36700 – Probability and Mathematical Statistics

Spring 2019

Homework 3

Due Friday, Feb 8th at 12:40 PM

All homework assignments shall be uploaded using Gradescope through the Canvas portal). Late submissions are not allowed.

1. Let $X \sim \text{Poi}(\lambda)$ and $Y \sim \text{Poi}(\mu)$ be independent. Let n be a positive integer. Find the conditional distribution of X given that $X + Y = n$.

2. Let X be a random variable. Assume $\mathbb{E}\left(e^{\frac{|X|}{c}}\right) \leq 2$ for a constant $c > 0$.

   (a) Show that $\mathbb{E}|X|^k \leq k!c^k$ for all $k \geq 2$.

   (b) Let $X_1, ..., X_n$ be iid copies of X. Assume further that $\mathbb{E}(X) = 0$. Find an upper bound (trivial ones don't count) of

   $$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq t\right)$$

   for $t > 0$.

3. Let X_1, ..., X_n be iid $\text{Ber}(p)$ random variables.

   (a) Find an upper bound of

   $$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - p\right| \geq t\right)$$

   for some arbitrary $t > 0$, using Chebyshev's inequality, Hoeffdings inequality, and Bernstein's inequality, respectively.

   (b) Now suppose p is very close to zero and n is very large. Compare the three upper bounds. Which inequality is the sharpest? Which is the weakest?

4. Assume that $X_n \xrightarrow{P} X$ and let $g(\cdot)$ be a continuous function. Prove that $g(X_n) \xrightarrow{P} g(X)$. (Remark: this result also holds for convergence in distribution. See Question 6(b) below.)

5. Show that if $X_n \rightsquigarrow X$ and $Y_n \xrightarrow{P} c$ (c is a constant), then $X_n + Y_n \rightsquigarrow c + X$ and $X_n Y_n \rightsquigarrow cX$.

6. Let $F_n \rightsquigarrow F$, where F_n and F are strictly increasing continuous CDF's. Let $U \sim U(0,1)$.

(a) Show that $F_n^{-1}(U) \overset{a.s.}{\to} F^{-1}(U)$. (Remark: this result indeed holds for general F_n, F.)

(b) Use the general version of the previous result to prove Question 4 in the case of convergence in distribution: If $X_n \rightsquigarrow X$ and $g(\cdot)$ is continuous, then $g(X_n) \rightsquigarrow g(X)$.

Optional: Prove Question 6 part (a) for general F_n, F, without assuming strict monotonicity or continuity.