36700 – Probability and Mathematical Statistics

Spring 2019

Homework 6

Due Friday, March 22nd at 12:40 PM

Notes

- All homework assignments shall be uploaded using Gradescope through the Canvas portal). Late submissions are not allowed.
- This assignment contains simulation problems. You are welcome to use your favorite programming language, such as Matlab, R, Python, Julia, c++ etc. Make sure
    1. Your code is submitted as part of your homework submission. Please print your code from a text file and save as pdf. Scanned paper with code written on it will not be accepted!
    2. Your code is self-contained, so that all functions are defined and required packages are loaded.
    3. Your code is as readable as possible, by following the good practices of writing codes.
    4. Your simulation results can be reproduced by hitting a single button. This usually requires to store the random number seeds used in your simulation.
- If you have never run numerical simulations, you can find a simple R tutorial on Canvas under the hw6 folder. If you need further help, please let the instructor or the TAs know.

Q1. Let $X_1, ...X_n \overset{iid}{\sim} \text{Bern}(p)$.

(a) Find the score function $\dot{\ell}(X_i; p)$ and the Fisher information $I_1(p)$.

(b) Find the MLE of p and its asymptotic distribution.

Q2. In the same context as in Q1, consider two-sided hypothesis test problem

$$H_0: \ p = 1/2, \quad \text{v.s.} \quad H_1 : p \neq 1/2\,,$$

using the test statistic given in Lecture Notes 12[1] and set $\alpha = 0.05$.

(a) Numerically approximate the power of the test when $n = 100$ and $p = 0.3, 0.4, 0.5, 0.6, 0.7$. [Hint: For each value of p you can generate a data set with $n = 100$, and conduct the hypothesis test. Repeat this many times (say, 1000 times) and record the proportion of rejection. The proportion of rejection can be viewed as an approximation of the power function at p (why?). Remember to store your random number seed so all the results can be reproduced exactly.]

[1] That is: $T = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}}$.

(b) Repeat the above simulation using the test developed for the one sided alternative $H_1 : p > 1/2$.

Q3. In the same context as in Q1, Q2, instead of using the test statistic given in the lecture notes (also given in the footnote on this page), one can also use the following

$$T_n' = \frac{\hat{p} - p_0}{\sqrt{\hat{p}(1-\hat{p})/n}} .$$

(a) Prove that under H_0, $T_n' \rightsquigarrow N(0,1)$.

(b) Repeat parts (a), (b) of Q2 for the test using T_n'.

(c) Do you see any benefits using T_n'? Briefly explain your answer.

Q4. In Q2, Q3, if you record the p-values of each test and each simulated data set, do you see any difference between the two test statistics? Explain your answer.

Q5. Let $X_1, ..., X_n$ be iid $N(\mu, \sigma^2)$ random variables. Consider

$$H_0 : \mu = \mu_0, \quad H_1 : \mu = \mu_1 .$$

(a) Assuming σ^2 is known, find the rejection region of the Neyman-Pearson test with Type I error controlled at level $\alpha \in (0,1)$.

(b) Now assume σ^2 is unknown. Consider

$$H_0 : \mu = \mu_0, \quad H_1 : \mu \neq \mu_0 .$$

Define the likelihood ratio for composite null hypotheses Θ_0 as

$$\Lambda(\vec{X}) = \frac{\sup_{\theta \in \Theta_0} L(\theta; \vec{X})}{\sup_{\theta \in \Theta} L(\theta; \vec{X})}$$

Note that in the denominator the supremum is taken over the entire parameter space Θ, not Θ_1!

Use the likelihood ratio principle to find a test statistic for the two-sided alternative. When n is large, how would you set the threshold in order to control type I error at level α?

Optional problem. Can you explain why the χ^2 test statistic for the multinomial distribution follows a χ^2_{k-1} distribution?