Jointly Distributed Random Variables

To specify the relationship between two random variables, we define the joint cumulative probability distribution function of X and Y by

$$F(x, y) = P\{X \leq x, Y \leq y\}$$

A knowledge of the joint probability distribution function often enable us to compute the probability of statements concerning the values of X and Y .

In the case where X and Y are both discrete random variables whose possible values are, respectively, $x_1, x_2, \ldots$, and $y_1, y_2, \ldots$, we define the *joint probability mass function* of X and Y, $p(x_i, y_j)$, by

$$p(x_i, y_j) = P\{X = x_i, Y = y_j\}$$

The individual probability mass functions of X and Y are easily obtained from the joint probability mass function by the following reasoning. Since Y must take on some value y_j, it follows that the event $\{X = x_i\}$ can be written as the union, over all j, of the mutually exclusive events $\{X = x_i, Y = y_j\}$. That is,

$$\{X = x_i\} = \bigcup_j \{X = x_i, Y = y_j\}$$

and so, using Axiom 3 of the probability function, we see that

$$P\{X = x_i\} = P\left(\bigcup_j \{X = x_i, Y = y_j\}\right)$$

$$= \sum_j P\{X = x_i, Y = y_j\}$$

$$\Rightarrow \quad P\{X = x_i\} = \sum_j p(x_i, y_j)$$

Similarly, we can obtain $P\{Y = y_j\}$ by summing $p(x_i, y_j)$ over all possible values of x_i, that is,

$$P\{Y = y_j\} = \sum_i P\{X = x_i, Y = y_j\}$$

$$\Longrightarrow \quad P\{Y = y_j\} = \sum_i p(x_i, y_j)$$

Hence, specifying the joint probability mass function always determines the individual mass functions. However, it should be noted that the reverse is not true. Namely, knowledge of $P\{X = x_i\}$ and $P\{Y = y_j\}$ does not determine the value of $P\{X = x_i, Y = y_j\}$.

Again, $\{(X, Y) = (x, y)\}$ are exhaustive and mutually exclusive events for different pairs (x, y), therefore,

$$\sum_{x} \sum_{y} \boldsymbol{p(x, y) = 1}$$

The above equation tell us that if we feed all possible x and y values to the joint PMF and sum up all the resulting values. It should be one. This is the underlying property of a valid joint PMF.

Example

Suppose that 3 batteries are randomly chosen from a group of 3 new, 4 used but still working, and 5 defective batteries. If X and Y denote, respectively, the number of new and used but still working batteries that are chosen, then the joint probability mass function of X and Y , $p(i, j) = P\{X = i, Y = j\}$, is given by

$$p(0,0) = \binom{5}{3} \Big/ \binom{12}{3} = 10/220$$

$$p(0,1) = \binom{4}{1}\binom{5}{2} \Big/ \binom{12}{3} = 40/220$$

$$p(0,2) = \binom{4}{2}\binom{5}{1} \Big/ \binom{12}{3} = 30/220$$

$$p(0,3) = \binom{4}{3} \Big/ \binom{12}{3} = 4/220$$

$$p(1,0) = \binom{3}{1}\binom{5}{2} \Big/ \binom{12}{3} = 30/220$$

$$p(1,1) = \binom{3}{1}\binom{4}{1}\binom{5}{1} \Big/ \binom{12}{3} = 60/220$$

$$p(1,2) = \binom{3}{1}\binom{4}{2} \Big/ \binom{12}{3} = 18/220$$

$$p(2,0) = \binom{3}{2}\binom{5}{1} \Big/ \binom{12}{3} = 15/220$$

$$p(2,1) = \binom{3}{2}\binom{4}{1}\Big/\binom{12}{3} = 12/220$$

$$p(3,0) = \binom{3}{3}\Big/\binom{12}{3} = 1/220$$

These probabilities can most easily be expressed in tabular form as shown in Table 4.1.

TABLE 4.1 $P\{X = i, Y = j\}$

$i \backslash j$	0	1	2	3	Row Sum $= P\{X = i\}$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
Column Sums $= P\{Y = j\}$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	

Because the individual probability mass functions of X and Y thus appear in the margin of such a table, they are often referred to as being the marginal probability mass functions of X and Y, respectively. It should be noted that to check the correctness of such a table we could sum the marginal row (or the marginal column) and verify that its sum is $\mathbf{1}$.

Problem

A program consists of two modules. The number of errors, X, in the first module and the number of errors, Y , in the second module have the joint distribution, $p(0,0) = p(0,1) = p(1,0) = 0.2$, $p(1,1) = p(1,2) = p(1,3) = 0.1$, $p(0,2) = p(0,3) = 0.05$. Find (a) the marginal distributions of X and Y , (b) the probability of no errors in the first module, and (c) the distribution of the total number of errors in the program.

[Remember: Collection of all the probabilities related to some random variable is sometime called the distribution of the random variable. i.e. joint distribution mean joint PMF etc.]

Solution

It is convenient to organize the joint pmf of X and Y in a table. Adding row wise and column wise, we get the marginal pmfs,

<table>
<tr><td colspan="2" rowspan="2">$p(x,y)$</td><td colspan="4">y</td><td rowspan="2">$p_X(x)$</td></tr>
<tr><td>0</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td rowspan="2">x</td><td>0</td><td>0.20</td><td>0.20</td><td>0.05</td><td>0.05</td><td>0.50</td></tr>
<tr><td>1</td><td>0.20</td><td>0.10</td><td>0.10</td><td>0.10</td><td>0.50</td></tr>
<tr><td colspan="2">$p_Y(y)$</td><td>0.40</td><td>0.30</td><td>0.15</td><td>0.15</td><td>1.00</td></tr>
</table>

This solves (a).

(b) $P_X(0) = 0.50$. (Because the probability of no errors in the first module mean $P\{X = 0\}$.

(c) Let Z = X + Y be the total number of errors. To find the distribution of Z, we first identify its possible values, then find the probability of each value. We see that Z can be as small as 0 and as large as 4. Then,

$$p_Z(0) = P\{X + Y = 0\} = P\{X = 0, Y = 0\} = p(0,0) = 0.20,$$

$$p_Z(1) = P\{X = 0, Y = 1\} + P\{X = 1, Y = 0\}$$

$$= p(0,1) + p(1,0) = 0.20 + 0.20 = 0.40,$$

$$p_Z(2) = p(0,2) + p(1,1) = 0.05 + 0.10 = 0.15,$$

$$p_Z(3) = p(0,3) + p(1,2) = 0.05 + 0.10 = 0.15,$$

$$p_Z(4) = p(1,3) = 0.10.$$

It is a good check to verify that

$$\sum_z p_Z(\mathrm{z}) = 1$$

Practice Problem

An internet service provider charges its customers for the time of the internet use rounding it up to the nearest hour. The joint distribution of the used time (X, hours) and the charge per hour (Y, cents) is given in the table below.

$p(x,y)$		x = 1	2	3	4
y	1	0	0.06	0.06	0.10
	2	0.10	0.10	0.04	0.04
	3	0.40	0.10	0	0

Each customer is charged $Z = X \cdot Y$ cents, which is the number of hours multiplied by the price of each hour. Find the distribution of Z.

[Hint: First find all the possible values of Z and then find its probabilities.]

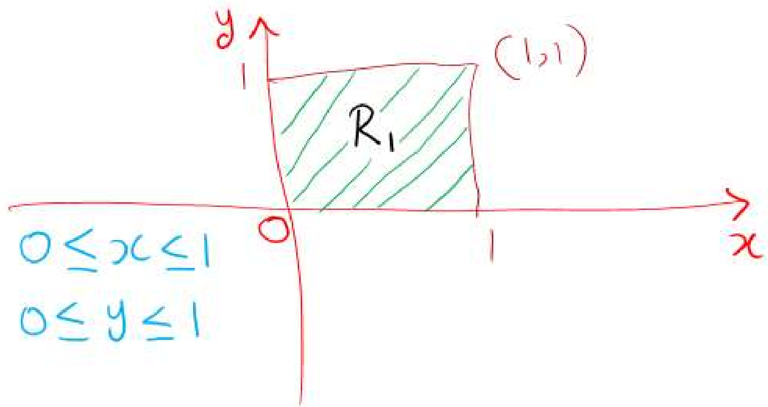
$\int_a^b f(x)\,dx$ is an integral defined over an interval $[a,b]$.

We also have integrals that are defined over a region R in xy-plane. e.g the double integral $\iint_R f(x,y)\,dA$

where R is some region in xy-plane.

The question is : How to evaluate a double integral?
Well! it depends upon the region R.

Consider region R_1 shown in the figure. In this case, we can evaluate the double integral quite easily. i.e.



$0 \le x \le 1$

$0 \le y \le 1$

$$\iint_{R_1} (x^2y + x)\, dA = \int_0^1 \int_0^1 (x^2y + x)\, dx\, dy$$

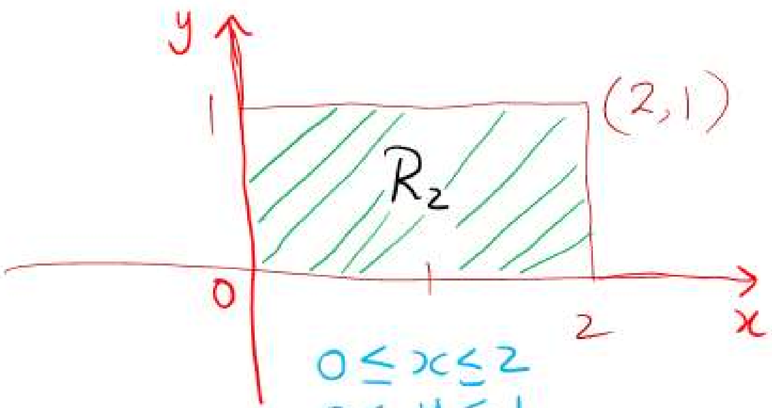
$$= \int_0^1 \left(\frac{x^3y}{3} + \frac{x^2}{2} \right) \Big|_0^1 dy = \int_0^1 \left(\frac{y}{3} + \frac{1}{2} \right) dy$$

$$= \left(\frac{y^2}{6} + \frac{1}{2}x \right) \Big|_0^1 = \frac{1}{6} + \frac{1}{2} = \frac{1+3}{6} = \frac{4}{6}$$

$$\Rightarrow \boxed{\iint_{R_1} (x^2y + x)\, dA = \frac{2}{3}}$$

Note: $\int_0^1 \int_0^1 (x^2y + x)\, dy\, dx$ also gives the same result.

Now consider the region shown in the figure.



$0 \le x \le 2$
$0 \le y \le 1$

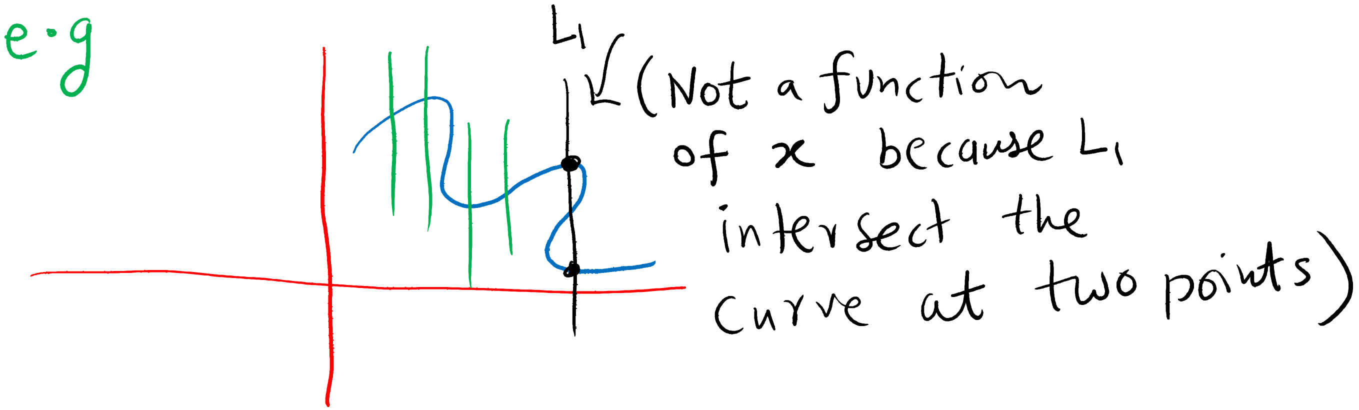
$$\iint_{R_2} (x^2 y + x)\, dA = \int_0^1 \int_0^2 (x^2 y + x)\, dx\, dy = \int_0^2 \int_0^1 (x^2 y + x)\, dy\, dx$$

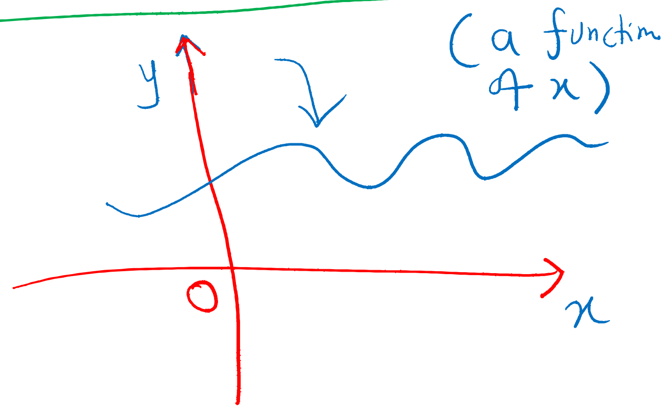
Vertical Line Test

We use V.L.T to check whether a curve is a graph of a function of x.

A curve will be a graph of a function of 'x' if no vertical line intersect the curve more than once.

e.g





Some Concepts from Multivariable Calculus

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.

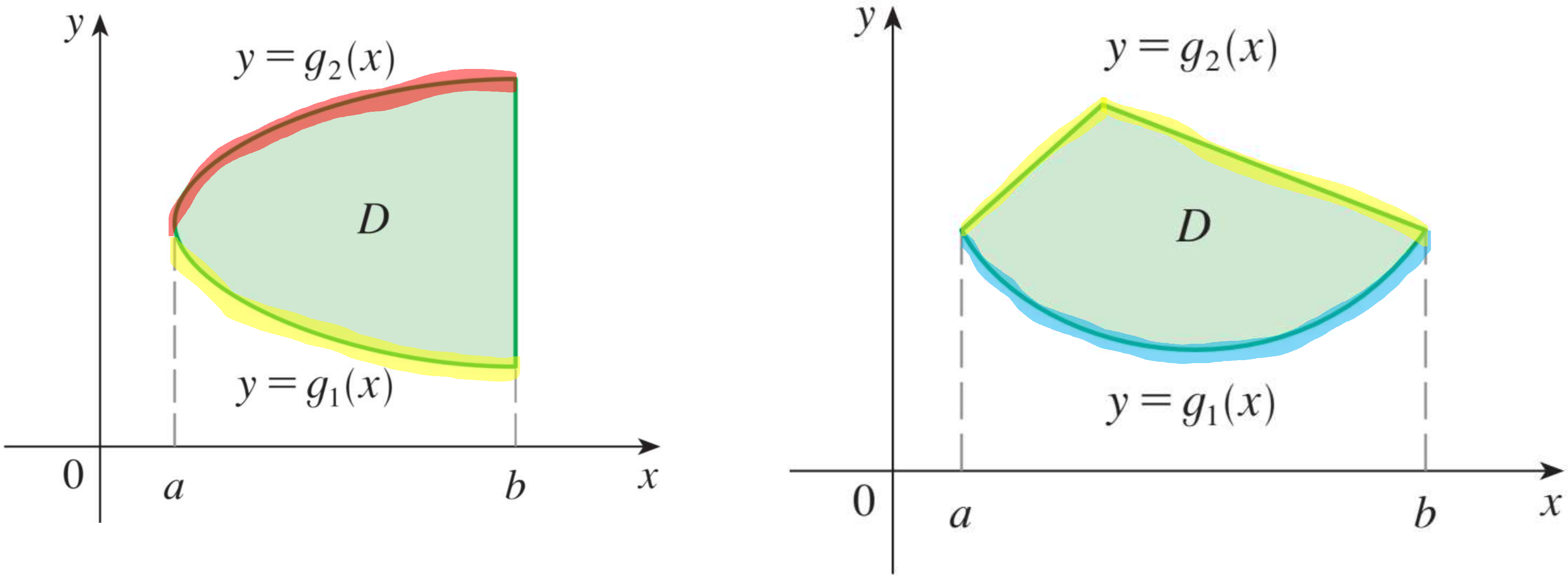


FIGURE 5

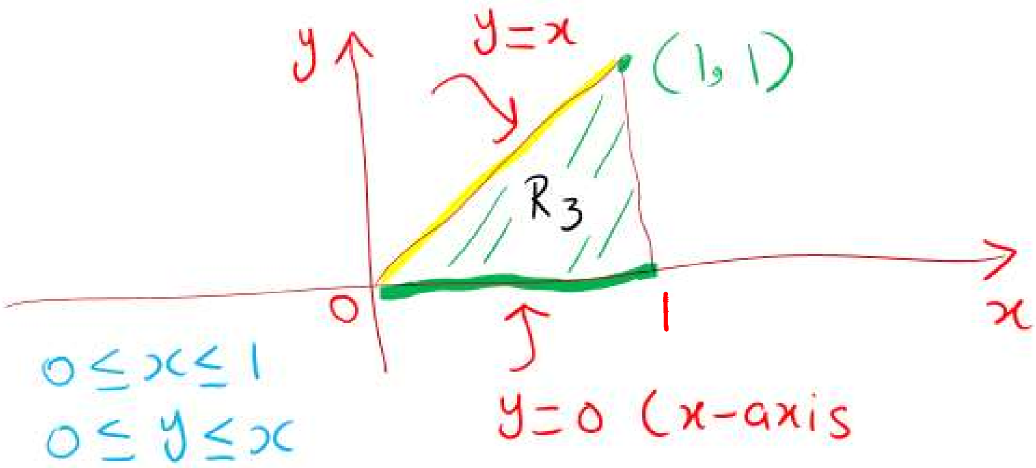
Some type I regions

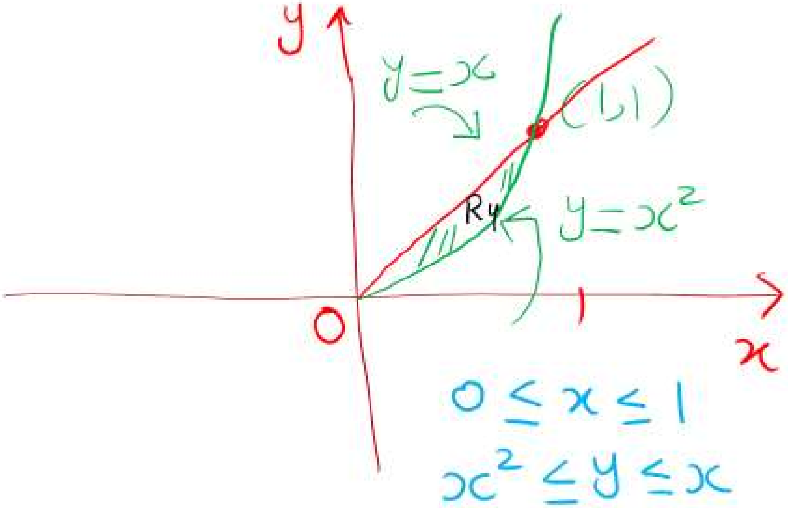
3 If f is continuous on a type I region D described by

$$D = \left\{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\right\}$$

then
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Let's evaluate $\iint_{R_3} (x^2y + x)\, dA$ & $\iint_{R_4} (x^2 + x)\, dA$, where R_3 & R_4 are shown in the figure. Both are type I region.





$$\iint_{R_3} (x^2y+x)\,dA = \int_0^1 \int_0^x (x^2y+x)\,dy\,dx = \int_0^1 \left(\frac{x^2y^2}{2} + xy \right)\Big|_0^x dx$$

$$= \int_0^1 \left(\frac{x^2 \cdot x^2}{2} + x \cdot x \right) dx = \int_0^1 \left(\frac{x^4}{2} + x^2 \right) dx$$

$$= \left(\frac{x^5}{10} + \frac{x^3}{3} \right)\Big|_0^1 = \frac{1}{10} + \frac{1}{3} = \frac{3+10}{30} = \frac{13}{30}$$

$$\Rightarrow \boxed{\iint_{R_3} (x^2y+x)\,dA = \frac{13}{30}}$$

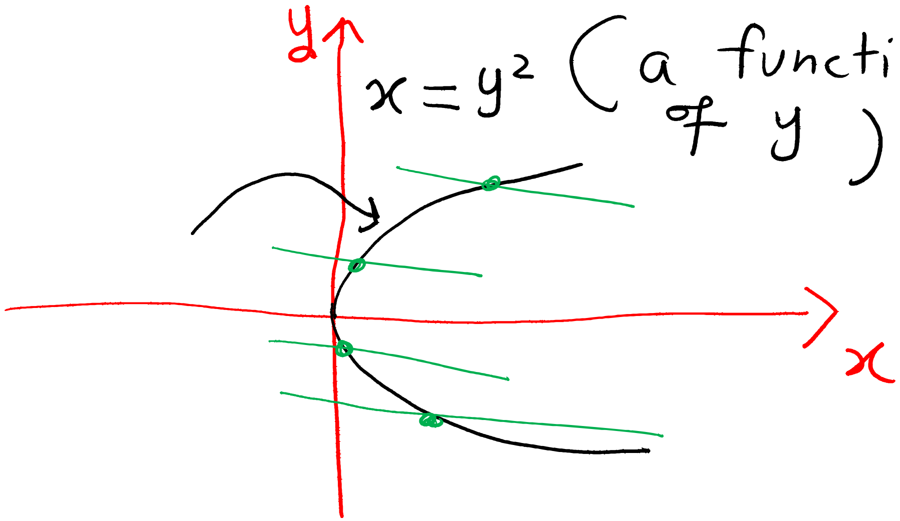
Similarly $\displaystyle\iint_{R_4} (x^2y+x)\,dx$

$$= \int_0^1 \int_{x^2}^{x} (x^2y+x)\,dy\,dx$$

Next,

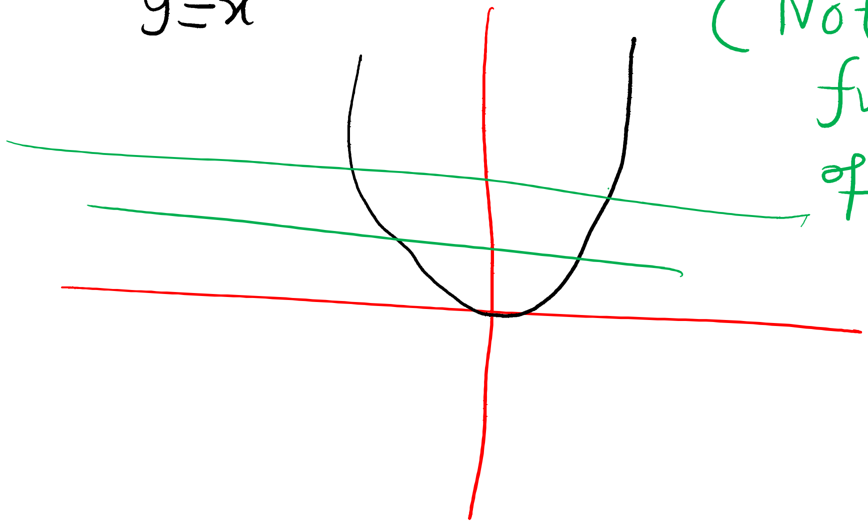
A curve will be a graph of a function of 'y' if no horizontal line intersect the curve more than once.

For example



$x = y^2$ (a function of y)

$y = x^2$



(Not a function of y)

We also consider plane regions of **type II**, which can be expressed as

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous. Three such regions are illustrated in Figure 7.

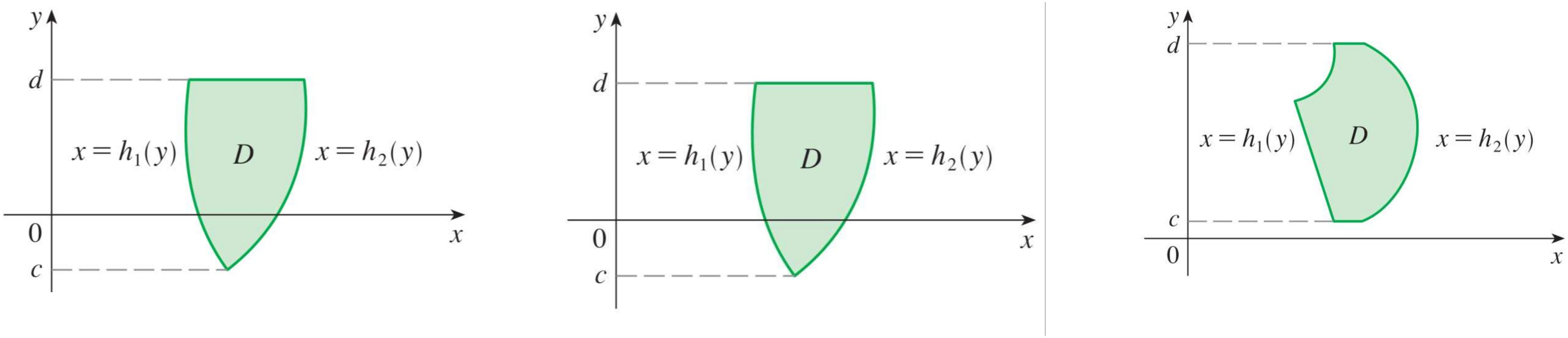


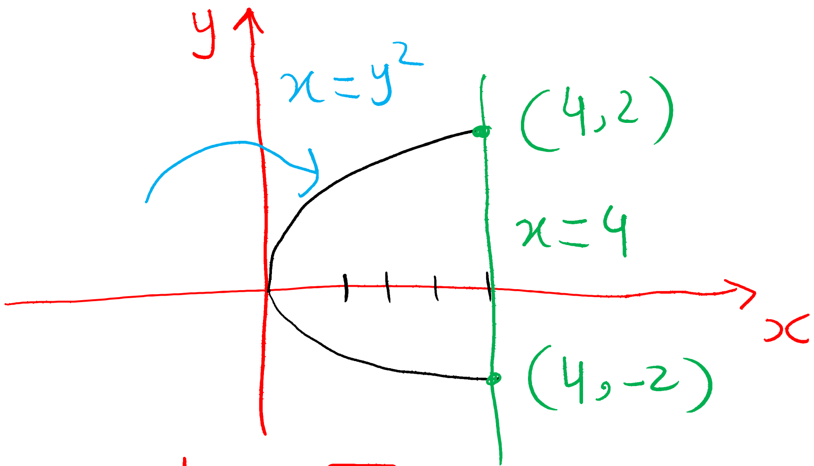
FIGURE 7
Some type II regions

4 If f is continuous on a type II region D described by

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

then
$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Let's evaluate $\iint_{R_5} (x^2 y + x)\, dA$, where R_5 is shown in the figure.



Clearly R_5 is a type II region $-2 \le y \le 2$ and $y^2 \le x \le 4$

So

$$\iint_{R_5} (x^2 y + x)\, dA = \int_{-2}^{2} \int_{y^2}^{4} (x^2 y + x)\, dx\, dy$$

Recall

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

We say that X and Y are *jointly continuous* if there exists a function $f(x, y)$ defined for all real x and y, having the property that for every set C of pairs of real numbers (that is, C is a set in the two-dimensional plane)

$$P\{(X, Y) \in C\} = \iint\limits_{(x,y)\in C} f(x, y)\, dx\, dy \qquad (4.3.3)$$

The function $f(x, y)$ is called the *joint probability density function* of X and Y. If A and B are any sets of real numbers, then by defining $C = \{(x, y) : x \in A, y \in B\}$, we see from Equation 4.3.3 that

$$P\{X \in A, Y \in B\} = \int_B \int_A f(x, y)\, dx\, dy \tag{4.3.4}$$

Because

$$F(a, b) = P\{X \in (-\infty, a], Y \in (-\infty, b]\}$$

$$= \int_{-\infty}^{b} \int_{-\infty}^{a} f(x, y)\, dx\, dy$$

it follows, upon differentiation, that

$$f(a, b) = \frac{\partial^2}{\partial a\, \partial b} F(a, b)$$

If X and Y are jointly continuous, they are individually continuous, and their probability density functions can be obtained as follows:

$$P\{X \in A\} = P\{X \in A, Y \in (-\infty, \infty)\} \tag{4.3.5}$$

$$= \int_A \int_{-\infty}^{\infty} f(x, y)\, dy\, dx$$

$$\Longrightarrow \quad P\{X \in A\} = \int_A f_X(x)\, dx$$

where

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$ is thus the probability density function of X.

Similarly, the probability density function of Y is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx \tag{4.3.6}$$

Note: A joint PDF must satisfy the equation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dy\, dx = 1$$

Because probability that x and y belongs to real numbers is 1, just like in case of one continuous random variable we have $\oint_{-\infty}^{\infty} f(x)dx = 1$.

Problem The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Compute **(a)** $P\{X > 1, Y < 1\}$; **(b)** $P\{X < Y\}$; and **(c)** $P\{X < a\}$.

Solution

(a)

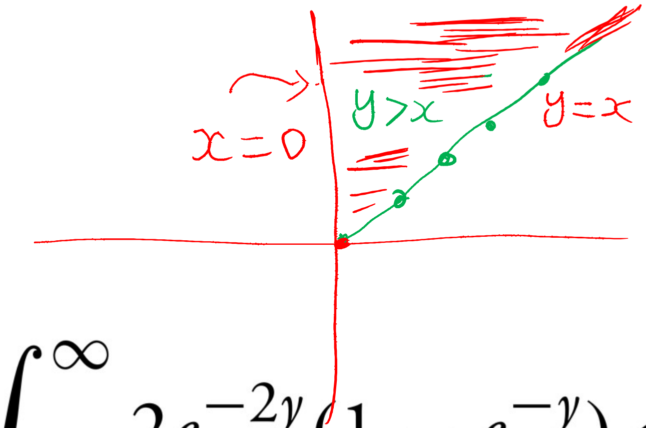
$$P\{X > 1, Y < 1\} = \int_0^1 \int_1^\infty 2e^{-x}e^{-2y}\,dx\,dy$$

$$= \int_0^1 2e^{-2y}(-e^{-x}|_1^\infty)\,dy$$

$$= e^{-1}\int_0^1 2e^{-2y}\,dy$$

$$\Longrightarrow \quad P\{X > 1, Y < 1\} = e^{-1}(1 - e^{-2})$$

$$P\{X < Y\} = \iint\limits_{(x,y):x<y} 2e^{-x}e^{-2y}\,dx\,dy$$



$$= \int_0^\infty \int_0^y 2e^{-x}e^{-2y}\,dx\,dy = \int_0^\infty 2e^{-2y}(1 - e^{-y})\,dy$$

$$= \int_0^\infty 2e^{-2y}\,dy - \int_0^\infty 2e^{-3y}\,dy \quad = 1 - \frac{2}{3}$$

$$\Longrightarrow \quad P\{X < Y\} = \frac{1}{3}$$

$$P\{X < a\} = \int_0^a \int_0^\infty 2e^{-2y} e^{-x} \, dy \, dx \; = \int_0^a e^{-x} \, dx$$

$$\Longrightarrow \; P\{X < a\} = 1 - e^{-a}$$

Practice Problem 1

Two continuous random variables X and Y have the joint density

$$f(x, y) = C(x^2 + y), \quad -1 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

(a) Compute the constant C.

(b) Find the marginal densities of X and Y.

[Hint: In part (a) set $\int_0^1 \int_{-1}^1 f(x, y) dx dy = 1$ and solve the resulting equation.]

Practice Problem 2

The joint probability density function of X and Y is given by

$$f(x, y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right), \quad 0 < x < 1, \quad 0 < y < 2$$

(a) Verify that this is indeed a joint density function.

(b) Compute the density function of X.

(c) Find $P\{X > Y\}$.

[Hint: (a) Since $f(x, y)$ is non negative and further show that $\int_0^2 \int_0^1 f(x, y)dxdy = 1$.]

Independent Random Variables

The random variables X and Y are said to be independent if for any two sets of real numbers A and B

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\}$$

When X and Y are discrete random variables, the condition of independence is equivalent to

$$p(x, y) = p_X(x)p_Y(y) \qquad \text{for all } x, y$$

where p_X and p_Y are the probability mass functions of X and Y.

In the jointly continuous case, the condition of independence is equivalent to

$$f(x, y) = f_X(x)f_Y(y) \qquad \text{for all } x, y$$

Loosely speaking, X and Y are independent if knowing the value of one does not change the distribution of the other. Random variables that are not independent are said to be dependent.

Expectation

If X is a discrete random variable taking on the possible values $x_1, x_2, \ldots ,$ then the expectation or expected value of X , denoted by E[X], is defined by

$$E[X] = \sum_i x_i P\{X = x_i\}$$

Problem

Find $E[X]$ where X is the outcome when we roll a fair die.

Solution

Since $p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = \frac{1}{6}$, we obtain that

$$E[X] = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = \frac{7}{2}$$

It is important to note that the expected value of X is not a value that X could possibly assume. (That is, in the previous problem, rolling a die cannot possibly lead to an outcome of $7/2$.) Thus, even though we call $E[X]$ the expectation of X, it should not be interpreted as the value that we expect X to have but rather as the average value of X in a large number of repetitions of the experiment. That is, if we continually roll a fair die, then after a large number of rolls the average of all the outcomes will be approximately $7/2$.

We can also define the expectation of a continuous random variable. Suppose that X is a continuous random variable with probability density function f then the expected value of X by

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

Problem

Suppose that you are expecting a message at some time past 5 P.M. From experience you know that X, the number of hours after 5 P.M. until the message arrives, is a random variable with the following probability density function:

$$f(x) = \begin{cases} \dfrac{1}{1.5} & \text{if } 0 < x < 1.5 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the expected amount of time past 5 P.M. until the message arrives.

Solution

The expected amount of time past 5 P.M. until the message arrives is given by

$$E[X] = \int_0^{1.5} \frac{x}{1.5}\, dx = .75$$

Hence, on average, you would have to wait three-fourths of an hour.

Problem Let X denote a random variable that takes on any of the values -1, 0, and respective probabilities

$$P\{X = -1\} = .2 \quad P\{X = 0\} = .5 \quad P\{X = 1\} = .3$$

Compute $E[X^2]$.

Solution

Let $Y = X^2$. Then the probability mass function of Y is given by

$$P\{Y = 1\} = P\{X = -1\} + P\{X = 1\} = .5$$
$$P\{Y = 0\} = P\{X = 0\} = .5$$

$$E[X^2] = E[Y] = 1(.5) + 0(.5) = .5$$

Expectation of a Function of a Random variable

(a) If X is a discrete random variable with probability mass function $p(x)$, then for any real-valued function g,

$$E[g(X)] = \sum_{x} g(x)p(x)$$

(b) If X is a continuous random variable with probability density function $f(x)$, then for any real-valued function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

If a and b are constants, then

$$\boxed{E[aX + b] = aE[X] + b}$$

If we take $a = 0$

$$E[b] = b$$

That is, the expected value of a constant is just its value. if we take $b = 0$, then we obtain

$$E[aX] = aE[X]$$

or, in words, the expected value of a constant multiplied by a random variable is just the constant times the expected value of the random variable.

The expected value of a random variable X , E[X], is also referred to as the mean or the first moment of X . The quantity $E[X^n]$, n ≥ 1, is called the nth moment of X .

$$E[X^n] = \begin{cases} \displaystyle\sum_x x^n p(x) & \text{if } X \text{ is discrete} \\ \displaystyle\int_{-\infty}^{\infty} x^n f(x)\, dx & \text{if } X \text{ is continuous} \end{cases}$$

In general, for any n,

$$E[X_1 + X_2 \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$$

Problem

A construction firm has recently sent in bids for 3 jobs worth (in profits) 10, 20, and 40 (thousand) dollars. If its probabilities of winning the jobs are respectively .2, .8, and .3, what is the firm's expected total profit?

Solution

Letting $X_i, i = 1, 2, 3$ denote the firm's profit from job i, then

$$\text{total profit} = X_1 + X_2 + X_3$$

and so

$$E[\text{total profit}] = E[X_1] + E[X_2] + E[X_3]$$

Now

$$E[X_1] = 10(.2) + 0(.8) = 2$$
$$E[X_2] = 20(.8) + 0(.2) = 16$$
$$E[X_3] = 40(.3) + 0(.7) = 12$$

and thus the firm's expected total profit is 30 thousand dollars.

Variance

If X is a random variable with mean μ, then the *variance* of X, denoted by $\mathrm{Var}(X)$, is defined by

$$\mathrm{Var}(X) = E[(X - \mu)^2]$$

An alternative formula for $\mathrm{Var}(X)$ can be derived as follows:

$$\begin{aligned}
\mathrm{Var}(X) = E[(X - \mu)^2] &= E[X^2 - 2\mu X + \mu^2] \\
&= E[X^2] - E[2\mu X] + E[\mu^2] = E[X^2] - 2\mu E[X] + \mu^2 \\
&= E[X^2] - \mu^2
\end{aligned}$$

That is,

$$\boxed{\mathrm{Var}(X) = E[X^2] - (E[X])^2}$$

or, in words, the variance of X is equal to the expected value of the square of X minus the square of the expected value of X. This is, in practice, often the easiest way to compute $\mathrm{Var}(X)$.

Problem Compute Var(X) when X represents the outcome when we roll a fair die.

Solution Since $P\{X = i\} = \frac{1}{6}, i = 1, 2, 3, 4, 5, 6$, we obtain

$$E[X^2] = \sum_{i-1}^{6} i^2 P\{X = i\} = 1^2\left(\tfrac{1}{6}\right) + 2^2\left(\tfrac{1}{6}\right) + 3^2\left(\tfrac{1}{6}\right) + 4^2\left(\tfrac{1}{6}\right) + 5^2\left(\tfrac{1}{6}\right) + 6^2\left(\tfrac{1}{6}\right)$$

$$= \frac{91}{6}$$

$$E[X] = \frac{7}{2}$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

A useful identity concerning variances is that for any constants a and b,

$$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$$

Specifying particular values for a and b in above equation leads to some important result. For instance, by setting $a = 0$

$$\mathrm{Var}(b) = 0$$

That is, the variance of a constant is 0.

Similarly, by setting $a = 1$ we obtain

$$\mathrm{Var}(X + b) = \mathrm{Var}(X)$$

That is, the variance of a constant plus a random variable is equal to the variance of the random variable.

Finally, setting $b = 0$ yields

$$\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$$

The square root of the $\mathrm{Var}(X)$ is called the *standard deviation* of X, and we denote it by $\mathrm{SD}(X)$. That is,

$$\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$$

Covariance

The *covariance* of two random variables X and Y, written $\mathrm{Cov}(X, Y)$, is defined by

$$\mathrm{Cov}(X, Y) = E[(X - \mu_x)(Y - \mu_y)]$$

where μ_x and μ_y are the means of X and Y, respectively.

A useful expression for $\mathrm{Cov}(X, Y)$ can be obtained by expanding the right side of the definition. This yields

$$
\begin{aligned}
\mathrm{Cov}(X, Y) &= E[XY - \mu_x Y - \mu_y X + \mu_x \mu_y] \\
&= E[XY] - \mu_x E[Y] - \mu_y E[X] + \mu_x \mu_y \\
&= E[XY] - \mu_x \mu_y - \mu_y \mu_x + \mu_x \mu_y \\
&= E[XY] - E[X]E[Y]
\end{aligned}
$$

From its definition we see that covariance satisfies the following properties:

$$\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$$

and

$$\mathrm{Cov}(X, X) = \mathrm{Var}(X)$$

In general, a positive value of Cov(X ,Y) is an indication that Y tends to increase as X does, whereas a negative value indicates that Y tends to decrease as X increases. The strength of the relationship between X and Y is indicated by the correlation between X and Y , a dimensionless quantity obtained by dividing the covariance by the product of the standard deviations of X and Y . That is,

$$\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

This quantity always has a value between −1 and +1.

Problem

A product is classified according to the number of defects it contains and the factory that produces it. Let X_1 and X_2 be the random variables that represent the number of defects per unit (taking on possible values of 0, 1, 2, or 3) and the factory number (taking on possible values 1 or 2), respectively. The entries in the table represent the joint possibility mass function of a randomly chosen product.

X_1 \ X_2	1	2
0	$\frac{1}{8}$	$\frac{1}{16}$
1	$\frac{1}{16}$	$\frac{1}{16}$
2	$\frac{3}{16}$	$\frac{1}{8}$
3	$\frac{1}{8}$	$\frac{1}{4}$

(a) Find the marginal probability distributions of X_1 and X_2.

(b) Find $E[(X_1)]$, $E[(X_2)]$, and $\mathrm{Cov}(X_1, X_2)$.

Solution

X_1 \ X_2	1	2	Row Sum $P\{X_1 = i\}$
0	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{8} + \frac{1}{16} = 3/16$
1	$\frac{1}{16}$	$\frac{1}{16}$	$1/8$
2	$\frac{3}{16}$	$\frac{1}{8}$	$5/16$
3	$\frac{1}{8}$	$\frac{1}{4}$	$3/8$
Column Sum $P\{X_2 = i\}$	$1/2$	$1/2$	

$$E[X_1] = \left(0 \times \frac{3}{16}\right) + \left(1 \times \frac{1}{8}\right) + \left(2 \times \frac{5}{16}\right) + \left(3 \times \frac{3}{8}\right)$$

$$E[X_1] = \frac{15}{8}$$

Similarly

$$E[X_2] = \left(1 \times \frac{1}{2}\right) + \left(2 \times \frac{1}{2}\right) = \frac{3}{2}$$

$$\mathrm{Cov}(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2]$$

$$E[X_1X_2] = \sum_{x_1}\sum_{x_2} x_1x_2\,p(x_1,x_2)$$

$$\sum_{x_1}\sum_{x_2} x_1x_2\,p(x_1,x_2)$$

$$= (0)(1)\left(\frac{1}{8}\right) + (0)(2)\left(\frac{1}{16}\right) + (1)(1)\left(\frac{1}{16}\right) +$$

$$(1)(2)\left(\frac{1}{16}\right) + (2)(1)\left(\frac{3}{16}\right) + (2)(2)\left(\frac{1}{8}\right) +$$

$$(3)(1)\left(\frac{1}{8}\right) + (3)(2)\left(\frac{1}{4}\right)$$

and so on.

Special Random Variables

Certain types of random variables occur over and over again in applications. Here we will study a variety of them.

The Bernoulli & Binomial Random Variables

Suppose that a trial, or an experiment, whose outcome can be classified as either a "success" or as a "failure" is performed. If we let X = 1 when the outcome is a success and X = 0 when it is a failure, then the probability mass function of X is given by

$$\boxed{\begin{aligned} P\{X=0\} &= 1-p \\ P\{X=1\} &= p \end{aligned}} \qquad \text{(i)}$$

where $p, 0 \leq p \leq 1$, is the probability that the trial is a "success." A random variable X is said to be a Bernoulli random variable (after the Swiss mathematician James Bernoulli) if its probability mass function is given by Equations (i) for some $p \in (0,1)$.

Its expected value is

$$E[X] = 1 \cdot P\{X = 1\} + 0 \cdot P\{X = 0\} = p$$

That is, the expectation of a Bernoulli random variable is the probability that the random variable equals 1.

Similarly by putting values in the formula of variance we get

$$Var(X) = p(1 - p)$$

Suppose now that n independent trials, each of which results in a "success" with probability p and in a "failure" with probability $1 - p$, are to be performed. If X represents the number of successes that occur in the n trials, then X is said to be a *binomial* random variable with parameters (n, p).

The probability mass function of a binomial random variable with parameters n and p is given by

The probability mass function of a binomial random variable with parameters n and p is given by

$$\boxed{P\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 0, 1, \ldots, n} \qquad \text{(ii)}$$

where $\binom{n}{i} = n!/[i!(n-i)!]$

If X is a binomial random variable with parameters n and p, then

$$\boxed{\begin{aligned} E[X] &= np \\ \mathrm{Var}(X) &= np(1 - p) \end{aligned}}$$

Problem

Five fair coins are flipped. If the outcomes are assumed independent, find the probability mass function of the number of heads obtained.

Solution

If we let X equal the number of heads (successes) that appear, then X is a binomial random variable with parameters $\left(n = 5,\ p = \frac{1}{2}\right)$. Hence,

$$P\{X = 0\} = \binom{5}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

$$P\{X = 1\} = \binom{5}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4 = \frac{5}{32}$$

$$P\{X = 2\} = \binom{5}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{10}{32}$$

$$P\{X = 3\} = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{10}{32}$$

$$P\{X = 4\} = \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1 = \frac{5}{32}$$

$$P\{X = 5\} = \binom{5}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0 = \frac{1}{32}$$

Problem

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) at least 10 survive, (b) from 3 to 8 survive, and (c) exactly 5 survive?

Solution

Let X be the number of people who survive. For any patient there are two cases: Either the patient will survive (Success) or will die (Failure).

Number of people having the disease $= n = 15$.

$P\{X \geq 10\} = ?$, $\quad P\{3 \leq X \leq 8\} = ?$, $\quad P\{X = 5\} = ?$.

Clearly we can see that X is a Binomial Random Variable.

Here X can take the values $0, 1, 2, \ldots, 15$. Also it is given that the probability of patient recovery (success) is 0.4 so $p = 0.4$. Thus the PMF of X is

$$P\{X = i\} = \binom{15}{i}(0.4)^{i}(0.6)^{15-i}, \quad i = 0, 1, 2, \ldots, 15.$$

(a) Now

$$P\{X \geq 10\} = P\{X = 10\} + P\{X = 11\} + \ldots + P\{X = 15\}$$

$$= \binom{15}{10}(0.4)^{10}(0.6)^{5} + \binom{15}{11}(0.4)^{11}(0.6)^{4} + \binom{15}{12}(0.4)^{12}(0.6)^{3}$$
$$+ \binom{5}{13}(0.4)^{13}(0.6)^{2} + \binom{15}{14}(0.4)^{14}(0.6)^{1} + \binom{15}{15}(0.4)^{15}(0.6)^{0}$$

$$\Rightarrow \boxed{P\{X \geq 10\} = 0.0338} \quad \text{Ans}$$

(b)

$$P\{3 \leq X \leq 8\} = P\{X=3\} + P\{X=4\} + \cdots + P\{X=8\}$$

$$= \binom{15}{3}(0.4)^3(0.6)^{12} + \cdots + \binom{15}{8}(0.4)^8(0.6)^7$$

$$\Rightarrow \boxed{P\{3 \leq X \leq 8\} = 0.8779} \quad \text{Ans}$$

(c) Similarly

$$\boxed{P\{X=5\} = 0.1859} \quad \text{Ans}$$

Practice Problem

A large chain retailer purchases a certain kind of electronic device from a manufacturer. The manufacturer indicates that the defective rate of the device is 3%.

(a) The inspector randomly picks 20 items from a shipment. What is the probability that there will be at least one defective item among these 20?

(b) Suppose that the retailer receives 10 shipments in a month and the inspector randomly tests 20 devices per shipment. What is the probability that there will be exactly 3 shipments each containing at least one defective device among the 20 that are selected and tested from the shipment?

Practice Problem

As part of a business strategy, randomly selected 20% of new internet service subscribers receive a special promotion from the provider. A group of 10 neighbors signs for the service. What is the probability that at least 4 of them get a special promotion?

The Poisson Random Variable

A random variable X, taking on one of the values 0, 1, 2, . . . , is said to be a Poisson random variable with parameter λ, $\lambda > 0$, if its probability mass function is given by

$$\boxed{P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!}, \qquad i = 0, 1, \ldots}$$

The symbol e stands for a constant approximately equal to 2.7183. It is a famous constant in mathematics, named after the Swiss mathematician L. Euler, and it is also the base of the so-called natural logarithm.

The Poisson probability distribution was introduced by S. D. Poisson in a book he wrote dealing with the application of probability theory to lawsuits, criminal trials, and the like.

Both the mean and the variance of a Poisson random variable are equal to the parameter λ.

The Poisson random variable has a wide range of applications in a variety of areas because it can be used as an approximation for a binomial random variable with parameters (n, p) when n is large and p is small.

If n independent trials, each of which results in a "success" with probability p, are performed, then when n is large and p small, the number of successes occurring is approximately a Poisson random variable with mean $\lambda = np$.

Some examples of random variables that usually obey, to a good approximation, the Poisson probability law (that is, they usually obey equation for some value of λ) are:

1. The number of misprints on a page (or a group of pages) of a book.
2. The number of people in a community living to 100 years of age.
3. The number of wrong telephone numbers that are dialed in a day.
4. The number of transistors that fail on their first day of use.

Problem

Suppose that the average number of accidents occurring weekly on a particular stretch of a highway equals 3. Calculate the probability that there is at least one accident this week.

Solution

Let X denote the number of accidents occurring on the stretch of highway in question during this week. Because it is reasonable to suppose that there are a large number of cars passing along that stretch, each having a small probability of being involved in an accident, the number of such accidents should be approximately Poisson distributed. Hence,

$$\begin{aligned} P\{X \geq 1\} &= 1 - P\{X = 0\} \\ &= 1 - e^{-3}\frac{3^0}{0!} \\ &= 1 - e^{-3} \approx .9502 \end{aligned}$$

Problem

Suppose that 1 person in 1000 makes a numerical error in preparing his or her income tax return. If 10,000 returns are selected at random and examined, find the probability that 6 of them contain an error.

Geometric Random Variable

In a series of Bernoulli trials (independent trials with constant probability p of success), let the random variable X denote the number of trials until the first success. Then, X is a geometric random variable with parameter p such that $0 < p < 1$ and the probability mass function of X is

$$P\{X = i\} = p(1-p)^{i-1}, \qquad i = 1,2,3,\ldots$$

The binomial and geometric distributions arise in very similar situations. The significant difference is that the number of trials in a binomial distribution is fixed from the start and the number of successes are counted, whereas, in a geometric distribution, trials are repeated as many times as necessary until the first success occurs.

Problem

Let X be a geometric random variable with $p = 0.25$. What is the probability that $X = 4$ (i.e. that the first success occurs on the 4th trial)?

Note: For X to be equal to 4, we must have had 3 failures, and then a success.

Solution

$$P\{X = 4\} = (0.25)(1 - 0.25)^{4-1} = 0.1055$$

Problem

For a certain manufacturing process, it is known that 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?

Solution

Let X is the number of items inspected before we found a defective item.

p =probability of being defective item= $1/100$, then X is a geometric R.V. with PMF given by

$$P\{X = i\} = (1/100)(1 - 1/100)^{i-1}, i = 1,2,3, \dots$$

Now

$$P\{X = 5\} = (1/100)(1 - 1/100)^{5-1} = 0.0096$$

The mean and variance of geometric random variable are given by:

$$E[X] = \frac{1}{p} \qquad \& \qquad Var(X) = \frac{1-p}{p^2}$$

Practice Problem

The probability that a student pilot passes the written test for a private pilot's license is 0.7. Find the probability that a given student will pass the test

(a) on the third try;

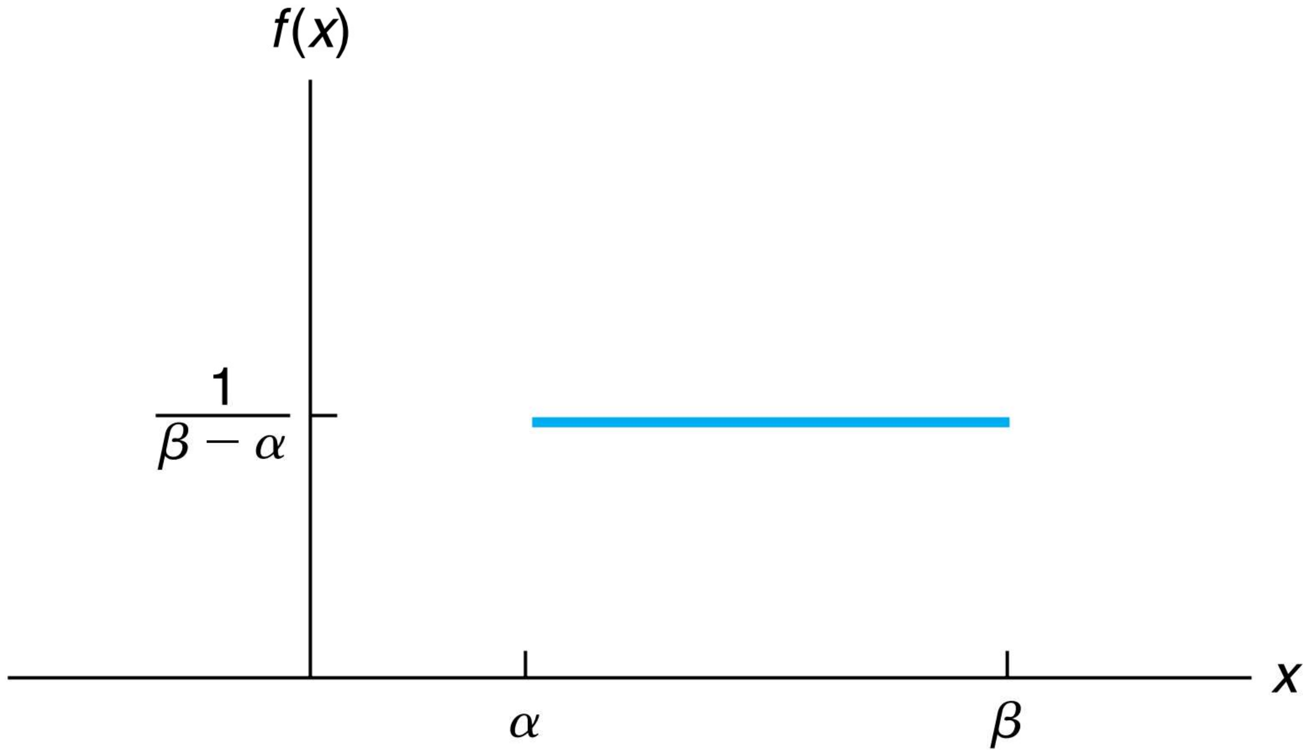
(b) before the fourth try.

The Uniform Random Variable

A random variable X is said to be uniformly distributed over the interval $[\alpha, \beta]$ if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & \text{if } \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$

A graph of this function is given in Figure 1 on the next slide.



Graph of f (x) for a uniform [α, β].

If we divide the domain of the uniform random variable in equal parts, then all parts will be equally likely. This mean that the probability that x lies in an interval of width Δx entirely contained in the interval from α to β is equal to $\Delta x/(\beta - \alpha)$, regardless of the exact location of the interval.

$$P\{a < X < b\} = \frac{1}{\beta - \alpha} \int_a^b dx = \frac{b-a}{\beta - \alpha}$$

The probability that X lies in any subinterval of $[\alpha, \beta]$ is equal to the length of that subinterval divided by the length of the interval $[\alpha, \beta]$.

$$E[X] = \frac{\alpha + \beta}{2} \qquad \& \qquad \mathrm{Var}(X) = \frac{(\beta - \alpha)^2}{12}$$

Problem

If X is uniformly distributed over the interval $[0, 10]$, compute the probability that (a) $2 < X < 9$, (b) $1 < X < 4$, (c) $X < 5$, (d) $X > 6$.

Solution

The respective answers are (a) $7/10$, (b) $3/10$, (c) $5/10$, (d) $4/10$.

Problem

Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits

(a) less than 5 minutes for a bus;

(b) at least 12 minutes for a bus

Solution

Let X denote the time in minutes past 7 A.M. that the passenger arrives at the stop. Since X is a uniform random variable over the interval (0, 30), it follows that the passenger will have to wait less than 5 minutes if he arrives between 7:10 and 7:15 or between 7:25 and 7:30. Hence, the desired probability for (a) is

$$P\{10 < X < 15\} + P\{25 < X < 30\} = \frac{5}{30} + \frac{5}{30} = \frac{1}{3}$$

Similarly, he would have to wait at least 12 minutes if he arrives between 7 and 7:03 or between 7:15 and 7:18, and so the probability for (b) is

$$P\{0 < X < 3\} + P\{15 < X < 18\} = \frac{3}{30} + \frac{3}{30} = \frac{1}{5}$$

Problem Beginning at 12:00 midnight, a computer center is up for 1 hour and down for 2 hours on a regular cycle. A person who doesn't know the schedule dials the center at a random time between 12:00 midnight and 5:00 AM. what is the probability that the center will be operating when he dials in?

Solution

Since person is equally likely to call between 12:00 midnight and 5:00 A.M. it is a uniform distribution. Let random variable Y be the time when the person calls.

Since the interval in which the person calls is 5 hours long, the interval will be $(0, 5)$.

Hence,

$$P(0 < Y < 1) + P(3 < Y < 4) = \int_0^1 f(y)dy + \int_3^4 f(y)dy$$

Since $f(y) = \dfrac{1}{5}\ \forall\ y \in (0, 5)$, hence

$$\begin{aligned} P(0 < Y < 1) + P(3 < Y < 4) &= \int_0^1 \left(\frac{1}{5}\right) dy + \int_3^4 \left(\frac{1}{5}\right) dy \\ &= \left(\frac{1}{5}\right) y\Big|_0^1 + \left(\frac{1}{5}\right) y\Big|_3^4 \\ &= \frac{1-0}{5} + \frac{4-3}{5} \ = 0.4 \end{aligned}$$

Practice Problem

Waiting Times You arrive at a bus stop to wait for a bus that comes by once every 30 minutes. You don't know what time the last bus came by. The time x that you wait before the bus arrives is uniformly distributed on the interval from 0 to 30 minutes.

a. What is the probability that you will have to wait longer than 20 minutes?

b. What is the probability that you will have to wait less than 10 minutes?

Normal Random Variables

A random variable is said to be normally distributed with parameters μ and σ^2, and we write $X \sim N(\mu, \sigma^2)$, if its density is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2}, \qquad -\infty < x < \infty$$

The normal density $f(x)$ is a bell-shaped curve that is symmetric about μ and that attains its maximum value of $\frac{1}{\sqrt{2\pi}\sigma} \approx \frac{0.399}{\sigma}$ at $x = \mu$ shown in figure 1 on the next slide.

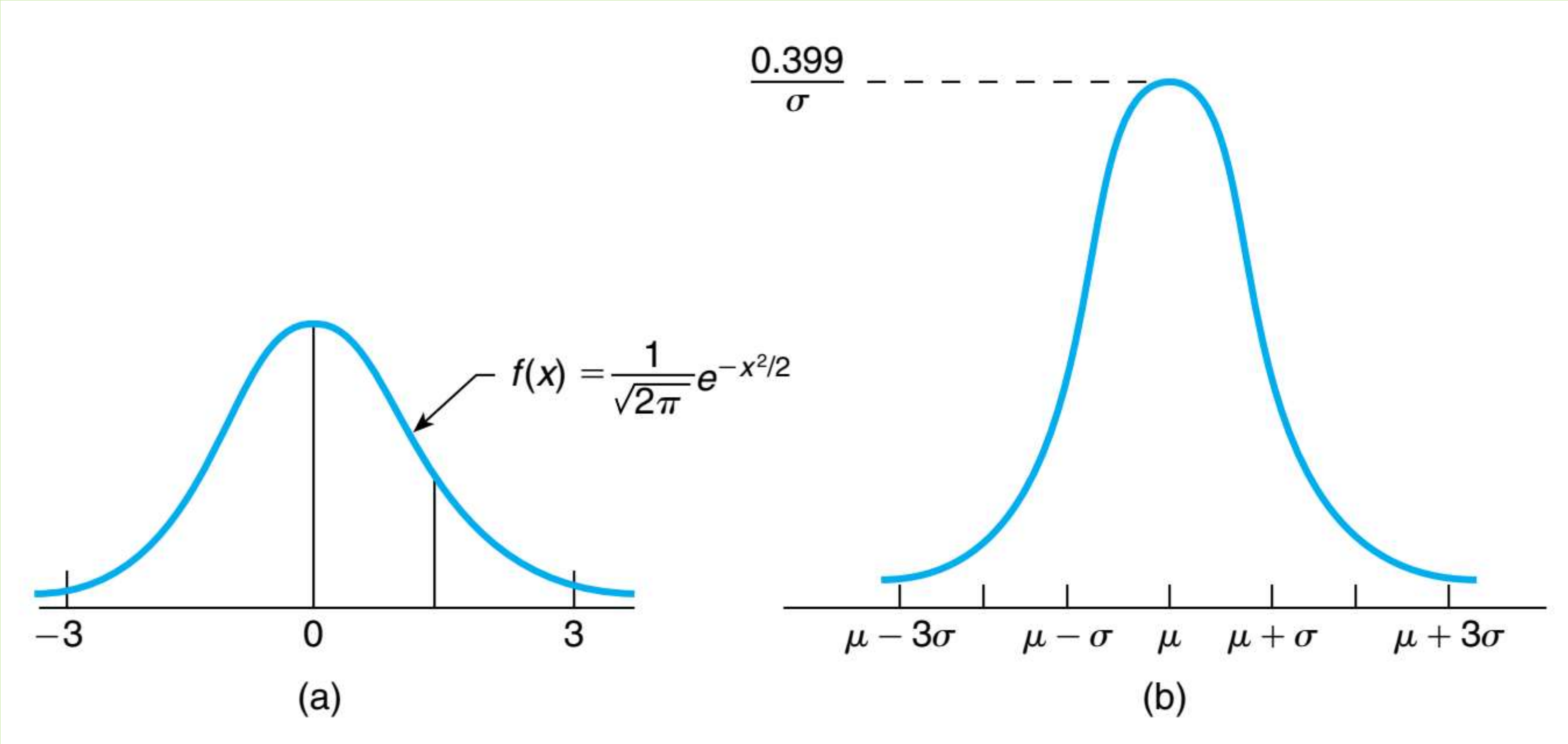
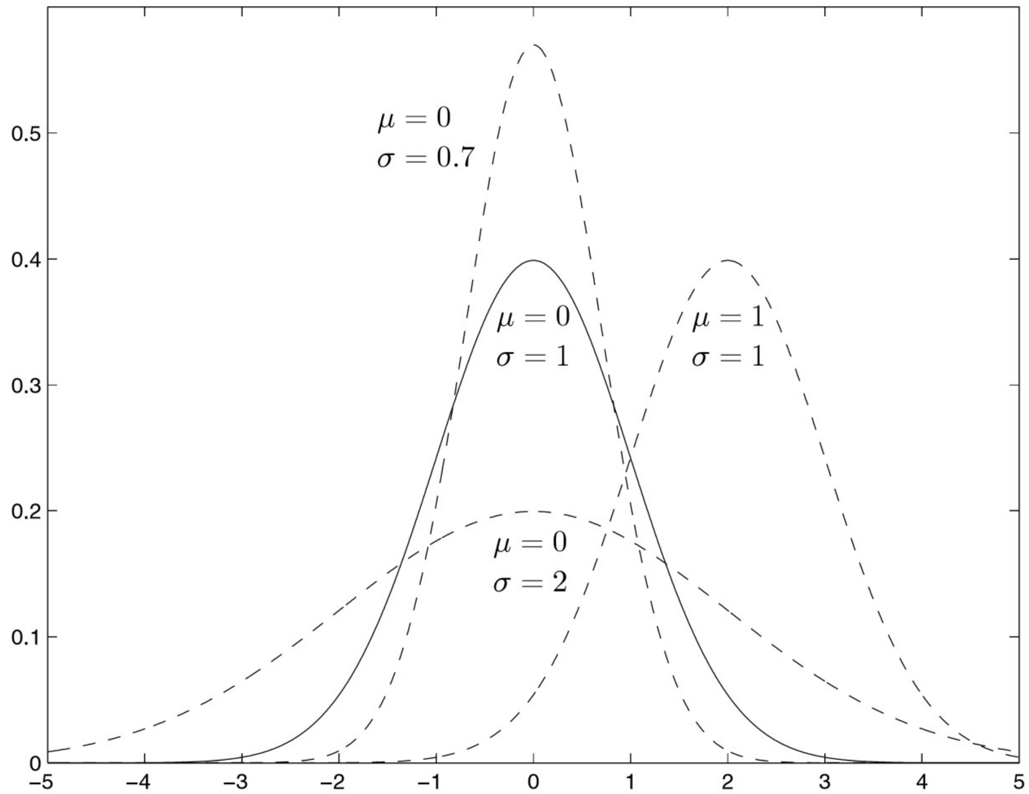


FIGURE *1 (a) with $\mu = 0, \sigma = 1$ and (b) with arbitrary μ and σ^2.*

The mean and variance of a normal random variable are given by:

$$E[X] = \mu \qquad \& \qquad Var(X) = \sigma^2$$

Changing μ shifts the curve to the left or to the right without affecting its shape, while changing σ makes it more concentrated or more flat. Often μ and σ are called location and scale parameters.



Normal densities with different location and scale parameters.

A normal random variable is called standard, or unit, normal random variable if it has mean 0 and variance 1 and it is denoted by $N(0,1)$. In case of standard r.v. we can calculate probabilities with the help of a table which is shown in the next slide.

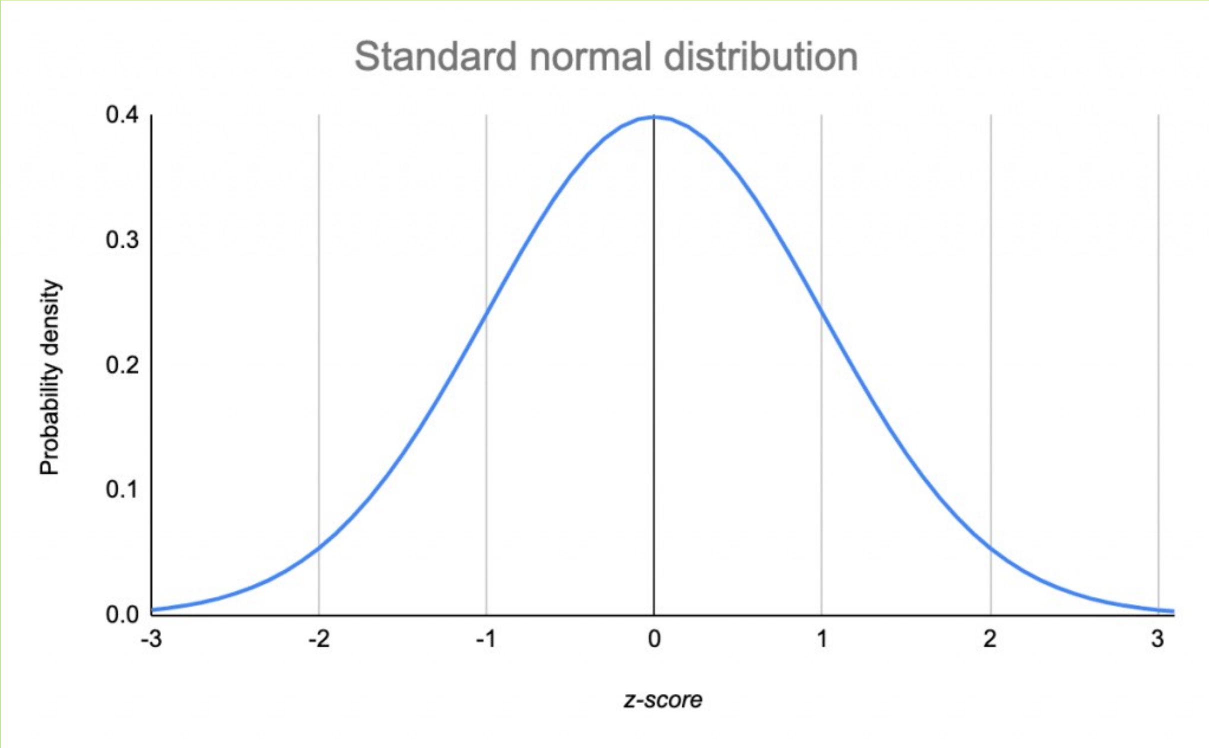


TABLE A1 *Standard Normal Distribution Function:* $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389

1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952

2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Problem

Given that Z has a normal distribution with $\mu = 0$ and $\sigma = 1$, Find (a) $P\{Z \leq 1.25\}$, (b) $P\{Z > 1.25\}$, (c) $P\{Z \leq -1.25\}$, (d) $P\{-0.38 \leq Z \leq 1.25\}$, and (e) $P\{Z \leq 5\}$.

Solution

(a) $P\{Z \leq 1.25\} = \varphi(1.25)$, a probability that is tabulated in standard normal table at the intersection of the row marked 1.2 and the column marked .05. The number there is .8944, so $P\{Z \leq 1.25\} = .8944$

(b) $P\{Z > 1.25\} = 1 - P\{Z \leq 1.25\} = 1 - \varphi(1.25) = 1 - 0.8944 = .1056$

(c) $P\{Z \leq -1.25\} = P\{Z \geq 1.25\}$ (By symmetry)

$$= .1056$$

(d) $P\{-0.38 \leq Z \leq 1.25\} = \varphi(1.25) - \varphi(-0.38) = 0.8944 - [1 - \varphi(0.38)] = 0.8944 - [1 - 0.6480]$

$$= 0.5424$$

(e) $P\{Z \leq 5\} \approx 1.$

A very important property of normal random variables is that if X is normal with mean μ and variance σ^2, then for any constants a and b, $b \neq 0$, the random variable $Y = a + bX$ is also a normal random variable with parameters

$a + b\mu$ and $b^2\sigma^2$

A Standard Normal variable, usually denoted by Z, can be obtained from a non-standard Normal(μ, σ) random variable X by *standardizing*, that is, subtracting the mean and dividing by the standard deviation,

$$Z = \frac{X - \mu}{\sigma}$$

Problem

If X is a normal random variable with mean $\mu = 3$ and variance $\sigma^2 = 16$, find

(a) $P\{X < 11\}$;
(b) $P\{X > -1\}$;
(c) $P\{2 < X < 7\}$.

Solution

(a)
$$P\{X < 11\} = P\left\{\frac{X-3}{4} < \frac{11-3}{4}\right\}$$
$$= \Phi(2)$$
$$= .9772$$

(b)

$$P\{X > -1\} = P\left\{\frac{X-3}{4} > \frac{-1-3}{4}\right\}$$

$$= P\{Z > -1\} = P\{Z < 1\}$$

$$= .8413$$

(c)

$$P\{2 < X < 7\} = P\left\{\frac{2-3}{4} < \frac{X-3}{4} < \frac{7-3}{4}\right\}$$

$$= \Phi(1) - \Phi(-1/4)$$

$$= \Phi(1) - (1 - \Phi(1/4))$$

$$= .8413 + .5987 - 1 = .4400 \quad \blacksquare$$

Thank you

Raamiz Ijaz
Changez Khan
Hassan Shah
Aimal Khan
Muhammad Ahmad
Afaq Ur
Rimsha Imran
Sarmad Shaukat
Fahim Baig
Fahad Khan
Hamza Raziq
Shaheryar Ashfaq
Ramzan Ali
Umer Saleem
Haris Ali
Hashim Khan
ShoaibAkhtar
Dawood Sarfraz
Azam Irfan

Muhammad Affan
Khafi Ul
Afaq Haider
Haider Ali
Ammaar Naeem
Armghan Ahmad
Urooj Afridi
Wajiha Mehboob
Taha Imran
Muhammad Ghazanfar
Furqan Manzoor
Taimoor Hassan
Saad Ali
Ali Gillani
Waleed Akram
Alishan Khattak
Lalain Noor