \dfrac{0.75}{0.9} = \underline{0.8333}.$

(b) $\boldsymbol{P}\{D \mid A\} = \dfrac{\boldsymbol{P}\{A \cap D\}}{\boldsymbol{P}\{A\}} = \dfrac{0.75}{0.8} = \underline{0.9375}.$

Problem

The concept of conditional probability has countless uses in both industrial and biomedical applications. Consider an industrial process in the textile industry in which strips of a particular type of cloth are being produced. These strips can be defective in two ways, length and nature of texture. For the case of the latter, the process of identification is very complicated. It is known from historical information on the process that 10% of strips fail the length test, 5% fail the texture test, and only 0.8% fail both tests. If a strip is selected randomly from the process and a quick measurement identifies it as failing the length test, what is the probability that it is texture defective?

Solution

Consider the events

$$L\text{: length defective,} \qquad T\text{: texture defective.}$$

Given that the strip is length defective, the probability that this strip is texture defective is given by

$$P(T|L) = \frac{P(T \cap L)}{P(L)} = \frac{0.008}{0.1} = 0.08.$$

$$P(A|B) = \frac{P(AB)}{P(B)} \Rightarrow \boxed{P(AB) = P(A|B)\,P(B)}$$

$$P(B|A) = \frac{P(BA)}{P(A)}$$

$$\Rightarrow P(B|A) = \frac{P(AB)}{P(A)} \Rightarrow \boxed{P(AB) = P(B|A)\,P(A)}$$

Note:

1) $P(AB) = P(A|B)\,P(B)$

2) $P(AB) = P(B|A)\,P(A)$

Problem

Suppose that two balls are to be selected at random, without replacement, from a box containing r red balls and b blue balls. What is the probability that the first ball will be red and the second ball will be blue?

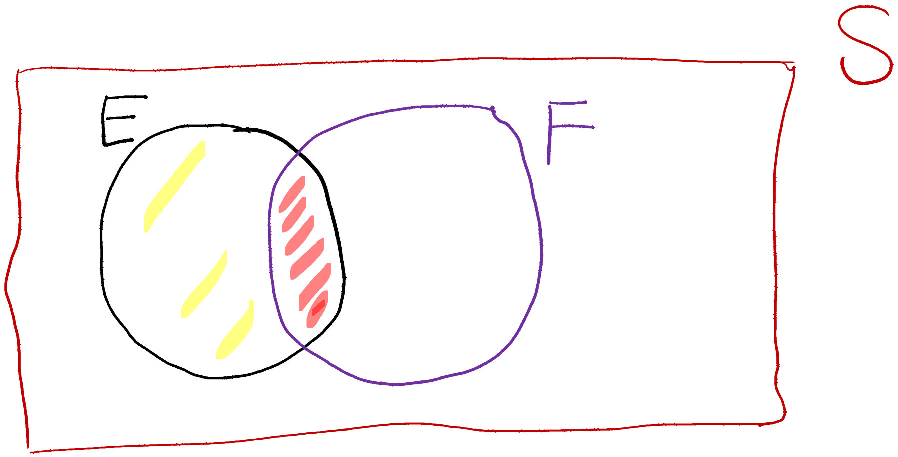
Solution

Let A be the event that the first ball is red, and let B be the event that the second ball is blue. Obviously, $P(A) = r/(r + b)$. Furthermore, if the event A has occurred, then one red ball has been removed from the box on the first draw. Therefore, the probability of obtaining a blue ball on the second draw will be

$$P(B|A) = \frac{b}{r + b - 1}$$

It follows that

$$P(AB) = \frac{b}{r+b-1} \cdot \frac{r}{r+b}$$

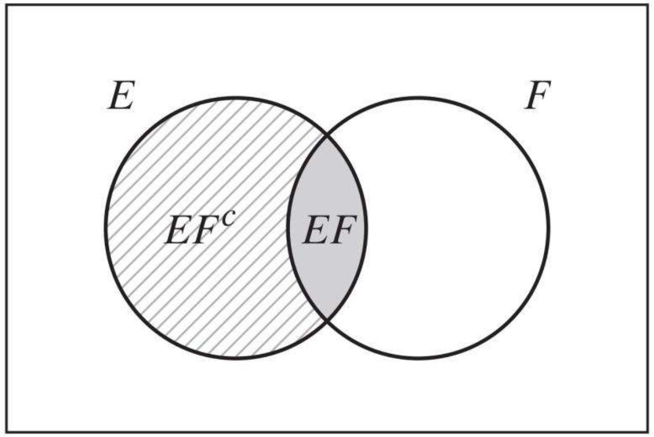


E = Yellow Region $\cup$ Red Region

$$\Rightarrow E = EF^c \cup EF$$

Let E and F be events. We may express E as

$$E = EF \cup EF^c$$



$E = EF \cup EF^c$. EF = Shaded Area; EF^c = Striped Area.

As EF and EF^c are clearly mutually exclusive, we have by Axiom 3 that

$$P(E) = P(EF) + P(EF^c)$$

$$= P(E|F)P(F) + P(E|F^c)P(F^c)$$

$$\Rightarrow \boxed{P(E) = P(E|F)P(F) + P(E|F^c)[1 - P(F)]}$$

Problem

A laboratory blood test is 99 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a “false positive” result for 1 percent of the healthy persons tested. (That is, if a healthy person is tested, then, with probability .01, the test result will imply he or she has the disease.) If .5 percent of the population actually has the disease, what is the probability a person has the disease given that his test result is positive?

Solution

Let D be the event that the tested person has the disease and E the event that his test result is positive. The desired probability P(D|E) is obtained by

$$
\begin{aligned}
P(D|E) &= \frac{P(DE)}{P(E)} \\
&= \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|D^c)P(D^c)}
\end{aligned}
$$

$$\Rightarrow \quad P(D|E) = \frac{(.99)(.005)}{(.99)(.005) + (.01)(.995)}$$

$$= .3322$$

Thus, only 33 percent of those persons whose test results are positive actually have the disease.

Suppose that $F_1, F_2, F_3, \ldots, F_n$ are mutually exclusive events such that

$$\bigcup_{i=1}^{n} F_i = S$$

In other words, exactly one of the events $F_1, F_2, F_3, \ldots, F_n$ must occur. By writing

$$E = \bigcup_{i=1}^{n} EF_i$$

and using the fact that the events EF_i, $i = 1, \ldots, n$ are mutually exclusive, we obtain that

$$P(E) = \sum_{i=1}^{n} P(EF_i) = \sum_{i=1}^{n} P(E|F_i)P(F_i)$$

OR

$$P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2) + \cdots + P(E|F_n)P(F_n)$$

The Law of Total Probability

Let $F_1, F_2, \dots, F_n$ be mutually exclusive and exhaustive events. Then for any other event E,

$$P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2) + \cdots + P(E|F_n)P(F_n)$$

$$= \sum_{i=1}^{n} P(E|F_i)P(F_i)$$

Events $F_1, F_2, \dots, F_n$ are called exhaustive events if $F_1 \cup F_2 \cup \cdots \cup F_n = S$.

Problem

In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Solution

$$\begin{aligned} A&: \text{ the product is defective,} \\ B_1&: \text{ the product is made by machine } B_1, \\ B_2&: \text{ the product is made by machine } B_2, \\ B_3&: \text{ the product is made by machine } B_3. \end{aligned}$$

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3).$$

$$P(B_1)P(A|B_1) = (0.3)(0.02) = 0.006,$$
$$P(B_2)P(A|B_2) = (0.45)(0.03) = 0.0135,$$
$$P(B_3)P(A|B_3) = (0.25)(0.02) = 0.005,$$

and hence

$$P(A) = 0.006 + 0.0135 + 0.005 = 0.0245.$$

Bayes' Formula

Suppose now that E has occurred and we are interested in determining which one of F_j also occurred. We have

$$P(F_j|E) = \frac{P(EF_j)}{P(E)}$$

$$\Rightarrow \boxed{P(F_j|E) = \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^{n} P(E|F_i)P(F_i)}}$$

In term of two events A and B we can work for Bayes Rule as follows :

$$P(A|\mathrm{B}) = \frac{P(AB)}{P(B)}$$

$$\Rightarrow P(A|\mathrm{B}) = \frac{P(B|A)P(A)}{P(B)}$$

Note:

Most of the time we don't have P(B) so we can use law of total probability to compute P(B).

Problem

Consider a test that can diagnose kidney cancer. The test correctly detects when a patient has cancer 90% of the time. Also, if a person does not have cancer, the test correctly indicates so 99.9% of the time. Finally, suppose it is known that 1 in every 10,000 individuals has kidney cancer. Find the probability that a patient has kidney cancer, given that the test indicates he does.

Solution

Let C denote the event that a patient has kidney cancer then C^c will denote the event that a patient has no kidney cancer.
Po denote the event that the patient tests positive for kidney cancer then $Po^c = N$ denote the event that the patient tests negative for kidney cancer.

Now we want $P(C|Po)$.

Given data is :

Test correctly detects when a patient has cancer 90% of the time : $P(Po|C) = \frac{90}{100} = 0.9$

If a person does not have cancer, the test correctly indicates so 99.9% of the time:

$$P(N|C^C) = \frac{99.9}{100} = 0.999 \Rightarrow P(Po|C^C) = 1 - 0.999 = 0.001$$

It is known that 1 in every 10,000 individuals has kidney cancer:

$P(C) = \frac{1}{10000} = 0.0001 \Rightarrow P(C^C) = 1 - P(C) = 0.9999.$

By Law of Total Probability

$$\begin{aligned} P(Po) &= P(Po|\mathrm{C})P(C) + P(Po|C^c)P(C^C) \\ &= (0.9 * 0.0001) + (0.001 * 0.9999) \\ &= 0.0010899 \end{aligned}$$

By Bayes Rule,

$$P(C|\mathrm{Po}) = \frac{P(Po|C)P(C)}{P(Po)} = \frac{0.9 * 0.0001}{0.0010899} \approx 0.08$$

This implies that only about 8% of patients that test positive under this particular test actually have kidney cancer.

Practice Problems

1. You ask your neighbor to water a sickly plant while you are on vacation. Without water it will die with probability 0.8; with water it will die with probability 0.15. You are 90 percent certain that your neighbor will remember to water the plant.

(a) What is the probability that the plant will be alive when you return?

(b) If it is dead, what is the probability your neighbor forgot to water it?

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2. There is a 60 percent chance that the event A will occur. If A does not occur, then there is a 10 percent chance that B will occur. What is the probability that at least one of the events A or B occurs?

Independent Events

E is independent of F if knowledge that F has occurred does not change the probability of E occurrence.

Since $P(E|F) = P(EF)/P(F)$, we see that E is independent of F if

$$P(EF) = P(E)P(F)$$

Since this equation is symmetric in E and F, it shows that whenever E is independent of F so is F of E.

Two events E and F are said to be independent if

$$P(EF) = P(E)P(F)$$

Two events E and F that are not independent are said to be dependent.

Problem

Suppose that we roll a pair of fair dice, so each of the 36 possible outcomes is equally likely. Let A denote the event that the first die lands on 3, let B be the event that the sum of the dice is 8, and let C be the event that the sum of the dice is 7.

(a) Are A and B independent?
(b) Are A and C independent?

Solution

(a) Since $A \cap B$ is the event that the first die lands on 3 and the second on 5, we see that

$$P(A \cap B) = P(\{(3, 5)\}) = \frac{1}{36}$$

On the other hand,

$$P(A) = P(\{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}) = \frac{6}{36}$$

and

$$P(B) = P(\{(2,6),(3,5),(4,4),(5,3),(6,2)\}) = \frac{5}{36}$$

Therefore, since $1/36 \neq (6/36) \cdot (5/36)$, we see that

$$P(A \cap B) \neq P(A)P(B)$$

and so events A and B are not independent.

Similar solve part (b).

Problem

Toss two coins and observe the outcome. Define these events:

A: Head on the first coin

B: Tail on the second coin

Are events A and B independent?

Solution

$$P(A)=\frac{1}{2}, P(B)=\frac{1}{2}, \text{and } P(A\cap B)=\frac{1}{4}.$$

Since $P(A)P(B)=\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=\frac{1}{4}$ and $P(A\cap B)=\frac{1}{4}$, we have $P(A)P(B)=P(A\cap B)$ and the two events must be independent.

Theorem

If A and B are independent events, then each of the following pair of events are also independent:

(i) A and B^C

(ii) A^C and B

(iii) A^C and B^c

Independency of Three Events

Three events E, F, and G are said to be independent if

$$
\begin{aligned}
P(EFG) &= P(E)P(F)P(G) \\
P(EF) &= P(E)P(F) \\
P(EG) &= P(E)P(G) \\
P(FG) &= P(F)P(G)
\end{aligned}
$$

Note that if E, F, and G are independent, then E will be independent of any event formed from F and G.

Definition of Independence of Several Events

We say that the events $A_1, A_2, \ldots, A_n$ are **independent** if

$$\mathbf{P}\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \mathbf{P}(A_i), \qquad \text{for every subset } S \text{ of } \{1, 2, \ldots, n\}.$$

Generalized Chain Rule

$$P(E_1 E_2 E_3 \ldots E_n)$$
$$= P(E_1)P(E_2|E_1)P(E_3|E_1 E_2) \ldots P(E_n|E_1 E_2 \ldots E_{n-1})$$

Practice Problem

1.

Of three cards, one is painted red on both sides; one is painted black on both sides; and one is painted red on one side and black on the other. A card is randomly chosen and placed on a table. If the side facing up is red, what is the probability that the other side is also red?

2.

If $B \subset A$, then show that $P(A \backslash \mathrm{B}) = P(A) - P(B)$.

[Hint: Use Venn diagram and third axiom of probability]

Practice Problem

Among employees of a certain firm, 70% know Java, 60% know Python, and 50% know both languages. What portion of programmers

(a) does not know Python?

(b) does not know Python and does not know Java?

(c) knows Java but not Python?

(d) knows Python but not Java?

(e) If someone knows Python, what is the probability that he/she knows Java too?

(f) If someone knows Java, what is the probability that he/she knows Python too?

Practice Problem

(Diagnostics of computer codes): A new computer program consists of two modules. The first module contains an error with probability 0.2. The second module is more complex; it has a probability of 0.4 to contain an error, independently of the first module. An error in the first module alone causes the program to crash with probability 0.5. For the second module, this probability is 0.8. If there are errors in both modules, the program crashes with probability 0.9. Suppose the program crashed. What is the probability of errors in both modules?

Practice Problem

A computer maker receives parts from three suppliers, S1, S2, and S3. Fifty percent come from S1, twenty percent from S2, and thirty percent from S3. Among all the parts supplied by S1, 5% are defective. For S2 and S3, the portion of defective parts is 3% and 6%, respectively.

(a) What is the probability that a random part is defective?

(b) A customer complains that a certain part in her recently purchased computer is defective. What is the probability that it was supplied by S1?

Practice Problem

1. Prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

2. A programming class is composed of 10 juniors, 30 seniors, and 10 graduate students. The final grades show that 5 of the juniors, 10 of the seniors, and 5 of the graduate students received an A for the course. If a student is chosen at random from this class and is found to have earned an A, what is the probability that he or she is a senior?

Practice Problems

1.

All athletes at the Olympic games are tested for performance-enhancing steroid drug use. The imperfect test gives positive results (indicating drug use) for 90% of all steroid users but also (and incorrectly) for 2% of those who do not use steroids. Suppose that 5% of all registered athletes use steroids. If an athlete is tested negative, what is the probability that he/she uses steroids?

2.

At a plant, 20% of all the produced parts are subject to a special electronic inspection. It is known that any produced part which was inspected electronically has no defects with probability 0.95. For a part that was not inspected electronically this probability is only 0.7. A customer receives a part and find defects in it. What is the probability that this part went through an electronic inspection?
