Lasso Regression:

Regularization for feature selection

CS229: Machine Learning

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Slides include content developed by and co-developed with Emily Fox

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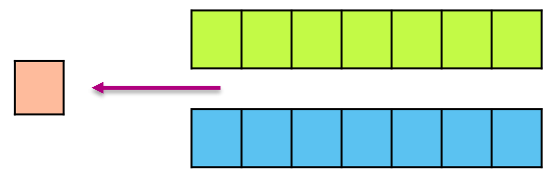
Feature selection task

Why might you want to perform feature selection?

Efficiency:

- If size($\mathbf{w}$) = 100B, each prediction is expensive
- If $\hat{\mathbf{w}}$ sparse , computation only depends on # of non-zeros

$$\hat{y}_i = \sum_{\hat{w}_j \neq 0} \hat{w}_j \, h_j(x_i)$$



Interpretability:

- Which features are relevant for prediction?

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Sparsity: Housing application

- Lot size
- Single Family
- Year built
- Last sold price
- Last sale price/sqft
- Finished sqft
- Unfinished sqft
- Finished basement sqft
- # floors
- Flooring types
- Parking type
- Parking amount
- Cooling
- Heating
- Exterior materials
- Roof type
- Structure style
- Dishwasher
- Garbage disposal
- Microwave
- Range / Oven
- Refrigerator
- Washer
- Dryer
- Laundry location
- Heating type
- Jetted Tub
- Deck
- Fenced Yard
- Lawn
- Garden
- Sprinkler System

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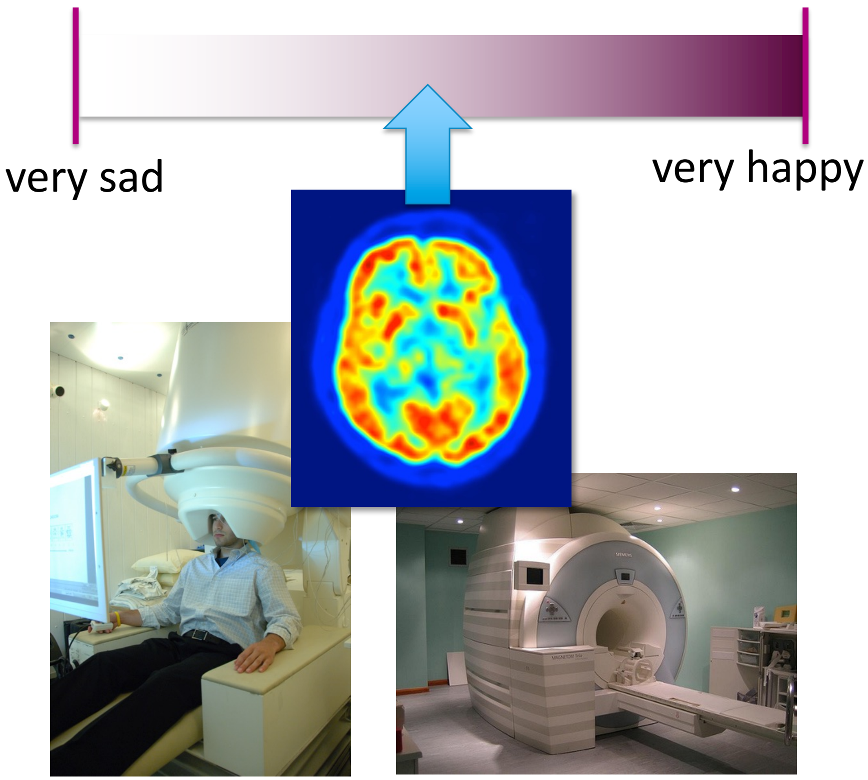
Sparsity: Reading your mind



Activity in which brain regions can predict happiness?

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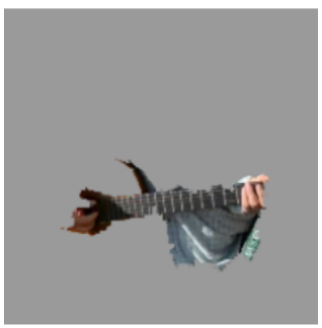
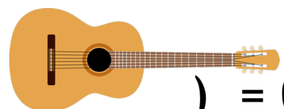
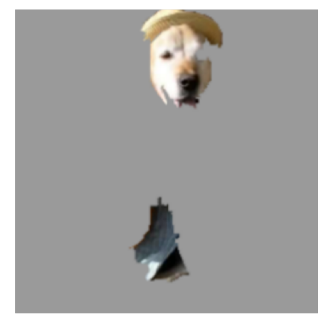
Explaining Predictions

P() = 0.32



P() = 0.24

P() = 0.21



“Why should I trust you?”: Explaining the Predictions of Any Classifier. Ribeiro, Singh & G. KDD 16

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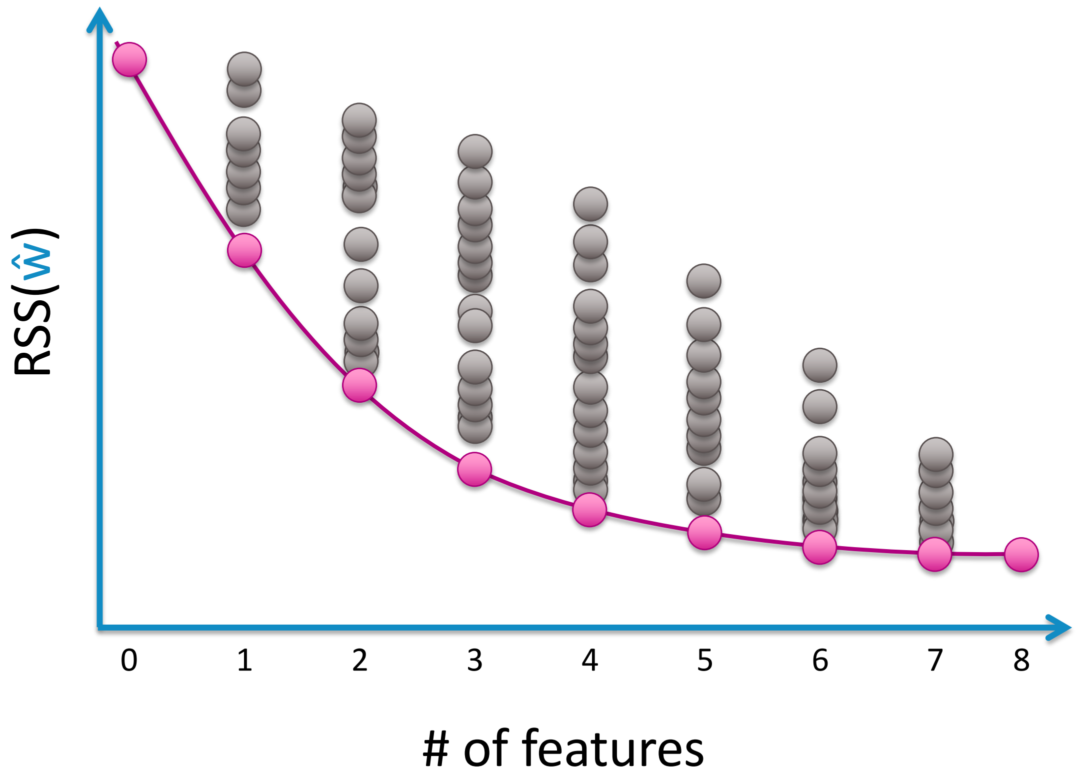
Option 1: All subsets or greedy variants

Find best model of for each size



- \# bedrooms
- \# bathrooms
- sq.ft. living
- sq.ft. lot
- floors
- year built
- year renovated
- waterfront

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Complexity of "all subsets"

$y_i = \varepsilon_i$	[0 0 0 … 0 0 0]	
$y_i = w_0 h_0(x_i) + \varepsilon_i$	[1 0 0 … 0 0 0]	
$y_i = w_1 h_1(x_i) + \varepsilon_i$	[0 1 0 … 0 0 0]	
$\vdots$	$\vdots$	2^D
$y_i = w_0 h_0(x_i) + w_1 h_1(x_i) + \varepsilon_i$	[1 1 0 … 0 0 0]	
$\vdots$	$\vdots$	
$y_i = w_0 h_0(x_i) + w_1 h_1(x_i) + \ldots + w_D h_D(x_i) + \varepsilon_i$	[1 1 1 … 1 1 1]	

$2^8 = 256$

$2^{30} = 1{,}073{,}741{,}824$

$2^{1000} = 1.071509 \times 10^{301}$

2^{100B} = HUGE!!!!!!

Typically, computationally infeasible

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Greedy algorithms

Forward stepwise:
Starting from simple model and iteratively add features most useful to fit

Backward stepwise:
Start with full model and iteratively remove features least useful to fit

Combining forward and backward steps:
In forward algorithm, insert steps to remove features no longer as important

Lots of other variants, too.

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Option 2: Regularize

Ridge regression: L_2 regularized regression

Total cost =

$$\underbrace{\text{measure of fit}}_{\text{RSS}(\mathbf{w})} + \lambda \underbrace{\text{measure of magnitude of coefficients}}_{||\mathbf{w}||_2^2 = w_0^2 + \ldots + w_D^2}$$

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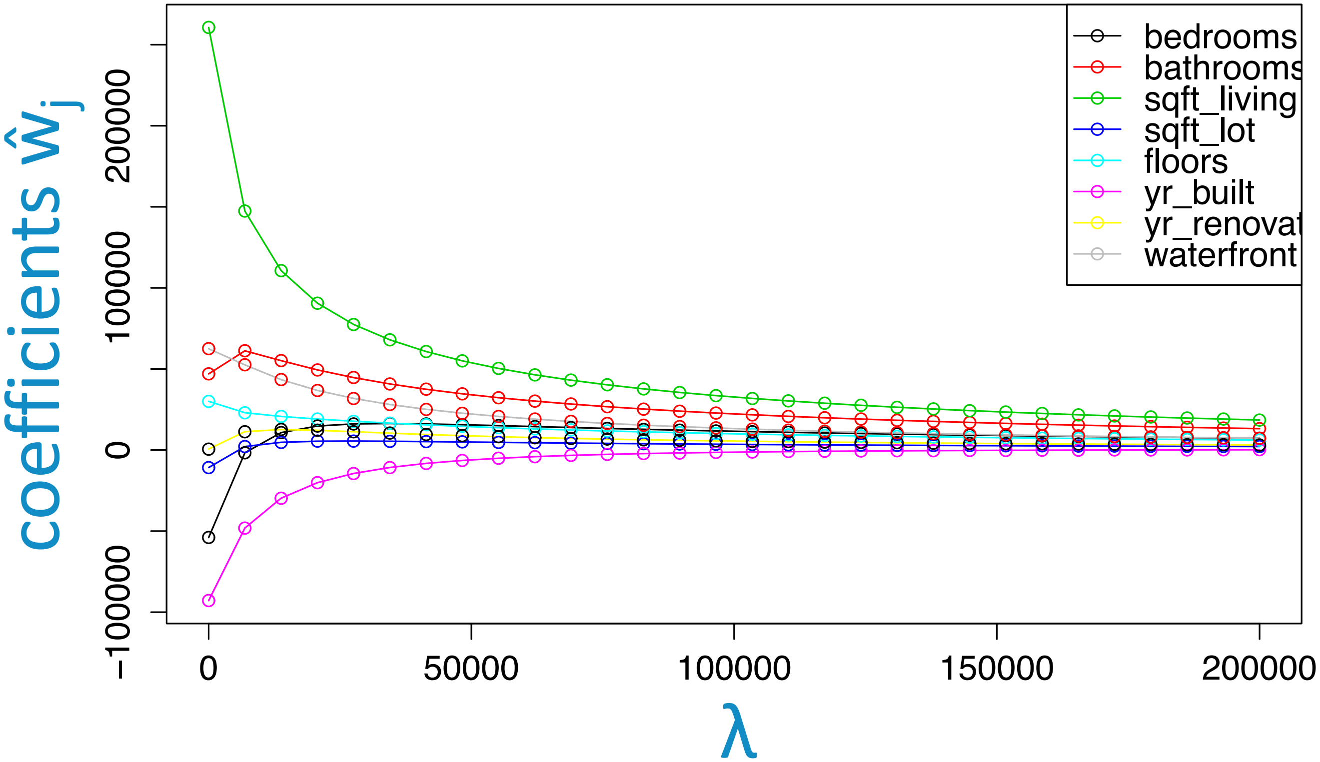
Coefficient path – ridge



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Using regularization for feature selection

Instead of searching over a discrete set of solutions, can we use regularization?

- Start with full model (all possible features)
- "Shrink" some coefficients *exactly* to 0
  - i.e., knock out certain features
- Non-zero coefficients indicate "selected" features

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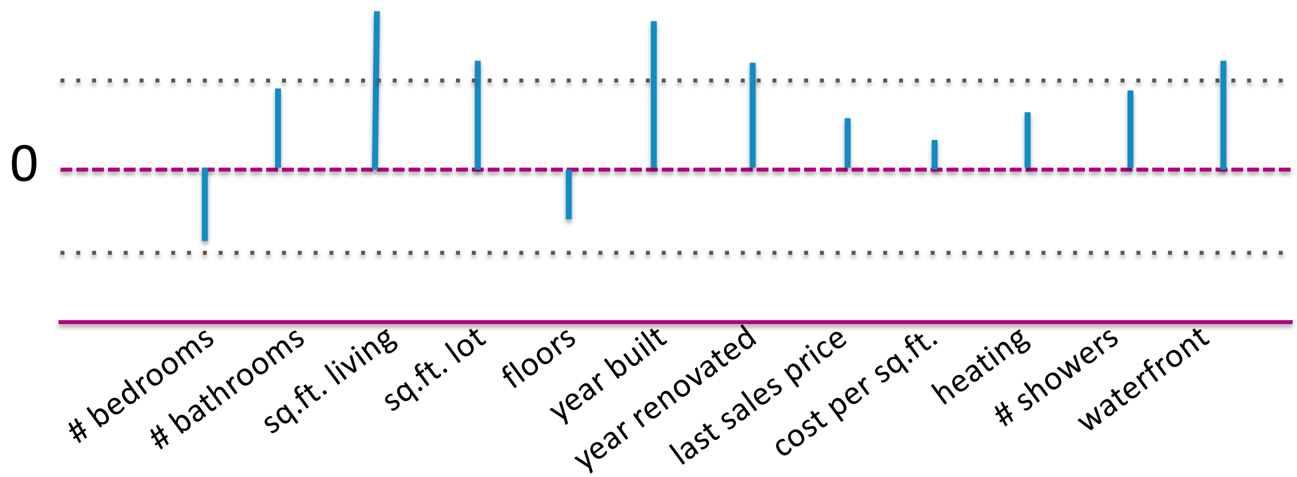
Thresholding ridge coefficients?

Why don't we just set small ridge coefficients to 0?



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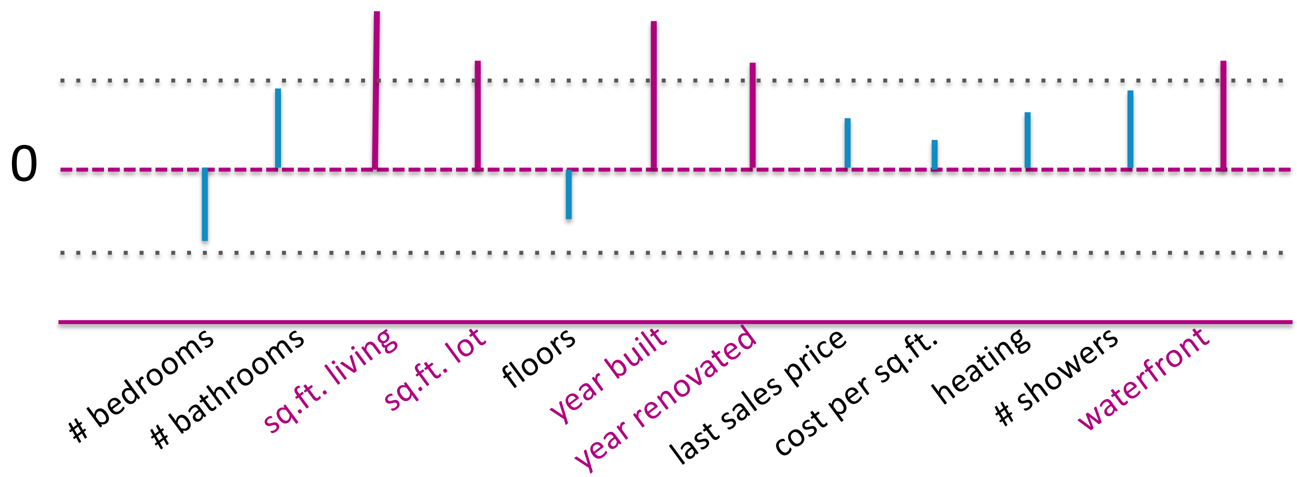
Thresholding ridge coefficients?

Selected features for a given threshold value



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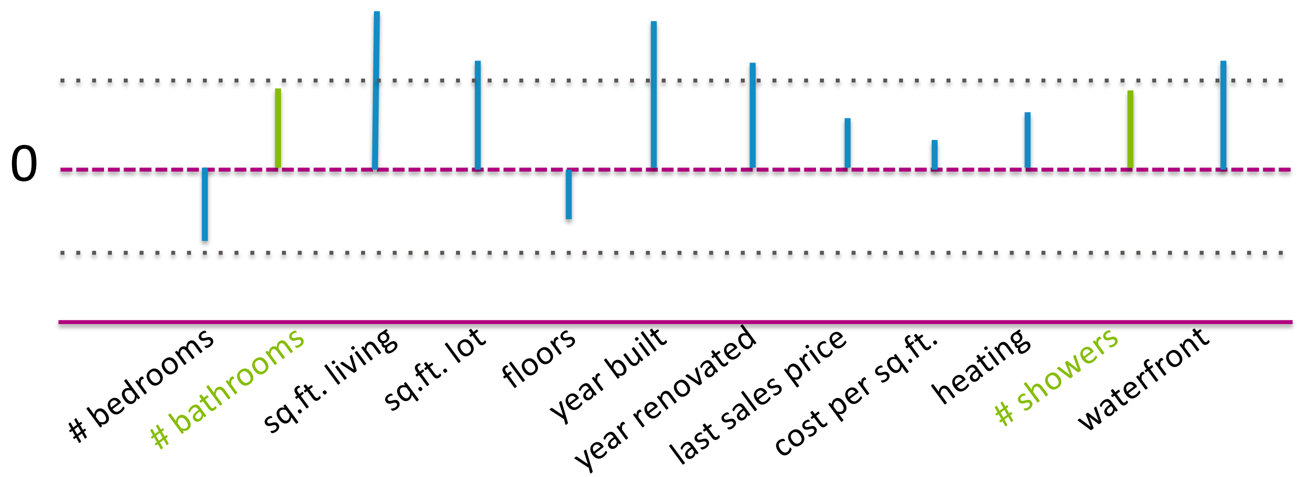
Thresholding ridge coefficients?

Let's look at two related features…



Nothing measuring bathrooms was included!

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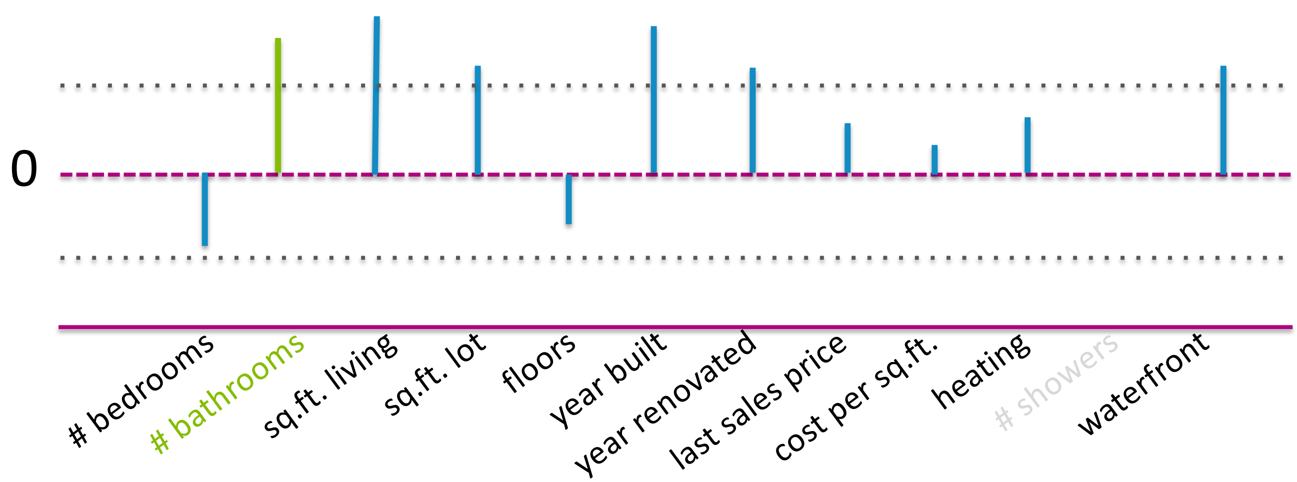
Thresholding ridge coefficients?

If only one of the features had been included…



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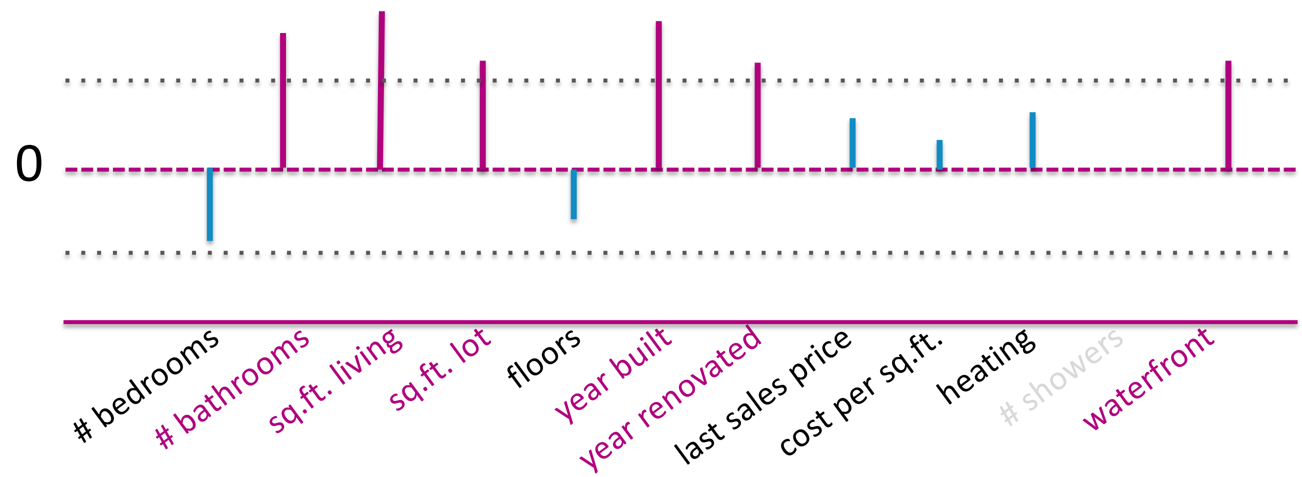
Thresholding ridge coefficients?

Would have included bathrooms in selected model



Can regularization lead directly to sparsity?

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Try this cost instead of ridge...

Total cost =

$$\underbrace{\text{measure of fit}}_{\text{RSS}(\mathbf{w})} + \lambda \underbrace{\text{measure of magnitude of coefficients}}_{||\mathbf{w}||_1 = |w_0| + \ldots + |w_D|}$$

Leads to sparse solutions!

Lasso regression
(a.k.a. L_1 regularized regression)

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Lasso regression: L_1 regularized regression

Just like ridge regression, solution is governed by a continuous parameter λ

$$\mathrm{RSS}(w) + \lambda ||w||_1$$

tuning parameter = balance of fit and sparsity

If λ=0:

If λ=∞:

If λ in between:

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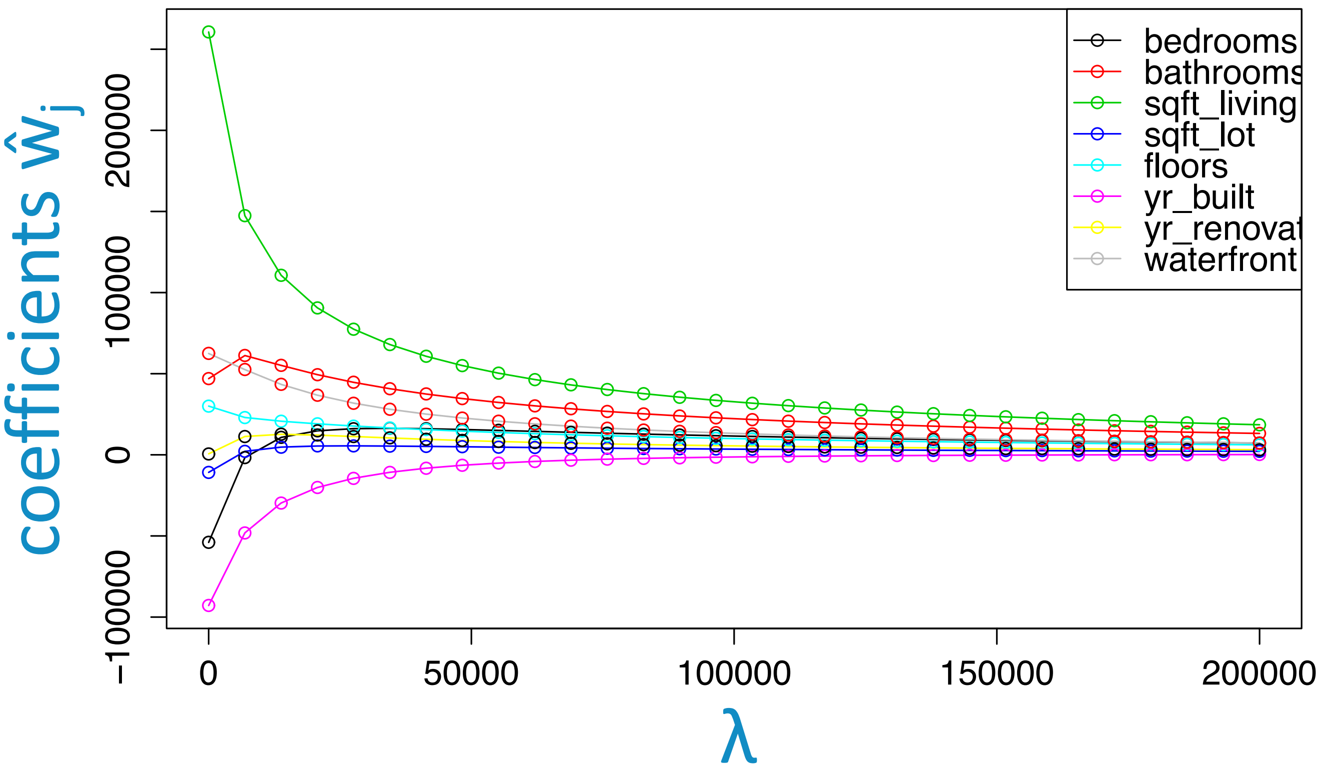
Coefficient path – ridge



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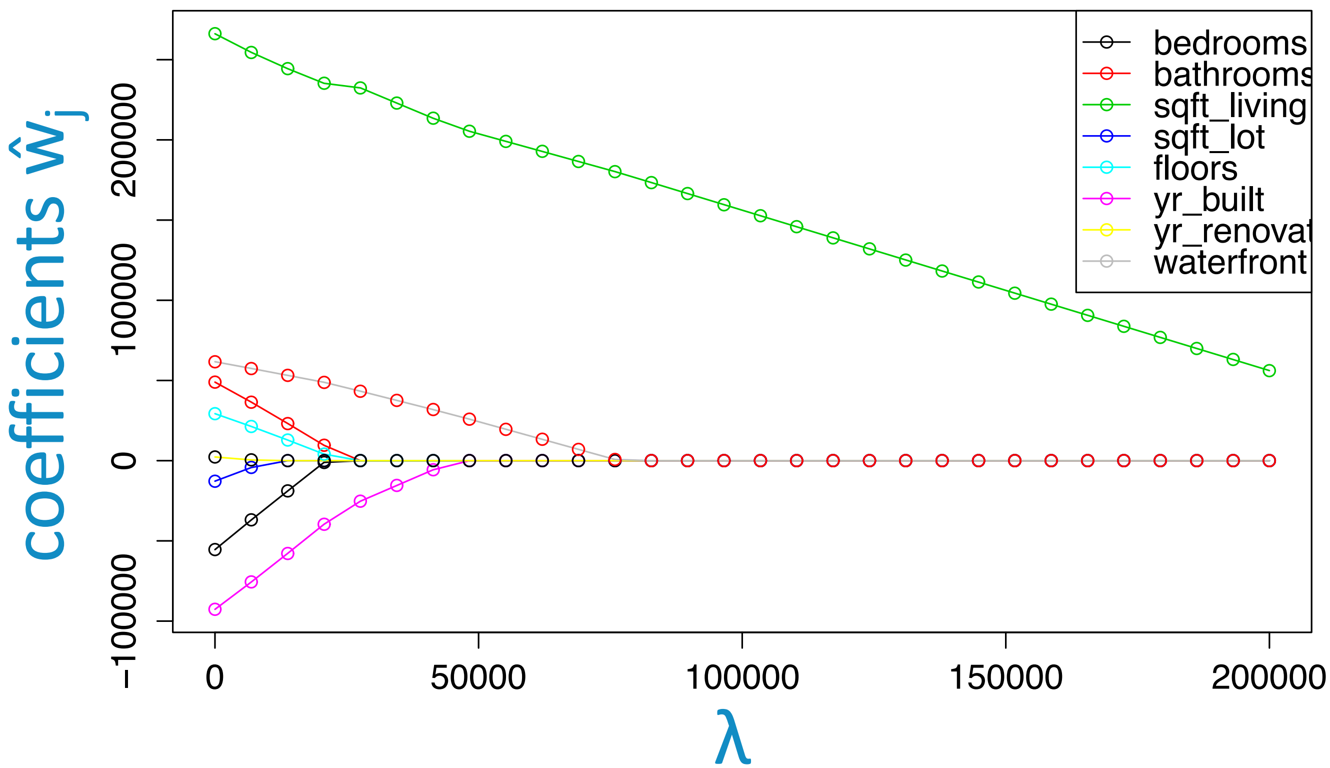
Coefficient path – lasso



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Practical concerns with lasso

Debiasing lasso

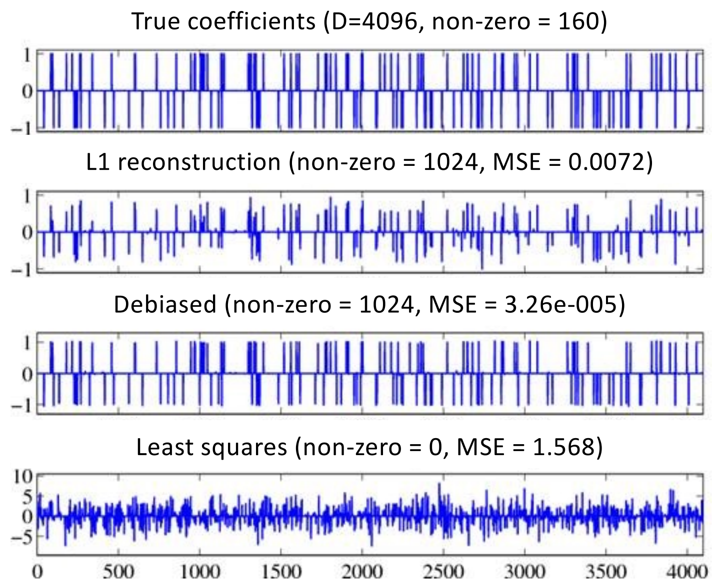
Lasso shrinks coefficients relative to LS solution

→ more bias, less variance

Can reduce bias as follows:

1. Run lasso to select features
2. Run least squares regression with only selected features

“Relevant” features no longer shrunk relative to LS fit of same reduced model



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Issues with standard lasso objective

1. With group of highly correlated features, lasso tends to select amongst them arbitrarily
   - Often prefer to select all together

2. Often, empirically ridge has better predictive performance than lasso, but lasso leads to sparser solution

Elastic net aims to address these issues

- hybrid between lasso and ridge regression
- uses L_1 and L_2 penalties

See Zou & Hastie '05 for further discussion

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Summary for feature selection and lasso regression

Impact of feature selection and lasso

Lasso has changed machine learning,
statistics, & electrical engineering

But, for feature selection in general, be careful about interpreting selected features

- selection only considers features included
- sensitive to correlations between features
- result depends on algorithm used
- there are theoretical guarantees for lasso under certain conditions

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What you can do now…

- Describe "all subsets" and greedy variants for feature selection
- Analyze computational costs of these algorithms
- Formulate lasso objective
- Describe what happens to estimated lasso coefficients as tuning parameter λ is varied
- Interpret lasso coefficient path plot
- Contrast ridge and lasso regression
- Implement K-fold cross validation to select lasso tuning parameter λ

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