**Details on $R_o$ you may or may not be interested in**

Interestingly, one way these epidemics are visualized and modeled is based on susceptibles contacting infected people at some average rate. These susceptibles are quantified as a percentage or fraction of the entire population[1] (which implies that some fraction of the population *isn't* susceptible, think vaccine or herd immunity). So, over time, these susceptible people are catching the disease at some rate. You could put it in pseudo-algebraic terms like this:

$$rate\ of\ change\ of\ susceptibles = contact\ rate\ X\ \ fraction\ X\ infectious$$ [2]

That needs to be refined, since the fraction of the population that are susceptibles will decrease, as they are converted to infectious. That means the rate is negative. So, add a negative sign:

$$susceptibles\ change = -contact\ rate\ X\ \ fraction\ X\ infectious$$

That's a bit cumbersome, so if you follow Leibniz' suggestion, and write a rate of change as the change divided by the corresponding change in time, where the symbol for change (difference) is a *d*:

$$\frac{dS}{dt} = -contact\ rate\ X\ \ fraction\ X\ infectious$$

People have come up with less cumbersome algebraic symbols, like $\beta$ for the contact rate, and a real fraction for the fraction, $S/N$, where $S$ is susceptibles and $N$ is the number of the total population. Likewise the average number of infectious would be well served with the letter I, so $I$.

$$\frac{dS}{dt} = -\beta\frac{S}{N}I$$

That's easier on the eye. It's important to remember all these quantities change over time, so they can be shown as functions of time, which may clutter things up a bit. For example, deaths from an epidemic (not shown here) would change the value $N$ over time.

$$\frac{dS(t)}{dt} = -\beta\frac{S(t)}{N(t)}I(t)$$

Furthermore, the number of infectious changes, too. So, another, separate equation:

$$\frac{dI(t)}{dt} = \beta\frac{S(t)}{N(t)}I(t) - \gamma I(t)$$

Notice the first term on the right side of the equation is just the negative of the change in susceptibles, since we're *adding* the same amount to infectious, that was *subtracting* from susceptibles before. So the change in infectious relates to the change in susceptibles, since members from one group are going over to the other. There is also the infectious period (not to be confused with contact rate). After a certain time, infectious either recover or die (this is kind of grim). So that has to be subtracted from the right side of the equation (the second term). $\gamma$ is a common symbol for the frequency associated with the period of infection. Therefore $\gamma^{-1}$ is the average period of infection. $\gamma$ is the third letter in the Greek alphabet, gamma. $\gamma I(t)$ should be recognized as the rate of infectious by death or recovery, taken away from the overall rate.

Here's where an alternative explanation for $R_o$ shows up. This is the basic reproduction "rate" (although it's not really a rate, more on that later). If you integrate the infectious rate equation above, you do the following:

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[1] Another way to look at this is the ratio of susceptibles to the whole population is the basis for assigning a *probability* of infection of anyone in the whole population, not just susceptibles: contact rate X probability X infectious. The program of compartmentalizing by susceptibles, infectious, and recovered is known as the SIR model.

[2] Contact rate is average number of people a person has contact with, per unit time. Hence contact tracing, that not only tells how many people a certain infected person has had contact with, but hopefully who they were.

$\frac{dI(t)}{dt} = \beta \frac{S(t)}{N(t)} I(t) - \gamma I(t)$ or $dI(t) = \left[ \beta \frac{S(t)}{N(t)} I(t) - \gamma I(t) \right] dt$ in alternative differential form. So, separating the $I$ variable, $\frac{dI(t)}{I(t)} = \beta \frac{S(t)}{N(t)} - \gamma\, dt$. In algebra, you do the same thing to both sides of an equation, so the next step just integrates both sides.

$\int \frac{dI(t)}{I(t)} = \int_t^{t+\tau} \beta \frac{S(x)}{N(x)} - \gamma\, dx$, using $x$ for dummy variable of integration. This adds up the two terms on the right side of the equation from some time $t$ to a little later time at $t + \tau$. The solution to the integral on the right can be visualized on a 3D cartesian coordinate graph with axes $x$, $S(x)$, and $N(x)$ as the volume between the surface $\beta S(x)/N(x) - \gamma$ and the plane defined by the line $S(x) = 0$ and the line $N(x) = 0$, from $x = t$ to $x = t + \tau$, from $S(t)$ to $= S(t + \tau)$, and from $N(t)$ to $= N(t + \tau)$.

$$\int \frac{dI(t)}{I(t)} = \int_t^{t+\tau} \frac{dI(x)}{I(x)} dx = \ln I(t+\tau) - \ln I(t) = \ln \frac{I(t+\tau)}{I(t)}$$

$\ln \frac{I(t+\tau)}{I(t)} = \int_t^{t+\tau} \beta \frac{S(x)}{N(x)} - \gamma\, dx$ And since $\exp\left[ \ln \frac{I(t+\tau)}{I(t)} \right] = \frac{I(t+\tau)}{I(t)}$, the following:

$$\frac{I(t+\tau)}{I(t)} = \exp\left[ \int_t^{t+\tau} \beta \frac{S(x)}{N(x)} - \gamma\, dx \right]$$

Finally, to $R_o$. It is expedient to combine $\beta$ and $\gamma$ into one, which simplifies this expression.[3]

$R_0 \equiv \frac{\beta}{\gamma}$, which makes the above expression $\frac{I(t+\tau)}{I(t)} = \exp\left[ \gamma \int_t^{t+\tau} R_0 \frac{S(x)}{N(x)} - 1\, dx \right]$. The basic reproduction "rate" is a combination of the contact rate and the infection period, in other words.

Such a thing is most likely to be numerically integrated, since it is rare to have an explicit algebraic form for $S(x)$ or $N(x)$ that can be integrated in a closed form. Or, if over the interval from $t$ to $t + \tau$, you have reason to believe $S/N$ is relatively constant, the integral simplifies to $\tau\gamma\,[R_0(S/N) - 1]$. Or

$\frac{I(t+\tau)}{I(t)} = \exp\left[ \tau\gamma \left( R_0 \frac{S}{N} - 1 \right) \right]$ for interval $\tau$. $R_0$ clearly governs growth of this exponential function, and is dimensionless, since it is a ratio of contact rate and infectious frequency with the same units, time$^{-1}$. That also explains why it is a bit of misnomer to call it a rate, when it has no unit dimensions, although it's common to hear it called a rate. The real trick still remains to find out the instantaneous reproduction number or the effective reproduction number.

What if the infectious period is 3 days, the contact period 4 days, and the ratio of susceptibles to the population is 3%, for the period of interest of 1 day? $\beta = ¼$, $\gamma = 1/3$, so $R_o = (1/4)/(1/3) = 0.75$. The growth multiplier is exp[1 X 1/3 X ( ¾ X 3% - 1)], or 0.72. This is negative growth, since it is less than 1. If you increase interval $\tau$ arithmetically, you will see the growth multiplier decrease exponentially.

Cases can be reported over time as a cumulative value, or a daily value. It's two ways of saying the same thing. The cumulative, or the total number of cases up to time $t$, $T(t)$, follows the equation $dT/dt = \beta S/N\, I$.

[3] As one would expect, there is more than one way to define $R_o$.