Weak Supervision Nuggets

- Indep case $\longrightarrow$ Simple estimation trick
- Correlations $\longrightarrow$ Inverse Covariance & Graph Structure.

Given: $x^{(1)} \ldots x^{(n)} \in \mathbb{R}^d$

$\lambda_1 \cdots \lambda_m \; : \; \lambda_i : \mathbb{R}^d \to \{-1, 1\} \cup \{\text{abstain}\}$

find $P(y \mid \bar{\lambda}, x^{(i)})$ $\qquad y \in \{-1, 1\}$

Idea: λ_i is a **noisy** function (inaccurate, incomplete)

λ_1 : "name in dictionary"

λ_2 : "upper case word"

... Programmatic labels ...

Model 0: No abstains, Independent **errors** (clearly correlated)

Each labeler has a hidden accuracy.

with prob p_i $\quad \lambda_j(x^{(i)}) = y^{(i)}$ $\qquad$ " λ_j is right "

$1-p_i$ $\quad \lambda_j(x^{(i)}) = -y^{(i)}$ $\qquad$ " λ_j is wrong "

Sadly, we don't see y, but we do see $\lambda_j(x^{(i)})$

λ_i, λ_j's errors independent and symmetric in the following sense

$$P(\lambda_j(x^{(i)}) = 1 \mid Y^{(i)} = 1) = P(\lambda_j(x^{(i)}) = -1 \mid Y^{(i)} = -1) = P(\lambda_j(x^{(i)}) Y = 1)$$

① $\mathbb{E}[\lambda_i Y]$ $\qquad$ observe if λ_i & Y agree value is 1

λ_i & Y, disagree value is -1

$$\therefore \; = p_i \, 1 + (1-p_i)(-1)$$

$$= 2p_i - 1 \triangleq q_i \quad \text{(define } q_i \text{ this way)} \qquad q_i \in [-1, 1]$$

② $\mathbb{E}[\lambda_i \lambda_j] = 1$ if $i=j$

$$\mathbb{E}[\lambda_i \lambda_j \mid Y=1] = P_1P_2 \cdot 1 + (1-P_1)(1-P_2)\cdot 1 \quad \text{"Agree"}$$

$$(1-P_1)P_2\cdot(-1) + P_1(1-P_2)(-1) \quad \text{"disagree"}$$

$$= a_1 \cdot a_2$$

NOTE we didn't use $\mid Y=1$, same true for $\mid Y=-1$

$$\mathbb{E}[\lambda_i\lambda_j] = \sum_{b\in\{-1,1\}} \mathbb{E}[\lambda_i\lambda_j \mid Y=b]\, P(Y=b) = a_1a_2 \underbrace{\sum_b P(Y=b)}_{1}$$

didn't need to know $P(Y=b)$.

We form a matrix $M \in \mathbb{R}^{m\times m}$ $\quad M_{ij} = \mathbb{E}[\lambda_i\lambda_j]$

<u>NB</u>: M can be estimated — unlike Y!

"Agreements and disagreements" $\Rightarrow$ <u>Key idea</u> Don't need to see y

<u>Simple Algorithm</u>: for any i,j,k distinct, $M_{ij}, M_{jk}, M_{ik} \neq 0$

$$\frac{M_{ij}M_{jk}}{M_{ik}} = \frac{a_i a_j^2 a_k}{a_j a_k} = a_j^2$$

So we can solve upto <u>sign</u> of a_i.

<u>NOTE</u>: If we knew $\text{sign}(a_1) = s$

then $M_{1k} = a_1 a_k$

$\text{sign}(M_{1k})\,\text{sign}(a_1) = \text{sign}(a_k)$ $\therefore$ can solve for all signs from one.

So a & $-a$ are solutions...

Assume $\sum_{i=1}^{n} a_i > 0$, breaks symmetry "good on average".

- What if $M_{ij} = 0$? $a_i = 0$ or $a_j = 0$.

this means $a_i = 0 \Rightarrow 2P_i - 1 = 0 \Rightarrow P_i = \frac{1}{2} \Rightarrow$ random noise!

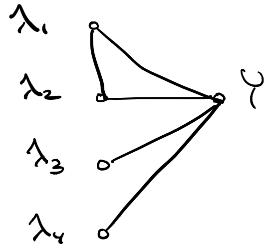
have every a_i bounded away from random noise

(CAN HANDLE w/ FANCIER tricks).

THEORY SAYS " let $\delta = \min_{j=1..m} |p_j - \frac{1}{2}|$, NEED SAMPLES PROPORTIONAL TO $\frac{1}{\delta^2}$ " (lot of work)

RECAP: Symmetry, simple ALGEBRAIC learns (so called coE GD)

WHAT if LABELERS ARE CORRELATED?

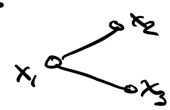


Key Concept: Probability distribution on Graphs.

$$E[\lambda_i \lambda_j | Y] = E[\lambda_i | Y] \, E[\lambda_j | Y]$$

if $(i,j) \notin$ Edge above.

Nugget Structure of INVERSE COVARIANCE for GAUSSIANS



$x_1 \sim N(0,1)$

$x_2 \sim N(x_1, 1)$ $\quad \epsilon_2 \sim N(0,1) \quad x_2 = x_1 + \epsilon_2$

$x_3 \sim N(x_2, 1)$ $\quad \epsilon_3 \sim N(0,1) \quad x_3 = x_1 + \epsilon_3$

1. $E[x_1] = 0 \qquad E[x_2] = \cancel{E[x_1]}^{\,0} + \cancel{E[\epsilon_2]}^{\,0} = 0$

$E[x_3] = 0.$

$E[x_1^2] = 1 \qquad E[x_2^2] = E[(x_1 + \epsilon_2)^2] = E[x_1^2] + 2\cancel{E[x_1 \epsilon_2]}^{\,0}$

$+ E[\epsilon_2^2]$

$= 2$

$E[x_1 x_2] = E[x_1^2 + x_1 \epsilon_1] = 1$

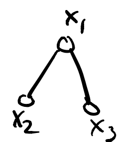
$E[x_2 x_3] = E[(x_1 + \epsilon_1)(x_1 + \epsilon_3)] = 1$

$$\Sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

NO CLEAR STRUCTURE?

$$\Sigma^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

No Edge



WE SAY A <u>probability distribution</u> $p : \mathbb{R}^d \to [0,1]$ agrees or factorizes wrt a graph $G = (V, E)$ if

$$p(x) = c \prod_{e=(v_i,v_j)\in E} p_e(x_i, x_j) \cdot \prod_{v \in V} p_v(x_i)$$

for some functions!

Normalization Constant

Now, let's look @ Gaussians over a graph.

$$\log \exp\{x^T \Sigma^{-1} x\} = \log \prod p_e(x_i,x_j) \prod_{v\in V} p_v(x_i) \quad \text{(ignore c)}$$

$$x^T \Sigma^{-1} x = \sum_{e \in E} \log p_e(x_i, x_j) + \sum_{v \in V} \log p_v(x_i)$$

let $A = \Sigma^{-1}$ for easy notation

$$\sum_{i,j} A_{ij} x_i x_j =$$

for $i,j \in E$ $\quad \nabla_{x_i x_j} = (A_{ij} + A_{ji}) = 0$ if $(i,j) \notin E$.

But $\Sigma^{-1} = A$ is symmetric, so $A_{ij} = 0$.

Thus, if a Gaussian factors on a graph, entries of inverse are 0!

More complex theory for discrete R.V.s — Loh & Wainwright 2014; Ratner et al 2018

Back to our Problem

Form "covariance matrix"

$$
\begin{array}{c|ccc|c}
 & \lambda_1 & \lambda_2 & \lambda_3 & Y \\
\hline
\lambda_1 & & & & \\
\lambda_2 & & \Sigma_O & & \\
\lambda_3 & & & & \\
\hline
Y & & & & 1
\end{array}
$$

We "see" Σ_O but not all of Sigma $\mathbb{E}[\lambda_i Y]$ is unobserved.

let $O = \{1, 2, 3\}$ ⌐ visible terms

$$(\Sigma^{-1})_O = (\Sigma_O - uu^T)^{-1}$$

$(\Sigma^{-1})_O$ ⟶ inverse on those entries

uu^T ⟶ some rank 1 vector.

let $B = \Sigma_O$

$$(B - uu^T)^{-1} = B^{-1} + \frac{B^{-1} u u^T B^{-T}}{1 - u^T B^{-1} u} \qquad z = \frac{B^{-1}u}{\sqrt{1 - u^T B^{-1} u}}$$

$$\text{So} \quad (\Sigma^{-1})_O = \Sigma_O^{-1} + z z^T$$

<u>Now</u>, if $(i,j) \notin E$ then we $(\Sigma^{-1})_{ij} = B_{ij} = 0$.

Hence $(\Sigma_O^{-1})_{ij} = -z_i z_j$

$$(B_{ij})^2 = z_i^2 z_j^2 \mapsto \log B_{ij}^2 = \log z_i^2 + \log z_j^2$$

This is a linear system in z_i^2 & z_j^2, and we can solve (if <u>enough</u> pairwise indep)

IN the notes:

- Higher rank versions (handle more correlations)
- How to *learn* graph structure
- How to handle sampling error (modern ML, lower bounds)

Recap:

- Weak Supervision formal theory
- Nuggets about Graphs & Prob. distributions (take graphical models!)
- "Method of moments" style Algorithms