36700 – Probability and Mathematical Statistics

Spring 2019

Homework 2

Due Friday, Feb 1st at 12:40 PM

All homework assignments shall be uploaded using Gradescope through the Canvas portal). Late submissions are not allowed.

1. Let $X \sim \text{Poi}(\lambda_1)$, $Y \sim \text{Poi}(\lambda_2)$. If X and Y are independent, what is the distribution of $X+Y$? Prove your claim.

2. Let (X,Y) be uniformly distributed on the unit disk: $\{(x,y) : x^2+y^2 \leq 1\}$. Let $R = \sqrt{X^2+Y^2}$. Find the CDF, pdf, and expected value of R.

3. Suppose $F(\cdot)$ is a continuous CDF.

   (a) Let $U \sim U(0,1)$ and $Y = F^{-1}(U)$. Find the CDF of Y.

   (b) Let X be a random variable with CDF F and $Z = F(X)$, find the CDF of Z.

   Can you extend the results to arbitrary CDFs (not necessarily continuous)?

4. Prove that $V(X) = E\left[V(X|Y)\right] + V\left[E(X|Y)\right]$.

5. Let $g(\cdot)$ be a convex function, X a random variable. Assume $E(g(X))$ and $E(X)$ exist.

   (a) Show that for all any real number x_0 there exist real numbers a, b (depend on x) such that $g(x_0) = ax_0 + b$ and $g(x) \geq ax + b$ for all real numbers x.

   (b) Prove Jensen's inequality by applying the previous part to a particular choice of x_0.

6. Prove that for all function $g(\cdot)$ and all random variables X, Y

   $$E\left[g(X)Y|X\right] = g(X)E(Y|X)\,.$$

   In other words, functions of X can be treated as "constants" when taking conditional expectation given X.

Optional problem. (Optional problems will not be graded, and require no submission. You work on it for fun. You are welcome to share your thoughts on Piazza.)

- Let X, Y be independent $N(0,1)$ random variables. Show that X/Y has a Cauchy distribution.