36700 – Probability and Mathematical Statistics

Spring 2019

Homework 4

Due Friday, Feb 22nd at 12:40 PM

All homework assignments shall be uploaded using Gradescope through the Canvas portal). Late submissions are not allowed.

1. Let $X_1, ...X_n \overset{iid}{\sim} U(a, b)$, where a, b are unknown parameters. Find the method of moment estimate of (a, b). (To unify the notation, let $m_1 = n^{-1}\sum_{i=1}^n X_i$, $m_2 = n^{-1}\sum_{i=1}^n X_i^2$. Your estimate shall be functions of m_1, m_2.)

2. Let $X_1, ...X_n \overset{iid}{\sim} U(a, b)$, where a, b are unknown parameters. Find the maximum likelihood estimate of (a, b).

3. A exponential distribution, denoted by $\text{Exp}(\beta)$, has CDF $F_\beta(x) = (1 - e^{-x/\beta})\mathbf{1}(x \geq 0)$. Let $X_1, ..., X_n$ be iid samples from F_β. Find the MoM and MLE estimates of β.

4. Let $f(\cdot)$, $g(\cdot)$ be two pdf's. Let X be a random variable with pdf $f(\cdot)$. Prove that $\mathbb{E}\log\left(\frac{f(X)}{g(X)}\right) \geq 0$. [Hint: use Jensen's inequality. This problem essentially says that $\mathbb{E}\log g(X)$ is uniquely maximized by setting $g = f$.]

5. Let $f_\theta(\cdot)$ be a pdf parameterized by θ. In particular, two distinct values θ_0 and θ_1 lead to two different pdf's f_{θ_0} and f_{θ_1}. Let $X_1, ..., X_n, ...$ be iid sample from density $f_{\theta_0}(\cdot)$. Let $\ell_n(\theta) = n^{-1}\sum_{i=1}^n \log f_\theta(X_i)$. Show that for all θ, there exists a constant $c \geq 0$ (possibly depending on θ and θ_0) such that

$$\ell_n(\theta_0) - \ell_n(\theta) \xrightarrow{P} c\,.$$

   How does this help justifying maximum likelihood estimators?

6. A Gamma distribution, denoted $\text{Gamma}(\alpha, \beta)$, has two parameters $\alpha > 0$, $\beta > 0$. The density function of $\text{Gamma}(\alpha, \beta)$ is

$$f(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}\mathbf{1}(y \geq 0)\,,$$

   where $\Gamma(\alpha)$ can be viewed as a constant to make the density integrate to 1. Now let $X_1, ..., X_n$ be iid samples from $\text{Poi}(\lambda)$, where λ has a prior distribution $\text{Gamma}(\alpha, \beta)$. Find the posterior distribution of λ.