Arithmetic and Number Properties

Types of Numbers

Integers:

Any counting number including negative numbers (e.g. -3, -1, 2, 7…but not 2.5)

Real Numbers:

Numbers that appear on the number line (i.e., one that is not imaginary) including pi, the square root of 2, etc.

A positive number is greater than 0, a negative number is less than 0.

Order of Operations: PEMDAS

Complete any arithmetical operation in the following order:

1. Parentheses
2. Exponents
3. Multiplication/Division
4. Addition/Subtraction

Example: $2+\frac{6}{2}\times(5-1)^2=2+\frac{6}{2}\times(4)^2=2+\frac{6}{2}\times 16=2+48=50$

You can remember PEMDAS as "Please Excuse My Dear Aunt Sally," or "Purple Eggplants Make Delicious Afternoon Snacks," or my personal favorite, "Pandas Explore Many Delightful Asian Scenes"

Commutative, Associative, and Distributive Properties

The Commutative Property:

$a+b=b+a$, $a\times b=b\times a$

The Associative Property:

$$(a+b)+c=a+(b+c),\ (a\times b)\times c=a\times(b\times c)$$

The Distributive Property:

$$a\times(b+c)=ab+ac,\ a\times(b-c)=ab-ac$$

The Commutative and Associate properties **do not work with subtraction or division.**

Prime Numbers

A prime number is one that is divisible only by itself and 1. In other words, a positive integer with exactly 2 positive divisors. This includes 2, 3, 5, 7, and 11, but not 9, because 9 = 3 x 3.

1 is not a prime. 2 is the smallest prime and the only even prime.

Memorize all primes below 60: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59…

Factorization

If X can be multiplied by Y to get Z, assuming all of these are positive integers, then X and Y are considered factors of Z.

The prime factorization of a number is dividing it into its constituent primes. So for 21, this is 3 x 7; for 60, 2 x 2 x 3 x 5. 7644 = 2 x 2 x 3 x 7 x 7 x 13. To find the prime factorization of 60, you can use 60 = 30 x 2 = 15 x 2 x 2 = 5 x 3 x 2 x 2.

16 has five positive divisors: 1, 2, 4, 8, 16.

40 has 8: 1, 2, 4, 5, 8, 10, 20, 40.

To find how many factors 720 has, first find its prime factorization: $2^4\times3^2\times5$. All of its factors will be of the form $2^a\times3^b\times5^c$. Now there are five choices for a (a= 0, 1, 2, 3, or 4), three choices for b (b = 0, 1, or 2), and two choices for c (c= 0 or 1). The total number of factors is therefore 5 x 3 x 2 = 30. 720 has 30 factors.

The greatest common factor (aka greatest common divisor) of two numbers is the biggest factor shared by two numbers. The GCF of 12 and 30 is 6 – it is the biggest divisor they both share. The easiest way to find the GCF is to take the prime factorization and multiply all of the primes that appear in both numbers. So since 56 = 2 x 2 x 2 x 7 and 70 = 2 x 5 x 7, the GCF is 2 x 7 = 14. If two numbers share no primes, the GCF is 1.

The least common multiple of two numbers is the smallest positive integer with both numbers as a factor. The LCM of 4 and 6 is 12 – it is the smallest number that has both 4 and 6 in its divisors. The LCM of 9 and 15 is 45; the LCM of 7 and 21 is 21, because 21's factors are 1, 3, 7, and 21. To find the LCM of any two numbers, take the prime factorization of each number, find what prime factors appear in both, and multiply one of each of the shared primes and then by all the unshared primes. So for example, 12 = 2 x 2 x 3, and 56 = 2 x 2 x 2 x 7, so the LCM of 12 and 56 is (2 x 2) [shared primes] x 3 [12's unshared primes] x (2*7) [56's unshared primes] = 168. The largest possible LCM for any two numbers is one multiplied by the other.

Divisibility

3 : sum of digits divisible by 3

4 : the last two digits of number are divisible by 4

5 : the last digit is either a 5 or zero

6 : even number and sum of digits is divisible by 3

8 : if the last three digits are divisible by 8

9: sum of digits is divisible by 9

Fast Fractions

$$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy} \longrightarrow \frac{1}{2}+\frac{1}{5}=\frac{2+5}{2\times 5}=\frac{7}{10}$$

Absolute Values

The absolute value of a number is its distance from a number line.

|x| = x, |-x| = x

Percentages

"Percent" = per 100; $19\% = \frac{19}{100}$; $0.43\% = \frac{0.43}{100} = \frac{43}{10000}$

To find what percent some part is of a whole, use $\frac{part}{whole} = \frac{percent}{100}$

For example, if 120 of 800 people in a town smoke, then $\frac{120}{800} = \frac{percent}{100} = \frac{15}{100} \rightarrow 15\%$ of the townspeople smoke. Most percentage problems break down into identifying the part, the percent, and the whole, one of which will be unknown.

If p percent of x is y, then $\frac{p}{100} = \frac{y}{x}$, so $\left(\frac{p}{100}\right) \times x = y$.

Percent change : % change = change/original value

If the price of something goes from \$40 to \$52, the percent change is $\frac{(52 - 40)}{40} = \frac{12}{40} = \frac{3}{10} = \frac{30}{100} = 30\%$. The price increases by 30%.

This can also be written as (change x 100) / original value. So here, it's $\frac{(52 - 40) \times 100}{40} = \frac{1200}{40} = 30\%$.

If something increases by 20%, then decreases by 5%, it is not the same as if it increased by 15%. For example: 100 -> 120 -> 114, whereas if 100 increased by 15% it would be 115.

If a price falls by 15%, you can multiply the original value by (1 – 0.15 = 0.85) to find the new value. 250% of the original price is the same as 150% more than the original price, and to find either you'd multiply the original price by 2.5.

Ratios

Ratios let us compare the proportions of two quantities. If there is a 2:5 ratio of boys to girls at a school, that means that for every 5 girls, there are 2 boys. So there could be 2 boys and 5 girls, 20 boys and 50 girls, 200 boys and 500 girls, etc.

Ratios are given by x:y, x to y, or x/y. If a question says "for every x there is/are a y," you are most likely dealing with a ratio question. Ratios can also be x:y:z.

Ratios can be simplified like fractions. 3:6 is the same as 1:2.

Remember that if there is a 2:5 ratio of boys to girls at a school, the ratio of boys to total students is 2:(5 + 2) = 2:7. 2/7 of the students are boys.

Powers and Roots

Exponents

Notation:

$$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32\,;\, x^3 = x \times x \times x\,;\, 4^x = 4 \times 4 \times 4 \times \ldots (x \; multiples \; of \; 4);\, x^1 = x$$

Exponent Laws:

$$x^A \times x^B = x^{(A+B)}$$

$$\frac{x^A}{x^B} = x^{(A-B)}$$

$$\left(x^A\right)^B = x^{(A \times B)}$$

1 and 0 as bases:

1 raised to any power is 1. 0 raised to any nonzero power is 0

Any nonzero number to the power of 0 is 1: $7^0 = 1$.

Fractions as exponents:

$$x^{\left(\frac{1}{2}\right)} = \sqrt{x}\,;\, x^{\left(\frac{2}{3}\right)} = \sqrt[3]{x^2}$$

Negative exponents:

$$x^{(-1)} = \frac{1}{x}\,;\, x^{(-2)} = \frac{1}{\left(x^2\right)}$$

Negative bases:

$$(-2)^4 = (-2)(-2)(-2)(-2) = 16 ; (-2)^5 = -32$$

A negative number raised to an even power is positive; a negative number raised to an odd power is negative.

Odd/even exponents:

$x^3 = 8 \rightarrow x = 2$, but $x^4 = 16 \rightarrow x = 2 \; and \; x = -2$

To raise 10 to any power, just put that many 0s after the 1: $10^5 = 100000$, a 1 with 5 zeros.

Roots

$\sqrt{49} = 7$, because $7^2 = 49$. Note that even though $(-7)^2 = 49$ as well, -7 is NOT considered a solution of $\sqrt{49}$; only the positive number counts in this case. In this case, -7 is known as an extraneous root. However, if you were given the question $x^2 = 49$, the answer would be $x = 7$ and $x = -7$.

Square roots of negative numbers (e.g., $\sqrt{-16}$):

They have no real solutions (they have imaginary solutions involving *i*, the square root of -1, but that definitely won't be on the GRE.)

Perfect squares:

Numbers with integers as their square roots: 4, 9, 16, etc.

To estimate square roots of numbers that aren't perfect squares, just examine the nearby perfect squares. For example, to find $\sqrt{50}$, you know that $\sqrt{49} = 7$ and $\sqrt{64} = 8$, so $\sqrt{50}$ must be between 7 and 8.

Cube roots:

$\sqrt[3]{n}$ = a number that, when cubed, equals n. $\sqrt[3]{-8} = -2$.

Simplifying roots:

Separate the number into its prime factors, and take out matching pairs:
$\sqrt{20}=\sqrt{2\times 2\times 5}=2\sqrt{5}$; $\sqrt{54}=\sqrt{9\times 6}=\sqrt{9}\times\sqrt{6}=3\sqrt{6}$; $\sqrt{72}=\sqrt{9\times 8}=3\sqrt{8}=6\sqrt{2}$

Adding roots:

$2\sqrt{7}+9\sqrt{7}=11\sqrt{7}$. Roots can be added like variables.

Algebra

Simplifying Expressions

Simplifying expressions:

$$(6xy+5x)-(4xy-3y)=2xy+5x+3y$$

Multiplying monomials:

$$\left(5y^3\right)\left(6y^2\right)=30y^5;-6y(5x+3y)=-30xy-18y^2$$

Multiplying polynomials using FOIL (First, Outer, Inner, Last)

$$(x+2)(x+7)=x\times x+x\times 7+2\times x+2\times 7=x^2+9x+14$$

$$(a+b)^2=a^2+2ab+b^2$$

$$(a-b)^2=a^2-2ab+b^2$$

$$\left(a^2-b^2\right)=(a+b)(a-b)$$

Factoring

Factoring using Greatest Common Factors:

$$6x^3+12x^2+33x=3x\left(2x^2+4x+11\right)$$

Factoring using difference of squares:

$$(a+b)(a-b)=a^2-b^2;(2x+5)(2x-5)=4x^2-25;4x^2-9y^2=(2x-3y)(2x+3y)$$

Factoring using quadratic polynomials:

$x^2 + ax + b = (x + m)(x + n)$, where a is the sum of m and n, and b is their product; for example, $x^2 + 5x - 14 = (x + 7)(x - 2)$

Combining methods:

$$2x^6 - 2x^2 = 2x^2\left(x^4 - 1\right) = 2x^2\left(x^2 + 1\right)\left(x^2 - 1\right) = 2x^2\left(x^2 + 1\right)(x + 1)(x - 1)$$

$$9x^3y^2 - 6x^2y^2 + xy^2 = xy^2\left(9x^2 - 6x + 1\right) = xy^2(3x - 1)^2$$

Factoring rational expressions (as long as x is not equal to 4):

$$\frac{6x^2 + 12x - 144}{2x^2 - 32} = \frac{6\left(x^2 + 2x - 24\right)}{2\left(x^2 - 16\right)} = \frac{3(x + 6)(x - 4)}{(x + 4)(x - 4)} = \frac{3(x + 6)}{x + 4}$$

Solving Equations

The golden rule of solving equations is, “What you do to one side of an equation, you must also do to the other”.

Eliminating fractions:

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = 1 ; \frac{2}{5}x = 8 \rightarrow \frac{5}{2}\frac{2}{5}x = \frac{5}{2}8 \rightarrow x = 20$$

Multiply by the LCD:

$$\frac{3x}{4} + \frac{1}{2} = \frac{x}{3} \rightarrow \times 12 \rightarrow \frac{36x}{4} + \frac{12}{2} = \frac{12x}{3} \rightarrow 9x + 6 = 4x \rightarrow x = -\frac{6}{5}$$

Cross-multiplication:

$$\frac{a}{b} = \frac{c}{d} \rightarrow ad = bc$$

$$\frac{7}{6x-6} = \frac{3}{2x+2} \rightarrow 3(6x-6) = 7(2x+2)$$

Quadratic equations:

$ax^2 + bx + c$, where a is not 0; if you can factor it to $(x+a)(x-b)=0$, then the solutions are -a and b.

Quadratic formula:

If $ax^2 + bx + c = 0$, and a is not 0, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Two variables/systems of equations (ex: $3x + y = 17$ and $2x - 2y = 6$):

Method 1: Substitution

$y = 17 - 3x$

$2x - 2(17 - 3x) = 6$

$2x + 6x - 34 = 6$

$x = 5$

Method 2: Elimination

$6x + 2y = 34$

$2x - 2y = 6$

add the two equations, so +2y and -2y eliminate one another

$8x = 40$

$x = 5$

A system of two equations with two unknowns can have 0, 1, or infinitely many solutions.

To solve a system of three equations with three variables, use substitution to reduce the problem to two equations with two variables, and solve from there.

Function notation: if given $f(x) = \ldots$ and asked what f(something else) is, simply replace every instance of x in the "…" expression with whatever is now in the parentheses

Similarly, if given a "strange operator" (a symbol you don't know– say, $x \Delta y$) and asked what $a \Delta 2x$ is, just replace "x" and "y" with "a" and "2x." So if $x \beta y = 3x + y^2$, then $5 \beta 2 = 3(5) + (2)^2$.

Inequalities: They can be treated like regular equations, with the following exception: multiplying or dividing an inequality by a negative number reverses the sign of the inequality.

If w < x and x < y, then w < y.

If a < b and c < d, then a + c < b + d. However, this does not hold for subtracting, multiplying, or dividing.

If |x| < 3, then -3 < x < 3; if |x| > 3, then x > 3 or x < -3.

If given a quadratic inequality (i.e., $ax^2 + bx + c < 0$, first solve for when the expression is equal to 0, then use a number line to check which values of x fulfill the inequality.

Geometry

Angles

A right angle is made up of 90 degrees

A straight line is made up of 180 degrees.

If two lines intersect, the sum of the resulting four angles equals 360.

Polygons

A polygon is any figure with three or more sides (e.g., triangles, squares, octagons, etc.).

$Total\ degrees = 180\,(n - 2)$, where n= # of sides

$$Average\ degrees\ per\ side\ or\ degree\ measure\ of\ congruent\ polygon = 180\frac{(n-2)}{n}$$

Triangles

Area = $\frac{1}{2}b \times h$

An isosceles right triangle (45-45-90) has sides in a ratio of $x : x : x\sqrt{2}$

A 30-60-90 triangle has sides in a ratio of $x:x\sqrt{3}:2x$, with the 1x side opposite the 30 degree angle.

An equilateral triangle has three equal sides. Each angle is equal to 60 degrees

Any given angle of a triangle corresponds to the length of the opposite side. The larger the degree measure of the angle, the larger the length of the opposite side.

A right triangle has a right angle (a 90 degree angle); the side opposite the right angle is called the hypotenuse, and is always the longest side.

For a right triangle with legs A and B and hypotenuse C: $A^2 + B^2 = C^2$. This is called the Pythagorean Theorem.

Each side of certain right triangles are integers. These sets of numbers are called Pythagorean triples, and you should memorize some of them: 3-4-5, 5-12-13, 8-15-17, 7-24-25. A multiple of a Pythagorean triple is a Pythagorean triple (e.g., 6-8-10).

The length of the longest side can never be greater than the sum of the two other sides.

The length of the shortest side can never be less than the positive difference of the other two sides.

Circles

$$Area = \pi r^2$$

$$Circumference = 2\pi r$$

A circle has 360 degrees. An arc is the portion of the circumference of a circle in x degrees of the circle.

$$Arc\ Length = \frac{x}{360} 2\pi r$$

$$Area\ of\ sector = \frac{x}{360} \pi r^2$$

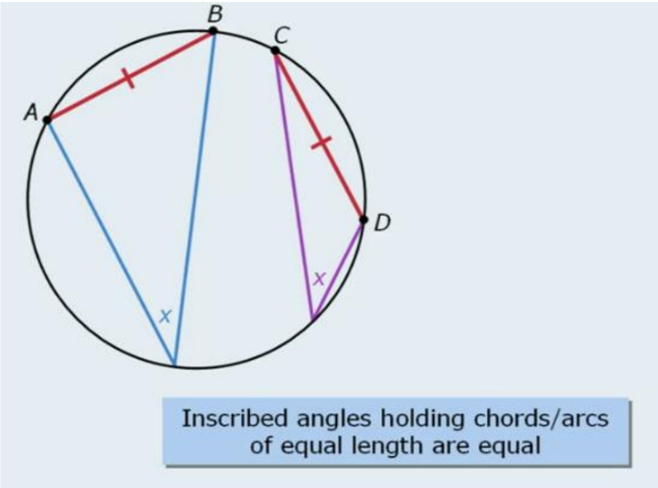
A fraction of the circumference of a circle is called an arc. To find the degree measure of an arc, look at the central angle.

A chord is a line segment between two points on a circle. A chord that passes through the middle of the circle is a diameter.

If two inscribed angles hold the same chord, the two inscribed angles are equal.

An inscribed angle holding the diameter is a right angle (90 degrees).

Inscribed angles holding chords/arcs of equal length are equal:



Squares

$Perimeter = 4s$, where s = side

$Area = s^2$

Rectangles

$Area = l \times w$, where l = length and w = width

$Perimeter = 2l + 2w$

Trapezoids

$$Area = \frac{Base_1 + Base_2}{2} \times height$$

Quadrilaterals

The area of a square is s^2 (s = side).

The diagonals of a square bisect one another, forming four 90 degree angles

The diagonals of a rhombus bisect one another, forming four 90 degree angles

The perimeter of a rectangle is twice its height plus twice its length (or, the sum of all its sides).

The area of a parallelogram can be found multiplying base x height (the base always forms a right angle with the height).

3-D Shapes

Cubes:

$$Volume = s^3$$

$$Surface\ Area = 6s^2$$

The volume of a cube and the surface area of a cube are equal when s = 6.

Rectangular Solids (including cubes):

$$Volume = height \times depth \times width$$

$$Surface\ Area = 2 \times height \times width + 2 \times width \times depth + 2 \times depth \times height = total\ of\ the\ areas\ of\ each\ rectangle$$

Cylinders:

$$Volume = r^2 \pi h$$

$$Surface\ Area = 2\pi r^2 + 2\pi r h = 2\pi r(r + h)$$

Coordinate Geometry

Lines

Any line can be represented by $y = mx + b$, where m is the slope and b is the y-intercept. This is called slope-intercept form.

The slope of a line can be found subtracting the y values of a pair of coordinates and dividing it by the difference in the x values: $slope = m = \dfrac{\left(y_2 - y_1\right)}{\left(x_2 - x_1\right)}$

To find the y-intercept plug in zero for x and solve for y

To find the x-intercept, plug in zero for y and solve for x

An equation like x = 3 is a vertical line at x = 3; an equation like y = 4 is a horizontal line at y = 4.

If given two points and asked to find the equation of a line that passes through them, first find the slope using the above formula, then plug one of the points into y = mx+b and solve for b.

The slopes of two lines which are perpendicular to each other are in the ratio of x : -1/x, where x is the slope of one of the lines (think: negative reciprocal).

The Distance Formula

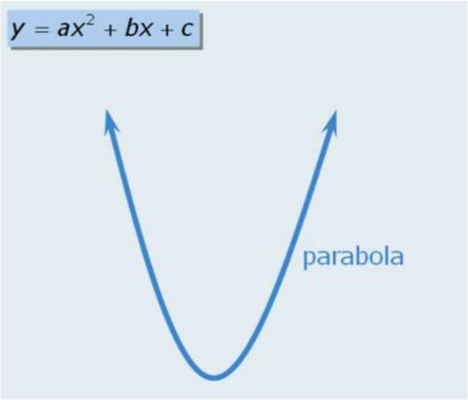
$$\sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

For finding the distance between (x_1, y_1) and (x_2, y_2)

Quadratics

This is the format of a quadratic equation: $y = ax^2 + bx + c$.

The graph of a quadratic equation is a symmetrical shape called a parabola, which open upwards if a > 0 and down if a < 0.



Word Problems

Distance, Rate, and Time

$Distance = Rate \times Time$

$rate = \frac{distance}{time}$

$time = \frac{distance}{rate}$

$average\ speed = \frac{total\ distance\ traveled}{total\ time}$

Work Rate

$\frac{1}{TotalWork} = \frac{1}{WorkRate_1} + \frac{1}{WorkRate_2}$

$output = rate \times time$

Sequences

$1 + 2 + 3 + \ldots + n = \frac{n\,(n+1)}{2}$

Interest

Simple Interest: $V = P\left(1 + \frac{rt}{100}\right)$, where P is principal, r is rate, and t is time

Compound Interest: $V = P\left(1 + \frac{r}{100\,n}\right)^{nt}$, where n is the number of times compounded per year

Statistics

Average or mean:

For a set of n numbers: total sum / n

Median:

Middlemost value when numbers are arranged in ascending order; for an even amount of numbers, take the average of the middle two

Mode:

The number that occurs most frequently

Example:

2, 3, 3, 4, 5, 6, 6, 6, 7: Mean = 42/9, Median = 5, Mode = 6

If the numbers in a set are evenly spaced, then the mean and median of the set are equal: {30, 35, 40, 45, 50, 55}

Weighted average:

(proportion) x (group A average) + (proportion) x (group B average) + …

Range:

Greatest value – least value

Standard deviation:

If you're given a set of n numbers a, b, c, … with a mean m:

$$SD = \sqrt{\frac{(a-m)^2 + (b-m)^2 + (c-m)^2 + \dots}{n}}$$

The standard deviation represents the average distance the data values are away from the mean.

Variance is the value inside the square root of the standard deviation = SD^2

If the standard deviation of a set of numbers is k, then k = 1 unit of standard deviation.

Counting

Fundamental Counting Principle: If a task is comprised of stages, where…

One stage can be accomplished in A ways

Another can be accomplished in B ways

Another can be accomplished in C ways

…and so on, then the total number of ways to accomplish the task is $A \times B \times C \times \ldots$

When tackling a counting problem:

Identify/list possible outcomes

Determine whether the task can be broken into stages

Determine the number of ways to accomplish each stage, beginning with the most restrictive stage(s)

Apply the Fundamental Counting Principle

Factorial notation: $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$

n unique objects can be arranged in n! ways. Example: There are 9 unique letters in the word wonderful, so we can arrange its letters in 9*8*7*… = 362,880 ways.

Restrictions:
$number\ of\ ways\ to\ follow\ a\ rule = number\ of\ ways\ to\ ignore\ the\ rule - number\ of\ ways\ to\ break\ the\ rule$

Arranging objects when some are alike:

$$\frac{n!}{(A!)(B!)(C!)\ldots}$$

Given n objects where A are alike, another B are alike, another C are alike and so on.

Combinations:

$$nCr = \frac{n!}{r!\left(n-r\right)!} \cdot 5C_3 = \frac{5!}{3!\left(2!\right)} = 10$$

When the order does not matter – for example, picking any 3 friends from a group of 5.

Permutations:

$$nPr = \frac{n!}{\left(n-r\right)!}$$

When the order does matter – for example, how many ways you could order 3 letters from the word PARTY?

Probability

The probability of an event:

0 = the event definitely won't occur

1 = the event definitely will occur

0.5 = there is a 50/50 chance the event will occur

Probability that event A will happen:

$$P(A)=\frac{number\ of\ outcomes\ where\ A\ occurs}{total\ number\ of\ outcomes}$$

The complement of an event:

The chance the event doesn't occur--so the complement of drawing a green ball is drawing a ball that isn't green.

P(event happens) + P(event does not happen) = 1

Mutually exclusive events:

Two events are mutually exclusive if they can't happen together: P (A and B) = 0

Events A and B (if they are independent events):

P(A and B) = P(A) x P(B)

Events A or B:

A happens, B happens, or both A and B happen.

P (A or B) = P(A) + P(B) – P(A and B)

Events A and B (if A and B are dependent events):

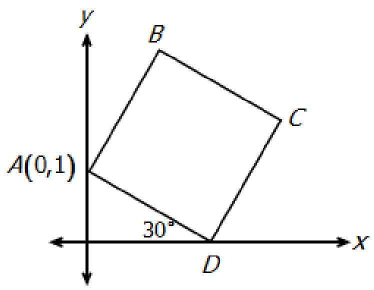
P(A and B) = P(A) x P(B|A)

P(B|A) is the probability that B occurs given that A occurs (example: the probability of drawing a heart, assuming you already drew a spade).

Practice Questions

Formulas to use: Triangles

If ABCD is a square, what are the coordinates of C?



- $(\sqrt{3}, \sqrt{3})$
- $(\sqrt{3}, 1+\sqrt{3})$
- $(2\sqrt{3}, \sqrt{3})$
- $(1+\sqrt{3}, \sqrt{3})$
- $(\sqrt{3}, 2\sqrt{3})$

The answer is D. $(1+\sqrt{3}, \sqrt{3})$.

Try the question online and watch the video explanation: http://gre.magoosh.com/questions/819

Formulas to use: Probability

Events A and B are independent.

The probability that events A and B both occur is 0.6

Column A	Column B
The probability that event A occurs	0.3

- The quantity in Column A is greater
- The quantity in Column B is greater
- The two quantities are equal
- The relationship cannot be determined from the information given

The answer is A. The quantity in Column A is greater.

Try the question online and watch the video explanation: http://gre.magoosh.com/questions/229