36700 – Probability and Mathematical Statistics

Spring 2019

Homework 5

Due Friday, March 1st at 12:40 PM

All homework assignments shall be uploaded using Gradescope through the Canvas portal). Late submissions are not allowed.

1. Let $X_1, ...X_n \overset{iid}{\sim} U(\theta, 1)$, where $\theta \in (\infty, 1)$ is an unknown parameter. In the last homework assignment we found the MLE of θ. Let $\hat{\theta}_n$ be the MLE. Is $\hat{\theta}_n$ asymptotically normal? If yes, find the asymptotic mean and variance. Otherwise, find a sequence r_n and a_n such that $r_n(\hat{\theta}_n - a_n)$ converges in distribution to a non-degenerate (not point mass) distribution.

2. Let X be a random variable (potentially multi-dimensional) with density function $f(\theta; x)$ where $\theta \in \mathbb{R}^1$ is a parameter. Assume that $L(\theta; x) = f(\theta; x)$ is a smooth function of θ (smooth enough such that all the continuity and differentiability involved in the score function and Fisher information are satisfied). Let $T(X) \in \mathbb{R}^1$ be a function of X. Let $\phi(\theta) = \mathbb{E}_\theta T(X)$. Assume that $\phi(\theta)$ is differentiable and $\mathrm{Var}_\theta(T(X)) < \infty$ for all θ. Prove that
$$\mathbb{E}_\theta \left[\dot{\ell}(\theta; X) T(X) \right] = \phi'(\theta)\,.$$

3. In the same context as the previous problem, assume in addition that $\mathbb{E}_\theta(T(X)) = \theta$ (in other words, $T(X)$ is an unbiased estimator of θ). Prove the **Cramer-Rao Lower Bound**
$$\mathrm{Var}_\theta(T(X)) \geq \frac{1}{\mathrm{Var}_\theta(\dot{\ell}(\theta; X))} = \frac{1}{I(\theta)}\,.$$

4. (This problem is worth 10 points) Let $X_1, ..., X_n$ be iid samples from $N(\mu, \sigma^2)$. Now (μ, σ^2) is a two-dimensional parameter. (It is indeed easier to view σ^2, not σ, as the parameter). Use notation $\theta = (\theta_1, \theta_2) = (\mu, \sigma^2)$.

   (a) Derive $\hat{\theta}_n$, the MLE of $\theta = (\theta_1, \theta_2)$.

   (b) Now the score function is a two-dimensional vector-valued function:
$$\dot{\ell}(\theta; X_i) = \nabla_\theta \ell(\theta; X_i) = \begin{bmatrix} \frac{\partial \log f(\theta; X_i)}{\partial \theta_1} \\ \frac{\partial \log f(\theta; X_i)}{\partial \theta_2} \end{bmatrix}.$$
   In this case the Fisher information becomes a matrix
$$I_1(\theta) = \mathbb{E}_\theta \left[(\nabla_\theta \ell)(\nabla_\theta \ell)^T \right] = -\mathbb{E}_\theta \nabla^2_\theta \ell\,,$$
   where $\nabla^2_\theta \ell$ is the Hessian matrix of $\ell(\theta; X_i)$, viewed as a function of θ.
   Find $I_1(\theta)$ for the normal distribution.

(c) Find the limiting distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$. You can directly use asymptotic normality of MLE.

(d) The quantity $\eta = \frac{\mu}{\sigma}$ is called the **coefficient of variation**. Find $\hat{\eta}_n$, the MLE of η, and the limiting distribution of $\sqrt{n}(\hat{\eta}_n - \eta)$.

5. Let $X_1, ..., X_n$ be iid samples from a common distribution. Let $\mu = \mathbb{E}X_1$ and $\sigma^2 = \mathrm{Var}(X_1)$, both finite. The MoM estimates of μ and σ^2 are

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} X_i\,, \qquad \hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu}_n)^2\,.$$

(a) Use multivariate CLT and results in homework 3 (e.g., continuous mapping and/or Slutsky's theorem) to derive the limiting distribution of

$$\sqrt{n}\begin{bmatrix} \hat{\mu}_n - \mu \\ \hat{\sigma}_n^2 - \sigma^2 \end{bmatrix}.$$

Do you need any additional assumptions for this result to hold?

(b) Compare the result in part (a) to part (c) of the previous problem.

(c) Find the limiting distribution of

$$\sqrt{n}\left(\frac{\hat{\mu}_n}{\hat{\sigma}_n} - \frac{\mu}{\sigma}\right),$$

and compare with the previous problem.

Optional problem. (This problem is not conceptually hard. But the derivation requires some careful book keeping.) Let $(X_1, Y_1), ..., (X_n, Y_n)$ be iid pairs from a joint distribution on $\mathbb{R}^2$. Assume that $\mathrm{Var}(X_1)$ and $\mathrm{Var}(Y_1)$ are both finite. Find an estimate of the correlation coefficient and prove asymptotic normality.