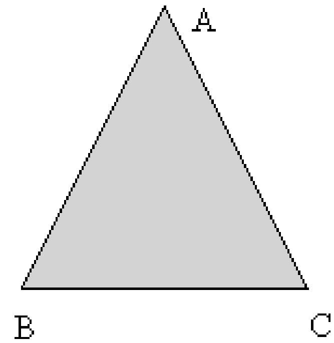
Triangles Part 1

Triangles



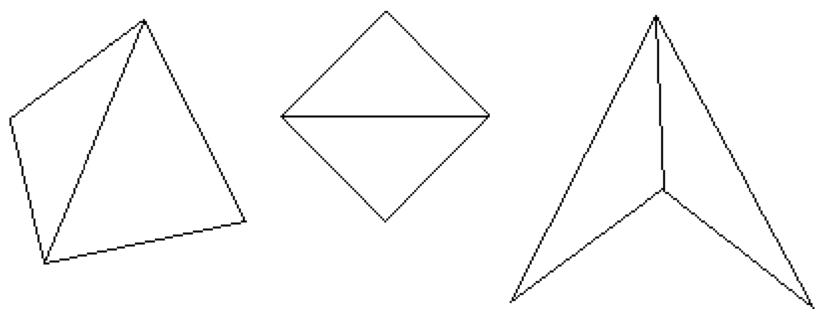
A **triangle** is the polygon with three sides and three vertices.

The adjoining figure shows the triangle ABC which is denoted by **Δ**ABC.

- AB, BC and AC are the sides of the triangle.
- ∟ABC , ∟BAC and ∟BCA denotes the interior angles of **Δ**ABC. These are also known as interior angles of **Δ**ABC.

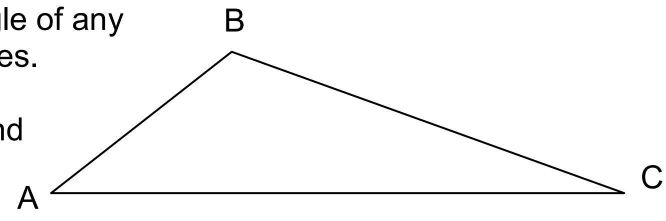
- The sum of the measures of the interior angles of a triangle is always 180^0 .
  ∟ABC + ∟BAC + ∟BCA = 180^0

Guess : What is sum the measures of the interior angles of a quadrilateral?(See the figure.)



Properties of a Triangle

1) The side facing to the bigger interior angle of any triangle is always bigger than other two sides.
Consider the following triangle.
In ΔABC, $\llcorner ABC$ has maximum degree and side opposite this angle is AC whose length is bigger than AB and BC.



2) Triangle Inequality: For any triangle, the sum of any two types of the triangle is always greater than the third side.

Hence in ΔABC,

$AB + BC > AC$ and
$AC + BC > AB$ and
$AB + AC > BC$

Foreg: There does not exit any triangle with length of the sides as 2, 2,10

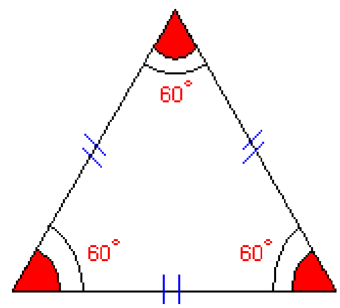
Types of Triangles

Triangles are of the following types.

Equilateral triangle

A triangle in which all sides have same length or all interior angle have equal measure, the triangle is said to be **equilateral triangle**.

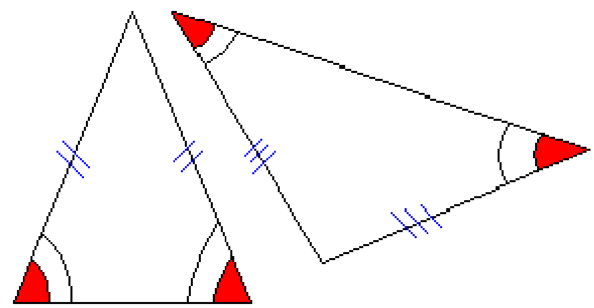
Note : Measure of an interior angle of an equilateral triangle is always 60^0.



Isosceles triangle

If a triangle has at least two sides of equal length or at least two interior angles of equal measure, then the triangle is said to be **isosceles triangle**.

Note: Every equilateral triangle is also an isosceles triangle. But every isosceles triangle need not be equilateral triangle.



Scalene triangle

In a triangle, if no two sides have equal length or no two interior angles have equal measure, then the triangle is said to be **scalene triangle**.

Note: Neither isosceles triangle nor an equilateral triangle can be a scalene triangle.

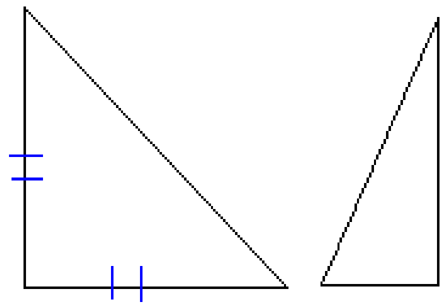
Right Triangle

Right angled triangle is a triangle in which one of the interior angle is a right angle.

- The side opposite to the right angle is known as **hypotenuse**.
- Hypotenuse is largest side of the right angles triangle.
- The two interior angles other than right angle are **complementary angles.**

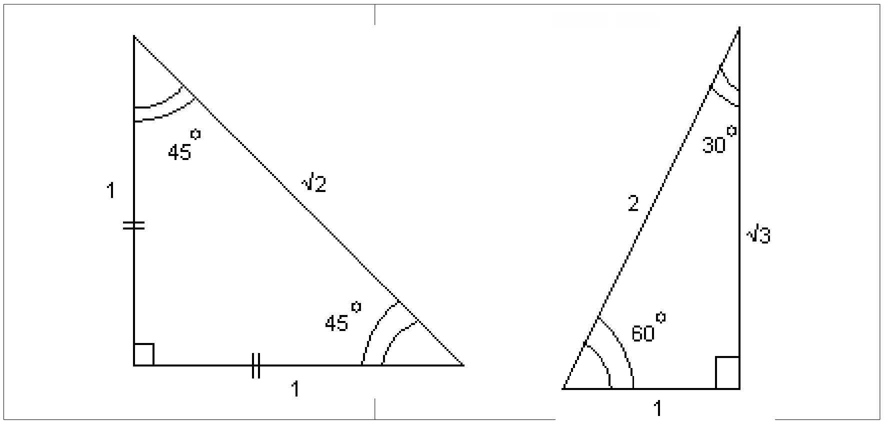
Note:

- Right angled triangle cannot be an equilateral triangle.
- But it can be either isosceles triangle or scalene triangle.



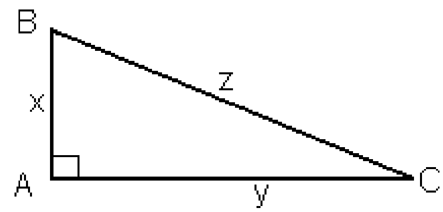
We shall consider two types of the right angle triangle.
(These triangles appear often in GRE questions. So understand their properties carefully. These will be useful for problem solving.)

45-45-90 right angled triangle	**30-60-90 right angled triangle**
The measure of the interior angles of this triangle are 45^0, 45^0, 90^0. This is an **isosceles** triangle.	The measure of the interior angles of this triangle are 30^0, 60^0, 90^0. This an **scalene** triangle
The ratio of the sides of this triangle is **$L_{45} : L_{45} : L_{90} = 1 : 1 : \sqrt{2}$** *where L_{45}, L_{45} and L_{90} are the length of the side opposite to the 45^0, 45^0, 90^0 angles respectively.*	The ratio of the sides of this triangle is: **$L_{30} : L_{60} : L_{90} = 1 : \sqrt{3} : 2$.** *where L_{30}, L_{60} and L_{90} are the length of the side opposite to the 30^0, 60^0, 90^0 angles respectively.*



Pythagorean theorem :

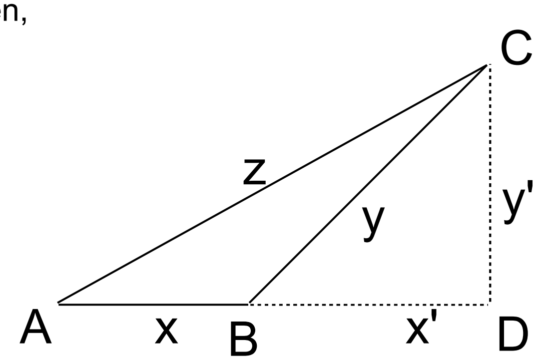
The square of the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides.



In ΔABC , interior angle BAC is a right angle.
Let length of side AB be x units, length of AC be y units and length of BC be z units.
By Pythagorean theorem, $z^2 = x^2 + y^2$.

Remember:

1) Suppose in ΔABC, $\angle ABC$ is an obtuse angle then, what can say about z^2 and $x^2 + y^2$?



Draw $CD \perp AB$.
Now ΔADC is a right angled triangle.
$z^2 = y'^2 + (x + x')^2$ -------------(1)

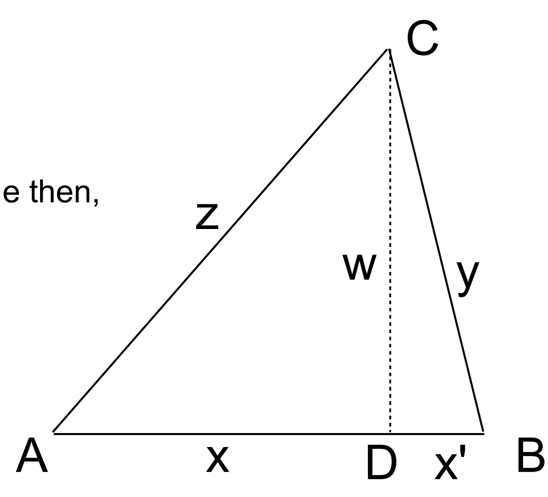
Also ΔBDC is a right angled triangle.
$y^2 = y'^2 + x'^2$ -------------(2)

From (1) we have

$z^2 = y'^2 + (x + x')^2$
$= y'^2 + x^2 + 2xx' + x'^2$ from(2)
$= (y'^2 + x'^2) + x^2 + 2xx'$
$= y^2 + x^2 + 2xx'$
$> y^2 + x^2$ .... as $2xx' > 0$

Hence $z^2 > y^2 + x^2$ when $\angle ABC$ is an obtuse

2) Suppose in ΔABC, $\angle ABC$ is an acute angle then, what can say about z^2 and $x^2 + y^2$?



Draw $CD \perp AB$.

Now ΔADC is a right angled triangle.
$z^2 \quad = w^2 + (x-x')^2$ -------------(1)

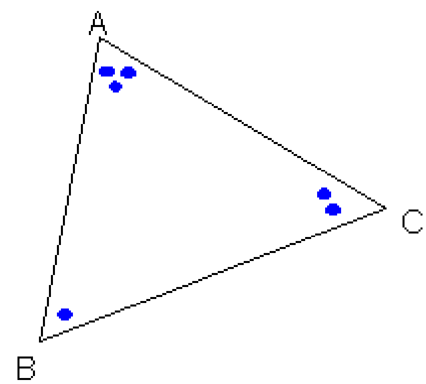
Also ΔBDC is a right angled triangle.
$y^2 \quad = w^2 + x'^2$ -------------(2)

From (1) we have
$z^2 \quad = w^2 + (x-x')^2$
$= y^2 - x'^2 + (x-x')^2$ from (2)
$< y^2 + (x+x')^2$ ....as $x'>0$

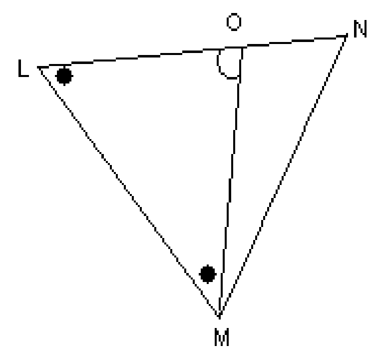
Hence $z^2 < y^2 + x^2$ when ∟ABC is an acute

Tutorials

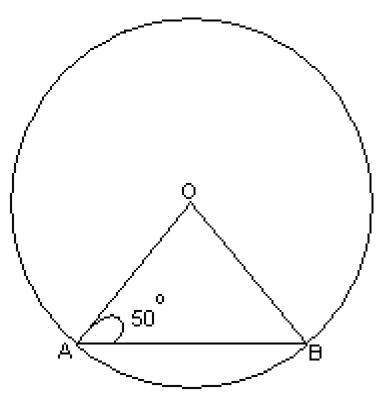
(Send your solutions to support@greedge.com)



1) In **Δ** MNO, ∟ABC = 41^0 , ∟ACB = 59^0.
What is the measure of the ∟BAC?



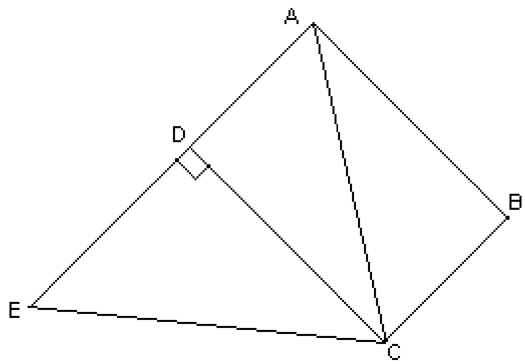
2) In the figure, **Δ**O LM is an isosceles triangle ∟NLM = 48^0 and ∟LMN = 68^0 . Find measure of the following angles:
- ∟LOM = ?
- ∟MON = ?
- ∟LNM. = ?



3) In the following figure O is the center of the circle
- What type of the triangle is Δ OAB?
- If the diameter of circle having center O is 2 cm, what is the length of the side OA?

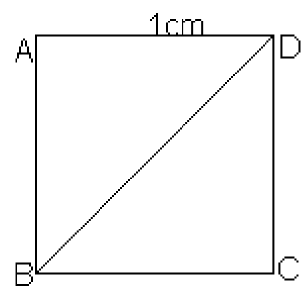
- Find the measure ∟OBA and ∟AOB.

4)

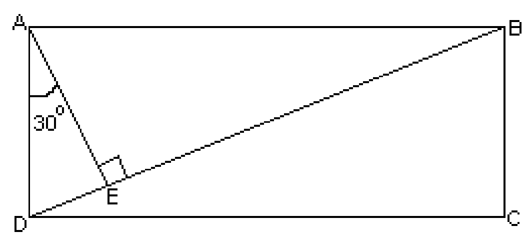


In the figure, ABCD is an rectangle,∟AEC = 50^0, ∟ACD = 30^0, find ∟BCE.

5) In the figure ABCD is the square , length of side AD is 1cm. Find the length of the diagonal.



6) In the figure ABCD is the rectangle, AE is perpendicular to BD (that is ∟AED = 90^0). ∟DAE = 30° and length of DE = 1 unit.
Find measures of



- ∟ADE

  = ?
- ∟BAE

  = ?
- ∟ABE

  = ?
- length of AD

  = ?
- length of AE

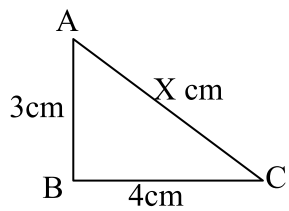
  = ?
- length of AB

  = ?
- length of BD

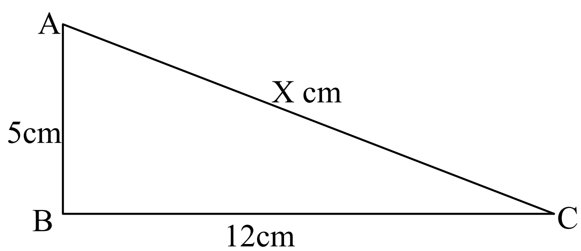
= ?

7) Find the value of X in the right angled triangle:

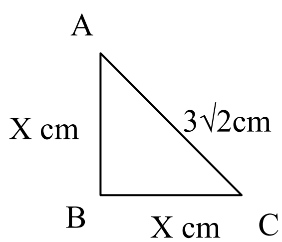
a)



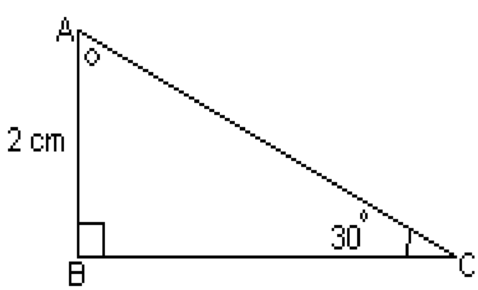
b)



c)

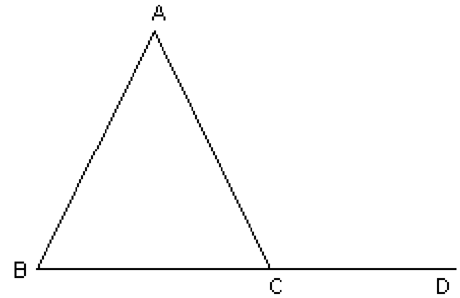


d) find the length of BC and AC.



Triangles Part 2

Exterior Angles:



Consider the figure ABC is a triangle.
∟ACD is known as exterior angle of the
Δ ABC.
Important property of exterior angle is :
Measure of the exterior is the sum of the two interior angles.
In the above figure,
∟ACD = ∟ABC + ∟BAC.

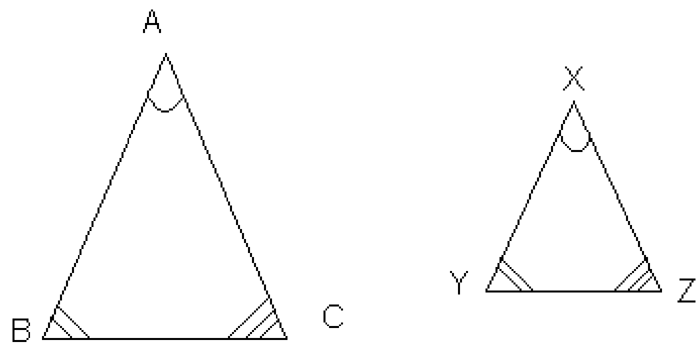
(this property is very useful in GRE problem solving)

Similar Triangles

Two triangle are said to be similar if

1) all the corresponding angles of both the triangles are same:
2) all the ratios of the corresponding sides are same.

In following figure there are two triangles namely ∆ABC and ∆XYZ.



We say ∆ABC is similar to ∆XYZ if

∟BAC = ∟YXZ;
∟ABC = ∟XYZ;
∟ACB = ∟XZY;

Or the corresponding sides are in same proportion, that is

AB / XY = BC / YZ
BC / YZ = AC / XZ
AC / XZ = AB / XY
=> AB / XY = BC / YZ = AC / XZ

Conditions for similar triangles:

1. Two triangles are similar if at least two corresponding angles are equal.
2. If two corresponding sides of two triangles are in proportion, and their included angles are equal, the triangles are similar.
3. If three sides of two triangles are in proportion, the triangles are similar.

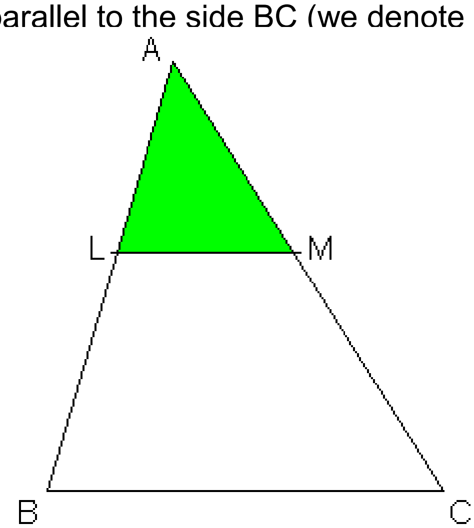
Hence for checking two triangles are similar you can check one of these conditions

Let us consider some different examples of similar triangles. These are typical types in which they appear in GRE questions.

Example 1

In the figure given a triangle **Δ** ABC and also LM is parallel to the side BC (we denote it as LM || BC) , which implies that

∟ ALM = ∟ABC and ∟AML = ∟ACB



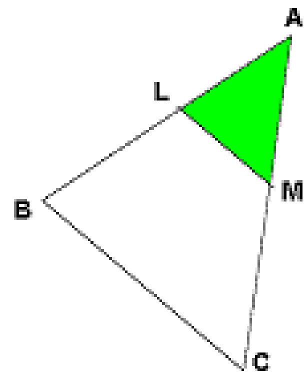
Now consider two triangles **Δ** ALM and **Δ** ABC
∟LAM = ∟BAC (They are same triangles)
∟ALM = ∟ABC (from above)
∟AML = ∟ACB (from above)

So first condition is satisfied.
Hence **Δ** ALM is similar to **Δ** ABC
First condition is also known as AA Test
Since **Δ** ALM is similar to **Δ** ABC we get
AL / LB = AM / MC = LM / BC.

Example 2

Consider the same figure , in which LM passes through the side AB and AC. Following information is provided:

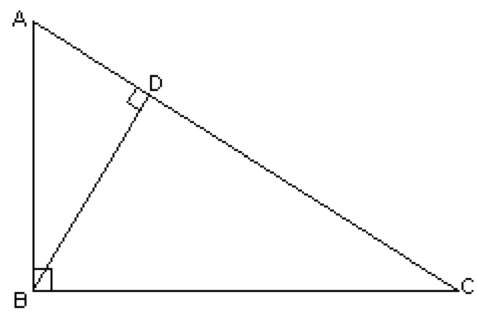
length of side AL = l (AL) = 1, l (LB) = 2,
l (AM) = 1.5 and l (MC) = 3.
(*Note, it is not given that LM is parallel to side BC).*



Now let check whether "**Δ** ALM is similar to **Δ** ABC"
Consider two triangles **Δ** ALM and **Δ** ABC
∟LAM = ∟BAC (They are same angles)
AL / AB = 1 / 3.
AM / AC = 1.5 / 4.5 = 1 / 3.
hence AL / AB = AM / AC
This satisfies the second condition .
Hence **Δ** ALM is similar to **Δ** ABC.
This second condition is known as SAS test.

Example 3:

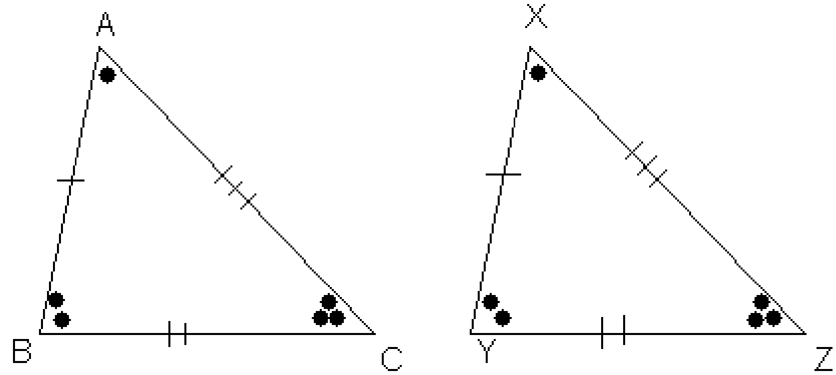
This is a little complex example.



Let us understand the figure.
Given **Δ** ABC is right angled triangle.
and BD is perpendicular to AC
which implies ∟BDA = 90^0 and also ∟BDC = 90^0.
We have three right angled triangles
Δ ABC , **Δ** ADB and **Δ** BDC
Now let us check similarities of **Δ** ABC and **Δ** ADB
BAD = BAC (common angles)
ABC = BDA (right angles)
Hence by AA test
Δ ABC is similar to **Δ** ADB
Similarly Check the similarities between ABC and BDC
Note that ABC is similar to ADB and ABC is similar to BDC which implies that ADB is similar to BDC. (Write the proof).

Congruent Triangles:

If given two similar triangles , the ratio of two corresponding sides is 1, then the two triangles are known as congruent triangles.



In other words we say two triangles are congruent if the corresponding angles and the corresponding sides are equal.

In the figure consider triangle
Δ ABC and **Δ** XYZ.
∟BAC = ∟YXZ
∟ABC = ∟XYZ
∟ACB = ∟XZY
and
l (AB) = l (XY)
l (BC) = l (YZ)
l (AC) = l (XZ)
hence **Δ** ABC is congruent to **Δ** XYZ (we write **Δ** ABC ≡ **Δ** XYZ).

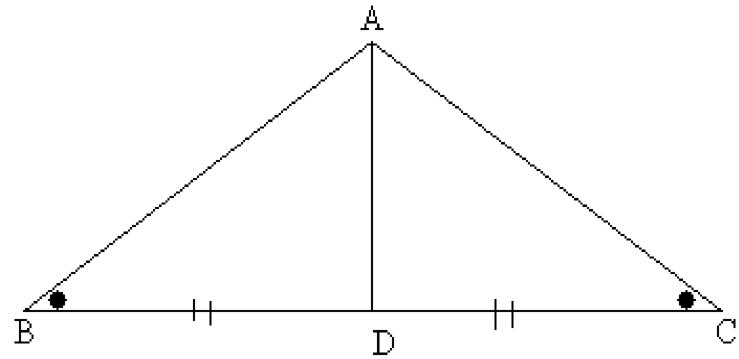
Conditions for congruent triangles:

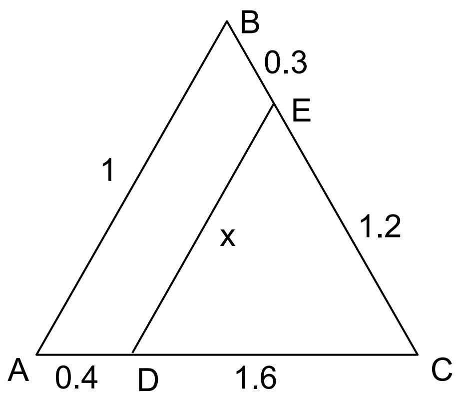
- Two triangles are said to be congruent if all the three corresponding sides are equal.
- Two triangles are said to be congruent if two corresponding sides are equal and the corresponding angle in between the two sides are equal.
- Two triangles are said to be congruent if two corresponding angles are equal and the corresponding side in between the two angles are equal.
- If the given two triangles are right angled triangles and if corresponding side other than the hypotenuse are equal the triangles are congruent.

Exercise:

(Send your solutions to support@greedge.com)

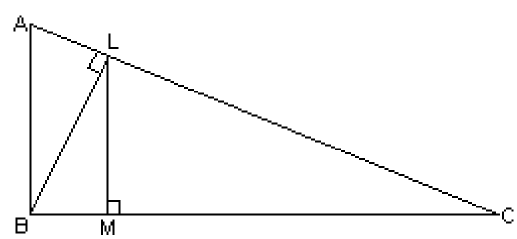
1) In the figure, Δ ABC is an isosceles triangle.
Show that Δ ABD ≡ Δ ACD.



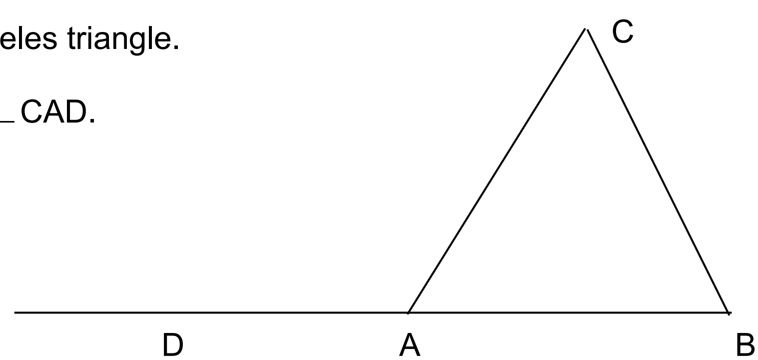
2) 

In the figure, DE || AB. Find the value of x.

3) In the figure Δ ABC is the right angled triangle.
Prove that Δ ABL is similar to ΔLMC

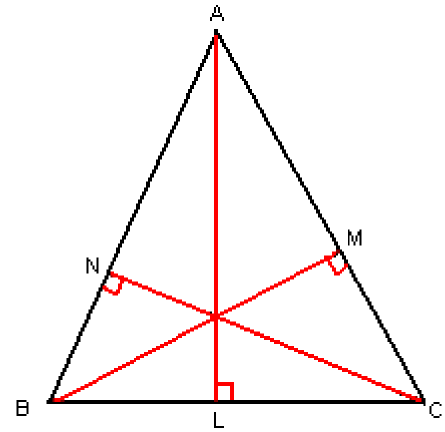


4) In the figure Δ ABC is the isosceles triangle.
AB = AC . If ∟ACB = 65°.
Find the measure of the angle ∟CAD.



Triangles Part 3

Let us consider some new definitions:



Altitude:

An **altitude** of a triangle is a straight line through a vertex and perpendicular to (i.e. forming a right angle with) the opposite side .
In the figure AL, BM and CN are the altitudes of **Δ** ABC.
In fact, AL gives height from the vertex A to the side BC.
BM gives the height from the vertex B to the side AC
and CN gives the height from the vertex C to the side AB.

MEDIAN

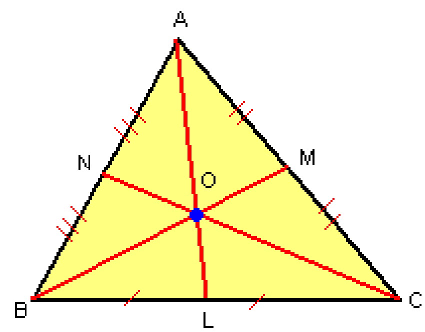
A **median** of a triangle is a line joining a vertex to the midpoint of the opposing side .

CENTROID

A centroid of a triangle is a point where all the three medians of the triangle meet.

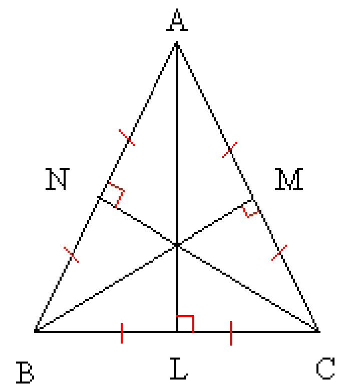
<u>Properties:</u>
A centroid of the triangle divides the median in the ratio of 2:1.

In the above figure
AL, BM and CN are the medians of **Δ** ABC.
Hence BL = LC,
AM = MC and AN = NB.
O is the centroid of the triangle.
AO : OL = 2 : 1
BO : OM = 2 : 1
CO : ON = 2 : 1

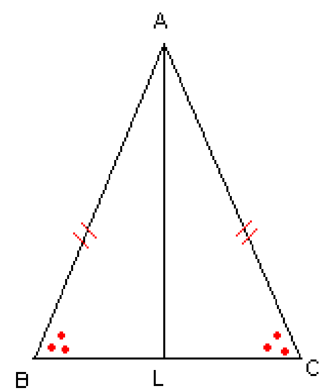


Note:

1) If the given triangle is an equilateral triangle then the median and the altitude of the triangle are same.

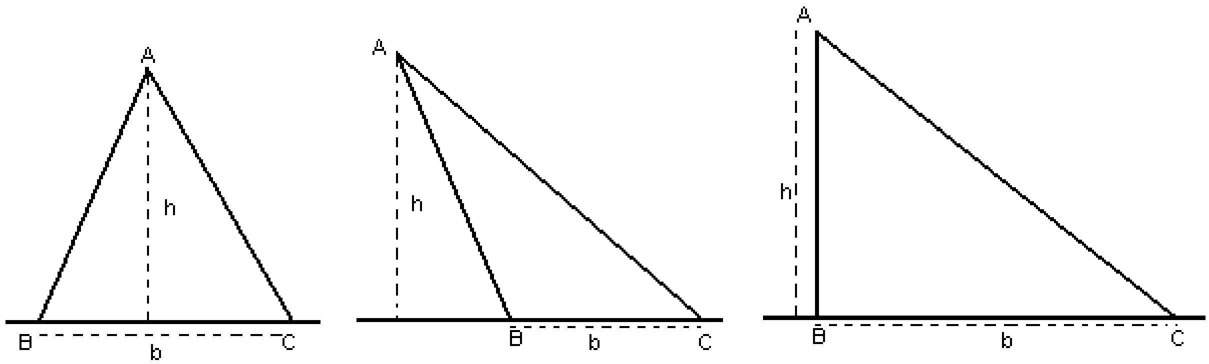


2) Suppose **Δ** ABC is an isosceles triangle, such that AC = AB and ∟ACB = ∟ABC.
AL is the line passing from A to the base BC and L is the midpoint of the BC.
Then AL is also an altitude(Try to prove.
Hint: Prove **Δ** ABL is congruent to **Δ** ACL.)



Area of the Triangle:

There are many ways to calculate the area of the triangle.
The area of any triangle is given by the given by the formula : ½ X **Base** X **Height**

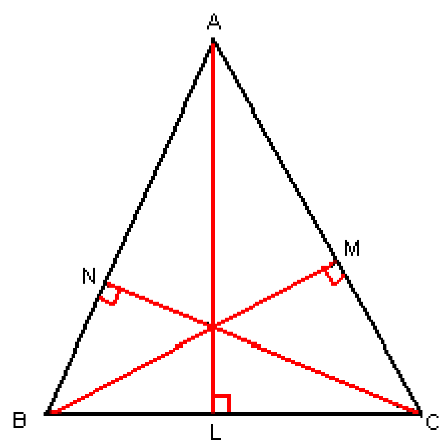


In the above figure h denotes the height of the triangle and b denotes the base of the triangle.

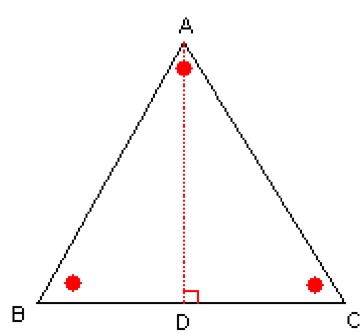
For a given triangle with the formula <u>½ X Base X Height</u> .
We can calculate its area in 3 ways.

Area of **Δ** ABC = ½ X AL X BC
= ½ X BM X AC
= ½ X CN X AB

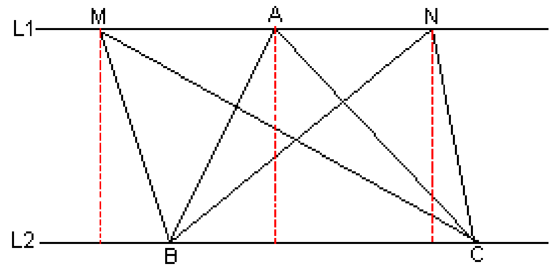
Since AL, BM and CN are the altitudes of **Δ** ABC with base BC, AC and AB.

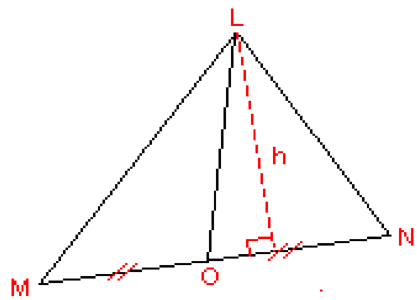


<u>Area of Equilateral triangle:</u>
Area of the equilateral triangle is given by
= √3 $(\text{side})^2/4$.
That is area of **Δ** ABC = √3 X BC $^2/4$.
Note that AB = BC= AC



The height between two parallel lines at any points is always same. Hence the area of the triangle with same base between two parallel lines is always same. Hence in the figure the area of **Δ** ABC , **Δ** MBC and **Δ** NBC are equal.





Consider the figure, LO is the median of the **Δ** LMN. MO = ON

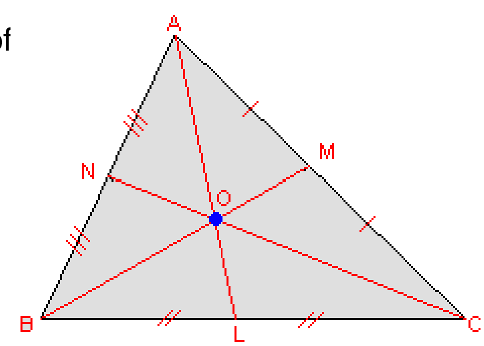
Area(**Δ** LMO) = 1/2 X height X base

= 1/2 X h X MO

similarly Area(**Δ** LNO) = 1/2 X height X base

= 1/2 X h X NO

We see height of both the triangles are same and also bases of both the triangles are equal hence area of both the triangles are same.



Consider the figure. BM, AL and CN are medians of Δ ABC and O is the centroid of Δ ABC . Hence

A(Δ ABC) = 6 X Δ A NO

= 6 X Δ NOB

= 6 X Δ BOL

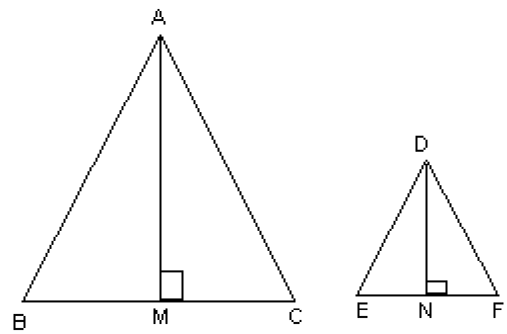
= 6 X Δ LOC

= 6 X ΔCOM

= 6 X Δ A OM

Application of similar triangles.

Let us see some applications of similar triangle in finding the areas.
Consider the figure, suppose it is given that
ΔABC and ΔDEF are similar.



We know that
$\angle BAC = \angle EDF$,
$\angle ABC = \angle DEF$ --------------(1)
and
$\angle ACB = \angle DFE$.

Also we have,
$AB/DE = AC/DF = BC/EF = m$(say) --------(2)
then we get,
$AB = m\ DE$-------------------------(*)
$AC = m\ DF$
$BC = m\ EF$

let AM be altitude(height) of ΔABC and DN be altitude(height) of ΔDEF.
Now,

consider ΔABM and ΔDEN
$\angle ABM = \angle DEN$ from (1)
$\angle AMB = \angle DNE = 90^o$.
Hence by AA test ΔABM is similar to ΔDEN

Hence by the properties of similar triangles we have
$AB/DE = BM/EN = AM/DN$

But by (2) we have
$AB/DE = m$
thus $AM/DN = m$
$\Rightarrow AM = m\ DN$-------------------(**)

But AM and DN are heights of ΔABM and ΔDEN.
Let us find area of ΔDEF and ΔABC

area(ΔDEF) $= 1/2 \times$ base $\times$ height
$= 1/2 \times EF \times DN$

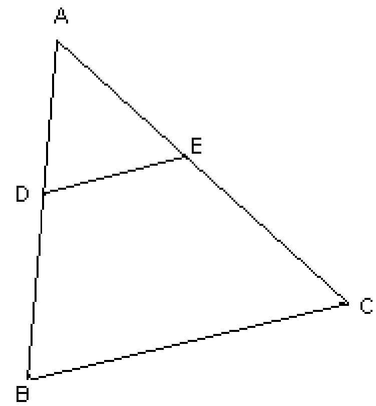
area(ΔABC) $= 1/2 \times$ base $\times$ height
$= 1/2 \times BC \times AM$
$= 1/2 \times (m\ EF) \times (m\ DN)$ (From (*) and (**))
$= m^2 \times 1/2 \times EF \times DN$

Hence area(ΔABC)$= m^2 \times$ (area(ΔDEF))

Question:

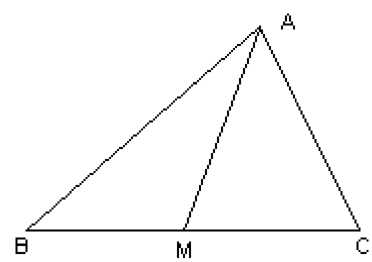
(Send your solutions to support@greedge.com)

1) Suppose DE|| BC and AB = 4 AD
Find the ratio of the area of ΔADE and area of ΔDBC.



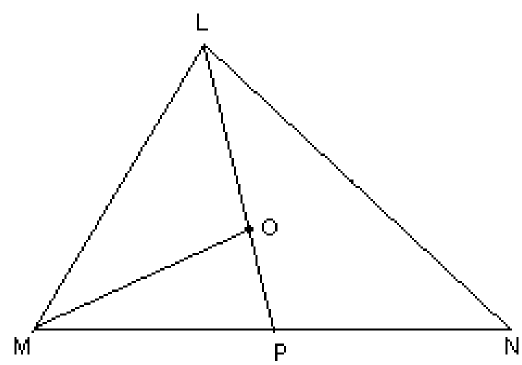
2) Let AM be the median of the ΔABC. If the distance of A from BC is 4 cm and BC = 4 cm then, find

- area(ΔABC) =?
- area(ΔABM) =?
- Compare area of ΔABM and area of ΔAMC

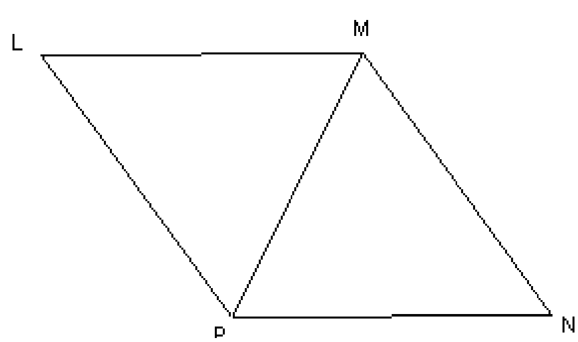


3) In the figure O is the centroid of ΔLMN and LP be the median of ΔLMN. If area of ΔLPN = 6, then find

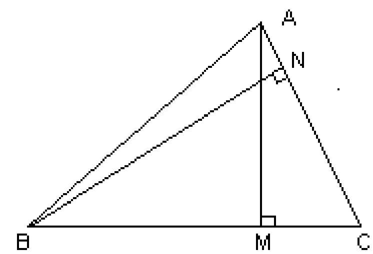
- area of ΔLMN
- find the ratio of the area of ΔMOP and area of ΔLOM.



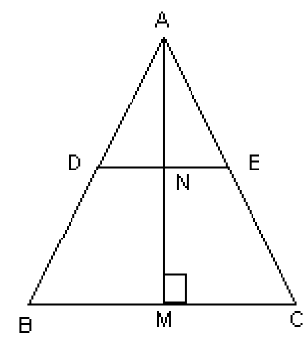
4) Consider a quadrilateral LMNP, suppose LM = MN = PN = LP = MP and let LM = 3 cm. Find the area of this quadrilateral.
(Hint: area of quadrilateral LMNP = area(ΔLMP) + area(ΔMPN))



5) In the figure AM and BN are the altitudes of ΔABC
If AM = 4 , BC = 5 and AC =5
Find area (ΔABC) and BN.



6) In the figure DE||BC and AM is altitude of Δ ABC .
Suppose DE : BC = 1 : 3 then find AN : NM.



7) Suppose DE || BC and AD = m AB.
then the following ratios:
a) area(Δ ADE) : area(Δ ABC)
b) area(Δ DBC) : area(Δ ABC)
c) area(Δ ADE) : (area Δ BDC) : area(Δ ABC)
Also compare the area of Δ DBC and
area of Δ BEC.